

UNIVERSAL ENRICHMENTS OF CATEGORIES (jmo Hyeon Tai Jung)

1) Introduction

In an ordinary caty \mathcal{C} , arrows from X to Y form a set $\mathcal{C}(X, Y)$.

In a (\mathcal{V}) -enriched caty \mathcal{C} , arrows from X to Y are an object $\mathcal{C}(X, Y) \in \mathcal{V}$, where \mathcal{V} is a "nice" monoidal caty.

Eg: $\mathcal{V} = \text{Set}, \text{Pos}, \text{Cat}, \text{CMet}, \text{CPO}_1, \dots$

Enriched categories are not categories. However, a \mathcal{V} -caty \mathcal{C} does have an underlying category \mathcal{C}_0 , with same objects, and $\mathcal{C}_0(X, Y) = \{I \rightarrow \mathcal{C}(X, Y) \mid I \text{ is unit for } \otimes \text{ in } \mathcal{V}\}$

So makes sense to talk of enriching an ordinary caty \mathcal{C} to a \mathcal{V} -caty: ie find a \mathcal{V} -cat \mathcal{C} st $\mathcal{C}_0 \cong \mathcal{C}$.

Eg: • \mathbb{C} -vs can be enriched over: Set, CMon, Ab, R-vs, C-vs
• $[\mathcal{C}^{\text{op}}, \text{Set}] \dots \dots \dots \text{Set}, ([\mathcal{C}^{\text{op}}, \text{Set}], \times), ([\mathcal{C}^{\text{op}} \times \mathcal{C}, \text{Set}], \circ)$
↑
profunctor comp.

Last example: $\text{Hom}(X, Y) : \mathcal{C}^{\text{op}} \times \mathcal{C} \rightarrow \text{Set} \quad (X, Y \in [\mathcal{C}^{\text{op}}, \text{Set}])$
 $(d, c) \mapsto [Xd, Yc]$

- R-Alg enriched over R-coalg (homs are Sweedler's measuring coalgebras)
- $\text{Mnd}(\text{Set})_{\text{acc}}$ enriched over $\text{Comonad}(\text{Set})_{\text{acc}}$ (Rivas, McDermott, Uustalu).

Q: is there a "most general" enrichment of a given caty \mathcal{C} ?

A: yes!

"nice and algebraic"

Thm Let \mathcal{C} be a locally presentable caty. The 2-caty of locally presentable enrichments of \mathcal{C} has a bi-initial object.

2) THE THEOREM

κ -small lim pres functs

A caty \mathcal{C} is locally presentable if $\mathcal{C} \simeq \text{Lim}_{\kappa}(\Pi, \text{Set})$ for some small Π with κ -small limits. There's a 2-caty $\text{Loc Pres}_{\otimes}$ with:

- objs: locally presentable monoidal closed catys
- maps $(\mathcal{V}, \otimes) \rightarrow (\mathcal{W}, \otimes)$: (lax) monoidal right adjoints, ie right adjoint in $\text{MonCat}_{\text{lax}}$.

ie, has a left adjoint which is actually strong monoidal (and not just oplax)

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Fix an (ordinary) locally presentable caty \mathcal{E} . Define $\text{Enr}_{\mathcal{E}}(\mathcal{V})$, for $\mathcal{V} \in \text{MonCat}_{\otimes}$, to be the caty with: \uparrow caty of enrichments of \mathcal{E} in \mathcal{V}

- objects, locally pres. \mathcal{V} -catys \mathcal{C} s.t. i.o. obs iso $\mathcal{C}_0 \cong \mathcal{E}$
- maps $\mathcal{C} \rightarrow \mathcal{C}'$ are i.o.o. \mathcal{V} -functors commuting with isos to \mathcal{E} .

Now we get 2-functor $\text{Enr}_{\mathcal{E}}(-): \text{Loc Pres}_{\otimes} \longrightarrow \text{Cat}$
which on kells: $F: \mathcal{V} \rightarrow \mathcal{W}$ mon. right adjoint induces

$$\begin{array}{ccc} \text{Enr}_{\mathcal{E}}(F): \text{Enr}_{\mathcal{E}}(\mathcal{V}) & \longrightarrow & \text{Enr}_{\mathcal{E}}(\mathcal{W}) \\ \mathcal{C} & \longmapsto & F_*(\mathcal{C}) \end{array}$$

where $F_*(\mathcal{C})$ is the \mathcal{W} -caty with same obs as \mathcal{C} , and $(F_*\mathcal{C})(x, y) = F(\mathcal{C}(x, y))$

Thm $\text{Enr}_{\mathcal{E}}$ is (bi-)representable.

Proof Let $\mathcal{U}(\mathcal{E})$ be $\text{Cocts}(\mathcal{E}, \mathcal{E})$, monoidal under composition.

This is locally presentable and monoidal (bi)closed! [Bird]

Now there is a monoidal action

$$\begin{aligned} *: \mathcal{U}(\mathcal{E}) \times \mathcal{E} &\longrightarrow \mathcal{E} \\ (F, X) &\longmapsto FX \end{aligned} \quad \leftarrow \begin{array}{l} \text{pres. colimits in} \\ \text{each variable.} \end{array}$$

So by adj functor then each functor $(-)*E: \mathcal{U}(\mathcal{E}) \rightarrow \mathcal{E}$
has right adj $\underline{\mathcal{E}}(E, -): \mathcal{E} \rightarrow \mathcal{U}(\mathcal{E})$.

So for every $E, E' \in \mathcal{E}$ have $\underline{\mathcal{E}}(E, E') \in \mathcal{U}(\mathcal{E})$ — these are the homs of an enrichment.

Claim this $\underline{\mathcal{E}}$ is a birepresenting element of $\text{Enr}_{\mathcal{E}}$.

