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Math 8230 Homework 1

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#6 Find the radius of convergence for each of the following power series:

(a) $\sum_{n=0}^{\infty} a^n z^n, a \in \mathbf{C}$

We have that for $a = 0$ then the sum is 0 and therefore the radius of convergence is ∞. Otherwise for $a \neq 0$, we have that $R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \left|\frac{a^n}{a^{n+1}}\right| = \lim_{n\to\infty} \left|a^{-1}\right| = |a|^{-1}$.

(b) $\sum_{n=0}^{\infty} a^{n^2} z^n, a \in \mathbf{C}$

We have that for $a = 0$ then the sum is 0 and therefore the radius of convergence is ∞. Otherwise for $a \neq 0$, we have that $R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty} \left|\frac{a^{n^2}}{a^{(n+1)^2}}\right| = \lim_{n\to\infty} \left|a^{-2n-1}\right|$.

For $|a| < 1$ we have that $R = \infty$ and for $|a| > 1$ then $R = 0$ and $|a| = 1$ then $R = 1$

(c) $\sum_{n=0}^{\infty} k^n z^n, k$ an integer $\neq 0$

We have that $R = \lim_{n\to\infty} \left|\frac{k^n}{k^{n+1}}\right| = |k|^{-1}$.

(d) $\sum_{n=0}^{\infty} z^{n!}$.
If we expand the series we have that $\sum_{n=0}^{\infty} z^{n!} = z + z + z^2 + z^6 + \ldots$. Therefore $a_n \in \{0, 1\}$ for $n > 1$. Therefore $\limsup a_n^{1/n} = 1$. Therefore $R = 1$.

#7 Show that the radius of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} z^{n(n+1)}$$

is 1, and discuss convergence for $z = 1, -1$, and i. (Hint: the n^{th} coefficient of this series is not $(-1)^n/n$.) We have that

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} z^{n(n+1)} = -z^2 + \frac{1}{2}z^6 - \frac{1}{3}z^12 + \ldots$$

We have have that $a_n = \frac{(-1)^m}{m} z^{m(m+1)}$ and solving for $n = m(m+1)$.
Hence we have that $\limsup_{n\to\infty} |a_n|^{1/n} = \limsup_{m\to\infty} \left|(-1)^m \frac{1}{m}\right|^{1/m(m+1)} = \lim_{m\to\infty} \frac{1}{m}^{\frac{1}{m(m+1)}}$. Using the continuity of log and L'Hopitals rule we have that $\log(\lim_{m\to\infty} \frac{1}{m}^{\frac{1}{m(m+1)}}) = \lim_{m\to\infty} \frac{-\log(m)}{m(m+1)} = \lim_{m\to\infty} \frac{-\frac{1}{m}}{2m+1} = 0$. Hence we have that $\limsup_{n\to\infty} |a_n|^{1/n} = 1$ or $R = 1$.

In the case of $z = 1$ we have that $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ which converges by the alternating series test to $-\log(2)$. In the case of $z = -1$ we have $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}(-1)^{n(n+1)} = \sum_{n=1}^{\infty} \frac{(-1)^{n(n+2)}}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$.

Hence it converges to the same number as $z = 1$. In the case of $z = i$, we have $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} i^{n(n+1)} = \sum_{n=1}^{\infty} \frac{(-1)^n(-1)^{\frac{n(n+1)}{2}}}{n} = \sum_{n=1}^{\infty} \frac{(-1)^{\frac{n(n+3)}{2}}}{n}$ which converges by the Dirchelet's convergence test as $\left|\sum_{n=1}^{m}(-1)^{\frac{n(n+3)}{2}} = 1 - 1 - 1 + 1 + 1 - 1 - 1 + 1 + 1 + \ldots\right| \in \{0, 1\}$ and $\lim 1/n = 0$.

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#3 Show that $\lim n^{1/n} = 1$.

We have that by the continuity of the limit that $\log(\lim_{n\to\infty} n^{1/n}) = \lim_{n\to\infty} \frac{\log(n)}{n} = 0$. Therefore $\lim_{n\to\infty} n^{1/n} = 1$.

