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Math 8090 Homework 2

1. Let $\{X_t\}$ be a stationary Gaussian time series with covariance function $\gamma(\cdot)$. Use Proposition A.3.1 to do the following:

   a. Check my work form class by finding $E[X_3|X_1, X_2]$. If there was an algebra error in my solution fix it.
   We have that

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \sim N_3\left(\mu, \begin{pmatrix} \gamma(0) & \gamma(1) & \gamma(2) \\ \gamma(1) & \gamma(0) & \gamma(1) \\ \gamma(2) & \gamma(1) & \gamma(0) \end{pmatrix}\right)$$

   Therefore using Proposition A.3.1 we have that

$$\begin{aligned} X_3|X_1, X_2 &\sim N(\mu + \Sigma_{21}\Sigma_{11}^{-1}(\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} - \mu), \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}) \\ &\sim N(\mu + \begin{pmatrix} \gamma(2) & \gamma(1) \end{pmatrix} \begin{pmatrix} \gamma(0) & \gamma(1) \\ \gamma(1) & \gamma(0) \end{pmatrix}^{-1} (\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} - \mu), \\ &\gamma(0) - \begin{pmatrix} \gamma(2) & \gamma(1) \end{pmatrix} \begin{pmatrix} \gamma(0) & \gamma(1) \\ \gamma(1) & \gamma(0) \end{pmatrix}^{-1} \begin{pmatrix} \gamma(2) \\ \gamma(1) \end{pmatrix}) \\ &\sim N(\mu + \begin{pmatrix} \gamma(2) & \gamma(1) \end{pmatrix} \frac{1}{\gamma(0)^2 - \gamma(1)^2} \begin{pmatrix} \gamma(0) & -\gamma(1) \\ -\gamma(1) & \gamma(0) \end{pmatrix} (\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} - \mu), \\ &\gamma(0) - \begin{pmatrix} \gamma(2) & \gamma(1) \end{pmatrix} \frac{1}{\gamma(0)^2 - \gamma(1)^2} \begin{pmatrix} \gamma(0) & -\gamma(1) \\ -\gamma(1) & \gamma(0) \end{pmatrix} \begin{pmatrix} \gamma(2) \\ \gamma(1) \end{pmatrix}) \\ &\sim N(\mu + \frac{1}{\gamma(0)^2 - \gamma(1)^2}((\gamma(0)\gamma(2) - \gamma(1)^2)(X_1 - \mu) + (-\gamma(2)\gamma(1) + \gamma(1)\gamma(0))(X_2 - \mu)), \\ &\gamma(0) - \frac{1}{\gamma(0)^2 - \gamma(1)^2}(\gamma(0)\gamma(2)^2 - \gamma(2)\gamma(1)^2 - \gamma(2)\gamma(1)^2 + \gamma(1)^2\gamma(0))) \end{aligned}$$

   Therefore $E[X_3|X_1, X_2] = \mu + \frac{((\gamma(0)\gamma(2) - \gamma(1)^2)(X_1 - \mu) + (-\gamma(2)\gamma(1) + \gamma(1)\gamma(0))(X_2 - \mu))}{\gamma(0)^2 - \gamma(1)^2}$.

   b. Sometimes we are missing a measurement. If we have X_2 and X_4, but missed X_3, we can try to predict X_3. Find a formula for $E[X_3|X_2, X_4]$.

   We have that

$$\begin{pmatrix} X_2 \\ X_3 \\ X_4 \end{pmatrix} \sim N_3\left(\mu, \begin{pmatrix} \gamma(0) & \gamma(1) & \gamma(2) \\ \gamma(1) & \gamma(0) & \gamma(1) \\ \gamma(2) & \gamma(1) & \gamma(0) \end{pmatrix}\right)$$

   or

$$\begin{pmatrix} X_3 \\ X_2 \\ X_4 \end{pmatrix} \sim N_3\left(\mu, \begin{pmatrix} \gamma(0) & \gamma(1) & \gamma(1) \\ \gamma(1) & \gamma(0) & \gamma(2) \\ \gamma(1) & \gamma(2) & \gamma(0) \end{pmatrix}\right)$$

Therefore using Proposition A.3.1 we have that

$$
\begin{aligned}
X_3|X_2,X_4 &\sim N(\mu + \Sigma_{12}\Sigma_{22}^{-1}(\begin{pmatrix} X_2 \\ X_4 \end{pmatrix} - \mu), \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}) \\
&\sim N(\mu + \begin{pmatrix} \gamma(1) & \gamma(1) \end{pmatrix} \begin{pmatrix} \gamma(0) & \gamma(2) \\ \gamma(2) & \gamma(0) \end{pmatrix}^{-1} (\begin{pmatrix} X_2 \\ X_4 \end{pmatrix} - \mu), \\
&\gamma(0) - \begin{pmatrix} \gamma(1) & \gamma(1) \end{pmatrix} \begin{pmatrix} \gamma(0) & \gamma(2) \\ \gamma(2) & \gamma(0) \end{pmatrix}^{-1} \begin{pmatrix} \gamma(1) \\ \gamma(1) \end{pmatrix}) \\
&\sim N(\mu + \begin{pmatrix} \gamma(1) & \gamma(1) \end{pmatrix} \frac{1}{\gamma(0)^2 - \gamma(2)^2} \begin{pmatrix} \gamma(0) & -\gamma(2) \\ -\gamma(2) & \gamma(0) \end{pmatrix} (\begin{pmatrix} X_2 \\ X_4 \end{pmatrix} - \mu), \\
&\gamma(0) - \begin{pmatrix} \gamma(1) & \gamma(1) \end{pmatrix} \frac{1}{\gamma(0)^2 - \gamma(1)^2} \begin{pmatrix} \gamma(0) & -\gamma(2) \\ -\gamma(2) & \gamma(0) \end{pmatrix} \begin{pmatrix} \gamma(1) \\ \gamma(1) \end{pmatrix}) \\
&\sim N(\mu + \frac{1}{\gamma(0)^2 - \gamma(2)^2}(\gamma(0)\gamma(1) - \gamma(1)\gamma(2))(X_2 + X_4 - 2\mu), \\
&\gamma(0) - \frac{2}{\gamma(0)^2 - \gamma(2)^2}(\gamma(0)\gamma(1)^2 - \gamma(2)\gamma(1)^2))
\end{aligned}
$$