Proof Let \mathcal{V} be a locally presentable \mathcal{V} -enrichment of \mathcal{E} .

Goal: a lax monoidal r. adj $\mathcal{U}(\mathcal{E}) \xrightarrow{F} \mathcal{V}$ st $F_*(\underline{\mathcal{E}}) = \mathcal{C}$.

Well, for each $E \in \mathcal{E}$, have $\underline{\mathcal{E}}(E, -): \mathcal{E} \rightarrow \mathcal{V}$ which preserves limits and is accurate, so has a left adjoint $(-)*E: \mathcal{V} \rightarrow \mathcal{E}$.

These give an action $*: \mathcal{V} \times \mathcal{E} \rightarrow \mathcal{E}$ which pres. colimits in each variable.

That's equiv to $\bar{*}: \mathcal{V} \xrightarrow{\text{"}\mathcal{U}(\mathcal{E})\text{"}} \text{Cocts}(\mathcal{E}, \mathcal{E})$ pres. colimits, strong monoidal.

$\Rightarrow \bar{*}$ has a right adjoint $F: \mathcal{U}(\mathcal{E}) \rightarrow \mathcal{E}$ lax monoidal right adjoint. \square

3) EXAMPLES

Set: $\mathcal{N}(\text{Set}) \simeq (\text{Set}, \times)$, $\underline{\text{Set}} = \text{Set}$

$\mathbb{C}\text{-vs}$: $\mathcal{U}(\mathbb{C}\text{-vs}) \simeq (\mathbb{C}\text{-vs}, \otimes)$, $\underline{\mathbb{C}\text{-vs}} = \text{canonical self-enrichment}$

Grp: $\mathcal{U}(\text{Grp})?$

$$\begin{aligned} \mathcal{U}(\text{Grp}) &= \text{Cochs}(\text{Grp}, \text{Grp}) \\ &\simeq \text{cogroups in Grp, ie} \end{aligned}$$

$$\{e\} \xleftarrow{i^0} G \xrightarrow{m^0} G + G \quad \leftarrow \text{free prod of groups}$$

$$\downarrow \cup_{(-)^{-1}0}$$

$$\simeq (\text{Set}, \times) \quad [\text{Kan}, 1958]$$

$[\mathcal{C}^{\text{op}}, \text{Set}]$: $\mathcal{N}([\mathcal{C}^{\text{op}}, \text{Set}]) \simeq ([\mathcal{C}^{\text{op}} \times \mathcal{C}, \text{Set}], 0)$. $[\mathcal{C}^{\text{op}}, \text{Set}]$ is as above.

How to compute enrichment? Well, if $X, Y \in [\mathcal{C}^{\text{op}}, \text{Set}]$,

$\underline{\mathcal{E}}(X, Y)(d, c)$ is just the set of

$$\begin{array}{ccc} y(d, c) \longrightarrow \underline{\mathcal{E}}(X, Y) & \text{in } [\mathcal{C}^{\text{op}} \times \mathcal{C}, \text{Set}] \\ \hline L \longrightarrow \underline{\mathcal{E}}(X, Y) & \text{in } \text{Cochs}([\mathcal{C}^{\text{op}}, \text{Set}], [\mathcal{C}^{\text{op}}, \text{Set}]) \\ \hline LX \longrightarrow Y & \text{in } [\mathcal{C}^{\text{op}}, \text{Set}] \\ \hline X \longrightarrow RY & \text{in } [\mathcal{C}^{\text{op}}, \text{Set}] \end{array}$$

Calculate $RY = [\mathcal{C}(d, -), Y_c]$ so,

$$\frac{\frac{x \longrightarrow Ry}{x_d \longrightarrow [e(d,2), y_c]} \quad nat \vdash x}{x_d \longrightarrow y_c}$$

$$s. \underline{[e^o, set]}(x, y)(d, c) = [x_d, y_c] \quad \checkmark$$

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More fun examples

Let $\mathcal{E} = \mathcal{I}/Set$ (I any set)

In this case $\mathcal{U}(\mathcal{E})$ has objects $(I \xrightarrow{f} X, f \int_{\substack{I \\ X \xrightarrow{e_i} I}}^I : i \in I)$

equivalently, $(I \xrightarrow{f} X \xrightarrow{e} [I, I] \mid ef = \pi_i : I \rightarrow [I, I])$

[Jibladze, 1995]. Enrichment of \mathcal{E} in $\mathcal{U}(\mathcal{E})$ has:

$$\underline{\mathcal{E}} \left(\begin{array}{c} I \\ x \downarrow \\ x \end{array}, \begin{array}{c} I \\ y \downarrow \\ y \end{array} \right) = I \xrightarrow{f} \left\{ \begin{array}{c} I \xrightarrow{g} I \\ x \downarrow \quad y \downarrow \\ x \xrightarrow{h} y \end{array} \right\} \xrightarrow{\pi_{dom}} [I, I]$$

$$i \longmapsto \begin{array}{c} I \xrightarrow{\Delta_i} I \\ f \downarrow \quad y \downarrow \\ x \xrightarrow{\Delta_{gi}} y \end{array}$$

Let $\mathcal{E} = Sh(X)$, X Stone space. In this case $\mathcal{U}(\mathcal{E}) =$ caty of spans

$$\begin{array}{ccc} & R & \otimes \\ \text{local} & \searrow & \\ \text{homeom} & & X \end{array} \quad \text{in Top.}$$

In this case $\underline{\mathcal{E}} \left(\begin{array}{c} A \\ x_1 \downarrow \\ x \end{array}, \begin{array}{c} B \\ x \downarrow \\ x \end{array} \right)$ is the span \otimes where $R \xrightarrow{lh} X$ has

local sections over U given by spans

