Paul Cubre

Math 8230 Final

pg. 83

#3 Let $p(z)$ be a polynomial of degree n and let $R > 0$ be sufficiently large so that p never vanishes in $\{z : z \geq R\}$. If $\gamma(t) = Re^{it}, 0 \leq t \leq 2\pi$, show that $\int_\gamma \frac{p'(z)}{p(z)} dz = 2\pi i n$.

We have that $p(z) = c(z - a_1) \cdots (z - a_n)$ by Corollary 3.6.

$$\begin{aligned}
\int_\gamma \frac{p'(z)}{p(z)} dz &= \int_\gamma \frac{(c(z-a_1)\cdots(z-a_n))'}{c(z-a_1)\cdots(z-a_n)} dz \\
&= \int_\gamma \frac{\sum_{i=1}^n (z-a_1)\cdots(z-a_{i-1})(z-a_{i+1})\cdots(z-a_n)}{(z-a_1)\cdots(z-a_n)} dz \\
&= \int_\gamma \sum_{i=1}^n \frac{1}{(z-a_i)} dz \text{ as } (z-a_j) \neq 0 \text{ for all } j \text{ on } \{z : z \geq R\} \\
&= \sum_{i=1}^n \int_\gamma \frac{1}{z-a_i} dz \\
&= \sum_{i=1}^n 2\pi i \text{ by Proposition 2.6} \\
&= 2\pi i n
\end{aligned}$$

pg.87

#6 Let f be analytic on $D = B(0;1)$ and suppose $|f(z)| \leq 1$ for $|z| < 1$. Show $|f'(0)| \leq 1$.

By Theorem 2.14 Cauchy's Estimate, we have the hypothesis such that $M = 1$, $R = 1$ and $a = 0$. Hence $|f'(0)| \leq 1$.

#8 Let G be a region and suppose $f_n : G \to \mathbb{C}$ is analytic for each $n \geq 1$. Suppose that $\{f_n\}$ converges uniformly to a function $f : G \to \mathbb{C}$. Show that f is analytic.

Since $\{f_n\}$ converges uniformly on G to f, then f is continuous on G. Now we must show $\int_T f = 0$ for every triangular path T in G. Let $\epsilon > 0$ and T be an arbitrary triangular path in G of length $|T|$. Then choose $\delta > 0$ such that $\delta \cdot |T| \leq \epsilon$. Then $\int_T f = \int_T f + 0 = \int_T (f - f_n)$ as $\int_T f_n = 0$ because f_n is analytic on G for every n. Also there exists a N such that $|f(z) - f_n(z)| \leq \delta$ for every z in G and $n \geq N$ by uniform convergence. Then we have the estimate for every $n \geq N$:

$$\begin{aligned}
\left| \int_T f(z) dz \right| &= \left| \int_T f(z) - f_n(z) dz \right| \\
&\leq \int_T |f(z) - f_n(z)| \, |dz| \\
&\leq \int_T \delta \, |dz| \\
&= \delta \, |T| \\
&\leq \epsilon
\end{aligned}$$

Therefore $\left|\int_T f(z)dz\right| \leq \epsilon$ for any ϵ and arbitrary T in G. Therefore f is continuous and $\int_T f = 0$ for any T in G and we have satisfied the conditions of Theorem 5.10 (Morera's Theorem) and we can conclude that f is analytic.

#9 Show that if $f : \mathbb{C} \to \mathbb{C}$ is a continuous function such that f is analytic off $[-1,1]$ then f is an entire function.

We have that f is continuous everywhere from the hypothesis of the problem. Therefore it is left to show that $\int_T f = 0$ for every arbitrary triangular path T in $\mathbb{C}$ to apply Morera's Theorem. Suppose T is an arbitrary triangular path in $\mathbb{C}$. If the triangle of T, does not contain $[-1,1]$ then $\int_T f = 0$.

Otherwise we have the triangle of T contains $[-1,1]$. Let T have three vertices's $\{z_1, z_2, z_3\}$. We are guaranteed one vertex not on $[-1,1]$. Call it z_1. Then we may parameterize $\overline{z_1z_2}$ as $z_1 + t(z_2 - z_1)$ for $t \in [0,1]$ and $\overline{z_2z_3}$ as $z_2 + t(z_3 - z_2)$ for $t \in [0,1]$ and $\overline{z_2z_1}$ as $z_1 + t(z_3 - z_1)$ for $t \in [0,1]$. Note the orientation is reversed for $z_2\bar{z}_1$. Let $\ell_2(\alpha) = z_1 + \alpha(z_2 - z_1)$ and $\ell_3(\alpha) = z_1 + \alpha(z_3 - z_1)$. Then define T_α as the triangle of $\{z_1, \ell_2(\alpha), \ell_3(\alpha)\}$. Then we may parameterize $\overline{z_1\ell_2(\alpha)}$ as $z_1 + t\alpha(z_2 - z_1)$ for $t \in [0,1]$ and $\overline{\ell_2(\alpha)\ell_3(\alpha)}$ as $z_1 + \alpha(z_2 - z_1) + t\alpha(z_3 - z_2)$ for $t \in [0,1]$ and $\overline{z_1\ell_3(\alpha)}$ by reverse orientation $z_1 + t\alpha(z_3 - z_1)$ for $t \in [0,1]$.