Therefore $E[X_3|X_2,X_4] = \mu + \frac{\gamma(1)(\gamma(0)-\gamma(2))(X_2+X_4-2\mu)}{\gamma(0)^2-\gamma(2)^2} = \mu + \frac{\gamma(1)(X_2+X_4-2\mu)}{\gamma(0)+\gamma(2)}$.

2. Let $X = (X_1, \ldots, X_n)$ be a random vector with mean vector μ and covariance matrix Σ. For matrix A and vector b of appropriate sizes, let $Y = b + AX$.

   a. Find the mean of Y in terms of b, A and μ. Justify your work using the linearity of expectation. We have that $Y_i = b_i + \sum_{j=1}^n A_{ij}X_j$. Therefore $E[Y_i] = b_i + \sum_{j=1}^n A_{ij}E[X_j] = b_i + A_{ij}\mu_j$ by linearity. Hence we have $E[Y] = b + AE[X] = b + A\mu$

   b. Find the covariance matrix of Y in terms of A and Σ. Use the definition of the covariance matrix of Y (pg. 376) to verify equation (A.2.5). Justify your work using valid linear algebra operations and properties of covariance. We have that

$$
\begin{aligned}
Cov(Y,Y) &= E(Y - E[Y])(Y - E[Y])^T \text{ by definition} \\
&= E[(b + AX - b - A\mu)(b + AX - b - A\mu)^T] \text{ by part a} \\
&= E[(AX - A\mu)(AX - A\mu)^T] \\
&= E[A(X - \mu)(X - \mu)^T A^T] \text{ by linearity of A and properties of transpose} \\
&= AE[(X - \mu)(X - \mu)^T]A^T \text{ by linearity of expectation} \\
&= A\Sigma A^T
\end{aligned}
$$

3. Let $Z \sim N_n(0, I)$. Let A and B be matrices of appropriate sizes. Show that $Y_1 = AZ$ and $Y_2 = BZ$ are independent if and only if $AB' = 0$ (a matrix of all 0's). You may use propositions in the back of the text.

We have that $Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} = \begin{pmatrix} AZ \\ BZ \end{pmatrix} = \begin{pmatrix} A \\ B \end{pmatrix} Z$. Thus by the previous problem $Cov(Y,Y) = \begin{pmatrix} A \\ B \end{pmatrix} Cov(Z,Z) \begin{pmatrix} A \\ B \end{pmatrix}^T$ or $Cov(Y,Y) = \begin{pmatrix} A \\ B \end{pmatrix} \begin{pmatrix} A' & B' \end{pmatrix}$. Furthermore we have that Y must follow a normal distribution with

$$\begin{aligned}
\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} &\sim N(0, \begin{pmatrix} Cov(Y_1, Y_1) & Cov(Y_1, Y_2) \\ Cov(Y_2, Y_1) & Cov(Y_2, Y_2) \end{pmatrix}) \\
&\sim N(0, \begin{pmatrix} AA' & AB' \\ BA' & BB' \end{pmatrix})
\end{aligned}$$

By proposition A.3.1, Y_1 and Y_2 are independent if and only if $Cov(Y_1, Y_2) = 0$ if and only if $AB' = 0$.

2.1 Suppose that $X_1, X_2, \ldots,$ is a stationary times series with mean μ and ACF $\rho(\cdot)$. Show that the best predictor of X_{n+h} of the form $aX_n + b$ is obtained by choosing $a = \rho(h)$ and $b = \mu(1 - \rho(h))$.

