Macroeconomic Dynamics Near the ZLB: A Tale of Two Countries *

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Abstract

We compute a sunspot equilibrium in an estimated small-scale New Keynesian model with a zero lower bound (ZLB) constraint on nominal interest rates and a full set of stochastic fundamental shocks. In this equilibrium a sunspot shock can move the economy from a regime in which inflation is close to the central bank's target to a regime in which the central bank misses its target, inflation rates are negative, and interest rates are close to zero with high probability. A nonlinear filter is used to examine whether the U.S. in the aftermath of the Great Recession and Japan in the late 1990s transitioned to a deflation regime. The results are somewhat sensitive to the model specification, but on balance, the answer is affirmative for Japan and negative for the U.S.

JEL CLASSIFICATION: C5, E4, E5

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1 Introduction

Japan has experienced near-zero interest rates and a deflation of about -1% since the late 1990s. In the U.S. the federal funds rate dropped below 20 basis points in December 2008 and has stayed near zero in the aftermath of the Great Recession through the end of 2015. Investors' access to money, which yields a zero nominal return, prevents interest rates from falling below zero and thereby creates a zero lower bound (ZLB) for nominal interest rates. The recent experiences of the U.S. and Japan have raised concern among policy makers about a long-lasting switch to a regime in which interest rates are zero, inflation is low, and conventional macroeconomic policies are less effective.[1] For instance, in the aftermath of the Great recession the president of the Federal Reserve Bank of St. Louis, James Bullard wrote: "During this recovery, the U.S. economy is susceptible to negative shocks that may dampen inflation expectations. This could push the economy into an unintended, low nominal interest rate steady state. Escape from such an outcome is problematic. [...] The United States is closer to a Japanese-style outcome today than at any time in recent history." (Bullard (2010))

The key contribution of this paper is to provide the first formal econometric analysis of the likelihood that the U.S. and Japan have transitioned to a long-lasting zero interest rate and low inflation regime. Starting point is a standard small-scale New Keynesian dynamic stochastic general equilibrium (DSGE) model. We explicitly impose the ZLB constraint on the interest rate feedback rule. We assume that in normal times monetary policy is active in the sense that the central bank changes interest rates more than one-for-one in response to deviations of inflation from the target. Moreover, fiscal policy is assumed to be passive in the sense that the fiscal authority uses lump-sum taxes to balance the government budget constraint in every period. It is well known, that in such an environment there are two steady states. In the targeted-inflation steady state, inflation equals the value targeted by the central bank and nominal interest rates are strictly positive. In the second steady state,

[1]Ueda (2012) provides a very thorough review of the policies used in the U.S. and Japan and he concludes that "the Japanese economy seems to be trapped in an 'equilibrium' whereby only exogenous forces generate movements to a better equilibrium with a higher rate of inflation."

the deflation steady state, nominal interest rates are zero and inflation rates are negative.[2]

To assess the likelihood of a transition to a deflation regime, we construct a stochastic equilibrium, in which agents react to fundamental shocks (e.g., technology shocks, demand shocks, monetary policy shocks) as well as to a Markov-switching sunspot shock that can move the economy between a targeted inflation regime and a deflation regime. This equilibrium offers two potential explanations for the recent experience in the U.S. and Japan: the economies may have been pushed to the ZLB either by a sequence of adverse fundamental shocks in the targeted-inflation regime or by a switch to the deflation regime. The type of explanation has important implications, not just for the central bank's ability to stimulate the economy using conventional interest rate policies, but also for the effectiveness of fiscal policy, as documented in Mertens and Ravn (2014). Aggregating results from different model specifications, we find that Japan shifted from the targeted-inflation regime into the deflation regime in 1999 and remained there until the end of our sample. The U.S., in contrast, remained in the targeted-inflation regime throughout its ZLB episode, with the possible exception of the first part of 2009, where the evidence is more mixed.

We use a first-order perturbation approximation of the targeted-inflation regime to estimate the model parameters based on U.S. and Japanese data on output, consumption-output ratio, inflation, and interest rates. Our estimation sample is chosen such that the observations pre-date the episodes of zero nominal interest rates and are consistent with the targeted-inflation regime. We estimate four model specifications that differ in terms of observables and interest rate feedback rule. Once the parameter estimates are obtained, we work directly with the nonlinear sunspot equilibrium. At a technical level, our paper is the first paper to use global projection methods to compute a sunspot equilibrium for a DSGE model with a full set of stochastic shocks that can be used to track macroeconomic time series.

[2] The second steady state is often called *undesirable*. However, in the context of a standard New Keynesian DSGE model with an explicit money demand motive the steady state is not necessarily bad in terms of welfare. While negative inflation rates in conjunction with a cost of adjusting nominal prices lead to an output loss, the zero interest rate implies that the welfare losses arising from the opportunity costs of holding real money balances are eliminated. We leave a careful normative analysis to future work.

Identifying the regime (targeted-inflation versus deflation) is not as easy as computing average inflation and interest rates and comparing them to the corresponding steady state values. Thus, for each model specification we use a nonlinear filter to extract the sequence of shocks that can explain the data. Most importantly, we obtain estimates of the probability that the economies were in either the targeted-inflation or the deflation regime. Due to the sequential nature of our filter conditional on parameter estimates, the results of the filter provide a quasi-real-time assessment of the state of the economy. We find that for each country three out of the four specifications agree, but there is considerable uncertainty to reach a definitive conclusion. Finally, we aggregate the results based on quasi posterior model probabilities that we obtain using predictive likelihoods for the set of observables that is common across specifications.

Our paper is related to three strands of the literature: sunspots and multiplicity of equilibria in New Keynesian DSGE models; global projection methods for the solution of DSGE models; and the use of particle filters to extract hidden states in nonlinear state-space models.

The relevance of sunspots in economic models was first discussed in Cass and Shell (1983), who define sunspots as "extrinsic uncertainty, that is, random phenomena that do not affect tastes, endowments, or production possibilities." Sunspot shocks can affect economic outcomes in environments in which there does not exist a unique equilibrium. A review of the sunspot literature in macroeconomics is provided by Benhabib and Farmer (1999). Subsequently, there has been extensive research on multiplicity of equilibria in New Keynesian DSGE models generated by so-called passive monetary policy rules that do not respond strongly enough to inflation deviations from target. Such policy rules are associated with undetermined local fluctuations in the neighborhood of the targeted-inflation steady state. An econometric analysis of this type of multiplicity is provided by Lubik and Schorfheide (2004).

If monetary policy is active instead of passive then the local dynamics near the targeted-inflation steady state are unique (subject to the caveats emphasized in Cochrane (2011)), but a second steady state arises from the kink in the monetary policy rule induced by the ZLB.

In this second steady state nominal interest rates are zero and inflation rates are negative. Because in the neighborhood of this second steady state the central bank is unable to lower interest rates in response to a drop in inflation, the local dynamics are indeterminate. Hirose (2014) estimates a linearized New Keynesian DSGE model by imposing that the economy is permanently at the ZLB and parameterizing the multiplicity of solutions as in Lubik and Schorfheide (2004).

Benhabib et al. (2001a,b) were the first to construct equilibria in which the economy transitions from the targeted-inflation steady state toward the deflation steady state. Recently, Armenter (2014) generalizes their results to a model in which monetary policy is not represented by a Taylor rule, but it is optimally chosen to maximize social welfare.

It should be clear from the above discussion that the model considered in our analysis has many equilibria. This opens the door for two research strategies: (i) characterize as many equilibria as possible and then examine which of these equilibria is consistent with the data. (ii) Choose one particular equilibrium and condition the empirical analysis on that equilibrium. The papers by Lubik and Schorfheide (2004) (studying inflation and interest rate dynamics pre and post Volcker disinflation) and Cochrane (2015) (studying inflation and interest rate dynamics during a ZLB episode and a subsequent exit from the ZLB) consider linearized DSGE models and are examples of the first approach. Our paper pursues the second avenue: we consider a particular equilibrium within which we can address the question whether an economy has transitioned into a long-lasting deflation regime. While there are other equilibria that allow for similar transitions, yet might exhibit different regime-conditional dynamics, at present it is computationally not feasible to enumerate. Thus, we focus our empirical analysis on an interesting equilibrium for which we do have a solution. Conditional on being in the targeted-inflation regime, the dynamics are very similar to the dynamics that arise in the targeted-inflation equilibrium, that is studied in, for instance, Maliar and Maliar (2015), Fernández-Villaverde et al. (2015), and Gust et al. (2012).

A sunspot equilibrium similar to ours has been recently analyzed by Mertens and Ravn (2014), but in a model with a much more restrictive exogenous shock structure. Related, Schmitt-Grohé and Uribe (2015) study an equilibrium in which confidence shocks, which resemble a change in regimes in our model, combined with downward nominal wage rigidity can deliver jobless recoveries near the ZLB in a mostly analytical analysis. More recently, Piazza (2015) shows that different monetary policy rules may change the path from the targeted-inflation steady state to the deflation steady state and demonstrates in a calibrated model that a combination of a perfect-foresight sunspot shock and a shock to the growth rate of the economy can generate dynamics similar to the Japanese experience. Our paper is the first to compute a sunspot equilibrium in a New Keynesian DSGE model that is rich enough to track macroeconomic time series and to use a filter to extract the evolution of the hidden sunspot shock.

In terms of solution method, our work is most closely related to the papers by Judd et al. (2010), Maliar and Maliar (2015), Fernández-Villaverde et al. (2015), and Gust et al. (2012).[3] All of these papers use global projection methods to approximate agents' decision rules in a New Keynesian DSGE model with a ZLB constraint. However, these papers solely consider an equilibrium in which the economy is always in the targeted-inflation regime – what we could call a targeted-inflation equilibrium – and some important details of the implementation of the solution algorithm are different.

To improve the accuracy of the model solution, we introduce two novel features. First, we use a piece-wise smooth approximation with two separate functions characterizing the decisions when the ZLB is binding and when it is not. This means all our decision rules allow for kinks at points in the state space where the ZLB becomes binding. Second, when constructing a grid of points in the model's state space for which the equilibrium conditions are explicitly evaluated by the projection approach, we combine draws from the ergodic distribution of the DSGE model with values of the state variables obtained by applying our

[3]Most of the other papers that study DSGE models with a ZLB constraint take various shortcuts to solve the model. In particular, following Eggertsson and Woodford (2003), many authors assume that an exogenous Markov-switching process pushes the economy to the ZLB. The subsequent exit from the ZLB is exogenous and occurs with a prespecified probability. The absence of other shocks makes it impossible to use the model to track actual data. Unfortunately, model properties tend to be very sensitive to the approximation technique and to implicit or explicit assumptions about the probability of leaving the ZLB, see Braun et al. (2012) and Fernández-Villaverde et al. (2015).

filtering procedure. This modification of the ergodic-set method proposed by Judd et al. (2010) ensures that the model solution is accurate in a region of the state space that is unlikely ex ante under the ergodic distribution of the model, but very important ex post to explain the observed data.

With respect to the empirical analysis, the only other papers that combine a projection solution with a nonlinear filter to track U.S. data throughout the Great Recession period are Gust et al. (2012) and Cuba-Borda (2014). Both papers restrict their attention to the targeted-inflation equilibrium. The first focuses on parameter estimation using post-2008 data in a New Keynesian model like ours and examine the extent to which the ZLB constrained the ability of monetary policy to stabilize the economy. The latter extracts fundamental shocks to account for the decline in economic activity during the U.S. Great Recession in a medium-scale model with investment. Our paper is the first to fit a nonlinear DSGE model with an explicit ZLB constraint to Japanese data.

The remainder of the paper is organized as follows. Section 2 presents a simple two-equation model that we use to illustrate the multiplicity of equilibria in monetary models with ZLB constraints. We also highlight the particular equilibrium studied in this paper. The New Keynesian model that is used for the quantitative analysis is presented in Section 3, and the solution of the model is discussed in Section 4. Section 5 describes the parameter estimates for the different model specifications in this paper and illustrates the dynamic properties of one of the estimated model specifications. Section 6 presents our main results regarding the identification of the sunspot regime in each country. Section 7 concludes. Detailed derivations, descriptions of algorithms, and additional quantitative results are summarized in an Online Appendix.

2 A Two-Equation Example

We begin with a simple two-equation example to characterize the sunspot equilibrium that we will study in the remainder of this paper in the context of a New Keynesian DSGE model with an interest-rate feedback rule and the ZLB constraint. Suppose that the economy can be described by a consumption Euler equation of the form

$$1 = \mathbb{E}_t \left[M_{t+1} \frac{R_t}{\pi_{t+1}} \right] \tag{1}$$

and the monetary policy rule

$$R_t = \max \left\{ 1, \ r\pi_* \left(\frac{\pi_t}{\pi_*} \right)^{\psi} \right\}, \quad \psi > 1. \tag{2}$$

In the fully-specified DSGE model introduced in Section 3 below, the stochastic discount factor M_{t+1} that appears in (1) is given by

$$M_{t+1} = \beta \frac{d_{t+1}}{d_t} \left(\frac{C_{t+1}}{C_t} \right)^{-\tau},$$

where C_t is consumption, d_t is a discount factor shock, and $1/\tau$ is the intertemporal elasticity of substitution. We define

$$r_t = 1/M_t, \tag{3}$$

where r_t can be interpreted as the real rate of return for a one-period asset. To keep the example simple, we assume that r_t exogenous and follows a stationary AR(1) process

$$\log \left(\frac{r_{t+1}}{r} \right) = \rho \log \left(\frac{r_t}{r} \right) + \sigma \epsilon_{t+1}, \quad \epsilon_{t+1} \sim iidN(0,1) \text{ and } \rho \in (0,1). \tag{4}$$

The parameter r corresponds to the steady state ($\sigma = 0$) of the real interest rate. We assume that the gross nominal interest rate is bounded from below by one, which is captured by the max operator in (2). Throughout the paper we refer to this bound as the ZLB because the net interest rate cannot fall below zero. The parameter π_* in the monetary policy rule represents the central bank's target inflation.

Loglinearizing around $\pi_t = \pi_*$, $r_t = r$ and $R_t = r\pi_*$ and using hats to denote percentage deviations from this point, yields the system

$$\hat{R}_t = \mathbb{E}_t \left[\hat{r}_{t+1} + \hat{\pi}_{t+1} \right] \tag{5}$$

$$\hat{R}_t = \max \left\{ -\log \left(r\pi_* \right), \psi \hat{\pi}_t \right\} \tag{6}$$

$$\hat{r}_{t+1} = \rho \hat{r}_t + \sigma \varepsilon_{t+1} \tag{7}$$

(5) is a version of the Fisher equation, which relates the nominal interest rate to the expected real interest rate and inflation. A similar system of equations arises from the log-linearized

equilibrium conditions of many monetary DSGE models. Combining (5) and (6) and using $\mathbb{E}_t[\hat{r}_{t+1}] = \rho\hat{r}_t$ yields the following expectational difference equation for inflation

$$\mathbb{E}_t\left[\hat{\pi}_{t+1}\right] = \max\left\{ -\log\left(r\pi_*\right) - \rho\hat{r}_t, \psi\hat{\pi}_t - \rho\hat{r}_t\right\}. \tag{8}$$

Just as the original system comprising (1) and (2), the linearized difference equation (8) has two steady states. In the targeted-inflation steady state inflation equals π_*, and the nominal interest rate is $R_* = r\pi_*$, so that $\hat{\pi} = \hat{R} = 0$. In the deflation steady state, $\hat{\pi} = \hat{R} = -\log\left(r\pi_*\right)$ and thus inflation equals $1/r$, and the nominal interest rate is at the ZLB.

The presence of two steady states suggests that the rational expectations difference equation (8) also has multiple stochastic solutions. We find solutions to this equation using a guess-and-verify approach (see Online Appendix for details). Suppose that we conjecture

$$\hat{\pi}_t = \theta_0 + \theta_1\hat{r}_t. \tag{9}$$

It can be verified that a solution that fluctuates around the targeted-inflation steady state (henceforth targeted-inflation equilibrium) is given by[4]

$$\theta_0^* = 0, \quad \theta_1^* = \frac{\rho}{\psi - \rho} > 0. \tag{10}$$

Because around the targeted-inflation steady state nominal interest rates respond to inflation more than one-for-one, the local dynamics are unique.

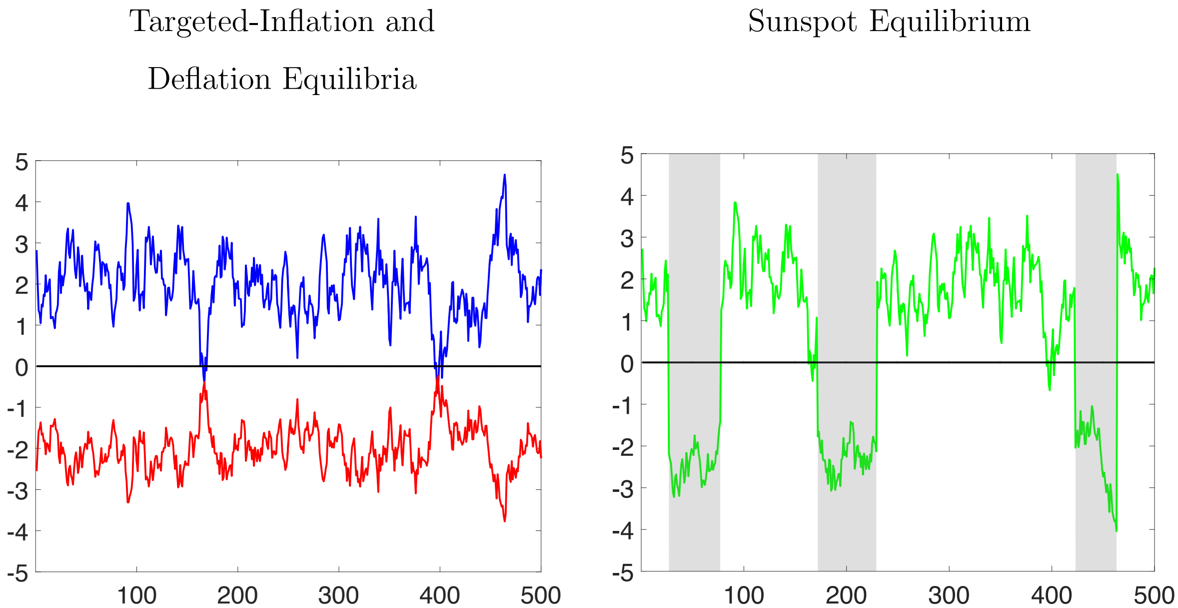
We can also obtain a solution that fluctuates around the deflation steady state (henceforth deflation equilibrium):

$$\theta_0^D = -\log(r\pi_*), \quad \theta_1^D = -1. \tag{11}$$

Because around the deflation steady state nominal interest rates do not respond to inflation, the local dynamics are indeterminate and one could construct other solutions (see, for instance, Lubik and Schorfheide (2004) as well as the Online Appendix) that may involve a sunspot shock ζ_t with the property that $\mathbb{E}_{t-1}[\zeta_t] = 0$ or the dependence of inflation

[4]It is assumed that the exogenous movements in $\widehat{r}_t$ are sufficiently small such that $(\psi\theta_1 * -\rho)\widehat{r}_t \geq -\log(r\pi_*)$ for all t.

Figure 1: Inflation Dynamics in the Two-Equation Model



Notes: The figure shows annualized net inflation rate, $400 \log \pi_t$. In the left panel, the blue line shows the targeted-inflation equilibrium, and the red line shows the deflation equilibrium. In the right panel, the shaded area corresponds to periods in which the system is in the deflation regime, $s_t = 0$.

on $\hat{r}_{t-1}$. However, we will restrict our attention to (11). Note that (10) and (11) have drastically different dynamics: inflation and real interest rates have a positive correlation in the targeted-inflation equilibrium, while this correlation switches signs in the deflation equilibrium.

