

UNIVERSAL ENRICHMENTS OF CATEGORIES (jmo Hyeon Tai Jung)

1) Introduction.

In an ordinary caty \mathcal{C} , arrows from X to Y form a set $\mathcal{C}(X,Y)$.

In a (V -)enriched caty \mathcal{C} , arrows from X to Y are an object $\mathcal{C}(X,Y) \in V$, where V is a "nice" monoidal caty.

Eg: $V = \text{Set}, \text{Pos}, \text{Cat}, \text{CMet}, \text{CPO}_\perp, \dots$

Enriched categories are not categories. However, a V -caty \mathcal{C} does have an underlying category \mathcal{C}_0 , with same objects, and $\mathcal{C}_0(X,Y) = \{I \rightarrow \mathcal{C}(X,Y)$
 unit for \otimes in $V\}$

So makes sense to talk of enriching an ordinary caty \mathcal{C} to a V -caty: ie find a V -cat \mathcal{C} s.t $\mathcal{C}_0 \cong \mathcal{C}$.

Eg:

- \mathbb{C} -vs can be enriched over: Set, CMon, Ab, IR-vs, \mathbb{C} -vs
- $[\mathcal{C}^{\text{op}}, \text{Set}] \dashv \vdash \dots \dashv \vdash \text{Set}, ([\mathcal{C}^{\text{op}}, \text{Set}], \times), ([\mathcal{C}^{\text{op}} \times \mathcal{C}, \text{Set}], \circ)$

profunctor comp.

Last example: $\text{Hom}(X,Y) : \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \text{Set} \quad (X, Y \in [\mathcal{C}^{\text{op}}, \text{Set}])$
 $(d, c) \mapsto [x_d, y_c]$

- IR-Alg enriched over IR-coalg (hom's are Sweedler's meaning coalgebras)

- $\text{Mnd}(\text{Set})_{\text{acc}}$ enriched over $\text{Comonad}(\text{Set})_{\text{acc}}$ (Rivas, McDermott, Uustalu).

Q: is there a "most general" enrichment of a given caty \mathcal{C} ?

A: yes!

Thm Let \mathcal{C} be a locally presentable caty. The 2-caty of locally presentable enrichments of \mathcal{C} has a bi-initial object.

"nice and algebraic"

2) THEOREM

k -small lim pres functors

A caty \mathcal{C} is locally presentable if $\mathcal{C} \simeq \text{Lim}_k(\mathbb{T}, \text{Set})$ for some small \mathbb{T} with k -small limits. There's a 2-caty $\underline{\text{Loc Pres}}_{\otimes}$ with:

- objs: locally presentable monoidal closed catys
- maps $(V, \otimes) \rightarrow (W, \otimes)$: (lax) monoidal right adjoints, ie
right adjoint in $\text{MonCat}_{\text{lax}}$.

ie, has a left adjoint which is actually strong monoidal (and not just oplax)

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Fix an (ordinary) locally presentable caty Σ . Define $\text{Enr}_{\Sigma}(V)$, for $V \in \text{MonCat}_{\otimes}$, to be the caty with: ^{caty of enrichments of Σ in V}
• objects, locally pres. V -catys \mathcal{C} +/w i.o. obs iso $\mathcal{C} \cong \Sigma$
• maps $\mathcal{C} \rightarrow \mathcal{C}'$ are i.o.o. V -functors commuting withisos to Σ .

Now we get 2-functor $\text{Enr}_{\Sigma}(-): \underline{\text{Loc Pres}}_{\otimes} \rightarrow \text{Cat}$

which on bcells: $F: V \rightarrow W$ mon. right adjoint induces

$$\begin{aligned}\text{Enr}_{\Sigma}(F): \text{Enr}_{\Sigma}(V) &\longrightarrow \text{Enr}_{\Sigma}(W) \\ \mathcal{C} &\longmapsto F_*(\mathcal{C})\end{aligned}$$

where $F_*(\mathcal{C})$ is the W -caty with same obs as \mathcal{C} , and $(F_*\mathcal{C})(x,y) = F(\mathcal{C}(x,y))$

THM $\text{Enr}_{\mathcal{E}}$ is (bi-)representable.

Proof Let $\mathcal{U}(\mathcal{E})$ be $\text{Cocts}(\mathcal{E}, \mathcal{E})$, monoidal under composition.

This is locally presentable, and monoidal (bi)closed!
[Bird]

Now there is a monoidal action

$$*: \mathcal{U}(\mathcal{E}) \times \mathcal{E} \longrightarrow \mathcal{E} \quad \begin{matrix} \leftarrow \\ \text{pres. colimits in} \end{matrix} \\ (F, x) \longmapsto Fx \quad \begin{matrix} \text{each variable.} \end{matrix}$$

So by adj functor theorem each functor $(-)*E: \mathcal{U}(\mathcal{E}) \rightarrow \mathcal{E}$
has right adj $\underline{\mathcal{E}}(E, -): \mathcal{E} \rightarrow \mathcal{U}(\mathcal{E})$.

So for every $E, E' \in \mathcal{E}$ have $\underline{\mathcal{E}}(E, E') \in \mathcal{U}(\mathcal{E})$ — these are the horns of
an enrichment.

Claim this $\underline{\mathcal{E}}$ is a birepresenting element of $\text{Enr}_{\mathcal{E}}$.

Proof Let \mathcal{C} be a locally presentable \mathcal{V} -enrichment of \mathcal{E} .