#6 Describe the following sets:

(a) $\{z : e^z = i\}$,

$$\begin{aligned} \{z : e^z = i\} &= \{z : e^z = e^{i\pi/2}\} \\ &= \{z : z = i\pi/2 + i2k\pi \text{ for } k \in \mathbf{Z}\} \end{aligned}$$

(b) $\{z : e^z = -1\}$,

$$\begin{aligned} \{z : e^z = -1\} &= \{z : e^z = e^{i\pi}\} \\ &= \{z : z = i\pi + i2k\pi \text{ for } k \in \mathbf{Z}\} \end{aligned}$$

(c) $\{z : e^z = -i\}$,

$$\begin{aligned} \{z : e^z = i\} &= \{z : e^z = e^{-i\pi/2}\} \\ &= \{z : z = -i\pi/2 + i2k\pi \text{ for } k \in \mathbf{Z}\} \end{aligned}$$

(d) $\{z : \cos z = 0\}$,

$$\begin{aligned} \{z : \cos z = 0\} &= \{z : \frac{1}{2}(e^{iz} + e^{-iz}) = 0\} \\ &= \{z : e^{iz} = -e^{-iz}\} \\ &= \{z : e^{iz} = e^{i\pi - iz}\} \\ &= \{z : iz = i\pi - iz + i2k\pi \text{ for } k \in \mathbf{Z}\} \\ &= \{z : z = \pi/2 + k\pi \text{ for } k \in \mathbf{Z}\} \end{aligned}$$

(e) $\{z : \sin z = 0\}$.

$$\begin{aligned} \{z : \sin z = 0\} &= \{z : \frac{1}{2i}(e^{iz} - e^{-iz}) = 0\} \\ &= \{z : e^{iz} = e^{-iz}\} \\ &= \{z : iz = -iz + i2k\pi \text{ for } k \in \mathbf{Z}\} \\ &= \{z : z = k\pi \text{ for } k \in \mathbf{Z}\} \end{aligned}$$

#7 Prove formulas for $\cos(z+w)$ and $\sin(z+w)$.

$$\begin{aligned}
\cos z\cos w - \sin z\sin w &= \left(\frac{1}{2}(e^{iz}+e^{-iz})\right)\left(\frac{1}{2}(e^{iw}+e^{-iw})\right) - \left(\frac{1}{2i}(e^{iz}-e^{-iz})\right)\left(\frac{1}{2i}(e^{iw}-e^{-iw})\right)\\
&= \frac{1}{4}\left(e^{iz}e^{iw}+e^{iz}e^{-iw}+e^{-iz}e^{iw}+e^{-iz}e^{-iw}\right)\\
&+ \frac{1}{4}\left(e^{iz}e^{iw}-e^{iz}e^{-iw}-e^{-iz}e^{iw}+e^{-iz}e^{-iw}\right)\\
&= \frac{1}{4}\left(2e^{i(z+w)}+2e^{-i(z+w)}\right)\\
&= \frac{1}{2}\left(e^{i(z+w)}+e^{-i(z+w)}\right)\\
&= \cos(z+w)
\end{aligned}$$

$$\begin{aligned}
\sin z\cos w + \cos z\sin w &= \left(\frac{1}{2i}(e^{iz}-e^{-iz})\right)\left(\frac{1}{2}(e^{iw}+e^{-iw})\right) + \left(\frac{1}{2}(e^{iz}+e^{-iz})\right)\left(\frac{1}{2i}(e^{iw}-e^{-iw})\right)\\
&= \frac{1}{4i}\left(e^{iz}e^{iw}+e^{iz}e^{-iw}-e^{-iz}e^{iw}-e^{-iz}e^{-iw}\right)\\
&+ \frac{1}{4i}\left(e^{iz}e^{iw}-e^{iz}e^{-iw}+e^{-iz}e^{iw}-e^{-iz}e^{-iw}\right)\\
&= \frac{1}{4i}\left(2e^{i(z+w)}-2e^{-i(z+w)}\right)\\
&= \frac{1}{2i}\left(e^{i(z+w)}-e^{-i(z+w)}\right)\\
&= \sin(z+w)
\end{aligned}$$

#8 Define $\tan(z) = \frac{\sin(z)}{\cos(z)}$; where is this function defined and analytic?

We have that $\tan(z) = \frac{\sin(z)}{\cos(z)} = \frac{\frac{1}{2i}(e^{iz}-e^{-iz})}{\frac{1}{2}(e^{iz}+e^{-iz})} = \frac{1}{i}\frac{e^{iz}-e^{-iz}}{e^{iz}+e^{-iz}}$. So long as $\cos(z) \neq 0$ the function is differentiable. Hence $e^{iz} = -e^{-iz} = e^{i\pi+iz}$ when $iz = i\pi - iz + i2\pi k$ for all $k \in \mathbf{Z}$. Therefore so long as $z \neq \frac{\pi}{2} + \pi k$ for all $k \in \mathbf{Z}$. Therefore $\tan(z)$ is analytic on $\mathbf{C} - \{\pi/2 + \pi k \text{ for } k \in \mathbf{Z}\}$.