In the remainder of the paper we will focus on an equilibrium in which a two-state Markov-switching sunspot shock $s_t \in \{0, 1\}$ triggers transitions from a targeted-inflation regime to a deflation regime and vice versa:

$$\hat{\pi}_t^{(s)} = \theta_0(s_t) + \theta_1(s_t)\hat{r}_t. \tag{12}$$

where $\theta_0(s_t)$ and $\theta_1(s_t)$ denote the regime-specific intercept and slope of the linear decision rule. Throughout this paper, we assume that the sunspot process s_t evolves independently from the fundamental shocks.[5] If the regimes are persistent, then the intercepts and slopes

[5]For toy models we were able to construct equilibria in which the Markov transition is triggered by ϵ_t. But we were unable to numerically construct such solutions for the DSGE model presented in Section 3.

Table 1: Decision Rule Coefficients

Targeted-Inflation Equilibrium	$\theta_0^* = 0$	$\theta_1^* = 1.5$
Deflation Equilibrium	$\theta_0^D = -0.01$	$\theta_1^D = -1$
Sunspot Equilibrium	$\theta_0(1) = -0.0002$	$\theta_1(1) = 1.4611$
	$\theta_0(0) = -0.0105$	$\theta_1(0) = -1.1295$

are similar in magnitude (but not identical) to the coefficients in (10) and (11), respectively. The precise values depend on the transition probabilities of the Markov switching process and ensure that (8) holds in every period t.

A numerical illustration is provided in Figure 1. We set $\pi_* = 1.005$, $\psi = 1.5$, $r = 1.005$, $\sigma = 0.0007$, $\rho = 0.9$, $p_{11} = 0.99$ and $p_{00} = 0.95$. The implied decision rule coefficients are summarized in Table 1. The left panel of Figure 1 compares the paths of annualized net inflation ($400 \log \pi_t$) under the targeted-inflation equilibrium (10) and the deflation equilibrium (11). The inflation paths are shifted by the difference between $400 \log \pi_*$ and $400 \log(1/r)$, which is 4%, and display perfect negative correlation. The right panel shows the sunspot equilibrium with visible shifts from the targeted-inflation regime to the deflation regime (shaded areas) and back.

We close this section with the following remarks: (i) The linearized two-equation model has many stochastic equilibria. (ii) We will focus on a particular sunspot equilibrium that is interesting for our empirical analysis because it can capture long lasting transitions into and out of a regime in which interest rates are zero and inflation rates are low. (iii) Our empirical analysis will be based on a small-scale DSGE model rather than the two-equation model presented in this section. (iv) We will not log-linearize the equilibrium conditions of the DSGE model, instead we will work with the nonlinear equilibrium conditions. (v) Unlike in the simple two-equation example we will not assume that interest rates are always strictly greater than zero in the targeted-inflation regime and always equal to zero in the deflation regime. Instead, our nonlinear decision rules imply that interest rates could be zero in the targeted-inflation regime and they could be strictly positive in the deflation regime. Likewise, in both regimes it is possible to observe both positive and negative inflation rates.

3 A Prototypical New Keynesian DSGE Model

Our quantitative analysis will be based on a small-scale New Keynesian DSGE model. Variants of this model have been widely studied in the literature and its properties are discussed in detail in Woodford (2003). The model economy consists of perfectly competitive final-goods-producing firms, a continuum of monopolistically competitive intermediate goods producers, a continuum of identical households, and a government that engages in monetary and fiscal policy. To keep the dimension of the state space manageable, we abstract from capital accumulation and wage rigidities. We describe the preferences and technologies of the agents in Section 3.1, and summarize the equilibrium conditions in Section 3.2.

3.1 Preferences and Technologies

Households. Households derive utility from consumption C_t relative to an exogenous habit stock and disutility from hours worked H_t. We assume that the habit stock is given by the level of technology A_t, which ensures that the economy evolves along a balanced growth path. We also assume that the households value transaction services from real money balances, detrended by A_t, and include them in the utility function. The households maximize

$$\mathbb{E}_t \left[\sum_{s=0}^{\infty} \beta^s d_{t+s} \left(\frac{(C_{t+s}/A_{t+s})^{1-\tau} - 1}{1-\tau} - \chi_H \frac{H_{t+s}^{1+1/\eta}}{1+1/\eta} + \chi_M V \left(\frac{M_{t+s}}{P_{t+s} A_{t+s}} \right) \right) \right], \tag{13}$$

subject to the budget constraint

$$P_t C_t + T_t + M_t + B_t = P_t W_t H_t + M_{t-1} + R_{t-1} B_{t-1} + P_t D_t + P_t SC_t.$$

Here β is the discount factor, d_t is a shock to the discount factor, $1/\tau$ is the intertemporal elasticity of substitution, η is the Frisch labor supply elasticity, and P_t is the price of the final good. The shock d_t captures frictions that affect intertemporal preferences in a reduced-form way. Fluctuations in d_t affect households patience and their desire to postpone consumption.

As we demonstrate below, and as is commonly exploited in the literature, a sufficiently large shock to d_t makes the central bank cut interest rates all the way to the ZLB. The households supply labor services to the firms in a perfectly competitive labor market, taking the real wage W_t as given. At the end of period t, households hold money in the amount of M_t. They have access to a bond market where nominal government bonds B_t that pay gross interest R_t are traded. Furthermore, the households receive profits D_t from the firms and pay lump-sum taxes T_t. SC_t is the net cash inflow from trading a full set of state-contingent securities.

Detrended real money balances $M_t/(P_tA_t)$ enter the utility function in an additively separable fashion. An empirical justification of this assumption is provided by Ireland (2004). As a consequence, the equilibrium has a block diagonal structure under the interest-rate feedback rule that we will specify below: the level of output, inflation, and interest rates can be determined independently of the money stock. We assume that the marginal utility $V'(m)$ is decreasing in real money balances m and reaches zero for $m = \bar{m}$, which is the amount of money held in steady state by households if the net nominal interest rate is zero. Since the return on holding money is zero, it provides the rationale for the ZLB on nominal rates. More specifically since households can hold as well as issue debt at the market rate R_t, their problem does not have a solution when $R_t < 1$. The ZLB ensures the existence of a monetary equilibrium.

Firms. The final-goods producers aggregate intermediate goods, indexed by $j \in [0, 1]$, using the technology:

$$Y_t = \left(\int_0^1 Y_t(j)^{1-\nu} dj \right)^{\frac{1}{1-\nu}}.$$

The firms take input prices $P_t(j)$ and output prices P_t as given. Profit maximization implies that the demand for inputs is given by

$$Y_t(j) = \left(\frac{P_t(j)}{P_t} \right)^{-1/\nu} Y_t.$$

Under the assumption of free entry into the final-goods market, profits are zero in equilibrium, and the price of the aggregate good is given by

$$P_t = \left(\int_0^1 P_t(j)^{\frac{\nu-1}{\nu}} dj \right)^{\frac{\nu}{\nu-1}}. \tag{14}$$

We define inflation as $\pi_t = P_t/P_{t-1}$.

Intermediate good j is produced by a monopolist who has access to the following production technology:

$$Y_t(j) = A_t H_t(j), \tag{15}$$

where A_t is an exogenous productivity process that is common to all firms and $H_t(j)$ is the firm-specific labor input. Intermediate-goods-producing firms face quadratic price adjustment costs of the form

$$AC_t(j) = \frac{\phi}{2}\left(\frac{P_t(j)}{P_{t-1}(j)} - \bar{\pi}\right)^2 Y_t(j),$$

where ϕ governs the price stickiness in the economy and $\bar{\pi}$ is a baseline rate of price change that does not require the payment of any adjustment costs. In our quantitative analysis, we set $\bar{\pi} = \pi_*$, where π_* is the target inflation rate of the central bank, which in turn is the steady state inflation rate in the targeted-inflation equilibrium. Firm j chooses its labor input $H_t(j)$ and the price $P_t(j)$ to maximize the present value of future profits

$$\mathbb{E}_t\left[\sum_{s=0}^{\infty}\beta^s Q_{t+s|t}\left(\frac{P_{t+s}(j)}{P_{t+s}}Y_{t+s}(j) - W_{t+s}H_{t+s}(j) - AC_{t+s}(j)\right)\right]. \tag{16}$$

Here, $Q_{t+s|t}$ is the time t value to the household of a unit of the consumption good in period $t+s$, which is treated as exogenous by the firm.

Government Policies. Monetary policy is described by an interest rate feedback rule. Because the ZLB constraint is an important part of our analysis we introduce it explicitly as follows:

$$R_t = \max\left\{1,\ R_t^* e^{\epsilon_{R,t}}\right\}. \tag{17}$$

Here R_t^* is the systematic part of monetary policy which reacts to the current state of the economy and $\epsilon_{R,t}$ is a monetary policy shock. We consider two specifications for R_t^*, which we refer to as *growth* and *gap* specifications. The growth specification takes the form

$$\text{Growth}: \quad R_t^* = \left[r\pi_*\left(\frac{\pi_t}{\pi_*}\right)^{\psi_1}\left(\frac{Y_t}{\gamma Y_{t-1}}\right)^{\psi_2}\right]^{1-\rho_R} R_{t-1}^{\rho_R}. \tag{18}$$

Here r is the steady-state real interest rate and π_* is the target-inflation rate. Provided that the ZLB is not binding, the central bank reacts to deviations of inflation from the target rate π_* and deviations of output growth from its long-run value γ.

Under the gap specification, the central bank reacts to a measure of the output gap in addition to inflation deviations from target:

$$\text{Gap}: \quad R_t^* = \left[r\pi_* \left(\frac{\pi_t}{\pi_*} \right)^{\psi_1} \left(\frac{Y_t}{Y_t^*} \right)^{\psi_2} \right]^{1-\rho_R} R_{t-1}^{\rho_R}, \tag{19}$$

where Y_t^* is the target level of output. In theoretical studies the targeted level of output often corresponds to the level of output in the absence of nominal rigidities and mark-up shocks, because from an optimal policy perspective, this is the level of output around which the central bank should stabilize fluctuations. However, historically, at least in the U.S., the central bank has tried to keep output close to the official measure of potential output, which is well approximated by a slow-moving trend. Thus we use exponential smoothing to construct Y_t^* directly from historical output data. It is given by

$$\log Y_t^* = \alpha \log Y_{t-1}^* + (1-\alpha) \log Y_t + \alpha \log \gamma. \tag{20}$$

where α is a parameter we calibrate such that $\log Y_t^*$ tracks an official measure of potential output.

We use two alternative policy rules in an attempt to capture the dynamics of R_t^*, which is in principle latent when the economy is at the ZLB. For instance, the U.S. experienced a large negative rate of output growth in 2008:Q4. Under the growth rule, this creates a large drop in R_t^*, but the drop is short-lived because output growth subsequently recovers. Under the gap rule, the reduction in R_t^* is more persistent, because the level of output stays below its historical average for a long period of time. Our analysis is sensitive to the desired interest rate, because R_t^* determines how constrained the central bank is by the ZLB and how likely it is that it will leave the ZLB in the subsequent quarters.

The government consumes a stochastic fraction of aggregate output. We assume that government spending evolves according to

$$G_t = \left(1 - \frac{1}{g_t}\right) Y_t. \tag{21}$$

The government levies a lump-sum tax T_t (or provides a subsidy if T_t is negative) to finance any shortfalls in government revenues (or to rebate any surplus). Its budget constraint is given by

$$P_t G_t + M_{t-1} + R_{t-1} B_{t-1} = T_t + M_t + B_t. \tag{22}$$

Exogenous shocks. The model economy is perturbed by four (fundamental) exogenous processes. Aggregate productivity evolves according to

$$\log A_t = \log \gamma + \log A_{t-1} + \log z_t, \text{ where } \log z_t = \rho_z \log z_{t-1} + \sigma_z \epsilon_{z,t}. \tag{23}$$

Thus, on average, the economy grows at the rate γ, and z_t generates exogenous stationary fluctuations of the technology growth rate around this long-run trend. We assume that the government spending shock follows the AR(1) law of motion

$$\log g_t = (1 - \rho_g) \log g_* + \rho_g \log g_{t-1} + \sigma_g \epsilon_{g,t}. \tag{24}$$

While we formally introduce the exogenous process g_t as a government spending shock, we interpret it more broadly as an exogenous demand shock that contributes to fluctuations in output. (21), (23) and (24) imply that log output and government spending are cointegrated and that the log government spending-output ratio is stationary. The shock to the discount factor evolves according to

$$\log d_t = \rho_d \log d_{t-1} + \sigma_d \epsilon_{d,t} \tag{25}$$

The monetary policy shock $\epsilon_{R,t}$ is assumed to be serially uncorrelated. We stack the four innovations into the vector $\epsilon_t = [\epsilon_{z,t}, \epsilon_{g,t}, \epsilon_{d,t}, \epsilon_{r,t}]'$ and assume that $\epsilon_t \sim iidN(0, I)$.

In addition to the fundamental shock processes, agents in the model economy observe an exogenous sunspot shock s_t, which follows a two-state Markov-switching process

$$\mathbb{P}\{s_t = 1\} = \begin{cases} (1 - p_{00}) & \text{if } s_{t-1} = 0 \\ p_{11} & \text{if } s_{t-1} = 1 \end{cases}. \tag{26}$$

3.2 Equilibrium Conditions

Because the exogenous productivity process has a stochastic trend, it is convenient to characterize the equilibrium conditions of the model economy in terms of detrended consumption

$c_t \equiv C_t/A_t$ and detrended output $y_t \equiv Y_t/A_t$. We write the consumption Euler equation (sometimes called the IS equation) as

$$c_t^{-\tau} = \beta R_t \mathcal{E}_t, \tag{27}$$

where

$$\mathcal{E}_t = \mathbb{E}_t \left[\frac{d_{t+1}}{d_t} \frac{c_{t+1}^{-\tau}}{\gamma z_{t+1} \pi_{t+1}} \right]. \tag{28}$$

The solution algorithm approximates the conditional expectation $\mathcal{E}_t$ using a Chebychev polynomial in terms of the state variables. In a symmetric equilibrium, in which all firms set the same price $P_t(j)$, the price-setting decision of the firms leads to the condition

$$\begin{aligned} &\phi\beta \mathbb{E}_t \left[\frac{d_{t+1}}{d_t} c_{t+1}^{-\tau} y_{t+1} (\pi_{t+1} - \bar{\pi}) \pi_{t+1} \right] \\ &= \; c_t^{-\tau} y_t \left\{ \frac{1}{\nu} \left(1 - \chi_h c_t^{\tau} y_t^{1/\eta} \right) + \phi(\pi_t - \bar{\pi}) \left[\left(1 - \frac{1}{2\nu} \right) \pi_t + \frac{\bar{\pi}}{2\nu} \right] - 1 \right\}. \end{aligned} \tag{29}$$

A log-linearization of (29) leads to the standard New Keynesian Phillips curve.

We show in the Online Appendix that the aggregate resource constraint can be expressed as

$$c_t = \left[\frac{1}{g_t} - \frac{\phi}{2} (\pi_t - \bar{\pi})^2 \right] y_t. \tag{30}$$

It reflects both government spending as well as the resource cost (in terms of output) caused by price changes. Finally, we reproduce the monetary policy rule

$$\begin{aligned} \text{Growth: } R_t = & \max \left\{ 1, \left[r\pi_* \left(\frac{\pi_t}{\pi_*} \right)^{\psi_1} \left(\frac{y_t}{y_{t-1}} z_t \right)^{\psi_2} \right]^{1-\rho_R} R_{t-1}^{\rho_R} e^{\sigma_R \epsilon_{R,t}} \right\}, \\ \text{Gap: } R_t = & \max \left\{ 1, \left[r\pi_* \left(\frac{\pi_t}{\pi_*} \right)^{\psi_1} \left(\frac{y_t}{y_{t-1}^*} z_t \right)^{\alpha\psi_2} \right]^{1-\rho_R} R_{t-1}^{\rho_R} e^{\sigma_R \epsilon_{R,t}} \right\}. \end{aligned} \tag{31}$$

where $y_t^* \equiv Y_t^*/A_t$. We do not use a measure of money in our empirical analysis and therefore drop the equilibrium condition that determines money demand.

As the two-equation model in Section 2, the New Keynesian model with the ZLB constraint has two steady states, which we refer to as the targeted-inflation and the deflation steady states. In the targeted-inflation steady state, inflation equals π_* and the gross interest rate equals $r\pi_*$, while in the deflation steady state, inflation equals $1/r$ and the interest rate is at the ZLB. Subsequently, we will focus on a stochastic sunspot equilibrium with a targeted-inflation regime ($s_t = 1$) and a deflation regime ($s_t = 0$).

4 Solution Algorithm

We now discuss some key features of the algorithm that is used to solve the nonlinear DSGE model presented in the previous section. Additional details can be found in the Online Appendix. We utilize a global approximation method following Judd (1992) where the decision rules are approximated by combinations of Chebyshev polynomials. The minimum set of state variables associated with our DSGE model is

$$\mathcal{S}_t = (R_{t-1}, y_{t-1}, d_t, g_t, z_t, \epsilon_{R,t}, s_t) \tag{32}$$

for the growth specifications and

$$\mathcal{S}_t = (R_{t-1}, y^*_{t-1}, d_t, g_t, z_t, \epsilon_{R,t}, s_t). \tag{33}$$

for gap specifications, and d_t is only relevant in the version with the discount factor shock. We included the regime-switching process s_t into the state vector because our goal is to characterize a sunspot equilibrium. An (approximate) solution of the DSGE model is a set of decision rules $\pi_t = \pi(\mathcal{S}_t; \Theta)$, $\mathcal{E}_t = \mathcal{E}(\mathcal{S}_t; \Theta)$, $c_t = c(\mathcal{S}_t; \Theta)$, $y_t = y(\mathcal{S}_t; \Theta)$, and $R_t = R(\mathcal{S}_t; \Theta)$ that solve the nonlinear rational expectations system given by (27) to (31), and the laws of motion of the exogenous processes. Note that conditional on $\pi(\mathcal{S}_t; \Theta)$ and $\mathcal{E}(\mathcal{S}_t; \Theta)$, Equations (27), (30) and (31) directly determine $c(\mathcal{S}_t; \Theta)$, $y(\mathcal{S}_t; \Theta)$, and $R(\mathcal{S}_t; \Theta)$. Thus, we only use Chebyshev polynomials to approximate $\pi(\mathcal{S}_t; \Theta)$ and $\mathcal{E}(\mathcal{S}_t; \Theta)$. In our notation the coefficient vector $\Theta \equiv \{\theta_i\}$, $i = 1, ..., N$, parameterizes all of the decision rules and N is the total number of coefficients.

The solution algorithm amounts to specifying a grid of points $\mathcal{G} = \{\mathcal{S}_1, \ldots, \mathcal{S}_M\}$ in the model's state space and determining Θ by minimizing the (unweighted) sum of squared residuals associated with (28) and (29). Because (28) and (29) are functions of $\mathcal{S}_t$, we are evaluating the residuals for each $\mathcal{S}_t \in \mathcal{G}$ and then sum the M squared residuals. There are two non-standard aspects of our solution method that we will now discuss in more detail: (i) the piecewise smooth representation of the functions $\pi(\cdot;\Theta)$ and $\mathcal{E}(\cdot;\Theta)$ and (ii) our iterative procedure of choosing grid points $\mathcal{G}$.