We wish to minimize $E[(aX_n + b - X_{n+h})^2]$. Thus we have equivalently

$$\begin{aligned}
&E[(aX_n + b)^2 - 2(aX_n + b)X_{n+h} + X_{n+h}^2] = E[a^2X_n^2 + 2abX_n + b^2 - 2aX_nX_{n+h} - 2bX_{n+h} + X_{n+h}^2] \\
&= E[a^2(X_n - \mu)^2 + a^2(2\mu X_n - \mu^2) + 2abX_n + b^2 - 2aX_nX_{n+h} - 2bX_{n+h} + (X_{n+h} - \mu)^2 + 2\mu X_{n+h} - \mu^2] \\
&= a^2Var(X_n) + Var(X_{n+h}) + E[a^2(2\mu X_n - \mu^2) + 2abX_n + b^2 - 2aX_nX_{n+h} - 2bX_{n+h} + 2\mu X_{n+h} - \mu^2] \\
&= a^2Var(X_n) + Var(X_{n+h}) + a^2\mu^2 + 2ab\mu + b^2 - 2b\mu + \mu^2 \\
&- 2aE[(X_n - \mu)(X_{n+h} - \mu) + \mu X_n + \mu X_{n+h} - \mu^2] \\
&= a^2Var(X_n) + Var(X_{n+h}) + a^2\mu^2 + 2ab\mu + b^2 - 2b\mu + \mu^2 - 2aCov(X_n, X_{n+h}) - 2a\mu^2 \\
&= a^2Var(X_n) + Var(X_{n+h}) - 2aCov(X_n, X_{n+h}) + \mu^2(a-1)^2 + 2b\mu(a-1) + b^2 \\
&= a^2Var(X_n) + Var(X_{n+h}) - 2aCov(X_n, X_{n+h}) + ((a-1)\mu + b)^2
\end{aligned}$$

Therefore we are minimized if $(a-1)\mu + b = 0$ or $b = (1-a)\mu$. Therefore we need to minimize $a^2Var(X_n) + Var(X_{n+h}) - 2aCov(X_n, X_{n+h})$ with respect to a. Since it is quadratic with respect to a the minimum is $a = \frac{Cov(X_n, X_{n+h})}{Var(X_n)} = \gamma(h)/\gamma(0) = \rho(h)$.

2.3 a. Find the ACVF of the time series $X_t = Z_t + .3Z_{t-1} - .4Z_{t-2}$, where $\{Z_t\} \sim WN(0, 1)$.

We have

$$\begin{aligned}
\gamma_X(h) &= Cov(X_t, X_{t+h}) \\
&= Cov(Z_t + .3Z_{t-1} - .4Z_{t-2}, Z_{t+h} + .3Z_{t+h-1} - .4Z_{t+h-2}) \\
&= Cov(Z_t, Z_{t+h}) + Cov(Z_t, .3Z_{t+h-1}) - Cov(Z_t, .4Z_{t+h-2}) \\
&+ Cov(.3Z_{t-1}, Z_{t+h}) + Cov(.3Z_{t-1}, .3Z_{t+h-1}) - Cov(.3Z_{t-1}, .4Z_{t+h-2}) \\
&- Cov(.4Z_{t-2}, Z_{t+h}) - Cov(.4Z_{t-2}, .3Z_{t+h-1}) + Cov(.4Z_{t-2}, .4Z_{t+h-2}) \\
&= \gamma_Z(h) + .3\gamma_Z(h-1) - .4\gamma_Z(h-2) \\
&+ .3\gamma_Z(h+1) + .09\gamma_Z(h) - .12\gamma_Z(h-1) \\
&- .4\gamma_Z(h+2) - .12\gamma_Z(h+1) + .16\gamma_Z(h)
\end{aligned}$$

Thus

$$\gamma_X(h) = \begin{cases} 1 + .09 + .16, & h = 0, \\ .3 - .12, & |h| = 1, \\ -.4, & |h| = 2, \\ 0, & \text{otherwise.} \end{cases}$$

b. Find the ACVF of the time series $Y_t = \tilde{Z}_t - 1.2\tilde{Z}_{t-1} - 1.6\tilde{Z}_{t-2}$, where $\{\tilde{Z}_t\} \sim WN(0, .25)$. Compare with the answer found in (a).

We have

$$\begin{aligned}
\gamma_Y(h) &= Cov(Y_t, Y_{t+h}) \\
&= Cov(Z_t - 1.2Z_{t-1} - 1.6Z_{t-2}, Z_{t+h} - 1.2Z_{t+h-1} - 1.6Z_{t+h-2}) \\
&= Cov(Z_t, Z_{t+h}) - Cov(Z_t, 1.2Z_{t+h-1}) - Cov(Z_t, 1.6Z_{t+h-2}) \\
&- Cov(1.2Z_{t-1}, Z_{t+h}) + Cov(1.2Z_{t-1}, 1.2Z_{t+h-1}) + Cov(1.2Z_{t-1}, 1.6Z_{t+h-2}) \\
&- Cov(1.6Z_{t-2}, Z_{t+h}) + Cov(1.6Z_{t-2}, 1.2Z_{t+h-1}) + Cov(1.6Z_{t-2}, 1.6Z_{t+h-2}) \\
&= \gamma_Z(h) - 1.2\gamma_Z(h-1) - 1.6\gamma_Z(h-2) \\
&- 1.2\gamma_Z(h+1) + 1.44\gamma_Z(h) + 1.92\gamma_Z(h-1) \\
&- 1.6\gamma_Z(h+2) + 1.92\gamma_Z(h+1) + 2.56\gamma_Z(h)
\end{aligned}$$

Thus

$$\gamma_Y(h) = \begin{cases} .25(1 + 1.44 + 2.56), & h = 0, \\ .25(-1.2 + 1.92), & |h| = 1, \\ .25(-1.6), & |h| = 2, \\ 0, & \text{otherwise.} \end{cases}$$

Clearly we have that $\gamma_X(h) = \gamma_Y(h)$.