Goal: a lax monoidal r. adj $\mathcal{U}(\mathcal{E}) \xrightarrow{F} \mathcal{V}$ st $F_*(\underline{\mathcal{E}}) = \mathcal{C}$.

Well, for each $E \in \mathcal{E}$, have $\mathcal{E}(E, -): \mathcal{E} \rightarrow \mathcal{V}$ which
preserves limits and is accessible, so has a left adjoint $(-)*E: \mathcal{V} \rightarrow \mathcal{E}$.

These give an action $*: \mathcal{V} \times \mathcal{E} \rightarrow \mathcal{E}$ which pres. colimits in
 $\mathcal{U}(\mathcal{E})$ each variable.

That's equiv to $\widehat{*}: \mathcal{V} \rightarrow \text{Cocts}(\mathcal{E}, \mathcal{E})$ pres. colimits, strong
monoidal.

$\Rightarrow *$ has a right adjoint $F: \mathcal{U}(E) \rightarrow E$ lax monoidal right adjoint. \square

3) EXAMPLES

Set: $\mathcal{V}(\text{Set}) \cong (\text{Set}, \times)$, $\underline{\text{Set}} = \text{Set}$

C-vs: $\mathcal{U}(\text{C-vs}) \cong (\text{C-vs}, \otimes)$, $\underline{\text{C-vs}} = \text{canonical self-enrichment}$

Grp: $\mathcal{U}(\text{Grp})?$

$$\mathcal{U}(\text{Grp}) = \text{Coch}(G, G)$$

\cong cogeoups in G , ie

$$\begin{array}{ccc} \{e\} & \xleftarrow{i^\circ} & G \xrightarrow{m^\circ} G + G \\ & & \downarrow \cup_{(+)^{-1}0} \end{array} \quad \text{free prod of groups}$$

$$\cong (\text{Set}, \times) \quad [\text{Kan, 1958}]$$

$[E^{\oplus}, \text{Set}]$: $\mathcal{V}([E^{\oplus}, \text{Set}]) \cong ([E^{\oplus} \times E, \text{Set}], \circ)$. $\underline{[E^{\oplus}, \text{Set}]}$ is as above.

How to compute enrichment? Well, if $X, Y \in [E^{\oplus}, \text{Set}]$,

$\underline{\Sigma}(X, Y)(d, c)$ is just the set of

$$\begin{array}{c} \underbrace{y(d, c)}_{L} \longrightarrow \underline{\Sigma}(X, Y) \quad \text{in } [E^{\oplus} \times E, \text{Set}] \\ \hline L \longrightarrow \underline{\Sigma}(X, Y) \quad \text{in } \text{Coch}([E^{\oplus}, \text{Set}], [E^{\oplus}, \text{Set}]) \\ \hline \begin{array}{ccc} LX & \longrightarrow & Y \\ X & \longrightarrow & Ry \end{array} \quad \begin{array}{l} \text{in } [E^{\oplus}, \text{Set}] \\ \text{in } [E^{\oplus}, \text{Set}] \end{array} \end{array}$$

Calculate $Ry = [\Sigma(d, -), Y_c]$ so,

$$\frac{x \longrightarrow Ry}{\begin{array}{c} x_a \longrightarrow [e(d_{1,2}), y_c] \\ \hline x_d \longrightarrow y_c \end{array}} \text{ nat } n x$$

So $\underline{[C^*, \text{Set}]}(X, Y)(d, c) = [x_d, y_c]$ ✓

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More fun examples

Let $\Sigma = \mathcal{I}/\text{Set}$ (\mathcal{I} any set)

In this case $\mathcal{U}(\Sigma)$ has objects $(\mathcal{I} \xrightarrow{f} X, f \downarrow \overset{\mathcal{I}}{\parallel} : i \in \mathcal{I})$

equivalently, $(\mathcal{I} \xrightarrow{f} X \xrightarrow{e} [\mathcal{I}, \mathcal{I}] \mid ef = \bar{\pi}_i : \mathcal{I} \rightarrow [\mathcal{I}, \mathcal{I}])$

[Jibladze, 1995]. Enrichment of Σ in $\mathcal{U}(\Sigma)$ has:

$$\Sigma \left(\underset{x}{\overset{\mathcal{I}}{\sqcup}}, \underset{y}{\overset{\mathcal{I}}{\sqcup}} f \right) = \mathcal{I} \xrightarrow{f} \left\{ \underset{x}{\overset{\mathcal{I}}{\sqcup}} \xrightarrow{g} \underset{y}{\overset{\mathcal{I}}{\sqcup}} \right\} \xrightarrow{\Pi_{\text{dom}}} [\mathcal{I}, \mathcal{I}]$$

$$i \longmapsto \begin{array}{c} \mathcal{I} \xrightarrow{\Delta_i} \mathcal{I} \\ f \downarrow \underset{x}{\overset{\mathcal{I}}{\sqcup}} \xrightarrow{\Delta g_i} \underset{y}{\overset{\mathcal{I}}{\sqcup}} \end{array}$$

Let $\Sigma = \text{Sh}(X)$, X Stone space. In this case $\mathcal{U}(\Sigma) = \text{category of spans}$

$$\begin{array}{ccc} & R & \\ \text{local homeom} & \swarrow & \searrow \\ X & & X \end{array} \text{ in Top.}$$

In this case $\Sigma \left(\underset{x}{\overset{A}{\sqcup}}, \underset{x}{\overset{B}{\sqcup}} \right)$ is the span \circledast where $R \xrightarrow{lh} X$ has

local sections over U given by spans