Piece-wise Smooth Decision Rules. The max operator in the monetary policy rule potentially introduces kinks in the decision rules $\pi(\mathcal{S}_t)$ and $\mathcal{E}(\mathcal{S}_t)$. While Chebyshev polynomials, which are smooth functions of the states, can in principle approximate functions with a kink, such approximations are quite inaccurate for low-order polynomials. Thus, unlike Judd et al. (2010), Fernández-Villaverde et al. (2015), and Gust et al. (2012), we use a piece-wise smooth approximation of the functions $\pi(\mathcal{S}_t)$ and $\mathcal{E}(\mathcal{S}_t)$ by postulating

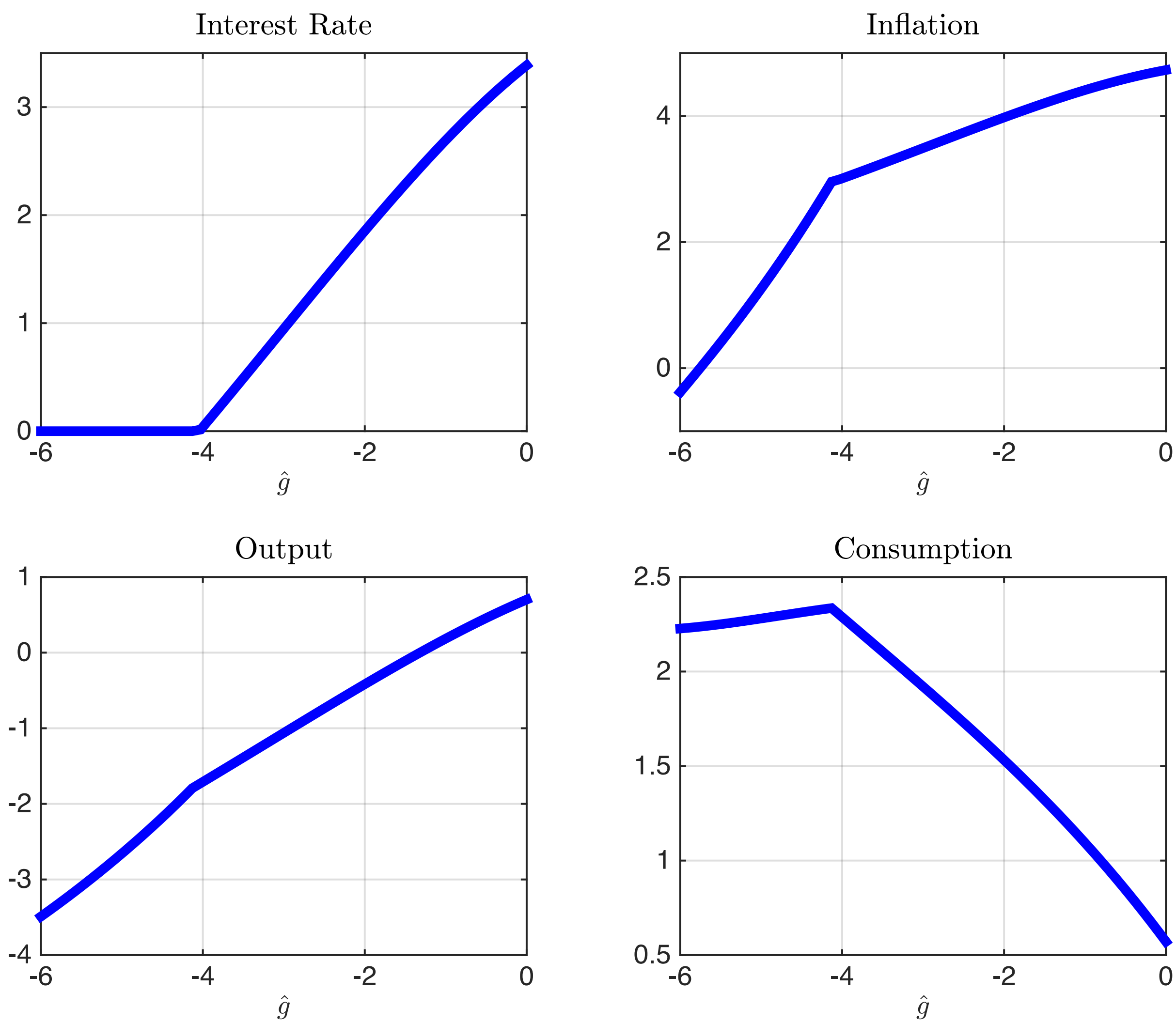
$$\pi(\mathcal{S}_t;\Theta) = \begin{cases} f_\pi^1(\mathcal{S}_t;\Theta) & \text{if } s_t = 1 \text{ and } R(\mathcal{S}_t) > 1 \\ f_\pi^2(\mathcal{S}_t;\Theta) & \text{if } s_t = 1 \text{ and } R(\mathcal{S}_t) = 1 \\ f_\pi^3(\mathcal{S}_t;\Theta) & \text{if } s_t = 0 \text{ and } R(\mathcal{S}_t) > 1 \\ f_\pi^4(\mathcal{S}_t;\Theta) & \text{if } s_t = 0 \text{ and } R(\mathcal{S}_t) = 1 \end{cases} \tag{34}$$

and similarly for $\mathcal{E}(\mathcal{S}_t,\Theta)$, where the functions $f_j^i(\cdot)$ are linear combinations of a complete set of Chebyshev polynomials up to fourth order.

In our experience, the flexibility of the piece-wise smooth approximation yields more accurate decision rules, especially for inflation. Figure 2 shows a slice of the decision rules. We vary g_t over a wide range where the ZLB is both slack and binding. To generate the figure we condition on $s_t = 1$, $R_{t-1} = 1$ and set y_{t-1}, z_t, and $\epsilon_{R,t}$ to their means conditional on $s_t = 1$. The monetary policy rule has kink due to the ZLB, while the decision rule for inflation has an apparent kink due to the piece-wise smooth approximation in (34). The decision rules for output and consumption inherit the kinks in the decision rule for inflation (and $\mathcal{E}(\mathcal{S}_t)$) and in the monetary policy rule. The kinks, especially the ones in the decision rules for inflation and consumption, are very severe. For instance, if the ZLB is binding, consumption is increasing in $\hat{g}$. If the ZLB is non-binding consumption falls as $\hat{g}$ rises. As a consequence, a smooth approximation obtained from a single Chebyshev polynomial would do a very poor job capturing the actual decision rules.[6]

[6]In an earlier version of the paper we indeed solved the model both ways and illustrated that the smooth

Figure 2: Sample Decision Rules



Note: This figure depicts the decision rules for 3vGrowth using parameter values estimated for the U.S. as described in Section 5.2. The x-axis corresponds to the state variable g_t, in percentage deviations from its steady state. The other state variables are fixed: $s_t = 1$, $R_{t-1} = 1$, and y_{t-1}, z_t, and $\epsilon_{R,t}$ set to their means conditional on $s_t = 1$.

Choice of Grid Points. With regard to the choice of grid points, projection methods that require the solution to be accurate on a fixed grid, e.g., a tensor product grid, become exceedingly difficult to implement as the number of state variables increases above three. While the Smolyak grid proposed by Krüger and Kubler (2004) can alleviate the curse of

approximation leads to approximation errors that are an order of magnitude larger relative to the piece-wise smooth approximation.

dimensionality to some extent, we build on recent work by Judd et al. (2010), which proposed to simulate the model to be solved, to distinguish clusters on the simulated series, and to use the clusters' centers as a grid for projections.[7] We modify their methodology significantly by combining simulated grid points with states obtained from the data using a nonlinear filter. Doing so is necessary to capture the behavior of the model in low probability regions of the state space that are important for our analysis. For example, when $s_t = 1$ ($s_t = 0$), negative (positive) inflation is typically outside the ergodic distribution of the model.

Because the set of simulated grid points that represent the ergodic distribution and the filtered states both depend on the solution of the model, some iteration of solution, on the one hand, and simulation and filtering, on the other hand, is required. For a given solution we simulate the model and get a set of points that characterize the ergodic distribution. We then run a particle filter, details of which are provided in the Online Appendix, to obtain the grid points which are consistent with data. We repeat this until we obtain a stable grid, which typically happens after three to five iterations.

We parameterize each $f_j^i(\cdot)$ in (34) for $i = 1, ..., 4$ and $j = \pi, \mathcal{E}$ with 210 parameters for a total of 1,680 elements in Θ and use $M = 880$ including the grid points from the ergodic distribution and the filtered states. For a given set of filtered states and simulated grid, the solution takes about six minutes on a single-core Windows-based computer using MATLAB where some computationally-intensive parts of the code are run using Fortran via mex files. The approximation errors are in the order of 10^{-4} on average, expressed in consumption units.

[7]The work by Judd, Maliar, and Maliar evolved considerably over time. We initially built on the working paper version, Judd et al. (2010). In the published version of the paper, Maliar and Maliar (2015), also consider ϵ-distinguishable (EDS) grids and locally-adaptive EDS grids. Their locally-adaptive grids are similar in spirit to our approach, which tries to control accuracy in a region of the state space that is important for the substantive analysis, even if it is far in the tails of the ergodic distribution.

5 Model Estimation and Dynamics

The data sets used in the empirical analysis are described in Section 5.1. In Section 5.2, we estimate the parameters of the DSGE model for the U.S. and Japan using data from before the economies reached the ZLB. These parameter estimates are the starting point for the subsequent analysis. We consider four different specifications for each country that differ in terms of number of observable variables used and the details of the monetary policy rule. We solve the model using the nonlinear methods outlined in the previous section and in Section 5.3, we illustrate the dynamic properties of one of the estimated models by focusing on the economy's ergodic distributions and by presenting regime-specific impulse responses.

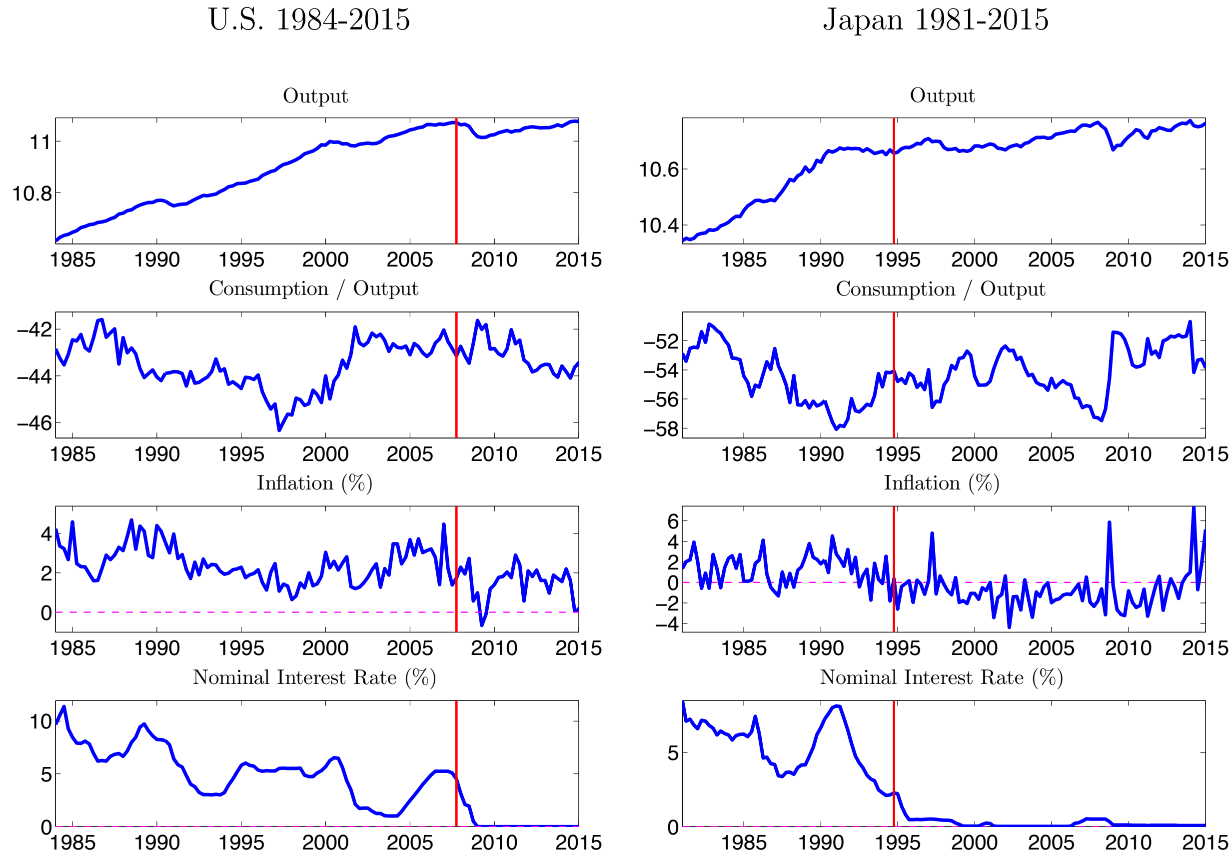
5.1 Data

The subsequent empirical analysis is based on log of real per-capita GDP, the log consumption-output ratio, GDP deflator inflation, and interest rates for the U.S. and Japan. The U.S. interest rate is the federal funds rate and for Japan we use the Bank of Japan's uncollateralized call rate. Consumption for the U.S. is the real personal consumption expenditures and real private consumption for Japan, where we normalize by an appropriate population measure to convert to per-capita terms. Further details about the data are provided in the Online Appendix.

The time series are plotted in Figure 3. The U.S. sample starts in 1984:Q1, after the Great Moderation and ends in 2015:Q2. The time series for Japan range from 1981:Q1 to 2015:Q1. The vertical lines denote the end of the estimation sample for each country, 2007:Q4 for the U.S. and 1994:Q4 for Japan. For the U.S. the fourth quarter of 2007 marks the beginning of the Great Recession, which was followed with a long-lasting spell of zero interest rate starting in 2009. In Japan, short-term interest rates dropped below 50 basis points in 1995:Q4 and have stayed at or near zero ever since. An important feature of the ZLB episode for Japan is consistently negative inflation rates – average inflation for Japan from 1999:Q1 to the end of the sample is nearly -1%.[8] This is in stark contrast with the

[8]The three positive spikes for Japanese GDP deflator inflation in 1997:Q2, 2008:Q4 and 2014:Q2 are

Figure 3: Data



Note: Output is the natural logarithm of per-capita output, consumption-output ratio is also in natural logarithm, scaled by 100, and inflation and nominal interest rate are in annualized percentage units. The vertical red line in each figure show the end of the estimation sample.

U.S., which experienced only two quarters of mildly negative inflation (2009:Q2 and Q3) and two quarters of inflation less than 0.5% at the very end of the sample. These features of the data are important (though not sufficient) for the identification of the sunspot regimes.

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unusual events that are not visible in, for instance, CPI inflation. The first and the third spike are due to increases in the value-added tax and the second is when a large decline in oil prices leads to a decrease in the import deflator which in turn generated a large jump in the GDP deflator. In our subsequent analysis we treat these observations as missing observations.

5.2 Model Estimation

For both the U.S. and Japan we estimate four versions of the DSGE model that differ in terms the monetary policy rules (growth vs. gap as in (31)) and the variables included in the estimation. For both countries, the first data set (three variables, henceforth 3v) comprises the log of output, inflation, and interest rates. The second data set (four variables, henceforth 4v) also includes the log consumption-output ratio.

For the U.S. the 3v data set is the standard data set for the estimation of small-scale DSGE models in the literature before the Great Recession, with the minor difference that we use the level of output instead of output growth. In this version we treat consumption as a latent variable and switch off the discount factor shock. Thus, the model is driven by technology growth, government spending (aggregate demand) and monetary policy shocks, which, again, is the typical specification for these models before the Great Recession. To estimate the DSGE model on the 4v data set, we activate the discount factor shock, which is widely used in the ZLB literature to drive model economies to the ZLB. Using a variant of consumption as an observable is natural, because the discount rate shock influences consumption directly.

During the estimation periods, it is clear that the data favor the targeted-inflation regime because both inflation and the nominal interest rate are positive. Moreover, we verify that the values of the state variables that are needed to rationalize the observations fall into a region of the state space in which the decision rules of the nonlinear model are well approximated by the decision rules obtained from a first-order perturbation solution of the DSGE model that ignores the ZLB. The first-order perturbation solution can be computed much faster and is numerically more stable than the global approximation to the sunspot equilibrium discussed in Section 4. We use a standard random walk Metropolis-Hastings (RWMH) algorithm to estimate the log-linearized DSGE models over the pre-ZLB sample periods. The implementation of the posterior sampler follows An and Schorfheide (2007) and is described in the Online Appendix.[9]

[9]The only somewhat nonstandard aspect of our methodology is the initialization of the Kalman filter to handle the nonstationarity in the log level of output.

We fix a subset of the parameters prior to the estimation. First, we want our model's average inflation conditional on being in the targeted-inflation regime to equal the average inflation in the estimation sample in each country. The former depends not only on π_* but also on the values for the sunspot transition parameters p_{11} and p_{00}. The latter two determine the expected durations of staying in each regime and therefore influence the long-run inflation expectations. We loosely calibrate these three parameters to match the following three observations: (i) average inflation conditional on $s_t = 1$ equals average inflation in the estimation sample; (ii) long-run inflation expectations when $s_t = 1$ are only slightly lower than average inflation; and (iii) when s_t transitions from one to zero, inflation expectations fall by about 1% in Japan and about 20 basis points in the U. S. Observations (ii) and (iii) are somewhat crude, obtained from long-run inflation expectations for Japan and the U.S. at the start of their ZLB experiences.[10] This procedure yields $p_{00} = 0.95$ and $p_{11} = 0.99$ for the U.S. and $p_{00} = 0.92$ and $p_{11} = 0.99$ for Japan. These values make the deflation regime ($s_t = 0$) less persistent than the targeted-inflation regime ($s_t = 1$) and imply that the unconditional probability of being in the deflation regime ($s_t = 0$) is 0.17 for the U.S. and 0.11 for Japan. Note that we identify the regime probabilities from the change in inflation expectations instead of the relative duration of the ZLB spell, which would be very sensitive to the start date of the estimation sample. The π_* values we use for each specification / country are tabulated in the Online Appendix.

We choose values for γ and β such that the steady state of the model matches the average output growth, and interest rates over the estimation sample period. The steady state government expenditure-to-output ratio is determined from national accounts data. Because our sample does not include observations on labor market variables, we fix the Frisch labor supply elasticity. Based on Ríos-Rull et al. (2012), who provide a detailed discussion of parameter values that are appropriate for DSGE models of U.S. data, we set $\eta = 0.72$ for

[10]This calibration involves fixing these three parameters, estimating the linear model to get values for the remaining parameters, solving the full nonlinear model to calculate the long-run averages of inflation and inflation expectations via simulations and iterating until we find a reasonable fit. The long-run inflation expectations are computed using the Consensus Forecast in Japan and taken from Aruoba (2014) for the U.S. (see Online Appendix).

the U.S. Our value for Japan is based on Kuroda and Yamamoto (2008) who use micro-level data to estimate labor supply elasticities along the intensive and extensive margin for males and females. The authors report a range of values which we aggregate into $\eta = 0.85$. The parameter ν, which captures the elasticity of substitution between intermediate goods, is set to 0.1. It is not separately identifiable from the price adjustment cost parameter ϕ. Finally, we calibrate the smoothing parameter α for trend output in (20) to make the implied trend output close to a measure from the data. For the U.S. we use the output gap measure produced by the Congressional Budget Office, and for Japan we use the potential growth rate from the Bank of Japan to construct an output gap measure.

For each country we estimate four DSGE model specifications: 3vGrowth, 4vGrowth, 3vGap, and 4vGap. The marginal prior distributions for τ, κ, ψ_1, ψ_2 and the parameters of the exogenous shock processes are tabulated in the Online Appendix. For the inverse IES τ we use Gamma distributions with mean 2 and standard deviations of 0.25 (U.S.) and 0.5 (Japan). We re-parametrize the price adjustment cost parameter ϕ in terms of the implied slope of the linearized New Keynesian Phillips curve: $\kappa = \tau(1-\nu)/(\nu\pi_*^2\phi)$. Our prior for κ has a mean of 0.3 and a standard deviation of 0.1, encompassing fairly flat and fairly steep Phillips curves. Our benchmark priors for the policy rule coefficients ψ_1 and ψ_2 are centered at 1.5 and 0.5, respectively, with standard deviations of 0.3 and 0.25, respectively. For the 3-variable specifications, the likelihood function was fairly uninformative about the policy rule coefficients.[11] Thus, we replaced the benchmark prior distributions for ψ_1 and ψ_2 with tighter prior distributions. For the 3vGap specifications the modified priors are centered at the parameter values obtained from the estimation of the corresponding 4-variable specification.