2.4 It is clear that the function $\kappa(h) = 1, h = 0, \pm 1, \ldots$, is an auto covariance function, since it is the auto covariance function of the process $X_t = Z, t = 0, \pm 1, \ldots$, where Z is a random variable with mean 0 and variance 1. By identifying appropriate sequences of random variables, show that the following functions are also auto covariance functions:

(a) $\kappa(h) = (-1)^{|h|}$
We may define $X_t = (-1)^{|t|}Z$ where Z is a random variable with mean 0 and variance 1. Then we have that $\gamma_X(|h|) = Cov(X_t, X_{t+|h|}) = Cov((-1)^t Z, (-1)^{t+|h|}Z) = (-1)^h Cov(Z, Z) = (-1)^{|h|}$

(b) $\kappa(h) = 1 + \cos\left(\frac{\pi h}{2}\right) + \cos\left(\frac{\pi h}{4}\right)$

Let Z_1, Z_2, Z_3, Z_4, Z_5 be independent random variable with mean 0 and variance 1. Then let $X_t = cos(\pi t/2)Z_1 + sin(\pi t/2)Z_2 + cos(\pi t/4)Z_3 + sin(\pi t/4)Z_4 + Z_5$. Then $\gamma(h, 0) = Cov(cos(\pi h/2)Z_1 + sin(\pi h/2)Z_2 + cos(\pi h/4)Z_3 + sin(\pi h/4)Z_4 + Z_5, Z_1 + Z_4 + Z_5) = cos(\pi h/2) + cos(\pi h/4) + 1 = \kappa(h)$.

(c) $\kappa(h) = \begin{cases} 1, & \text{if } h = 0, \\ 0.4, & \text{if } h = \pm 1, \\ 0, & \text{otherwise.} \end{cases}$ For the MA(1) process we have $X_t = Z_t + \theta Z_{t-1}$. Then $\gamma(0) = \sigma^2(1 + \theta^2)$ and $\gamma(1) = \sigma^2\theta$. Letting $\sigma^2 = 1/(1 + \theta^2)$ gives use $\kappa(0)$ and solving $\theta/(1 + \theta^2) = .4$ yields $\theta = 2$ or $1/2$.

2.5 Suppose that $\{X_t, t = 0, \pm 1, \dots\}$ is stationary and that $|\theta| < 1$. Show that for each fixed n the sequence

$$S_m = \sum_{j=1}^{m} \theta^j X_{n-j}$$

is convergent absolutely and in mean square (see Appendix C) as $m \to \infty$.

We compute $E[|S_m|] \leq E[\sum j = 1^n |\theta|^j |X_{n-j}|] = \mu_t \sum_{j=1}^{n} |\theta|^j = \mu_t \frac{|\theta|(1-|\theta|^n)}{1-|\theta|} < \infty$. Therefore it converges absolutely.

We compute $E[(S_k - S_m)^2]$ for $k > m$.

$$\begin{aligned} E[(S_k - S_m)^2] &= E[(\sum_{i=m+1}^{k} \theta^i X_{n-i})^2] \\ &= E[(\theta^{m+1} \sum_{j=0}^{k-(m+1)} \theta^j X_{n-j-(m+1)})^2] \end{aligned}$$

2.8 Show that the autoregressive equations

$$X_t = \phi_1 X_{t-1} + Z_t, t = 0, \pm 1, \dots,$$

where $\{Z_t\} \sim WN(0, \sigma^2)$ and $|\phi| = 1$, have no stationary solution. HINT: Suppose there does exist a stationary solution $\{X_t\}$ and use the autoregressive equation to derive an expression for the variance of $X_t - \phi_1^{n+1} X_{t-n-1}$ that contradicts the stationarity assumption.

Assume that we have a stationary solution.

We have that

$$\begin{aligned} Var(X_t - \phi_1^{n+1} X_{t-n-1}) &= Var(X_t - \phi_1^n (X_{t-n} - Z_{t-n})) \text{ by substitution of eqn} \\ &= \dots = Var(\sum_{j=0}^{n} \phi_1^j Z_t - j) \\ &= \sum_{j=0}^{n} \phi_1^{2j} Var(Z_{t-j}) \text{ by defn of white noise} \\ &= (n+1)\sigma^2 \end{aligned}$$

But we also have that

$$\begin{aligned} Var(X_t - \phi_1^{n+1} X_{t-n-1}) &= Var(X_t) - 2\phi_1^{n+1} Cov(X_t, X_{t-n-1}) + \phi_1^{2(n+1)} Var(X_{t-n-1}) \text{ by defn of stationarity} \\ &= \gamma(0) - 2\phi_1^{n+1}\gamma(n+1) + \phi_1^{2(n+1)}\gamma(0) \end{aligned}$$

Thus we have that

$$\begin{aligned} (n+1)\sigma^2 &= \gamma(0) - 2\phi_1^{n+1}\gamma(n+1) + \phi_1^{2(n+1)}\gamma(0) \\ &= \gamma(0) - 2\phi_1^{n+1}\gamma(n+1) + \gamma(0) \\ &\leq 4\gamma(0) \end{aligned}$$

But this is true for all n therefore $\gamma(0)$ is unbounded a contradiction.

2.10 (Solve "by hand" rather than using ITSM software). Use the program ITSM to compute the coefficients ψ_j and π_j, $j = 1, \ldots, 5$, in the expansions

$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}$$

and

$$Z_t = \sum_{j=0}^{\infty} \pi_j X_{t-j}$$

for the ARMA(1, 1) process defined by the equations

$$X_t - 0.5X_{t-1} = Z_t + 0.5Z_{t-1}, \{Z_t\} \sim WN(0, \sigma^2).$$

(Select File¿Project¿New¿Univariate, then Model¿Specify. In the resulting dialog box enter 1 for the AR and MA orders, specify $\phi(1) = \theta(1) = 0.5$, and click OK. Finally, select Model¿AR/MA Infinity¿Default lag and the values of ϕ_j and π_j will appear on the screen.) Check the results and those obtained in Section 2.3.