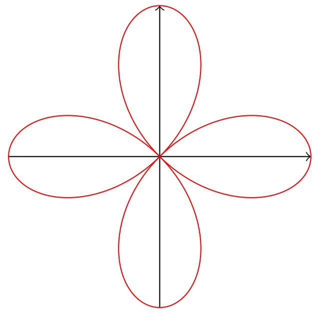
Then we have that

$$
\begin{aligned}
g(\alpha) &= \int_{T_\alpha} f(z)dz \\
&= \int_{\overline{z_1\ell_2(\alpha)}} f(z)dz + \int_{\overline{\ell_2(\alpha)\ell_3(\alpha)}} f(z)dz + \int_{\overline{\ell_3(\alpha)z_1}} f(z)dz \\
&= \int_{\overline{z_1\ell_2(\alpha)}} f(z)dz + \int_{\overline{\ell_2(\alpha)\ell_3(\alpha)}} f(z)dz - \int_{\overline{z_1\ell_3(\alpha)}} f(z)dz \\
&= \int_0^1 f(z_1 + t\alpha(z_2 - z_1))\alpha(z_2 - z_1)dt + \int_0^1 f(z_1 + \alpha(z_2 - z_1) + t\alpha(z_3 - z_2))\alpha(z_3 - z_2)dt \\
&- \int_0^1 f(z_1 + t\alpha(z_3 - z_1))\alpha(z_3 - z_1)dt \\
&= \int_0^1 f(z_1 + t\alpha(z_2 - z_1))\alpha(z_2 - z_1) + f(z_1 + \alpha(z_2 - z_1) + t\alpha(z_3 - z_2))\alpha(z_3 - z_2) \\
&- f(z_1 + t\alpha(z_3 - z_1))\alpha(z_3 - z_1)dt
\end{aligned}
$$

Clearly since z_1 is not on $[-1,1]$ then there exists some $\alpha^* \in [0,1]$ such that T_{α^*} does not contain $[-1,1]$. Hence $g(\alpha^*) = \int_{T_{\alpha^*}} f = 0$ by Cauchy's Theorem. Then $M_1 = f(z_1+t(\alpha+h)(z_2-z_1)) - f(z_1+t\alpha(z_2-z_1))$, $M_2 = f(z_1+(\alpha+h)(z_2-z_1)+t(\alpha+h)(z_3-z_2)) - f(z_1+\alpha(z_2-z_1)+t\alpha(z_3-z_2))$, $M_3 = f(z_1+t(\alpha+h)(z_3-z_1)) - f(z_1+t\alpha(z_3-z_1))$ is continuous with respect to t and α. Moreover when $h \to 0$ then $M_i \to 0$. Then $\left|\frac{g(\alpha+h)-g(\alpha)}{h}\right| \le \int_0^1 |M_1(z_2 - z_1) + M_2(z_3 - z_2) + M_3(z_3 - z_1)| \left|\frac{\alpha+h-\alpha}{h}\right| dt = \int_0^1 |M_1(z_2 - z_1) + M_2(z_3 - z_2) + M_3(z_3 - z_1)| \, dt$. Therefore as $\lim_{h\to 0} \left|\frac{g(\alpha+h)-g(\alpha)}{h}\right| = 0$ Hence $g'(\alpha) = 0$ for all $\alpha \in [0,1]$. Hence $0 = g(1) = \int_T f$ and by Morera's Theorem f is analytic on $\mathbb{C}$.

pg.96

#5 Evaluate the integral $\int_\gamma \frac{dz}{z^2+1}$ where $\gamma(\theta) = 2\left|\cos 2\theta\right| e^{i\theta}$ for $0 \le \theta \le 2\pi$.



The path is the following:

We can see that the poles $-i$ and i are surrounded by one loop. Then by the most important integral formula we have:

$$\begin{aligned}\int_\gamma \frac{dz}{z^2+1} &= \int_\gamma \frac{\frac{i}{2}}{z+i} - \frac{\frac{i}{2}}{z-i} dz \\ &= \frac{i}{2}\int_\gamma \frac{dz}{z-(-i)} - \frac{i}{2}\int_\gamma \frac{dz}{z-i} \\ &= \frac{i}{2}2\pi i - \frac{i}{2}2\pi i \\ &= 0\end{aligned}$$

pg.111

#15 Let f be analytic in $G = \{z : 0 < |z-a| < r\}$ except that there is a sequence of poles $\{a_n\}$ in G with $a_n \to a$. Show that for any ω in $\mathbb{C}$ there is a sequence $\{z_n\}$ in G with $a = \lim z_n$ and $\omega = \lim f(z_n)$.