We truncate the prior distribution at the boundary of the determinacy region associated with the linearized version of the DSGE model. Thus, we are essentially imposing the existence of a second steady state (which requires that $\psi_1 > 1$) when we are estimating the model. In view of the empirical results in Lubik and Schorfheide (2004) who estimate a similar model without imposing determinacy on post-1982 data and find no evidence in favor of $\psi_1 < 1$, the $\psi_1 > 1$ restriction strikes us as reasonable.

[11]This problem is well recognized in the literature; see, for instance, Cochrane (2011) for a theoretical appraisal and Mavroeidis (2010) for identification-robust inference in single-equation estimation settings.

Table 2: Posterior of DSGE Model Parameters

	US								Japan							
Parameter	3vGrowth		4vGrowth		3vGap		4vGap		3vGrowth		4vGrowth		3vGap		4vGap	
τ	1.92	(1.56, 2.33)	2.10	(1.72, 2.51)	2.03	(1.66, 2.47)	2.28	(1.90, 2.68)	0.96	(0.59, 1.45)	1.21	(0.71, 1.84)	1.23	(0.79, 1.79)	1.73	(1.10,2.48)
κ	0.26	(0.17, 0.38)	0.31	(0.20, 0.44)	0.32	(0.18, 0.50)	0.27	(0.14, 0.44)	0.48	(0.29, 0.70)	0.48	(0.30, 0.69)	0.55	(0.37, 0.78)	0.39	(0.23,0.58)
ψ_1	1.47*	(1.38, 1.56)*	2.67	(2.21, 3.17)	2.50*	(2.42, 2.58)*	2.55	(2.17, 2.99)	1.43*	(1.19, 1.66)*	1.80	(1.43, 2.22)	1.67*	(1.59, 1.75)*	1.69	(1.36,2.10)
ψ_2	0.80*	(0.79, 0.82)*	0.90	(0.61, 1.24)	0.36*	(0.34, 0.37)*	0.35	(0.26, 0.44)	0.44	(0.28, 0.64)	0.42	(0.24, 0.63)	0.15*	(0.14, 0.17)*	0.15	(0.09,0.22)
ρ_r	0.67	(0.61, 0.73)	0.80	(0.76, 0.84)	0.73	(0.67, 0.78)	0.81	(0.77, 0.84)	0.64	(0.49, 0.75)	0.73	(0.61, 0.82)	0.73	(0.64, 0.80)	0.78	(0.70,0.85)
ρ_g	0.87	(0.84, 0.90)	0.93	(0.88, 0.97)	0.79	(0.75, 0.82)	0.87	(0.82, 0.93)	0.86	(0.80, 0.91)	0.94	(0.90, 0.98)	0.86	(0.81, 0.90)	0.92	(0.86,0.97)
ρ_z	0.10	(0.03, 0.20)	0.16	(0.05, 0.29)	0.09	(0.03, 0.17)	0.36	(0.17, 0.56)	0.03	(0.01, 0.08)	0.06	(0.02, 0.12)	0.07	(0.02, 0.13)	0.14	(0.04,0.29)
ρ_d			0.93	(0.90, 0.96)			0.94	(0.92, 0.97)			0.90	(0.84, 0.95)			0.90	(0.85,0.95)
$100\sigma_r$	0.22	(0.18, 0.26)	0.16	(0.14, 0.19)	0.19	(0.16, 0.23)	0.14	(0.12, 0.16)	0.24	(0.18, 0.32)	0.21	(0.16, 0.28)	0.21	(0.17, 0.26)	0.19	(0.15,0.23)
$100\sigma_g$	0.61	(0.51, 0.73)	0.46	(0.41, 0.52)	1.09	(0.94, 1.26)	0.47	(0.42, 0.53)	0.96	(0.68, 1.34)	0.75	(0.64, 0.88)	1.43	(1.10, 1.87)	0.77	(0.66,0.90)
$100\sigma_z$	0.64	(0.56, 0.74)	0.46	(0.41, 0.52)	0.57	(0.48, 0.68)	0.39	(0.33, 0.45)	1.22	(1.03, 1.45)	1.10	(0.94, 1.29)	1.01	(0.83, 1.23)	1.09	(0.93,1.28)
$100\sigma_d$			1.84	(1.30, 2.60)			2.47	(1.66, 3.84)			1.10	(0.71, 1.58)			1.37	(0.93,1.95)

Notes: The estimation samples are 1984:Q1-2007:Q4 for the U.S. and 1981:Q1-1994:Q4 for Japan. We report posterior means and 90% credible intervals (5th and 95th percentile of the posterior distribution) in parentheses. All results are based on the last 50,000 draws from a RWMH algorithm, after discarding the first 50,000 draws. Entries with a * indicate that we replaced the benchmark prior for this parameter with a tighter prior. See Online Appendix for further details.

The resulting posterior estimates reported in Table 2 are in line with the estimates reported elsewhere in the literature. Most notable are the implicit estimates of the slope of the New Keynesian Phillips curve, which are around 0.3 for the U.S. and 0.5 for Japan, implying fairly flexible prices and relatively small real effects of unanticipated interest rate changes.[12] The posterior distributions for most of the estimated parameters move somewhat significantly away from their priors, or at least they get much tighter. A notable exception is the elasticity of intertemporal substitution parameter τ for the U.S., which remains near the prior mean of 2.

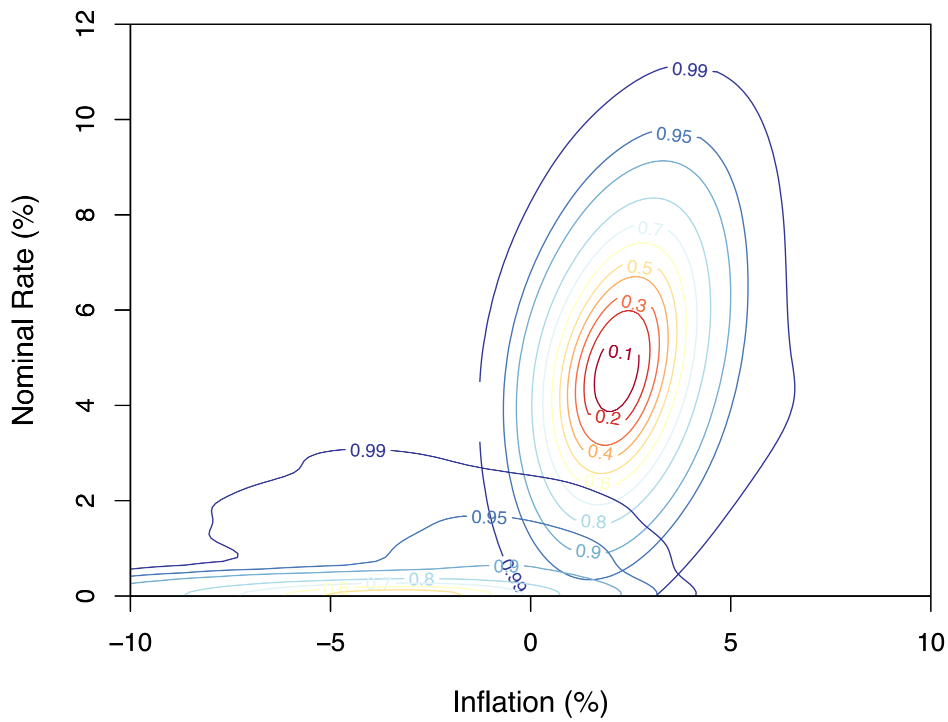
5.3 Equilibrium Dynamics

In this section we discuss the dynamics of the estimated specifications. Since we have a total of eight estimated specifications, we focus on the 4vGrowth specification for the U.S. All models behave qualitatively similarly, though sometimes there are quantitative differences, which we point out when relevant. We start with an illustration of the ergodic distribution by simulating a long sequence of observations. Given our choices of p_{00} and p_{11} for the U.S., approximately 17% of the observations are associated with the deflation regime, whereas the remaining 83% are associated with the targeted-inflation regime. Figure 4 depicts contour plots for the joint probability density function of inflation and interest rates conditional on the regimes $s_t = 0$ and $s_t = 1$, respectively. Formally, we show $p(R_t, \pi_t | s_t = j)$ for $j = 0, 1$, which means that the two sets of contours are not weighted by the unconditional probabilities $\mathbb{P}\{s_t = j\}$. In the contour plots each line represents one percentile with the outermost line showing the 99^{th} percentile. Under the deflation regime there is a high probability that the interest rate is equal to zero, which leads to a point mass on the x-axis and is not reflected in the contour plot.

As expected, the two regime-conditional distributions are approximately centered near the respective steady state values. Average inflation when $s_t = 1$ is slightly above π_* (2.5% versus 2.4%) and average inflation conditional on $s_t = 0$ is below inflation at the deflation

[12] A survey of DSGE-model-based New Keynesian Phillips curve is provided in Schorfheide (2008). Our estimates fall within the range of the estimates obtained in the literature.

Figure 4: Regime-Conditional Ergodic Distribution: 4vGrowth, U.S. Data



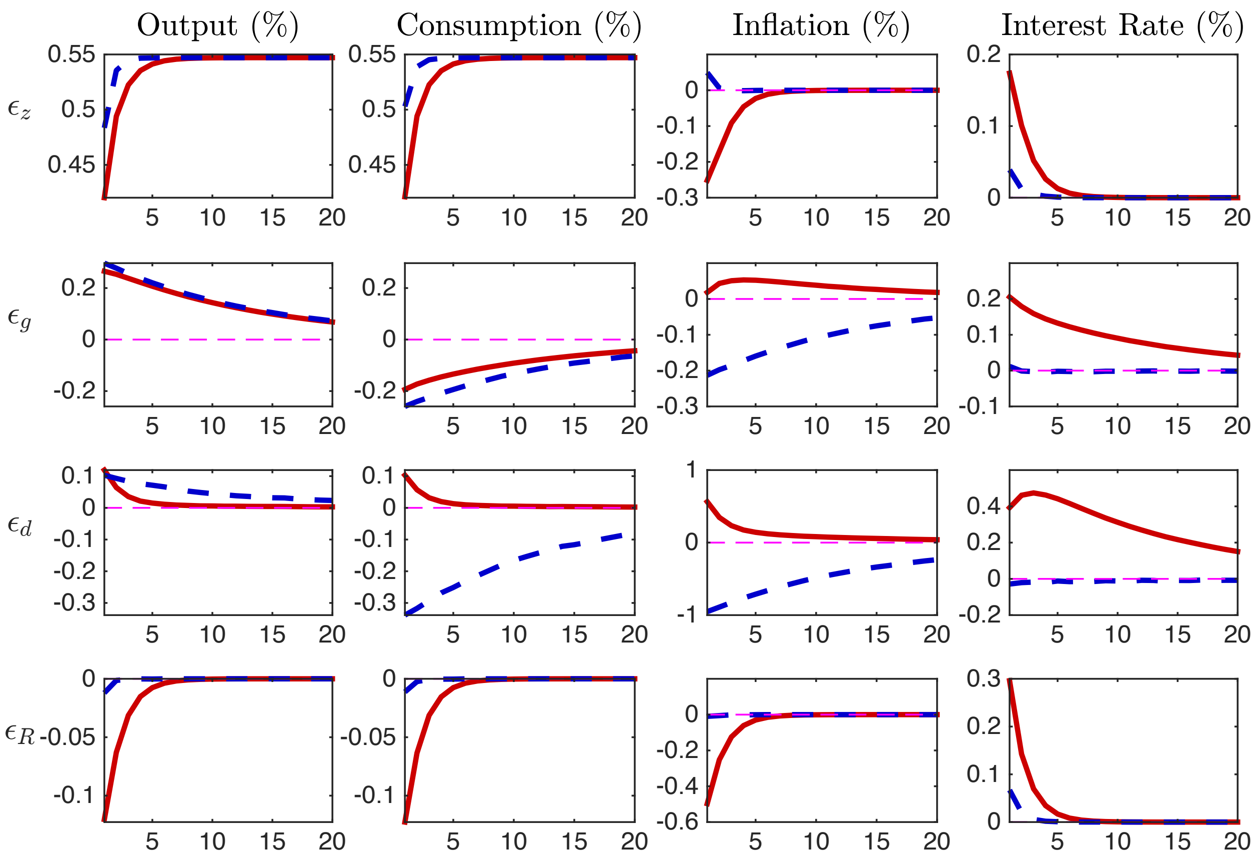
Notes: Figure depicts the joint probability density function (kernel density estimate) of annualized net inflation and interest rates conditional on the targeted-inflation regime and the deflation regime, respectively. Formally, the two sets of contours correspond to $p(R_t, \pi_t | s_t = j)$ for $j = 0, 1$.

steady state (-4.2% versus -2.8%). Under the targeted-inflation regime, inflation is positive with probability 99.7%. The probability of reaching the ZLB given $s_t = 1$ is virtually zero given the shock processes estimated based on the pre-Great-Recession sample. This means that rationalizing the post-2008 U.S. experience with the targeted-inflation regime requires large shocks that are unlikely in view of the pre-2008 data. Under the deflation regime, on the other hand, interest rates are zero with 89% probability – even in the absence of extreme shocks – and inflation rates are negative with 97.6% probability.[13]

To better understand how the economy evolves in each regime, we compute impulse response functions (IRFs) to one standard deviation shocks conditional on remaining in the same regime throughout the response. Prior to the shock the economy is assumed to be at

[13]Because average inflation in the estimation sample and hence π_* is lower in Japan, it is more likely to observe deflation when $s_t = 1$ but virtually impossible to observe positive inflation when $s_t = 0$.

Figure 5: Impulse Response Functions: 4vGrowth, U.S. Data



Notes: The figure depicts impulse response functions to one-standard deviation shocks conditional. The economy is assumed to be at the mean of the regime-conditional distribution when the shocks hit and to stay in the regime in the remaining periods. Solid lines depict the responses for $s_t = 1$ and dashed lines show the responses for $s_t = 0$. For output and consumption the figure shows percentage deviations from the baseline path. For the interest rate and inflation it shows differences in annualized percentages relative to the baseline path.

the mean of the regime-conditional ergodic distribution. The IRFs are plotted in Figure 5. Each column corresponds to one of the variables of interest (output, consumption, inflation, and interest rate) and each row corresponds to one of the structural innovations ($\epsilon_{z,t}$, $\epsilon_{g,t}$, $\epsilon_{d,t}$, and $\epsilon_{R,t}$).

The responses conditional on $s_t = 1$ are standard. The shock to technology raises output and consumption permanently. Because it is a supply shock, prices and quantities move in opposite directions. The reaction to the positive output growth dominates in the monetary policy rule and therefore the interest rate rises. The government spending shock acts like an aggregate demand shock, increasing output and inflation temporarily. In response the central bank raises interest rates. Because nominal interest rates rise more strongly than inflation, the real interest rate increases, which reduces consumption.

To understand the response to the discount factor shock innovation $\epsilon_{d,t}$, recall that the stochastic discount factor M_{t+1} is a function of $\beta d_{t+1}/d_t$. In log-linear terms, an unanticipated rise in $\hat{d}_t$ implies that $\mathbb{E}_t[\hat{d}_{t+1} - \hat{d}_t] = (\rho_d - 1)\hat{d}_t$ is negative, because $\hat{d}_t$ follows an AR(1) process. Thus, a positive $\hat{d}_t$ shock makes the households less patient. This induces an increase in consumption and output, and an associated rise in inflation. The central bank reacts to these by increasing the interest rate, dampening the effect of the discount factor shock. The discount factor can be interpreted as an aggregate demand shock in the sense that it generates positive comovement between output and inflation. Unlike an expansionary g_t shock, however, the d_t shock raises consumption.

A shock to monetary policy that increases the interest rate has the usual effects: output and inflation fall and, because the real interest rate rises, consumption falls as well. According to our estimates, the degree of price stickiness is relatively small and therefore the New Keynesian Phillips curve is relatively steep. Thus, the real effect of an unanticipated monetary policy shock is small (output and consumption drop by about 10 basis points) in comparison to the inflation response (annualized inflation falls by about 40 basis points).

The IRFs conditonal on the $s_t = 0$ regime display some important differences. In this case, a positive technology shock increases inflation slightly. On the other hand, positive government spending and discount factor shocks reduce inflation. Thus, the signs of the inflation responses switch, compared to the $s_t = 1$ regime. This result is linked to the findings of Eggertsson (2011) and Mertens and Ravn (2014), who show that positive demand shocks may lead to a negative comovement of prices and output in the deflation regime.[14] The sign switching for the inflation response to the discount factor shock is the same phenomenon that we demonstrated in Section 2 for the simple model, in which inflation responds positively

[14]More specifically, Mertens and Ravn (2014) show that the EE curve, which plots inflation versus output using the relationship in (27) with necessary substitutions, has two segments, one downward sloping and one upward sloping. If the equilibrium is in the upward-sloping portion, then a positive demand shock may generate a decrease in inflation while increasing output.

to a real-rate shock in the $s_t = 1$ regime but negatively in the $s_t = 0$ regime. We also observe in Figure 5 that consumption falls instead of rises in response to a discount factor shock because of the decline in the real interest rate. Finally, monetary policy is much less effective in the deflation regime.

6 Evidence of a Sunspot Switch

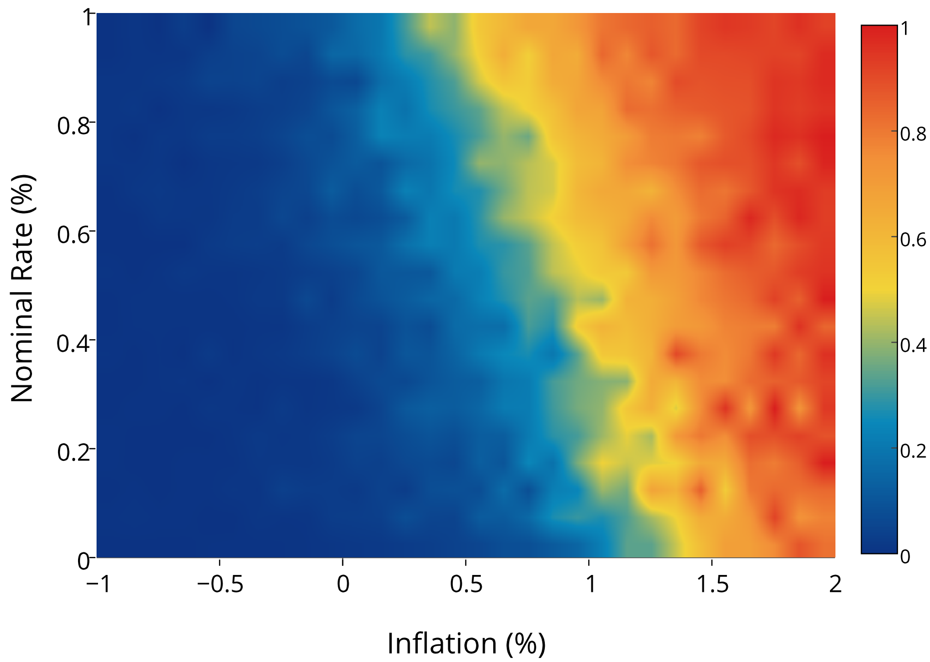
We are now ready to address our main empirical question: did the U.S. and Japan experience a change in regimes due to a switch in the sunpot variable at or near the beginning of their ZLB episodes? We do this in multiple steps. First, we examine the evidence that individual pairs of inflation and interest rate observations provide about the prevailing sunspot regime in Section 6.1. Next, we use a nonlinear filter in Section 6.2 to track the sunspot regime over time, which brings in information from other variables and allows for dynamics to matter. The analysis up to this point is using all four specifications for each country. Finally, we aggregate the filtering results across the different model specifications in Section 6.3.