We have that the equations can be rewritten as $(1 - 1/2B)X_t = (1 + 1/2B)Z_t$ using B as the back shift operator. We have the two identities:

$$(1 + \frac{B}{2} + \left(\frac{B}{2}\right)^2 + \left(\frac{B}{2}\right)^3 + \ldots)(1 - \frac{B}{2}) = 1$$
$$(1 - \frac{B}{2} + \left(\frac{B}{2}\right)^2 - \left(\frac{B}{2}\right)^3 + \ldots)(1 + \frac{B}{2}) = 1$$

Hence we may write

$$(1 + \frac{B}{2} + \left(\frac{B}{2}\right)^2 + \left(\frac{B}{2}\right)^3 + \ldots)(1 - 1/2B)X_t$$
$$= (1 + \frac{B}{2} + \left(\frac{B}{2}\right)^2 + \left(\frac{B}{2}\right)^3 + \ldots)(1 + 1/2B)Z_t$$
$$(1 - \frac{B}{2} + \left(\frac{B}{2}\right)^2 - \left(\frac{B}{2}\right)^3 + \ldots)(1 - 1/2B)X_t$$
$$= (1 - \frac{B}{2} + \left(\frac{B}{2}\right)^2 - \left(\frac{B}{2}\right)^3 + \ldots)(1 + 1/2B)Z_t$$

Reducing by the identities we have that

$$X_t = (1 + \frac{B}{2} + \left(\frac{B}{2}\right)^2 + \left(\frac{B}{2}\right)^3 + \ldots)(1 + 1/2B)Z_t$$
$$(1 - \frac{B}{2} + \left(\frac{B}{2}\right)^2 - \left(\frac{B}{2}\right)^3 + \ldots)(1 - 1/2B)X_t$$
$$= Z_t$$

Simplifying

$$X_t = (1 + \left(\frac{1}{2} + \frac{1}{2}\right) B + \left(\frac{1}{4} + \frac{1}{4}\right) B^2 + \left(\frac{1}{8} + \frac{1}{8}\right) B^3 + \dots) Z_t = Z_t + \sum_{j=1}^{\infty} \frac{1}{2^{j-1}} Z_{t-j}$$

$$Z_t = (1 - \left(\frac{1}{2} + \frac{1}{2}\right) B + \left(\frac{1}{4} + \frac{1}{4}\right) B^2 - \left(\frac{1}{8} + \frac{1}{8}\right) B^3 + \dots) X_t = X_t - \sum_{j=1}^{\infty} \frac{1}{(-2)^{j-1}} X_{t-j}$$

Thus we have that $\psi_t = \begin{cases} \frac{1}{2^{j-1}}, & \text{for } j \geq 1, \\ 1, & \text{for } j = 0. \end{cases}$ and $\pi_t = \begin{cases} \frac{(-1)^j}{2^{j-1}}, & \text{for } j \geq 1, \\ 1, & \text{for } j = 0. \end{cases}$

2.11 Assume normality and use the finite version of the variance formula (2.4.2) Suppose that in a sample of size 100 from an AR(1) process with mean μ, $\phi = .6$, and $\sigma^2 = 2$ we obtain $\bar{x}_{100} = .271$. Construct an approximate 95% confidence interval for μ. Are the data compatible with the hypothesis that $\mu = 0$?

Since we have an AR(1), then $\gamma_X(h) = \frac{\phi^{|h|}\sigma^2}{1-\phi^2}$. Since 100 is a large sample then we may approximate the distribution of $\bar{X}_n$ using the normal distribution. Then by Prop 2.4.1 we have that $\bar{X}_n \sim N(\mu, \frac{1}{n}\sum_{|h|<\infty}\gamma(h))$. Thus the variance is

$$\begin{aligned} \frac{\sigma^2}{n(1-\phi^2)}\left(1 + 2\sum_{h=1}^{\infty}\phi^h\right) &= \frac{\sigma^2}{n(1-\phi^2)}\frac{-\phi+1+2\phi}{1-\phi} \\ &= \frac{\sigma^2}{n(1-\phi^2)}\frac{\phi+1}{1-\phi} \\ &= \frac{\sigma^2}{n(1-\phi)^2} \end{aligned}$$

Thus $\bar{X}_{100} \sim N(\mu, \frac{2}{100(1-.6)^2})$ and the CI is $[\bar{X}_{100} \pm z_{.025}\sqrt{\frac{2}{100(1-.6)^2}}] = [.271 \pm 0.6929519]$. Therefore since 0 is in the CI, the hypothesis is plausible.

2.12 Suppose that in a sample of size 100 from an MA(1) process with mean μ, $\phi = .6$, and $\sigma^2 = 1$ we obtain $\bar{x}_{100} = .157$. Construct an approximate 95% confidence interval for μ. Are the data compatible with the hypothesis that $\mu = 0$?

Since this is a MA(1) process, then $\gamma_X(h) = 0$ for $|h| > 1$, $\phi\sigma^2$ for $|h| = 1$ and $(1+\phi)\sigma^2$ for $h = 0$. Since 100 is a large sample then we may approximate the distribution of $\bar{X}_n$ using the normal distribution.