#14 Suppose $f : G \to \mathbf{C}$ is analytic and that G is connected. Show that if $f(z)$ is real for all $z \in G$ then f is constant.

We have the f is continuously differentiable by definition so f is continuous and $f(G)$ is connected. Therefore $f(z) = u(z) + iv(z)$ where u, v have are real and have continuous partial derivatives. Therefore we have $v(z) = 0$ and $0 = v_x = v_y = u_x = -u_y$ by the Cauchy-Riemann equations. Therefore we have that $f'(z) = 0$. Then by 3.2.10, f is constant.

#15 For $r > 0$ let $A = \left\{\omega : \omega = \exp\left(\frac{1}{z}\right) \text{ where } 0 < |z| < r\right\}$; determine the set A.

$$
\begin{aligned}
A &= \left\{\omega : \omega = \exp\left(\frac{1}{z}\right) \text{ where } 0 < |z| < r\right\} \\
&= \left\{\omega : |\omega|\, e^{i\theta} = e^{\mathrm{Re}(\frac{1}{z}) + i\mathrm{Im}(\frac{1}{z})} \text{ where } 0 < |z| < r\right\} \\
&= \left\{\omega : |\omega| = e^{\mathrm{Re}(1/z)} \text{ and } \theta + 2\pi k = \ \mathrm{Im}(1/z) \text{ where } 0 < |z| < r\right\} \\
&= \{\omega : \log|\omega| = \mathrm{Re}(1/z) \text{ and } \theta + 2\pi k = \ \mathrm{Im}(1/z) \text{ where } 0 < |z| < r\} \\
&= \left\{\omega : \log|\omega| = \mathrm{Re}(s) \text{ and } \theta + 2\pi k = \mathrm{Im}(s) \text{ where } |s| > \frac{1}{r}\right\} \\
&= \left\{\omega : \log|\omega| = x \text{ and } \theta + 2\pi k = y \text{ where } x^2 + y^2 > \frac{1}{r^2}\right\}
\end{aligned}
$$

Hence when we pick $\omega \in \mathbf{C} - 0$ then choose $x = \log|\omega|$. We may then choose k large enough such that $x^2 + (\theta + 2\pi k)^2 > \frac{1}{r^2}$ and $y = \theta + 2\pi k$. Hence $\omega \in A$. Therefore we have shown that $\mathbf{C} - 0 \subset A$. Since $e^{\mathrm{Re}(\frac{1}{z})} \neq 0$ for $0 < |z| < r$. Then we have equality and $\mathbf{C} - 0 = A$.

#16 Find an open connected set $G \subset \mathbf{C}$ and two continuous functions f and g defined on G such that $f(z)^2 = g(z)^2 = 1 - z^2$ for all z in G. Can you make G maximal? Are f and g analytic?

Let $G = \mathbf{C} \backslash (-\infty, -1] \cup [1, \infty)$. Then $(G)^2$ is $\mathbf{C} \backslash [1, \infty)$. Taking $1 - (G)^2$ then gives $\mathbf{C} \backslash (-\infty, 0]$. This is the maximal domain for the square root function which is $\sqrt{z} = e^{\frac{1}{2}\mathrm{Log}(z) + ik\pi}$ for $k \in \mathbf{Z}$. There are two values for $e^{ik\pi} = \pm 1$. Hence we may let $f(z) = \sqrt{1 - z^2}$ and $g(z) = -\sqrt{1 - z^2}$. We have analytic since f and g are compositions of Log and e which are analytic. We have $1 - (G)^2$ is a maximal connected set for the square root function.

#17 Give the principal branch of $\sqrt{1 - z}$.

We have that $\sqrt{1 - z} = \exp(\frac{1}{2}\mathrm{Log}(1 - z)) = \exp(\frac{1}{2}\ln|(1 - z)| + \frac{1}{2}i\mathrm{Arg}(1 - z)) = \sqrt{|1 - z|}\exp(i\mathrm{Arg}(1 - z)/2)$. Therefore we have $G = \mathbf{C} - \{1\}$.

#19 Let G be a region and define $G^* = \{z : \bar{z} \in G\}$. If $f : G \to \mathbf{C}$ is analytic prove that $f^* : G^* \to \mathbf{C}$, defined by $f^*(z) = \overline{f(\overline{z})}$, is also analytic.