6.1 Static Analysis: Evidence from Inflation and Interest Rate Observations

Our goal in this section is to conduct inference on the hidden process s_t. In Figure 4 we showed the bivariate ergodic distribution of (π_t, R_t) for the targeted-inflation and the deflation regimes. Glancing at Figure 4, it seems clear that an observation of a 3% inflation and a 6% interest rate is strong evidence in favor of $s_t = 1$. Conversely, zero interest rates combined with an inflation rate of -5% provides evidence for $s_t = 0$. However, if the interest rate is zero and inflation is low, as it has been the case for the U.S. since 2009, it is more difficult to determine by visual inspection which regime is favored by the data. The heatmap in Figure 6 shows

$$\mathbb{P}\{s_t = 1 | R_t, \pi_t\} = \frac{p(R_t, \pi_t | s_t = 1)\mathbb{P}\{s_t = 1\}}{p(R_t, \pi_t | s_t = 1)\mathbb{P}\{s_t = 1\} + p(R_t, \pi_t | s_t = 0)\mathbb{P}\{s_t = 0\}}. \tag{35}$$

Figure 6: $\mathbb{P}\{s_t = 1|\pi_t, R_t\}$: 4vGrowth, U.S. Data



Notes: The legend for the colors is to the right of the heatmap.

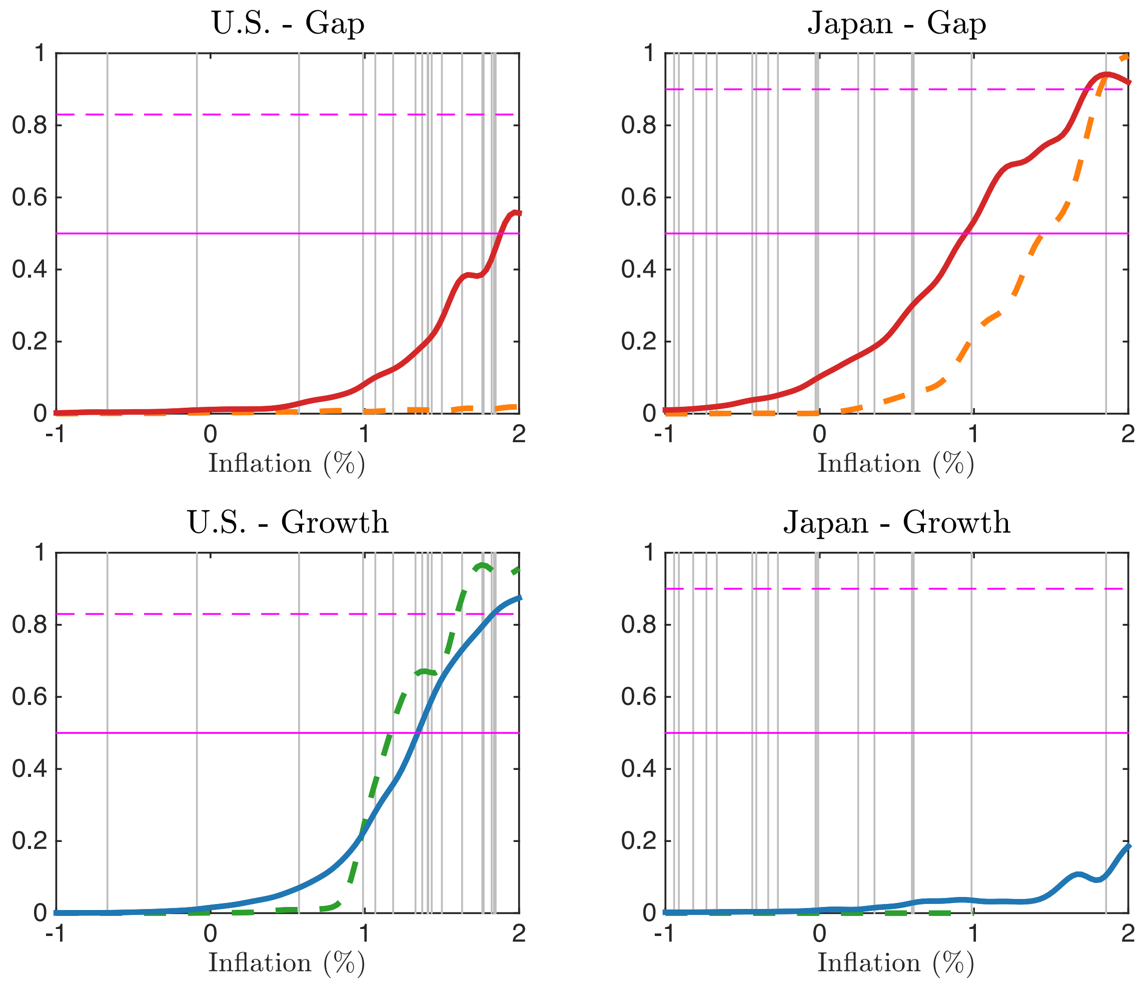
for a section of the (π_t, R_t) space, for which the evidence about the sunspot shock based on the contour plot in Figure 4 is ambiguous.[15] Suppose interest rates are around 25 basis points. Then an inflation rate of more than 1.5% would be interpreted as some evidence for $s_t = 1$ (indicated by the warm colors), whereas an inflation rate below 0.75% would be evidence in favor of $s_t = 0$ (indicated by dark blue).

Because our main interest is to infer the sunspot regime during the respective ZLB episodes of the U.S. and Japan, we want to zoom in to the bottom part of the heatmap figure. Thus, we now compute $\mathbb{P}\{s_t = 1|ZLB, \pi_t\}$, where we interpret interest rates in the range from 0% to 0.25% as the ZLB being binding. Results are depicted in Figure 7. Unlike in the heatmap we now apply a kernel smoother to approximate the probabilities. In this figure vertical lines correspond to the inflation value of a ZLB observation for the country.

Not surprisingly, all probabilities start at zero for low inflation observations and are

[15]To generate the heatmap we define bins for inflation and interest rates and count the number of realizations within each bin based on a long simulation from the model. The probability that $s_t = 1$ in a bin simply is the fraction of $s = 1$ observations in that bin.

Figure 7: $\mathbb{P}\{s_t = 1|ZLB, \pi_t\}$



Notes: For the purposes of this figure ZLB is defined as interest rate being between 0% and 0.25%. In each panel, solid lines show the 4-variable specification and dashed lines show the 3-variable specification. Each vertical line shows the inflation value for a data point in the ZLB sample for the country. The horizontal dashed line shows the country-specific threshold $\mathbb{P}\{s_t = 1\}$, which is 0.83 for the U.S. and 0.89 for Japan. The horizontal solid line is at 0.5.

increasing as inflation increases. The probabilities can be compared to three thresholds: 0.5, $\mathbb{P}\{s_t = 1\}$, and $\mathbb{P}\{s_t = 1|ZLB\}$. The first threshold is natural to construct a point estimator of s_t that is restricted to the set $\{0, 1\}$. As soon as the posterior probability of s_t exceeds 0.5, the point estimate (under a 0-1 loss function) is $\hat{s}_t = 1$. The second threshold is the prior probability of being in the targeted-inflation regime, which is 0.83 for the U.S. and 0.89 for Japan. If the inflation and interest rate pair exceeds the second threshold, then the data provide additional evidence in favor of the targeted-inflation regime. If the third threshold is exceeded, then the inflation observation increases the evidence against the deflation regime conditional on the economy being at the ZLB. The first two thresholds are shown by horizontal lines in Figure 7. The third threshold conditions on being at the ZLB. The probabilities $\mathbb{P}\{s_t = 1|ZLB\}$ are close to zero for all specifications and are not shown.

As we saw in Figure 3, inflation rates in the U.S. were mostly positive and inflation rates in Japan were mostly negative during the ZLB period. The plots in Figure 7 suggest, ignoring the fact that evidence from multiple observations should be aggregated, that the Japanese inflation and interest rate data imply evidence in favor of the deflation regime: at the observed π_t's the posterior probabilities of $s_t = 1$ are very close to zero, substantially below any of the three thresholds, with the exception of the 4vGap specification for Japan, which clears the third threshold but not the other two.

For the U.S. the conclusion depends on the model specification and the threshold used. Using 0.5 as the cutoff for a point estimate of s_t that is restricted to zero or one, the growth specifications imply that most of the observations favor the targeted-inflation regime, while the gap specifications favor the deflation regime. Relative to the prior distribution $\mathbb{P}\{s_t = 1\}$ the evidence in almost all of the inflation and interest rate observations leads to a downward revision of the probability that the economy is in the targeted-inflation regime. However, this downward revision is not as strong as in the case of Japan because U.S. inflation rates remained mostly positive.

While individual inflation and interest rate observation provide some evidence about the regime, this evidence does not suffice to determine whether the U.S. or Japan did transition to the deflation regime. First, the economy evolves dynamically and the probability of being in one regime or another depends not only on the observed variables but also on the state of the economy, including the history of s_t. Second, variables other than inflation may contain key information that may help distinguish the two regimes – this is evident from Figure 6 by the wide yellow-colored region where the probability of being in the two regimes are about the same. Third, the four different specifications may and do disagree. In the next two sections we tackle these issues to obtain a single and clear answer to the question of which regime the two countries were in their ZLB episodes.

6.2 Dynamic Analysis: Evidence from a Nonlinear Filter

We now use a nonlinear filter to conduct inference about the hidden state s_t. The filter addresses two of the above-mentioned shortcomings of the static analysis: it accounts for the state of the economy in period $t-1$ and it also uses information from output and the consumption-output ratio (4-variable specifications). The DSGE model has a nonlinear state-space representation of the form

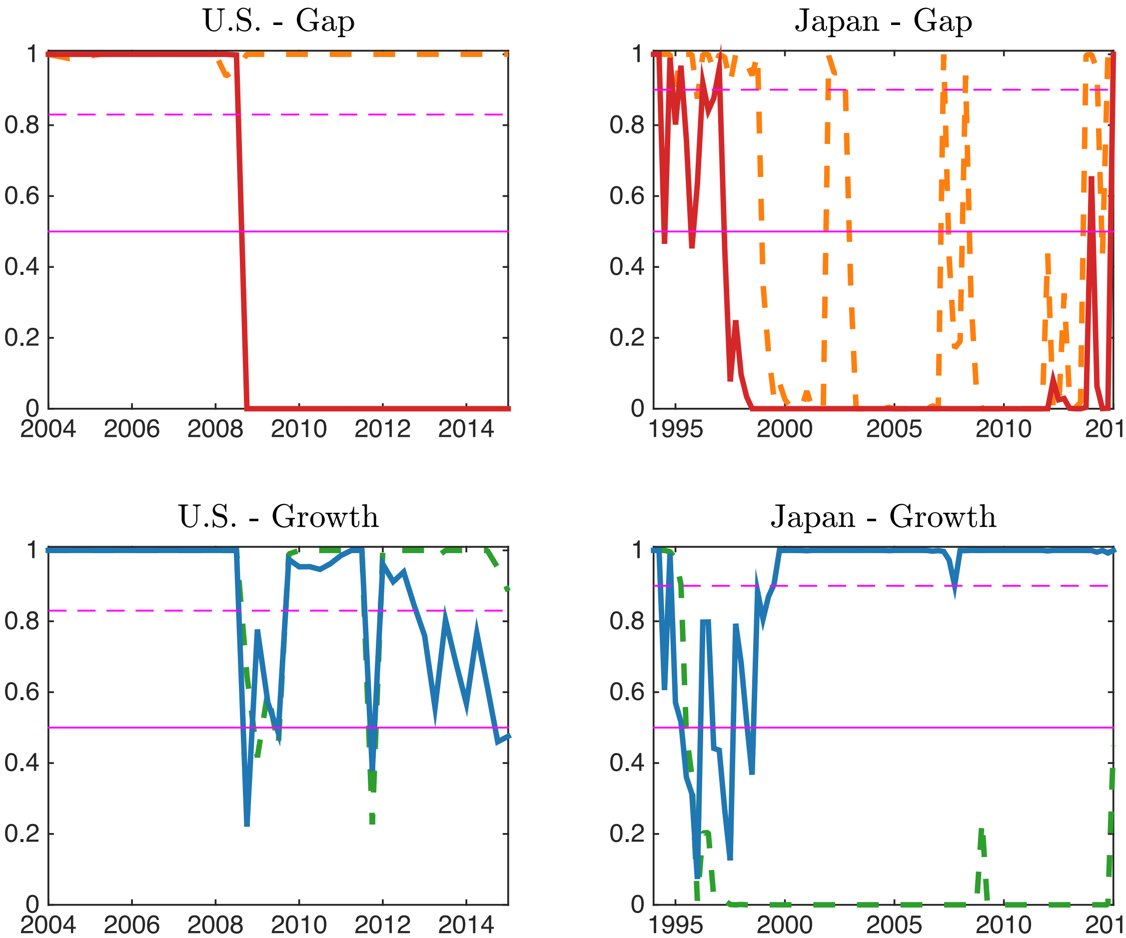
$$\begin{aligned} y_t^o &= \Psi(x_t) + \nu_t \\ x_t &= F_{s_t}(x_{t-1}, \epsilon_t) \\ \mathbb{P}\{s_t = 1\} &= \begin{cases} (1-p_{00}) & \text{if } s_{t-1} = 0 \\ p_{11} & \text{if } s_{t-1} = 1 \end{cases} \end{aligned} \tag{36}$$

Here y_t^o is the vector of observables. We use the o superscript to distinguish the vector of observables from detrended output in our DSGE model. For the 3-variable specifications it consists of log of output, inflation, and nominal interest rates. For the 4-variable specifications the vector also includes the log consumption-output ratio. $Y_{1:t}^o$ is the sequence $\{y_1^o, \ldots, y_t^o\}$. The vector x_t stacks the continuous state variables, which are given by $x_t = [R_t, y_t, y_t^*, y_{t-1}, d_t, z_t, g_t, A_t]'$, and $s_t \in \{0, 1\}$ is the Markov-switching process, where y_{t-1} is only necessary in the growth specifications, y_t^* is only necessary in the gap specifications, and d_t is only relevant for the 4-variable specifications.[16] The lower case output variables in the state vector are detrended by the level of technology A_t. The first equation in (36) is the measurement equation, where $\nu_t \sim N(0, \Sigma_\nu)$ is a vector of measurement errors. The second equation corresponds to the law of motion of the continuous state variables. The vector $\epsilon_t \sim N(0, I)$ stacks the innovations $\epsilon_{d,t}$, $\epsilon_{z,t}$, $\epsilon_{g,t}$, and $\epsilon_{R,t}$, where once again $\epsilon_{d,t}$ is used only in the 4-variable specifications. The functions $F_0(\cdot)$ and $F_1(\cdot)$ are generated by the model solution procedure described in Section 4. The third equation represents the law of motion of the Markov-switching process.

Given the system in (36) and conditioning on the posterior mean estimates obtained in Section 5.2, we use a sequential Monte Carlo filter (also known as the particle filter) to

[16]The econometric state variables x_t of the state-space representation are slightly different from the economic state variables $\mathcal{S}_t$ that appear in the solution.

Figure 8: Filtered Probability of Targeted-Inflation Regime



Notes: The figure shows the filtered probabilities $\mathbb{P}\{s_t = 1|Y^o_{1:t}\}$ for each specification and country, starting five years prior to the start of the ZLB episode for the country. In each panel, solid lines show the 4-variable specification and dashed lines show the 3-variable specification. The dashed horizontal line shows the country-specific threshold $\mathbb{P}\{s_t = 1\}$, which is 0.83 for the U.S. and 0.89 for Japan. The solid horizontal line shows 0.5.

extract estimates of the sunspot shock process s_t, and the latent state x_t.[17] Because the filter is sequential, the results of this filter can be thought of as a quasi-real-time assessment of the probability of a sunspot switch.[18] Figure 8 depicts the filtered probabilities $\mathbb{P}\{s_t = 1|Y^o_{1:t}\}$. As in Figure 7, we plot the prior $\mathbb{P}\{s_t = 1\}$ as a dashed horizontal line in each panel.

[17]This filter is described in the Online Appendix. A more detailed exposition is provided in Herbst and Schorfheide (2015).

[18]We use the qualifier "quasi" because the data we use is not the real-time data but what is available as of the date we write the paper.

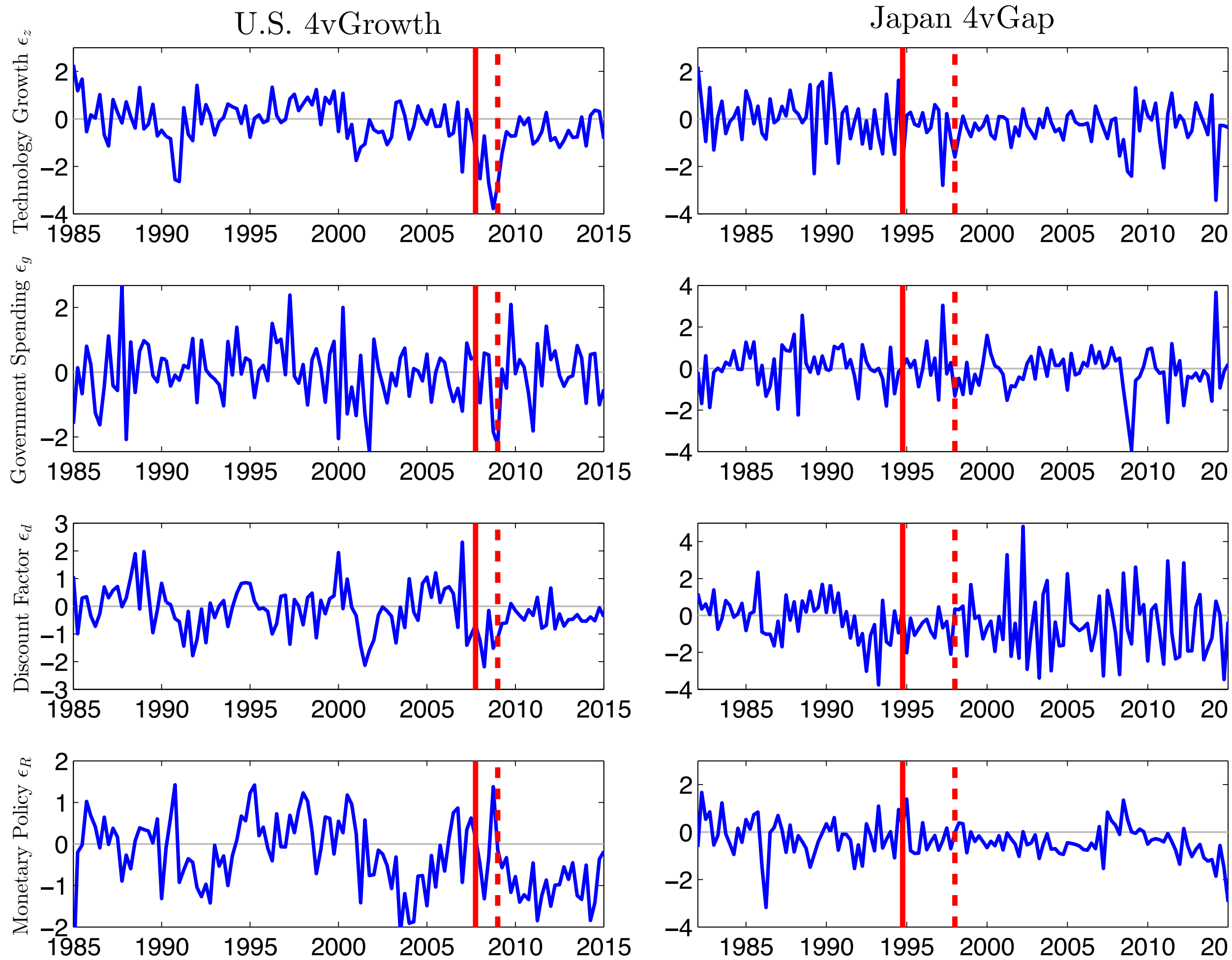
Using the simple rule by which $\mathbb{P}\{s_t = 1 | Y^o_{1:t}\} > \mathbb{P}\{s_t = 1\}$ is interpreted as evidence in favor of $s_t = 1$, we find that three out of four specifications for the U.S. indicate that the economy stayed in the targeted-inflation regime after 2008, although the conclusions for 4vGrowth is somewhat less strong. For Japan, we draw the opposite conclusions. Three out of four specifications suggest that Japan transitioned to the deflation regime in the late 1990s. The exceptions are the 4vGap specification for the U.S. and 4vGrowth specification for Japan. It is interesting to note that across the four specifications for the U.S. there is some uncertainty which vindicates Bullard (2010)'s concern of the possibility of a shift to the deflationary regime.