Then by Prop 2.4.1 we have that $\bar{X}_n \sim N(\mu, \frac{1}{n}\sum_{|h|<\infty}\gamma(h))$. Thus the variance is $\frac{\sigma^2(1+\phi^2+2\phi)}{n}$. Thus $\bar{X}_{100} \sim N(\mu, \frac{1+2(.6)+.6^2}{100})$. and the CI is $[\bar{X}_{100} \pm z_{.025}\frac{1+.6}{10}] = [.157 \pm 0.3135942]$. Therefore since 0 is in the CI, the hypothesis is plausible.

2.13 Suppose that in a sample of size 100, we obtain $\hat{\rho}(1) = .438$ and $\hat{\rho}(2) = .145$.

a. Assuming that the data were generated from an AR(1) model, construct approximate 95% confidence intervals for both $\rho(1)$ and $\rho(2)$. Based on these two confidence intervals, are the data consistent with an AR(1) model with $\phi = .8$?

For an AR(1) model we have that $w_{11} = \sum_{k=1}^{\infty} 2\{\rho(k+1) + \rho(k-1) - 2\rho(1)\rho(k)\}$ and $w_{22} = \sum_{k=1}^{\infty} 2\{\rho(k+2) + \rho(k-2) - 2\rho(2)\rho(k)\}$. Furthermore $\hat{\rho(i)} \sim N(\rho(i), \frac{1}{W_{ii}})$ by equation 2.4.9. Computing we have $w_{ii} = (1-\phi^{2i})(1+\phi^2)(1-\phi^2)^{-1} - 2i\phi^{2i}$ by 2.4.12. With $\phi = .8$ then, $w_{11} = .36$ and $w_{22} = 1.0512$. Therefore we have that the CI for $\hat{\rho(1)}$ and $\hat{\rho(2)}$ is approximately $[\hat{\rho(i)} \pm z_{.025}\sqrt{w_{ii}/n}]$ which is $[.438 \pm 0.1175978]$ and $[.145 \pm 0.2009513]$ respectively. The true value for $\rho(1) = .8$ and $\rho(2) = .64$ are both not in the CI. Therefore the hypothesis of $\phi = .8$ is rejected.

b. Assuming that the data were generated from an MA(1) model, construct approximate 95% confidence intervals for both $\rho(1)$ and $\rho(2)$. Based on these two confidence intervals, are the data consistent with an MA(1) model with $\theta = .6$?

For a MA(1) model we have $w_{11} = 1 - 3\rho^2(1) + 4\rho^4(1)$ and $w_{22} = 1 + 2\rho^2(1)$ with $\rho(1) = .6/(1 + .6^2)$. As we have $\hat{\rho(i)} \sim N(\rho(i), \frac{1}{W_{ii}})$ by equation 2.4.9, we can construct the CI for $[\hat{\rho(i)} \pm z_{.025}\sqrt{w_{ii}/n}]$ which is $[.438 \pm 0.1476653]$ and $[.145 \pm 0.2310159]$ for $\hat{\rho(1)}$ and $\hat{\rho(2)}$ respectively. The true value for $\rho(1) = 0.4411765$ and $\rho(2) = 0$ which are both in their respective CI. Therefore the hypothesis of $\theta = .6$ is plausible.

2.15 Suppose that $\{X_t, t = 0, \pm 1, \dots\}$ is a stationary process satisfying the equations

$$X_t = \phi_1 X_{t-1} + \cdot + \phi_p X_{t-p} + Z_t,$$

where $\{Z_t\} \sim WN(0, \sigma^2)$ and Z_t is uncorrelated with X_s for each $s < t$. Show that the best linear predictor $P_n X_{n+1}$ of X_{n+1} in terms of $1, X_1, \dots, X_n$, assuming $n > p$, is

$$P_n X_{n+1} = \phi_1 X_n + \cdots + \phi_p X_{n+1-p}.$$

What is the mean squared error of $P_n X_{n+1}$?

We have that $E[X_t] = 0$. Let $P_n X_{n+1} = a + b_n X_1 + \cdots + b_1 X_n$. Then we have that $a = 0$ and $Cov(X_{n+1} - \hat{X}_{n+1}, X_i) = Cov(\phi_1 X_{t-1} + \dot{+} \phi_p X_{t-p} + Z_t - b_1 X_n - \cdots - b_n X_1), X_i) = 0$ for $1 \leq i \leq n$ by work in class. Therefore the choice of $b_i = \phi_i$ for $1 \leq i \leq p$ and $b_i = 0$ for $i > p$ satisfies the equations and the mean squared error is $E[(X_{n+1} - \hat{X}_{n+1})^2] = E[Z_{n+1}^2] = \sigma^2$.

2.18 Let $\{X_t\}$ be the stationary process defined by the equations

$$X_t = Z_t - \theta Z_{t-1}, t = 0, \pm 1, \dots,$$

where $|\theta| < 1$ and $\{Z_t\} \sim WN(0, \sigma^2)$. Show that the best linear predictor $\tilde{P}_n X_{n+1}$ of X_{n+1} based on $\{X_j, -\infty < j \leq n\}$ is

$$\tilde{P}_n X_{n+1} = -\sum_{j=1}^{\infty} \theta^j X_{n+1-j}.$$

What is the mean squared error of the predictor $\tilde{P}_n X_{n+1}$?