We have that $f(x, y) = u(x, y) + iv(x, y)$ where $(x, y) \in G$ u and v are real valued functions with continuous partials. Therefore $f^*(x, y) = u^*(x, y) + iv^*(x, y) = u(x, -y) - iv(x, -y)$ for $(x, y) \in G^*$ or $(x, -y) \in G$. Hence for $(x, -y) \in G$ we have $u_x^*(x, y) = u_x(x, -y), u_y^*(x, -y) = -u_y(x, -y), v_x^*(x, y) = -v_x(x, -y)$, and $v_y^*(x, y) = v_y(x, -y)$ by chain rule. Additionally since u_x, u_y, v_x, and v_y are continuous on G then v^* and u^* have continuous partials on G^*. Then we have that $u_x^* = u_x = v_y = v_y^*$ on G^* and $u_y^* = -u_y = v_x = -v_x^*$ on G^*. Hence we have f^* satisfies the Cauchy Riemann conditions with continuous partial derivatives, hence it is analytic.

#20 Let $z_1, z_2, \ldots, z_n$ be complex numbers such that $\mathrm{Re}(z_k) > 0$ and $\mathrm{Re}(z_1 \ldots z_k) > 0$ for $1 \leq k \leq n$. Show that $\log(z_1 \ldots z_n) = \log z_1 + \cdots + \log z_n$, where $\log z$ is the principal branch of the logarithm. If the restrictions on the z_k are removed, does the formula remain valid?

We have that $Re(z_k), Re(z_1 \dots z_k) > 0$ then $\arg(z_k), \arg(z_1 \dots z_k) \in (-\pi/2, \pi/2)$. We will proceed by induction.

We have that

$$\begin{aligned}
\log(z_1 z_2) &= \log|z_1 z_2| + i\arg(z_1 z_2), \arg(z_1 z_2) \in (-\pi, \pi] \\
&= \log|z_1||z_2| + i\arg(z_1) + i\arg(z_2), \text{ for } \arg(z_i) \in (-\pi/2, \pi/2) \\
&= \log|z_1| + \log|z_2| + i\arg(z_1) + i\arg(z_2), \text{for } \arg(z_i) \in (-\pi/2, \pi/2) \\
&= \log(z_1) + \log(z_2)
\end{aligned}$$

Thus we have shown the base case. Under the induction hypothesis we have that $\log(z_1 \dots z_{n-1}) = \log(z_1) + \dots + \log(z_{n-1})$. Therefore we will show the induction step.

$$\begin{aligned}
\log(z_1 \dots z_n) &= \ln|z_1 \dots z_n| + i\arg(z_1 \dots z_n) \text{ for } \arg(z_1 \dots z_n) \in (-\pi, \pi] \\
&= \ln|z_1 \dots z_{n-1}| + \log|z_n| + i\arg(z_1 \dots z_{n-1}) + i\arg(z_n) \\
&\text{for } \arg(z_1 \dots z_{n-1}), \arg(z_n) \in (-\pi/2, \pi/2) \\
&= \log(z_1 \dots z_{n-1}) + \log(z_n)
\end{aligned}$$

Thus we have shown the formula. If the restrictions are removed we may observe $z_1 = -1$ and $z_2 = i$. Then $\log(-1) = \ln 1 + i\pi, \log(i) = \ln 1 + i\pi/2, \log(-i) = \ln 1 - i\pi/2$. Therefore $\log(-1) + \log(i) = i3\pi/2 \neq -i\pi/2 = \log(-i)$.

#21 Prove that there is no branch of the logarithm defined on $G = \mathbf{C} - \{0\}$. (Hint: Suppose such a branch exists and compare this with the principal branch.)

Suppose there is a continuous function g that is a branch of the logarithm defined on G. Then $z = \exp(g(z))$ for all $z \in G$. Let $\text{Log}(z) = \ln|z| + i\text{Arg}(z)$ on $H = \mathbf{C} - \{\mathbf{R} \leq 0\}$. Thus $g(z)|_H = \text{Log}(z) + 2\pi ki = \ln|z| + i\text{Arg}(z) + 2\pi ki$.

Since $g(z)$ is continuous on G, then $\lim_{h\to 0} g(z+h) = g(z)$ for all $z \in G$. Choose $z = x + i0 \in G\backslash H$ for $x < 0$. and $h = iy$. Then $\lim_{y\to 0^+} g(x+iy) = \lim_{y\to 0^+} \ln|x+iy| + i\text{Arg}(x+iy) + 2\pi ki = \ln|x| + i\pi + 2\pi ki$ and $\lim_{y\to 0^-} g(x+iy) = \lim_{y\to 0^-} \ln|x+iy| + i\text{Arg}(x+iy) + 2\pi ki = \ln|x| - i\pi + 2\pi ki$. Hence the right and left limits are different and g is not continuous. Therefore we have a contradiction that $z \in G\backslash H$. Hence $G = H$.