The filter also generates estimates of the exogenous shock processes and their innovations. The subsequent discussion focuses on the 4vGrowth specification for the U.S. and the 4vGap specification for Japan. We will see in Section 6.3 that the inference about s_t from these two specifications is by and large consistent with the conclusions drawn after aggregation across the four specifications for each country. Time series plots for the filtered innovations ϵ_t are provided in Figure 9. Recall that in our model $\log C_t/Y_t \approx -\log g_t$ (in a first-order approximation of the aggregate resource constraint the relationship holds exactly). Thus, the government spending shock by construction tracks the consumption-output ratio. In the last quarter of 2008, the U.S. experienced a large drop in output, which turned out to be permanent. In our model, this is captured by a negative technology growth shock of roughly 3.5 standard deviations. In addition, the aggregate demand shock g_t dropped by about 2 standard deviations and the discount factor innovation is also negative in 2008:Q4. All three adverse shocks generate a drop in interest rates (see Figure 5) and push the economy toward the ZLB. While the two adverse demand shocks are deflationary, the adverse technology shock is inflationary. This is consistent with the modest decline in inflation.[19]

After 2008:Q4, the technology growth shocks stay slightly negative on average, depressing output growth and preventing a quick and full recovery. The discount factor shock

[19]Due to the simplicity of the DSGE model, the shock decomposition is not refined enough to generate a more detailed narrative of the recent U.S. experience that emphasizes the disruption in financial intermediation. Shocks to the financial system and nonlinearities generated by its disruption, are interpreted as large technology or discount factor shocks by our model.

Figure 9: Filtered Shock Innovations ϵ_t



Note: Innovations are shown in multiples of their standard deviations. The solid vertical line shows the end of the estimation sample and the dashed vertical line shows the beginning of the ZLB episode.

innovations also remain on average negative, delaying the mean reversion of the d_t process and keeping the economy near the ZLB. Moreover, the filtered sequence of monetary policy shocks is mostly negative. In the absence of these shocks interest rates would have been between 0.75% and 1%. Thus, from the perspective of the DSGE model, U.S. monetary policy is more expansionary in the aftermath of the Great Recession than what is implied by the systematic part of the interest rate feedback rule.

Given the long-lasting drop in output, one might expect the Phillips curve relationship in the DSGE model to imply a significant deflation, which did not occur in the U.S.. In this regard, our simple DSGE models works similarly as the richer DSGE model studied in Del Negro et al. (2015). One important reason for why the New Keynesian Phillips curve embedded in the DSGE model does not predict deflation is that the Phillips curve is forward looking. Inflation depends on the sum of discounted expected future marginal costs. Because the model has a fairly strong mean reversion, it predicts marginal costs to rise in the medium run, which allows the model to explain what the literature has termed the "missing deflation" in the U.S. Moreover, the expansionary monetary policy contributed to the positive inflation rates.

The 4vGap specification for Japan implies that the economy transitioned to the deflation regime in the late 1990s. In the deflation regime the negative inflation rates generate a non-negligible resource cost and the approximation $\log C_t/Y_t \approx -\log g_t$ is no longer accurate. The discount factor shock also affects the consumption-output ratio. As is apparent from the impulse responses in Figure 5 none of the shocks has a significant impact on the interest rates, which are with high probability zero. The filter essentially inverts these relationships. Most notably, the slow growth of the Japanese economy since the late 1990s maps into technology growth innovations that are on average negative. An inspection of the regime-conditional ergodic distributions drawn in Figure 4 indicates that inflation rates in the deflation regime are with high probability less than -4%.[20] Actual Japanese inflation, while being negative, has always been above -4%, which is translated by the filter in a sequence of discount factor innovations that are fairly volatile and on average below zero.

6.3 Aggregating the Results

We now formally aggregate the results from the four different specifications for each country by computing weights for each specification that are related to the goodness of fit. The obvious difficulty here is that the four specifications do not share a common dataset. In order to compare 3-variable and 4-variable specifications, we follow the approach in Del Negro et al. (2016) and construct one-step-ahead predictive densities for the subset of common

[20]The contours for 4vGap-Japan look similar to the contours for 4vGrowth-U.S., which are shown in the figure.

observations. Let z_t^o be the 3×1 vector of output, inflation, and interest rates. These three variables are the core variables that most New-Keynesian models aim to capture. Moreover, let $p(z_t^o|Y_{1:t-1}^o, \mathcal{M}_j)$ be the predictive density for z_t^o given specification $\mathcal{M}_j$ and the $t-1$ information set $Y_{1:t-1}^o$.[21] Based on the predictive densities, we can define the quasi model probabilities

$$\tilde{p}_t(\mathcal{M}_j) = \frac{\prod_{t=T_0}^{T} p(z_t^o|Y_{1:t-1}^o, \mathcal{M}_j)}{\sum_{j=1}^{4} \prod_{t=T_0}^{T} p(z_t^o|Y_{1:t-1}^o, \mathcal{M}_j)} \tag{37}$$

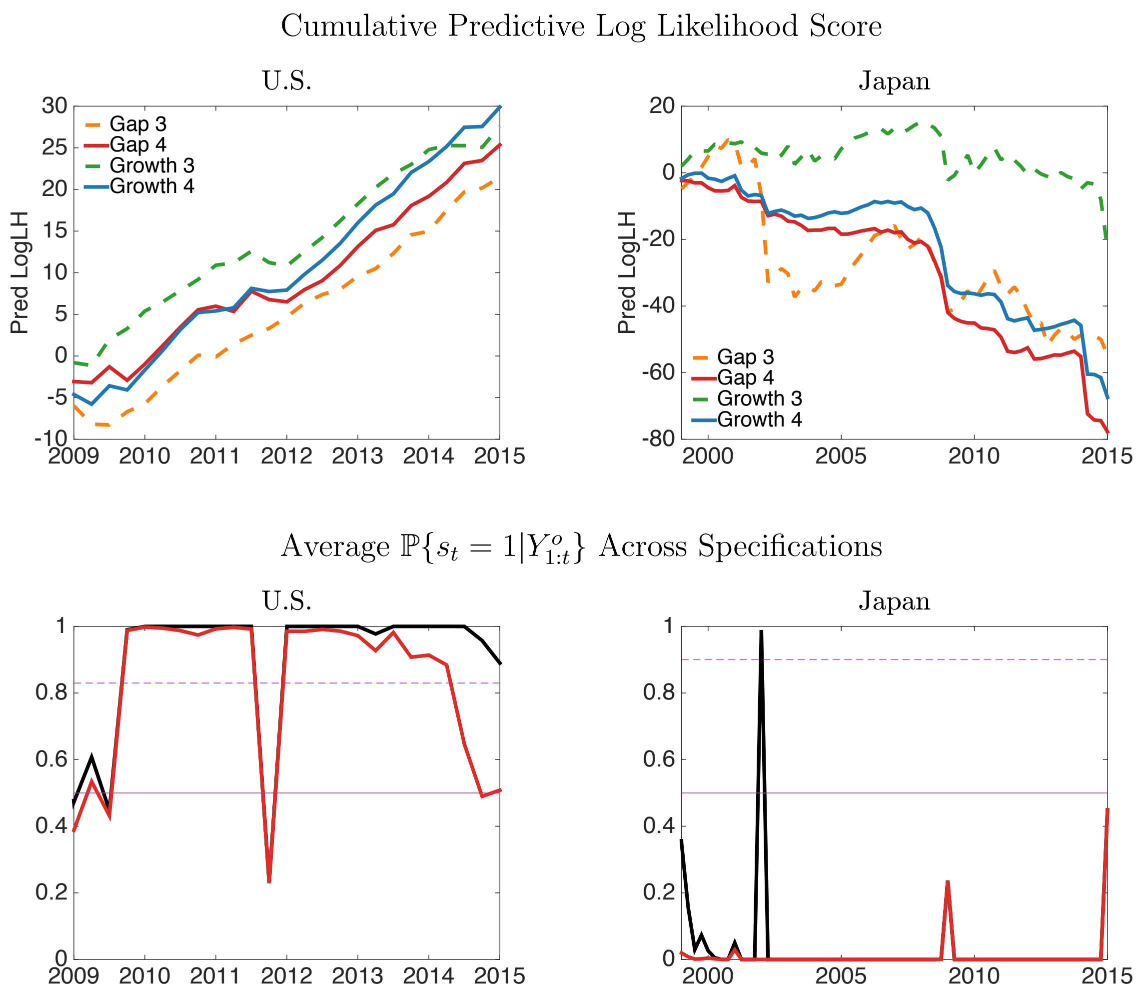
and use them to create weighted averages of $\mathbb{P}\{s_t = 1|Y_{1:t}^o, \mathcal{M}_j\}$.

The results are presented in Figure 10. For each specification, we plot the log of the numerator of (37) in the top two panels. We take T_0 to be the beginnings of the respective ZLB periods. Each line can be interpreted as running predictive score of a model specification. For the U.S. the difference in fit between the four specifications is relatively small. The performance differential between the best and the worst specification does not significantly widen over time, though the relative rankings change. Until the end of 2013 the 3vGrowth specification dominates, whereas after 2014, the 4vGrowth specification attains the highest predictive score. The 3vGap specification is the least preferred. For Japan the 3vGrowth specification also is best, though by a much larger margin. Unlike for the U.S., in the case of Japan the gap between the 3vGrowth specification and the three other specifications widens toward the end of the sample.

From an ex ante perspective, the relative ranking of the 3- and 4-variable specifications based on the predictive likelihood for is unclear. The 3-variable models are optimized to track the variables included in the z_t^o vector. The 4-variable models, on the one hand, have an additional degree of freedom, namely, the latent discount factor shock d_t, which can improve the tracking of the trivariate vector z_t^o. On the other hand, the 4-variable specifications also have to track the consumption-output ratio. Doing so may lead to a deterioration of the one-step-ahead forecast performance for z_t^o. Ex post, it turns out that one of the 3-variable specifications, namely 3vGrowth, dominates the 4-variable specifications. However, for both the U.S. and Japan, the 4-variable specifications are competitive with the 3vGap

[21]The conditioning information set here differs across the 3-variable and 4-variable specifications. To avoid an overly tedious notation, we did not introduce a j index for the information set.

Figure 10: Combined Filtered Probabilities



Notes: Top panels: the log predictive score is cumulative. T_0 is the beginning of the ZLB episodes. Bottom panels: we convert the cumulative log predictive score into quasi model probabilities (see (37)) and use them to compute a weighted average of $\mathbb{P}\{s_t = 1 | Y^o_{1:t}\}$ across the four specifications. We consider two choices of T_0: the beginning of the ZLB episodes (red) and the beginning of the estimation sample (black). The horizontal dashed line shows the country-specific threshold $\mathbb{P}\{s_t = 1\}$, which is 0.83 for the U.S. and 0.89 for Japan. The solid line indicates the 0.5 threshold.

specification. Even though we are considering a period in which interest rates are zero, there seems to be information about the policy-rule specification (gap versus growth). This information arises from the fact that even if interest rates are currently zero, beliefs about the future conduct of monetary policy affect current output and inflation.

The bottom panels of Figure 10 show the quasi model probabilities computed for two choices of T_0: the beginning of the ZLB episodes (red) and the beginning of the estimation sample (black). The first choice is consistent with the predictive scores depicted in the top panels of the figure. The second choice of T_0 also factors in the fit of the model specifications prior to the ZLB episodes and thereby places more weight on the 3-variable specifications. Using horizontal lines, we depict two of the thresholds discussed in Section 6.1: 0.5 and $\mathbb{P}\{s_t = 1\}$. After aggregating the information from the four specifications, we conclude that the U.S. has remained in the targeted-inflation regime in the aftermath of the Great Recession and that Japan's ZLB experience is best described by a switch to the deflation regime.

For the U.S. there is significant uncertainty about the regime at the beginning of 2009. However, subsequently, there is only a single quarter, 2011:Q4, in which the probability of being in the targeted-inflation regime falls below 0.5. This quarter exhibits an unusually low inflation rate. In 2014, using the weights based on T_0=2009:Q1 the probability of the targeted-inflation regime falls toward 0.5, because the 4vGrowth specification starts to dominate the weighted average. Recall from the bottom right panel of Figure 8 that the filtered probability of $s_t = 1$ drops from 1 to about 0.5 between 2012 and the end of the sample. For Japan there is only one quarter in which the probability of being in the targeted-inflation regime clears all thresholds. This happens in 2002:Q1, when inflation is positive and seems like an outlier relative to the period before and after. Except in 2000 and in 2002:Q1 the black and the red lines are on top of each other, implying that the inference is not sensitive to the choice of T_0.

7 Conclusion

The recent experiences of the U.S. and Japan have raised concern among policy makers about a long-lasting switch to a regime in which interest rates are zero, inflation is low, and conventional macroeconomic policies are less effective. We solve a small-scale New Keynesian DSGE model imposing the ZLB constraint and introducing a non-fundamental Markov sunspot shock that can move an economy between a targeted-inflation regime and a deflation regime. An economy may be pushed to the ZLB either by successive fundamental shocks (e.g., an adverse discount factor shock) in the targeted-inflation regime or by a switch to the deflation regime. We develop a quantitative framework that can distinguish these two possibilities.

Our empirical analysis focuses on the U.S. and Japan and utilizes four different DSGE model specifications for each country that differ in terms of the observables used and the monetary policy rule. Using a nonlinear filter, we find that for each country three of the four specifications agree: the U.S. remained in the targeted-inflation regime during its ZLB episode, with the possible exception of the early part of 2009 where evidence is more mixed. Japan switched to the deflation regime in 1999 and remained there until the end of our sample. We aggregate our results using quasi model probabilities and the final results confirm the above conclusions.

Our model is silent as to why the two experiences are different because the sunspot process in our model is purely exogenous. In a richer, but computationally much more challenging specification, the coordination of beliefs may be correlated with fundamentals and be affected by central bank actions. Perhaps one key difference between Japan in 1999 and the U.S. in 2009 is in their conduct of monetary policy. Ito and Mishkin (2006), who provide a summary of the actions taken by the Bank of Japan and the Japanese government conclude that "(...) mistakes in the management of expectations [by the Bank of Japan] are a key reason why Japan found itself in a deflation that it is finding very difficult to get out of". The actions of U.S. policymakers contrast greatly with those of the Bank of Japan. The Federal Reserve and in general policy makers in the U.S. enacted unconventional policies such as quantitative easing and forward guidance and evidently did a good job in coordinating inflation expectations near its target. We leave a formal quantitative analysis for future research.

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Online Appendix to "Macroeconomic Dynamics Near the ZLB: A Tale of Two Countries"

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This Appendix consists of six sections:

A. Solving the Two-Equation Model of Section 2

B. Equilibrium Conditions for the Model of Section 3

C. Model Solution Algorithm

D. Data

E. DSGE Model Estimation

F. Particle Filter For Sunspot Equilibrium

A Solving the Two-Equation Model of Section 2

The model is characterized by the nonlinear difference equation

$$\mathbb{E}_t\left[\hat{\pi}_{t+1}\right] = \max\Big\{ -\log\left(r\pi_*\right) - \rho\hat{r}_t, \psi\hat{\pi}_t - \rho\hat{r}_t\Big\} \tag{A.1}$$

We assume that $r\pi_* \geq 1$ and $\psi > 1$.

The Targeted-Inflation Equilibrium. Consider a solution to (A.1) that takes the following form

$$\hat{\pi}_t = \theta_0 + \theta_1\hat{r}_t \tag{A.2}$$

We now determine values of θ_0 and θ_1 such that (A.1) is satisfied. We begin by calculating the following expectation

$$\begin{aligned} \mathbb{E}_t[\hat{\pi}_{t+1}] &= \mathbb{E}_t[\theta_0 + \theta_1\hat{r}_{t+1}] \\ &= \theta_0 + \theta_1\rho\hat{r}_t \end{aligned}$$

Combining this expression with (A.1) yields

$$\theta_0 + \theta_1\rho\hat{r}_t = \max\left\{-\log\left(r\pi_*\right) - \rho\hat{r}_t, \psi\theta_0 + (\psi\theta_1 - \rho)\hat{r}_t\right\} \tag{A.3}$$

The targeted-inflation equilibrium is obtained by equating the left-hand side of (A.3) with the second term in the right-hand-side max-operator, which leads to

$$\theta_0^* = 0, \quad \theta_1^* = \frac{\rho}{\psi - \rho}. \tag{A.4}$$

The derivation is completed by noting that in the targeted-inflation equilibrium the second term in the max operator in (A.3) is greater than the first term:

$$\frac{\psi\rho}{\psi-\rho}\hat{r}_t \geq -\log\left(r\pi_*\right), \tag{A.5}$$

provided that σ is small and thus $\hat{r}_t \approx 0$. Because $\psi > 1$, there is a locally unique stable targeted-inflation equilibrium.

Deflation Equilibria. A deflation equilibrium is obtained by equating the left-hand side of (A.3) with the first term in the right-hand-side max-operator, which leads to

$$\theta_0^D = -\log\left(r\pi_*\right), \quad \theta_1^D = -1, \tag{A.6}$$

The derivation is completed by noting that under this solution the first term in the max operator in (A.3) is greater than the second term:

$$(\psi - 1)\log(r\pi_*) \geq -\psi \hat{r}_t, \tag{A.7}$$

provided that σ is sufficiently small and thus $\hat{r}_t \approx 0$.

Locally around the deflation steady state, interest rates do not respond to inflation rates and it is well-known that the system is locally indeterminate. This suggests that we can construct alternative solutions to (A.1). Consider the following conjecture for inflation which is a generalized version of (A.2)

$$\hat{\pi}_t^D = -\log(r\pi_*) + \theta_1^D \hat{r}_t + \theta_2^D \hat{r}_{t-1} + \theta_3^D \zeta_t, \tag{A.8}$$

where we allow the solution to depend on $\hat{r}_{t-1}$, and a sunspot variable ζ_t which has the property $\mathbb{E}_{t-1}[\zeta_t] = 0$.

Under this conjecture, the expected inflation becomes

$$\begin{aligned} \mathbb{E}_t[\hat{\pi}_{t+1}] &= \mathbb{E}_t\left[-\log(r\pi_*) + \theta_1^D \hat{r}_{t+1} + \theta_2^D \hat{r}_t + \theta_3^D \zeta_{t+1}\right] && \text{(A.9)} \\ &= -\log(r\pi_*) + \left(\rho\theta_1^D + \theta_2^D\right)\hat{r}_t && \text{(A.10)} \end{aligned}$$

Assuming once again that this expectation is equal to the first term in the max operator, we need to solve

$$-\log(r\pi_*) + \left(\rho\theta_1^D + \theta_2^D\right)\hat{r}_t = -\log(r\pi_*) - \rho\hat{r}_t \tag{A.11}$$

which requires $\rho\theta_1^D + \theta_2^D = -\rho$. This shows that there are two indeterminacies. First, the value of θ_3^D is arbitrary. Second, for any choice of θ_2^D we can find a θ_1^D that is a valid solution. In the main text we set $\theta_3^D = 0$ and chose $\theta_2^D = 0$ to yield $\theta_1^D = -1$, but clearly a continuum of other values are admissible.