We have that $Z_{n+1} = X_{n+1} + \theta Z_n$. Recursively substituting we have $Z_{n+1} = X_{n+1} + \sum_{j=1}^{\infty} \theta^j X_{n+1-j}$. Therefore Using the properties of $\tilde{P}_n$ we have that $\tilde{P}_n Z_{n+1} = E[Z_{n+1}]$ as $Cov(Z_{n+1}, X_j) = 0$ for $j \leq n$. Thus we have $0 = \tilde{P}_n X_{n+1} + \sum_{j=1}^{\infty} \theta X_{n+1-j}$ which shows our desired best linear predictor.

Then we have that $E[(X_{n+1} - \tilde{P}_n X_{n+1})^2] = E[Z_{n+1}^2] = \sigma^2$.

2.19 If $\{X_t\}$ is defined as in Problem 2.18 and $\theta = 1$, find the best linear predictor $P_n X_{n+1}$ of X_{n+1} in terms of $X_1, \dots, X_n$. What is the corresponding mean squared error?

We have that $Cov(X_{t+h}, X_t) = Cov(Z_{t+h} - Z_{t+h-1}, Z_t - Z_{t-1})$ yields $2\sigma^2$ if $h = 0$, $-\sigma^2$ if $|h| = 1$ and 0 otherwise. Thus we have Γ is a trilinear matrix and the solution to $\Gamma a = (-\sigma^2, 0, \dots, 0)'$ is $(\frac{n}{n+1}, \frac{n-1}{n+1}, \dots, \frac{1}{n+1})$. Therefore we have that the best linear predictor is $P_n X_{n+1} = \frac{1}{n+1}\sum_{i=1}^{n} iX_i$. This gives the mean square error as $E[(X_{n+1} - P_n X_{n+1})^2] = \gamma(0) - (\frac{n}{n+1}, \frac{n-1}{n+1}, \dots, \frac{1}{n+1})(-\sigma^2, 0, \dots, 0)' = 2\sigma^2 - \frac{n\sigma^2}{n+1}$.

2.20 In the innovations algorithm, show that for each $n \geq 2$, the innovation $X_n - \hat{X}_n$ is uncorrelated with $X_1, \dots, X_{n-1}$. Conclude that $X_n - \hat{X}_n$ is uncorrelated with the innovations $X_1 - \hat{X}_1, \dots, X_{n-1} - \hat{X}_{n-1}$.

2.21 Let X_1, X_2, X_4, X_5 be observations from the MA(1) model

$$X_t = Z_t + \theta Z_{t-1}, \{Z_t\} \sim WN(0, \sigma^2).$$

We have that for MA(1), the $\gamma(|\pm 1|) = \theta\sigma^2$, $\gamma(0) = (1+\theta^2)\sigma^2$, and 0 otherwise.

a. Find the best linear estimate of the missing value X_3 in terms of X_1 and X_2.

We have that

$$\Gamma a = \begin{pmatrix} (1+\theta^2)\sigma^2 & \theta\sigma^2 \\ \theta\sigma^2 & (1+\theta^2)\sigma^2 \end{pmatrix} a = \begin{pmatrix} \gamma(1) \\ \gamma(1) \end{pmatrix} = \begin{pmatrix} \theta\sigma^2 \\ \theta\sigma^2 \end{pmatrix}$$

Thus $a_i = \theta/(1+\theta+\theta^2)$. Hence $\hat{X}_3 = \theta/(1+\theta+\theta^2)(X_1 + X_2)$.

b. Find the best linear estimate of the missing value X_3 in terms of X_4 and X_5.

We have that

$$\Gamma a = \begin{pmatrix} (1+\theta^2)\sigma^2 & \theta\sigma^2 \\ \theta\sigma^2 & (1+\theta^2)\sigma^2 \end{pmatrix} a = \begin{pmatrix} \gamma(1) \\ \gamma(2) \end{pmatrix} = \begin{pmatrix} \theta\sigma^2 \\ 0 \end{pmatrix}$$

Thus $a_1 = \frac{\theta}{\theta^4+\theta^2+1}(\theta^2+1)$ and $a_2 = \frac{\theta}{\theta^4+\theta^2+1}(-\theta)$. Hence $\hat{X}_3 = \frac{\theta(\theta^2+1)X_4}{\theta^4+\theta^2+1} + \frac{-\theta^2 X_5}{\theta^4+\theta^2+1}$.

c. Find the best linear estimate of the missing value X_3 in terms of X_1, X_2, X_4, and X_5.

We have that

$$\Gamma a = \begin{pmatrix} (1+\theta^2)\sigma^2 & \theta\sigma^2 & 0 & 0 \\ \theta\sigma^2 & (1+\theta^2)\sigma^2 & 0 & 0 \\ 0 & 0 & (1+\theta^2)\sigma^2 & \theta\sigma^2 \\ 0 & 0 & \theta\sigma^2 & (1+\theta^2)\sigma^2 \end{pmatrix} a = \begin{pmatrix} \gamma(2) \\ \gamma(1) \\ \gamma(1) \\ \gamma(2) \end{pmatrix} = \begin{pmatrix} 0 \\ \theta\sigma^2 \\ \theta\sigma^2 \\ 0 \end{pmatrix}$$

Thus $a_1 = a_4 = \frac{-\theta^2}{\theta^4+\theta^2+1}$ and $a_2 = a_3 = \frac{\theta(\theta^2+1)}{\theta^4+\theta^2+1}$. Hence $\hat{X}_3 = \frac{\theta}{\theta^4+\theta^2+1}(-\theta(X_1+X_5) + (\theta^2+1)(X_2+X_4))$.

d. compute the mean squared errors for each of the estimates in (a), (b), and (c).