Suppose there exists no sequence $\{z_n\}$ with $z_n \to a$ such that $\lim f(z_n) = \omega$. Therefore there exists a $R < r$ such that $B(a,R)$ then $f(z) - \omega \neq 0$ otherwise we could find a sequence z_n such that $f(z_n) = \omega$ for z_n in $B(a, R_n)$. Define $g(z) = \frac{1}{f(z)-\omega}$ is analytic on $B(a,R)\backslash \cup \{a\}$ as $g(a_n) = 0$. Therefore $\lim_{n\to\infty} g(a_n) = 0$. Hence $\{z : g(z) = 0\}$ has a limit point on $B(a,R)$. Hence $g(z) \equiv 0$ on $B(a,R)$. This is a contradiction that f has isolated singularities on $B(a,R)\backslash\{a\}$.

pg.126

#2 Suppose f is analytic on $\bar{B}(0;1)$ and satisfies $|f(z)| < 1$ for $|z| = 1$. Find the number of solutions (counting multiplicities) of the equation $f(z) = z^n$ where n is an integer larger than or equal to 1.

Let $\tilde{f}(z) = f(z) - z^n$ and $\tilde{g}(z) = z^n$. Then $\left|\tilde{f} + \tilde{g}\right| = |f(z)| < 1 = |z^n| = |\tilde{g}|$ for $|z| = 1$. Then by the weak Rouche's Theorem, we have that $Z_{\tilde{f}} = Z_{\tilde{g}} = n$ as $\tilde{f}$ and $\tilde{g}$ are analytic and do not have poles on the unit disk. Hence $f(z) = z^n$ has n solutions.

#9 Let $\lambda > 1$ and show that the equation $\lambda - z - e^{-z} = 0$ has exactly one solution i the half plane $\{z : Rez > 0\}$. Show that this solution must be real. What happens to the solution as $\lambda \to 1$?

We let $\tilde{f}(z) = \lambda - z - e^{-z}$ and $\tilde{g} = -\lambda + z$. Then $\left|\tilde{f}(z) + \tilde{g}(z)\right| = |-e^{-z}| = \left|e^{-x-iy}\right| = |e^{-x}| < 1 < |\lambda| < |\lambda - z| = |\tilde{g}|$ for all z in $\{z : Rez > 0\}$. Therefore for any compact set including λ, $\{z : Rez > 0\}$ we have that $Z_{\tilde{f}} = Z_{\tilde{g}} = 1$ as both are analytic functions with no poles. Hence there is one solution by Rouche's theorem.

Suppose that z is a solution to $\tilde{f}(z) = 0$. Then $\lambda - x - iy - e^{-x}(\cos(y) - i\sin(y)) = 0$, then $\lambda - x - e^{-x}\cos(y) = 0$ and $-y + \sin(y)e^{-x} = 0$. We have that $\alpha\sin(y) - y = 0$ for $y = 0$ for all $\alpha = e^{-x} \leq 1$ which is true on the positive real half plane. Hence the imaginary part of z must be zero.

As $\lambda \to 1$. Then we approach $1 - z - e^{-z} = 0$ which has a solution 0 which is outside the half plane.

#10 Let f be analytic in a neighborhood of $D = \bar{B}(0;1)$. If $|f(z)| < 1$ for $|z| = 1$, show that there is a unique z with $|z| < 1$ and $f(z) = z$. If $|f(z)| \leq 1$ for $|z| = 1$, what can you say?

Let $\tilde{f}(z) = f(z) - z$ and $\tilde{g}(z) = z$. Then $\left|\tilde{f} + \tilde{g}\right| = |f(z)| < 1 = |z| = |\tilde{g}|$ for $|z| = 1$. By Rouche's Theorem, we have that $Z_{\tilde{f}} = Z_{\tilde{g}} = 1$ as $\tilde{f}$ and $\tilde{g}$ are analytic and don't have poles. Hence $f(z) = z$ has 1 solutions. Hence a unique z in the disk.

We have that if $f(z) = z$ or $f(z) = z^2$ then there are infinitely many, and one solution. Thus we no longer can apply Rouche's Theorem.

pg.301

#2 Show that if f is a one-one entire function then $f(z) = az + b$ for some constants a and b, $a \neq 0$.

Since f is entire and one to one, then f must be a polynomial by Corollary 4.4. The only entire, $1-1$ polynomial is linear. Furthermore $a \neq 0$ otherwise f would be a constant, a contradiction to f being $1-1$.