A Sunspot Equilibrium. Let $s_t \in \{0, 1\}$ denote the Markov-switching sunspot process. Assume the system is in the targeted-inflation regime if $s_t = 1$ and that it is in the deflation regime if $s_t = 0$. The probabilities of staying in state 0 and 1, respectively, are denoted by p_{00} and p_{11}. We conjecture that the inflation dynamics follow the process

$$\hat{\pi}_t^{(s)} = \theta_0(s_t) + \theta_1(s_t)\hat{r}_t. \tag{A.12}$$

In this case condition (A.3) turns into

$$\begin{aligned}
\left[p_{11}\theta_0(1) + (1-p_{11})\,\theta_0(0)\right] + \rho\left[p_{11}\theta_1(1) + (1-p_{11})\,\theta_1(0)\right]\hat{r}_t &= \psi\left(\theta_0(1) + \theta_1(1)\hat{r}_t\right) - \rho\hat{r}_t \\
\left[p_{00}\theta_0(0) + (1-p_{00})\,\theta_0(1)\right] + \rho\left[p_{00}\theta_1(0) + (1-p_{00})\,\theta_1(1)\right]\hat{r}_t &= -\log\left(r\pi_*\right) - \rho\hat{r}_t
\end{aligned}$$

which yield a system of four linear equations by collecting the constants and slopes of each side, in the four unknowns, $(\theta_0(0), \theta_1(0), \theta_0(1), \theta_1(1))$ that can easily be solved numerically. To complete the solution, we need to verify that under the solutions we found, the right-side of the max operator is active in the $s = 1$ solution, and the left-side is active in the $s = 0$ solution. This requires checking

$$\psi\theta_0(1) + \left(\psi\theta_1(1) - \rho\right)\hat{r}_t \geq -\log\left(r\pi_*\right) - \rho\hat{r}_t \tag{A.13}$$

$$-\log\left(r\pi_*\right) - \rho\hat{r}_t \geq \psi\theta_0(0) + \left(\psi\theta_1(0) - \rho\right)\hat{r}_t \tag{A.14}$$

when $\hat{r}_t \approx 0$ which would hold when σ is small.

B Equilibrium Conditions for the Model of Section 3

In this section we sketch the derivation of the equilibrium conditions presented in Section 3.

B.1 Households

The representative household solves

$$\max_{\{C_{t+s},H_{t+s},B_{t+s},M_{t+s}\}} \mathbb{E}_t\left[\sum_{s=0}^{\infty}\beta^s d_{t+s}\left(\frac{(C_{t+s}/A_{t+s})^{1-\tau}-1}{1-\tau}-\chi_H\frac{H_{t+s}^{1+1/\eta}}{1+1/\eta}+\chi_M V\left(\frac{M_{t+s}}{P_{t+s}A_{t+s}}\right)\right)\right],$$

subject to:

$$P_tC_t+T_t+B_t+M_t=P_tW_tH_t+M_{t-1}+R_{t-1}B_{t-1}+P_tD_t+P_tSC_t,$$

Consumption and bond holdings. Let $\beta^s d_{t+s}\lambda_{t+s}$ be the Lagrange multiplier on the household budget constraint. Then the first-order condition with respect to consumption and bond holdings are given by:

$$\begin{aligned} P_t\lambda_t &= \left(\frac{C_t}{A_t}\right)^{-\tau}\frac{1}{A_t} \\ \lambda_t &= \beta\frac{d_{t+1}}{d_t}R_t\lambda_{t+1}. \end{aligned}$$

Combining the two equation of the bond holding first order condition, leads to the consumption Euler equation:

$$1=\beta\mathbb{E}_t\left[\frac{d_{t+1}}{d_t}\left(\frac{C_{t+1}/A_{t+1}}{C_t/A_t}\right)^{-\tau}\frac{1}{\gamma z_{t+1}}\frac{R_t}{\pi_{t+1}}\right],$$

where $\gamma z_{t+1}=A_{t+1}/A_t$. We define the stochastic discount factor as:

$$Q_{t+1|t}=\frac{d_{t+1}}{d_t}\left(\frac{C_{t+1}/A_{t+1}}{C_t/A_t}\right)^{-\tau}\frac{1}{\gamma z_{t+1}}.$$

Labor-Leisure Choice. Taking first-order conditions with respect to H_t yields the standard intratemporal optimality condition for the allocation of labor

$$\frac{W_t}{A_t}=\chi_H\left(\frac{C_t}{A_t}\right)^{\tau}H_t^{1/\eta}.$$

B.2 Intermediate Goods Firms

Each intermediate good producer buys labor services $H_t(j)$ at the real wage W_t. Firms face nominal rigidities in terms of price adjustment costs. The adjustment cost, expressed as a fraction of firms' real output, is given by the function $\Phi_p\left(\frac{P_t(j)}{P_{t-1}(j)}\right)$. We assume that the adjustment cost function is twice-continously differentiable, weakly increasing and weakly convex, $\Phi_p' \geq 0$ and $\Phi_p'' \geq 0$. The firm maximizes expected discounted real profits with respect to $H_t(j)$ and $P_t(j)$:

$$\mathbb{E}_t \sum_{s=0}^{\infty} \beta^s Q_{t+s|t} \left(\frac{P_{t+s}(j)}{P_{t+s}} A_{t+s} H_{t+s}(j) - \Phi_p\left(\frac{P_{t+s}(j)}{P_{t+s-1}(j)}\right) A_{t+s} H_{t+s}(j) - W_{t+s} H_{t+s}(j) \right),$$

subject to

$$A_t H_t(j) = \left(\frac{P_t(j)}{P_t}\right)^{-1/\nu} Y_t.$$

We use $\mu_{t+s}\beta^s Q_{t+s|t}$ to denote the Lagrange multiplier associated with this constraint. In equilibrium, the firms use the households' stochastic discount factor to discount future profits.

Price setting decision. Setting $Q_{t|t} = 1$, the first-order condition with respect to $P_t(j)$ is given by:

$$\begin{aligned} 0 &= \frac{A_t H_t(j)}{P_t} - \Phi_p'\left(\frac{P_t(j)}{P_{t-1}(j)}\right) \frac{A_t H_t(j)}{P_{t-1}(j)} - \frac{\mu_t}{\nu} \left(\frac{P_t(j)}{P_t}\right)^{-1/\nu - 1} \frac{Y_t}{P_t} \\ &\quad + \beta \mathbb{E}_t \left[Q_{t+1|t} \Phi_p'\left(\frac{P_{t+1}(j)}{P_t(j)}\right) A_{t+1} H_{t+1}(j) \frac{P_{t+1}(j)}{P_t^2(j)} \right]. \end{aligned}$$

Firms' labor demand. Taking first-order conditions with respect to $H_t(j)$ yields

$$W_t = \frac{P_t(j)}{P_t} A_t - \Phi_p\left(\frac{P_t(j)}{P_{t-1}(j)}\right) A_t - \mu_t A_t.$$

Symmetric equilibrium. We restrict attention to a symmetric equilibrium where all firms choose the same price $P_t(j) = P_t \, \forall j$. This assumption implies that in equilibrium all firms face identical marginal costs and demand the same amount of labor input. Combining the firms' price setting and labor demand first order conditions and assuming that the price adjustment costs are quadratic, i.e.,

$$\Phi_p\left(\frac{P_t(j)}{P_{t-1}(j)}\right) = \frac{\phi}{2}\left(\frac{P_t(j)}{P_{t-1}(j)} - \bar{\pi}\right)^2,$$

we obtain:

$$(1-\nu)-\chi_H\left(\frac{C_t}{A_t}\right)^{\tau}H_t^{1/\eta}-\frac{\phi}{2}\left(\frac{P_t}{P_{t-1}}-\bar{\pi}\right) \quad +$$
$$\nu\phi\left(\frac{P_t}{P_{t-1}}-\bar{\pi}\right)\frac{P_t}{P_{t-1}}=\nu\beta\mathbb{E}_t\left[Q_{t+1|t}\frac{P_{t+1}}{P_t}\Phi_p'\left(\frac{P_{t+1}}{P_t}\right)\frac{Y_{t+1}}{Y_t}\right].$$

B.3 Equilibrium Conditions

Resource constraint. The derivation of the aggregate resource constraint is straightforward. In equilibrium real profits by intermediate producers is given by:

$$D_t=Y_t-\Phi_p\left(\pi_t\right)Y_t-W_tH_t$$

Substituting this into the household budget constraint we obtain:

$$C_t+\left[\frac{T_t}{P_t}+\frac{M_t}{P_t}+\frac{B_t}{P_t}-\frac{M_{t-1}}{P_t}-\frac{R_{t-1}B_{t-1}}{P_t}\right]=W_tH_t+Y_t-\Phi_p\left(\pi_t\right)Y_t-W_tH_t$$

From the government budget constraint in (22) we can see that the term in square brackets corresponds to real government expenditure G_t. Simplifying yields:

$$C_t+G_t=\left[1-\Phi_p\left(\pi_t\right)\right]Y_t$$

The technology process introduces a long-run trend in the variables of the model. To make the model stationary we use the following transformations: $y_t=Y_t/A_t$, $c_t=C_t/A_t$, and note that $Y_t/Y_{t-1}=\frac{y_t}{y_{t-1}}\gamma z_t$. We also define the gross inflation rate $\pi_t=P_t/P_{t-1}$. The equilibrium conditions shown in the main text follow inmediately:

C Model Solution Algorithm

Algorithm 1 (Model Solution)

1. Start with a guess for Θ. For the targeted-inflation regime ($s_t = 1$), this guess is obtained from a first-order linear approximation around the targeted-inflation steady state. For the deflation regime ($s_t = 0$), it is obtained by assuming constant decision rules for inflation and $\mathcal{E}$ at the deflation steady state.

2. Given this guess, simulate the sunspot model for a large number of periods. This simulation also includes the simulated path of the sunspot variable s_t. We use 10,000 simulations after a burn-in period of 150 observations.

3. Given the simulated path, obtain the grid for the state variables over which the approximation needs to be accurate. We use a time-separated grid algorithm, that is we sample from the simulation at fixed intervals to deliver the grid points that represent the ergodic set. For a fourth order approximation we construct the ergodic distribution setting M = 880 for both countries and obtain 50% of these points conditioning on $s_t = 1$ and the remaining are conditioned on $s_t = 0$. This grid oversamples points from the $s_t = 0$ regime to increase the accuracy of the solution.

   (a) For the U.S. we replace 356 grid points of the ergodic grid with filtered states. We use 36 filtered states corresponding to the period 2000:Q1-2008Q4, 160 points corresponding to filtered states, using multiple particles per period from 2009:Q1-2015:Q2 conditioning on $s_t = 1$, and 160 points corresponding to filtered states using multiple particles per period from 2009:Q1-2009:Q4 conditioning on $s_t = 0$.

   (b) For Japan we replace 260 grid points of the ergodic grid with filtered states using multiple particles per period from 1999:Q1 - 2015:Q1. We obtain 50% of these filtered states forcing $s_t = 1$ and the remaining 50% forcing $s_t = 0$.

4. Solve for the Θ by minimizing the sum of squared residuals using Algorithm 2 and a standard nonlinear solver.

5. Repeat steps 2-4 a sufficient number of times so that both the ergodic distribution of the sunspot model and the filtered states used in the solution grid remain unchanged from one iteration to the next.

Algorithm 2 (Determinining the Approximate Decision Rules)

1. For a generic grid point $\mathcal{S}_i$ and the current value for Θ, compute $f^1_\pi(\mathcal{S}_i;\Theta)$, $f^2_\pi(\mathcal{S}_i;\Theta)$, $f^1_\mathcal{E}(\mathcal{S}_i;\Theta)$, and $f^2_\mathcal{E}(\mathcal{S}_i;\Theta)$.

2. Assume $\zeta_i \equiv I\{R(\mathcal{S}_i,\Theta) > 1\} = 1$ and compute π_i, and $\mathcal{E}_i$, as well as c_i and y_i using (27) and (30).

3. Compute R_i based on (17) using π_i and y_i obtained in Step 2. If R_i is greater than unity, then ζ_i is indeed equal to one. Otherwise, set $\zeta_i = 0$ (and thus $R_i = 1$) and recompute all other objects.

4. The final step is to compute the residual functions. In each regime $s_t = \{0,1\}$ there are four residuals, corresponding to the four functions being approximated. For a given set of state variables $\mathcal{S}_i$, only two of them will be relevant since we either need the constrained decision rules or the unconstrained ones. Taking into account the transition of the sunspot the residual functions will be given by

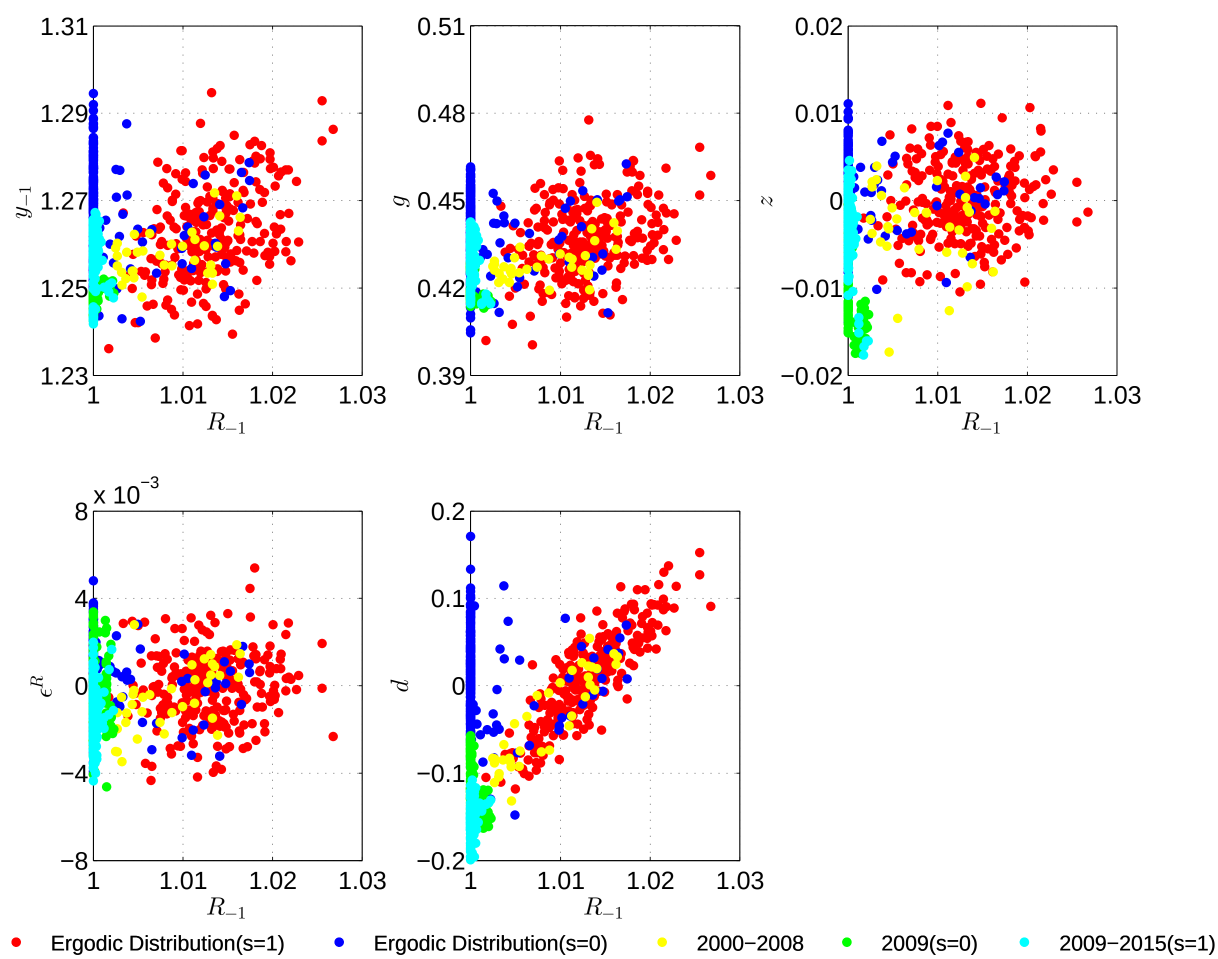
$$\begin{aligned}
\mathcal{R}^1(\mathcal{S}_i) &= \mathcal{E}_i - \int\!\!\int\!\!\int\!\!\int\!\!\int \frac{d'}{d}\frac{c(\mathcal{S}')^{-\tau}}{\gamma z'\pi(\mathcal{S}')} dF\left(d'\right) dF\left(g'\right) dF\left(z'\right) dF\left(\epsilon'_R\right) dF\left(s'\right) && \text{(A.15)}\\
\mathcal{R}^2(\mathcal{S}_i) &= \xi\left(c_i,\pi_i,y_i\right) && \text{(A.16)}\\
&- \phi\beta \int\!\!\int\!\!\int\!\!\int\!\!\int \frac{d'}{d} c(\mathcal{S}')^{-\tau} y(\mathcal{S}')\left[\pi(\mathcal{S}')-\bar{\pi}\right]\pi(\mathcal{S}') dF\left(d'\right) dF\left(g'\right) dF\left(z'\right) dF\left(\epsilon'_R\right) dF\left(s'\right)
\end{aligned}$$

where

$$\xi\left(c_i,\pi_i,y_i\right) \equiv c_i^{-\tau} y_i \left\{ \frac{1}{\nu}\left(1-\chi_h c_i^\tau y_i^{1/\eta}\right) + \phi(\pi_i-\bar{\pi})\left[\left(1-\frac{1}{2\nu}\right)\pi_i + \frac{\bar{\pi}}{2\nu}\right] - 1 \right\}.$$

Note that this step involves computing $\pi(\mathcal{S}')$, $y(\mathcal{S}')$, $c(\mathcal{S}')$, and $R(\mathcal{S}')$ which is done following steps 1-3 above for each value of $\mathcal{S}'$. We use a non-product monomial integration rule to evaluate these integrals. The integrals are written to accommodate the 4-variable versions and when the discount factor shock isn't in the model we drop the integration with respect to d.

Figure A-1: Solution Grid for the Sunspot Equilibrium - 4vGrowth U.S.

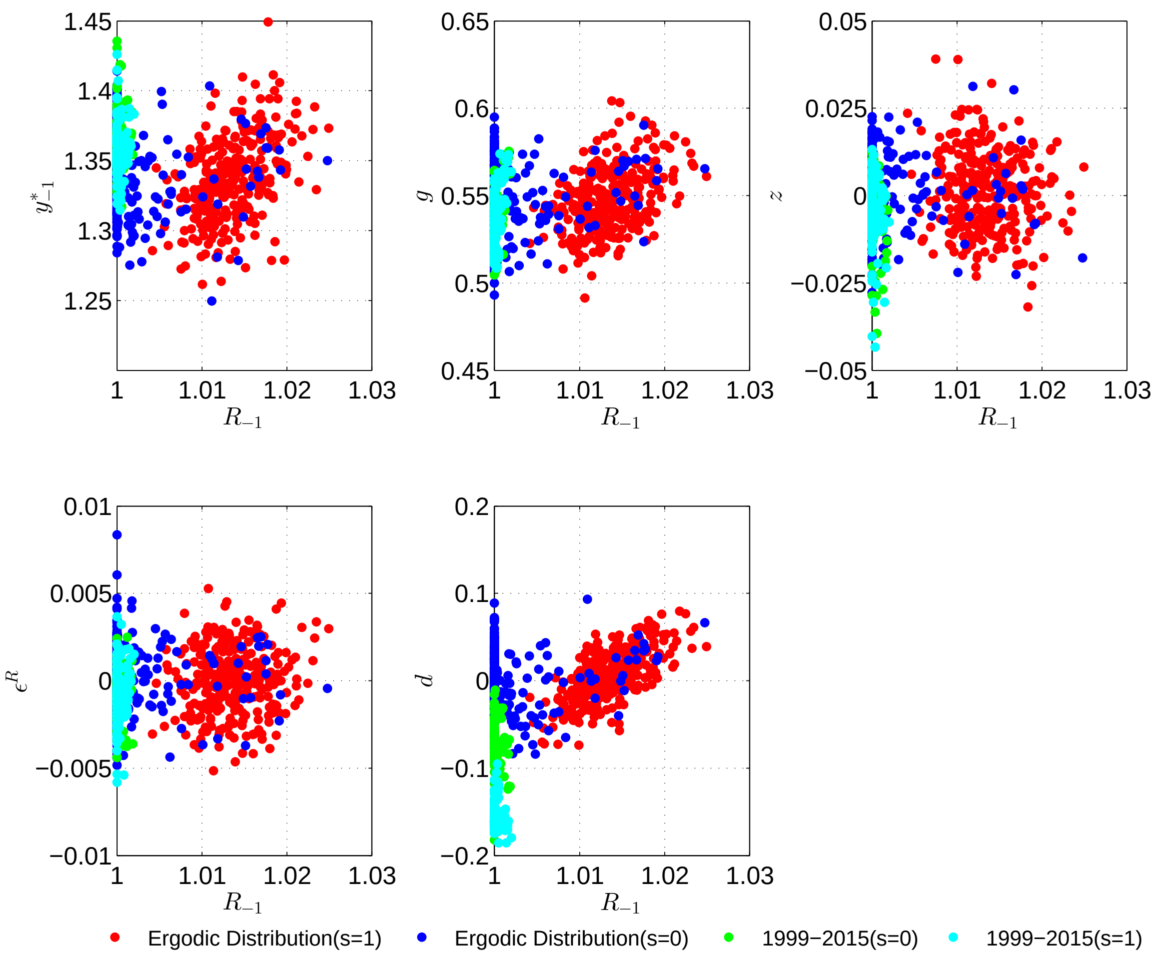


5. The objective function to be minimized is the sum of squared residuals obtained in Step 4.

We use analytical derivatives of the objection function, which speeds up the solution by two orders of magnitude. As a measure of accuracy, we compute the approximation errors from (A.15) and (A.16), converted to consumption units. The approximation errors are in the order of 10^{-4} on average.