The mean square errors are $\gamma(0) - a'\gamma_n(h)$. Therefore we have

$$
\begin{aligned}
MSE_{(a)} &= (1+\theta^2)\sigma^2 - \sigma^2 \frac{2\theta^2}{1+\theta+\theta^2} \\
MSE_{(b)} &= (1+\theta^2)\sigma^2 - \sigma^2 \frac{\theta^2(\theta^2+1)}{1+\theta^2+\theta^4} \\
MSE_{(c)} &= (1+\theta^2)\sigma^2 - \sigma^2 \frac{2\theta^2(\theta^2+1)}{1+\theta^2+\theta^4}
\end{aligned}
$$

2.22 Repeat parts (a)-(d) of Problem 2.21 assuming now that the observations X_1, X_2, X_4, X_5 are from the causal AR(1) model

$$X_t = \phi X_{t-1} + Z_t, \{Z_t\} \sim WN(0, \sigma^2).$$

a. Find the best linear estimate of the missing value X_3 in terms of X_1 and X_2.

We have that

$$\Gamma a = \begin{pmatrix} \frac{\sigma^2}{1-\phi^2} & \frac{\phi\sigma^2}{1-\phi^2} \\ \frac{\phi\sigma^2}{1-\phi^2} & \frac{\sigma^2}{1-\phi^2} \end{pmatrix} a = \begin{pmatrix} \gamma(1) \\ \gamma(1) \end{pmatrix} = \begin{pmatrix} \frac{\phi\sigma^2}{1-\phi^2} \\ \frac{\phi\sigma^2}{1-\phi^2} \end{pmatrix}$$

Thus $a_i = \frac{\phi}{\phi+1}$. Hence $\hat{X}_3 = \frac{\phi}{\phi+1}(X_1 + X_2)$.

b. Find the best linear estimate of the missing value X_3 in terms of X_4 and X_5.

We have that

$$\Gamma a = \begin{pmatrix} \frac{\sigma^2}{1-\phi^2} & \frac{\phi\sigma^2}{1-\phi^2} \\ \frac{\phi\sigma^2}{1-\phi^2} & \frac{\sigma^2}{1-\phi^2} \end{pmatrix} a = \begin{pmatrix} \gamma(1) \\ \gamma(2) \end{pmatrix} = \begin{pmatrix} \frac{\phi\sigma^2}{1-\phi^2} \\ \frac{\phi^2\sigma^2}{1-\phi^2} \end{pmatrix}$$

Thus $a_1 = \phi$ and $a_2 = 0$. Hence $\hat{X}_3 = \phi X_4$.

c. Find the best linear estimate of the missing value X_3 in terms of X_1, X_2, X_4, and X_5.

We have that

$$\Gamma a = \begin{pmatrix} \frac{\sigma^2}{1-\phi^2} & \frac{\phi\sigma^2}{1-\phi^2} & \frac{\phi^3\sigma^2}{1-\phi^2} & \frac{\phi^4\sigma^2}{1-\phi^2} \\ \frac{\phi\sigma^2}{1-\phi^2} & \frac{\sigma^2}{1-\phi^2} & \frac{\phi^2\sigma^2}{1-\phi^2} & \frac{\phi^3\sigma^2}{1-\phi^2} \\ \frac{\phi^3\sigma^2}{1-\phi^2} & \frac{\phi^2\sigma^2}{1-\phi^2} & \frac{\sigma^2}{1-\phi^2} & \frac{\phi\sigma^2}{1-\phi^2} \\ \frac{\phi^4\sigma^2}{1-\phi^2} & \frac{\phi^3\sigma^2}{1-\phi^2} & \frac{\phi\sigma^2}{1-\phi^2} & \frac{\sigma^2}{1-\phi^2} \end{pmatrix} a = \begin{pmatrix} \gamma(2) \\ \gamma(1) \\ \gamma(1) \\ \gamma(2) \end{pmatrix} = \begin{pmatrix} \frac{\phi^2\sigma^2}{1-\phi^2} \\ \frac{\phi\sigma^2}{1-\phi^2} \\ \frac{\phi\sigma^2}{1-\phi^2} \\ \frac{\phi^2\sigma^2}{1-\phi^2} \end{pmatrix}$$

Thus $a_1 = a_4 = 0$ and $a_2 = a_3 = \frac{\phi}{\phi^2+1}$. Hence $\hat{X}_3 = \frac{\phi}{\phi^2+1}(X_2 + X_4))$.

d. compute the mean squared errors for each of the estimates in (a), (b), and (c).

The mean square errors are $\gamma(0) - a'\gamma_n(h)$. Therefore we have

$$
\begin{aligned}
MSE_{(a)} &= \frac{\sigma^2}{1-\phi^2} - \sigma^2 \frac{2\phi^2}{(\phi+1)(1-\phi^2)} \\
MSE_{(b)} &= \frac{\sigma^2}{1-\phi^2} - \sigma^2 \frac{\phi^2}{1-\phi^2} = \sigma^2 \\
MSE_{(c)} &= \frac{\sigma^2}{1-\phi^2} - \sigma^2 \frac{2\phi^2}{1-\phi^4}
\end{aligned}
$$