Figures A-1 and A-2 show the solution grid for the sunspot equilibrium for 4vGrowth model for the U.S. and 4vGap model for Japan. For each panel, we have R_{t-1} on the x axis and one of the other exogenous state variables on the y axis. The red (blue) dots are the

Figure A-2: Solution Grid for the Sunspot Equilibrium - 4vGap Japan



grid points that represent the ergodic distribution conditional on $s_t = 1$ ($s_t = 0$). For both countries we include filtered grid points used in the construction of the grid. For the U.S. the yellow dots denote filtered states between 2000 to 2008; the green dots represent filtered states from 2009 conditioning on $s_t = 0$ and the turquoise dots represent filtered states from 2009 to 2015 conditioning on $s_t = 1$. For Japan the green (turquoise) dots represent filtered states from 1999 to 2015 conditioning on $s_t = 0$ ($s_t = 1$). In both cases the filtered states lie in the tails of the ergodic distribution of the sunspot equilibrium. By adding these filtered states to the grid points, we ensure that our approximation will be accurate in these low-probability regions.

D Data

D.1 United States

Real per capita GDP: We obtain real GDP (GDPC96) and convert into per capita terms using the Civilian Noninstitutional Population (CNP16OV). We smooth the population series applying an eight-quarter backward-looking moving average filter. Source: FRB St. Louis FRED database.

Real per capita consumption: We obtain real personal consumption expenditures (PCECC96) and convert into per capita terms using the Civilian Noninstitutional Population (CNP16OV). Source: FRED.

GDP Deflator Inflation: Log difference of GDP deflator (GDPDEF), multiplied by 400 to convert it into annualized percentages. Source: FRED.

Interest Rate / Monetary Policy Rate: Effective federal funds rate (FEDFUNDS) averaged over each quarter. Source: FRED.

Potential GDP: We use real potential GDP (GDPPOT) produced by the U.S. Congressional Budget Office. Source: FRED.

D.2 Japan

Real per capita GDP: We collect real GDP (RGDP) from the Cabinet Office's National Accounts. We use the statistical release of benchmark year 2005 that covers the period 1994:Q1 - 2015:Q1. To extend the sample we collect RGDP figures from the benchmark year 2000 and construct a series spanning the period 1981:Q1-2015:Q1 using the quarterly growth rate of the RGDP benchmark year 2000. Our measure of per-capita output is RGDP divided by the total population of 15 years and over. We smooth the quarterly growth of the population series using an eight quarter backward-looking moving average filter. We obtain population data from the Statistics Bureau of the Ministry of Foreign Affairs Historical data Table b-1.

Real per capita consumption: We collect real Private Consumption data from the Cabinet Office's National Accounts and follow the same procedure as for real GDP to convert it into per capita terms.

GDP deflator inflation: Log difference of GDP deflator, multiplied by 400 to convert it into annualized percentages. We use the implicit GDP deflator index from the Cabinet Office. We also extend the benchmark year 2005 release using the growth rate of the index from the benchmark year 2000 figures.

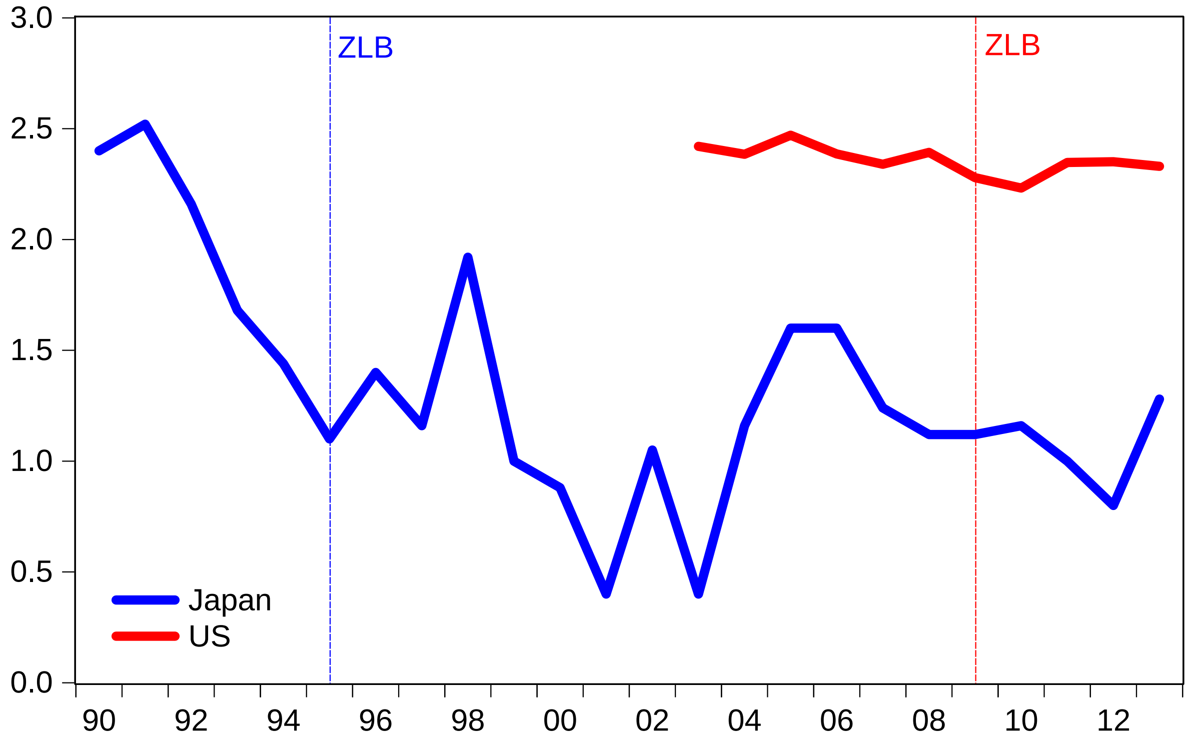
Interest Rate / Monetary Policy Rate: For the nominal interest rate we use the Bank of Japan's uncollateralized call rate (STSTRACLUCON) from 1986:M7-2015:M3. To complete the series from 1981.M1 - 1985.M6 we use the monthly average of the collateralized overnight call rate (STSTRACLCOON). Finally we transform monthly figures using quarterly averages over the sample period.

Potential GDP: We construct a measure of potential GDP using the output gap estimated by the Bank of Japan and semi-annual potential growth rates from the Cabinet Office. We set our measure of potential GDP equal to observed real GDP in 1981.Q1.

D.3 Inflation Expectations

In Figure A-3 we plot 10-year inflation expectations for Japan and the U.S. starting five years prior to each country's respective ZLB episode. For Japan we use the Consensus Forecasts and for the U.S. we use the results from Aruoba (2014), which are based on surveys. The vertical lines in the figure depict the start of the ZLB episode of the two countries. For the U.S., long-run inflation expectations simply do not move during or after the financial crisis and they show small fluctuations around 2.3%. For Japan, the expectations are around 2.5% prior to the burst of the housing price bubble and they gradually decline to 0.5% by 2003.

Figure A-3: 10-Year Inflation Expectations



Notes: Units are annualized percentage. Vertical lines show the quarter where interest rates fall to the ZLB in each country.

Table A-1: Parameters Fixed During Estimation

	US				Japan			
Parameter	3vGrowth	4vGrowth	3vGap	4vGap	3vGrowth	4vGrowth	3vGap	4vGap
$400\ln(r^*)$	0.86	0.86	0.86	0.86	1.88	1.88	1.88	1.88
$400\ln(\pi^*)$	2.72	2.38	2.46	2.46	1.27	1.25	1.28	1.27
$100\ln(\gamma)$	0.50	0.50	0.50	0.50	0.56	0.56	0.56	0.56
c/y	0.65	0.65	0.65	0.65	0.58	0.58	0.58	0.58
ν	0.10	0.10	0.10	0.10	0.10	0.10	0.10	0.10
η	0.72	0.72	0.72	0.72	0.85	0.85	0.85	0.85
α			0.90	0.90			0.85	0.85

Notes: The steady state ratio $c/y = 1/g_*$. The sunspot shock transition probability parameters p_{00} and p_{11} do not affect the estimation because it is based on a first-order perturbation approximation of the targeted-inflation equilibrium.

E DSGE Model Estimation

E.1 Priors for estimation

Table A-1 lists the parameters that were fixed during the estimation. Marginal prior distributions the remaining DSGE model parameters are summarized in Table A-2. We assume that the parameters are *a priori* independent. Thus, the joint prior distribution is given by the product of the marginals. We truncate the joint distribution to ensure local determinacy of the targeted inflation equilibrium. For the 3-variable specifications we used alternative priors for ψ_1 and ψ_2, which are summarized in Table A-3.

E.2 Posterior Simulator

We estimate a first-order approximation of the targeted-inflation equilibrium of the DSGE model, ignoring the presence of the ZLB. Thus, the likelihood function can be evaluated with the Kalman filter which speeds up the estimation. For the the output growth specifications we initialize the initial level of technology, A_0, in the Kalman filter such that $\log Y_1$ is identical

Table A-2: Benchmark Priors for Estimation

			US		Japan	
Parameter	Description	Density	P(1)	P(2)	P(1)	P(2)
τ	Inverse IES	$\mathcal{G}$	2.0	.25	2.0	0.5
κ	Slope (linearized) Phillips curve	$\mathcal{G}$	0.3	0.1	0.3	0.1
ψ_1	Taylor rule: weight on inflation	$\mathcal{G}$	1.5	0.3	1.5	0.3
ψ_2	Taylor rule: weight on output	$\mathcal{G}$	0.5	.25	0.5	.25
ρ_r	Interest rate smoothing	$\mathcal{B}$	0.5	0.2	0.6	0.2
ρ_g	Persistence: demand shock	$\mathcal{B}$	0.8	0.1	0.6	0.2
ρ_z	Persistence: technology shock	$\mathcal{B}$	0.2	0.1	0.6	0.2
ρ_d	Persistence: discount factor shock	$\mathcal{B}$	0.8	0.1	0.6	0.2
$100\sigma_r$	Std dev: monetary policy shock	$\mathcal{IG}$	0.3	4.0	0.3	4.0
$100\sigma_g$	Std dev: demand shock	$\mathcal{IG}$	0.4	4.0	0.4	4.0
$100\sigma_z$	Std dev: technology shock	$\mathcal{IG}$	0.4	4.0	0.4	4.0
$100\sigma_d$	Std dev: discount factor shock	$\mathcal{IG}$	0.4	4.0	0.4	4.0

Notes: $\mathcal{G}$ is Gamma distribution; $\mathcal{B}$ is Beta distribution; and $\mathcal{IG}$ is Inverse Gamma distribution. P(1) and P(2) are the mean and the standard deviations for Beta and Gamma distributions; s and ν for the Inverse Gamma distribution, where $p_{\mathcal{IG}}(\sigma|\nu, s) \propto \sigma^{-\nu-1}e^{-\nu s^2/2\sigma^2}$.

to the data in the first quarter of our estimation sample. All other states are initialized using their unconditional distribution. For the output gap specifications we initialize A_0 such that if we use $\log Y_t$ as observable the model tracks $\log Y_t^*$ computed from actual output data.

Draws from the posterior distribution are generated using the random walk Metropolis algorithm (RWM) described in An and Schorfheide (2007). The covariance matrix of the proposal distribution is scaled such that the RWM algorithm has an acceptance rate of approximately 50%. We generate 100,000 draws from the posterior distribution and discard the first 50,000 draws. Summary statistics of the posterior distribution are based on the last 50,000 draws of the sequence.

Table A-3: Modified Priors for 3-Variable Models

Parameter	Description	Density	US P(1)	US P(2)	Japan P(1)	Japan P(2)
			3vGrowth Specification			
ψ_1	Taylor rule: weight on inflation	$\mathcal{G}$	1.50	0.05	1.50	0.15
ψ_2	Taylor rule: weight on output	$\mathcal{G}$	0.80	0.01	(Benchm.)	
			3vGap Specification			
ψ_1	Taylor rule: weight on inflation	$\mathcal{G}$	2.55	0.05	1.70	0.05
ψ_2	Taylor rule: weight on output	$\mathcal{G}$	0.35	0.01	0.15	0.01

Notes: $\mathcal{G}$ is Gamma distribution.

F Particle Filter For Sunspot Equilibrium

The particle filter is used to extract information about the state variables of the model from data on log output, log consumption-output ratio (for 4v specifications) inflation, and nominal interest rates over the periods 1984:Q1 to 2015:Q4 (U.S.) and 1981:Q1 to 2015:Q4 (Japan).

F.1 State-Space Representation

Let y_t^o be the $n \times 1$ vector of observables, where n is either 3 or 4. The vector x_t stacks the continuous state variables, which are given by $x_t = [R_t, y_t, y_t^*, y_{t-1}, d_t, z_t, g_t, A_t]'$, and $s_t \in \{0, 1\}$, is the Markov-switching process. Note that the output measures here are detrended by the level of technology A_t and only y_{t-1} is necessary for the growth specifications and y_{t-1}^* is necessary for the gap specifications.

$$y_t^o = \Psi(x_t) + \nu_t \tag{A.17}$$

$$\mathbb{P}\{s_t = 1\} = \begin{cases} (1 - p_{00}) & \text{if } s_{t-1} = 0 \\ p_{11} & \text{if } s_{t-1} = 1 \end{cases} \tag{A.18}$$

$$x_t = F_{s_t}(x_{t-1}, \epsilon_t) \tag{A.19}$$

The first equation is the measurement equation, where $\nu_t \sim N(0, \Sigma_\nu)$ is a vector of measurement errors. The second equation represents the law of motion of the Markov-switching process. The third equation corresponds to the law of motion of the continuous state variables. The vector $\epsilon_t \sim N(0, I)$ stacks the innovations $\epsilon_{z,t}$, $\epsilon_{g,t}$, and $\epsilon_{R,t}$. The functions $F_0(\cdot)$ and $F_1(\cdot)$ are generated by the model solution procedure. Measurement and state transition equations can be summarized by the densities $p(y_t^o|x_t)$, $p(s_t|s_{t-1})$, and $p(x_t|x_{t-1}, s_t)$.

F.2 Particle Filtering

Let $\varsigma_t = [x_t', s_t]'$ and $Y_{t_0:t_1}^o = \{y_{t_0}^o, \ldots, y_{t_1}^0\}$. Particle filtering relies on sequential importance sampling approximations. The distribution $p(\varsigma_t|Y_{1:t}^o)$ is approximated by a set of pairs

$\{(\varsigma_t^{(j)}, W_t^{(j)})\}_{j=1}^M$ in the sense that

$$\frac{1}{M}\sum_{j=1}^{N} h(\varsigma_t^{(j)})W_t^{(j)} \xrightarrow{a.s.} \mathbb{E}[h(\varsigma_t)|Y_{1:t}^o], \tag{A.20}$$

where $\varsigma_t^{(j)}$ is the j'th particle, $W_t^{(j)}$ is its weight, and M is the number of particles. Our implementation of the particle filters follows Algorithm 12 of Herbst and Schorfheide (2015). In the remainder of this section we provide a few implementation details.

Measurement Errors. The baseline measurement error standard deviations are as follows. Log output: 0.0016617; log consumption output ratio 0.0015; inflation 0.30560; interest rates 0.75056. We then reduce the baseline standard deviations by 0.7, 0.7, 0.7, and 0.2, respectively. For comparison, in our data set the standard deviations for output growth (not in percentages), the log consumption-output ratio, inflation, and interest rates are 0.011, 0.018, 2.02, 2.81 for Japan; and 0.006, 0.0098, 0.99, 3.01 for the U.S.

Initialization. The filter requires an initial distribution $p(\varsigma)$. Because we are operating under the assumption that the economies are in the targeted-inflation regime ($s_t = 1$) at the beginning of the sample (and during the estimation period) and because in the relevant region of the state space the decision rules are well approximated by a first-order perturbation solution, we use the linear representation of the targeted-inflation equilibrium to construct an initial distribution.

Let $t = 1$ correspond to the first observation in the sample. Because the law of motion for the detrended state variables (all elements of x_t except A_t) is stationary, we can initialize the distribution in period $t = 1$ using the implied ergodic distribution of the linearized system. A_1 is treated as follows. We set $\log A_1 = (\log Y_1 + \log Y_1^* - 2\log y_*)/2$, where y_* is the model implied steady state of Y_t/A_t. Conditional on $\log A_1$ we can compute $\log(Y_1/A_1)$ and $\log(Y_1^*/A_1)$. We treat these objects as "known," setting their variance equal to zero. We then run the Kalman filter based on the observations y_2^o to y_4^o of the sample. We use the posterior distribution of $x_4|Y_{1:4}^o$ obtained from the Kalman filter iterations and $s_4 = 1$ to initialize the nonlinear filter. In particular, we generate particles from $p(\varsigma_4)$ by *iid* sampling. This initialization ensures that the model tracks the levels of Y_t and Y_t^*.

Proposal Distribution. States for the next period are drawn from the proposal distribution

$$g(\varsigma_t|\varsigma_{t-1}, y_t) = \begin{cases} g_0(x_t|x_{t-1}, y_t^o, s_t = 0)\lambda(\varsigma_{t-1}, y_t^o) & \text{if } s_t = 0 \\ g_1(x_t|x_{t-1}, y_t^0, s_t = 1)(1 - \lambda(\varsigma_{t-1}, y_t^o)) & \text{if } s_t = 1 \end{cases},$$

Rather than drawing x_t directly from a distribution, we draw the innovation ϵ_t from a proposal distribution and then simulate the state-transition forward (see Algorithm 14 in Herbst and Schorfheide (2015)): $\epsilon_t^{(j)} \sim N\big(\mu(s_t), \Sigma(s_t)\big)$. For $\Sigma(s_t)$ we use the identity matrix and we determine $\mu(s_t)$ by grid search, maximizing $p(y_t^o|\bar{x}_{t-1}, y_t^o, \epsilon_t)p(\epsilon_t)$ with respect to ϵ_t. Here $p(\epsilon_t)$ is the DSGE model implied distribution of the shock innovations. Executing the grid search for each particle conditional on $\bar{x}_{t-1}^{(j)}$ rather than the average across particles $\bar{x}_{t-1}$ turned out to be too time consuming. We set $\lambda(\varsigma_{t-1}, y_t^o) = 1/2$.

Number of Particles. We set $M = 100,000$.