Paul Cubre
Collaborators: Kara Stasikelis

Math 8230 Homework 2

p.54

#1 Find the image of $\{z : Rez < 0, |Imz| < \pi\}$ under the exponential function.

#4 Discuss the mapping properties of z^n and $z^{1/n}$ for $n \geq 2$. (Hint: use polar coordinates)

We have that $z = re^{i\theta}$, then $z^n = r^n e^{in\theta}$.

We have that $z = re^{i\theta}$, then $z^{1/n} = e^{\log(z^{1/n})} = e^{\frac{1}{n}\log|z| + \frac{1}{n}iArg(z) + \frac{1}{n}i2\pi k} = |z|^{1/n} e^{\frac{i}{n}(Arg(z)+2\pi k)} = |z|^{1/n} e^{\frac{i}{n}Arg(z)}\zeta_n$, where $\zeta_n = e^{\frac{i2\pi k}{n}}$.

#6(d) Evaluate the following cross ratios: $(i-1, \infty, 1+i, 0)$.

We have that $\frac{(i-1)-(1+i)}{(i-1)-0} / \frac{\infty-(1+i)}{\infty-0} = \frac{(i-1)-(1+i)}{(i-1)-0} = \frac{-2}{i-1} = \frac{-2(i+1)}{-1-1} = i+1$ by p.48.

#7 If $Tz = \frac{az+b}{cz+d}$ find z_2, z_3, z_4 (in terms of a, b, c, d) such that $Tz = (z, z_2, z_3, z_4)$.

We have by definition that $T(z_2) = 1, T(z_3) = 0$ and $T(z_4) = \infty$. Therefore $\frac{az_2+b}{cz_2+d} = 1$, $\frac{az_3+b}{cz_3+d} = 0$ and $\frac{az_4+b}{cz_4+d} = \infty$. Therefore we have that $z_2 = \frac{d-b}{a-c}$, $az_3 + b = 0$ or $z_3 = -\frac{b}{a}$, and $cz_4 + d = 0$ or $z_4 = -\frac{d}{c}$.

#9 If $Tz = \frac{az+b}{cz+d}$, find necessary and sufficient conditions that $T(\Gamma) = \Gamma$ where Γ is the unit circle $\{z : |z| = 1\}$.

We have that if $z \in \Gamma$ and $T(z) \in \Gamma$ then $|z| = 1$ and $|T(z)| = 1$.

$$\begin{aligned}
1 &= |T(z)|^2 \\
&= \left|\frac{az+b}{cz+d}\right|^2 \\
&= \frac{|az+b|^2}{|cz+d|^2}
\end{aligned}$$

Thus

$$\begin{aligned}
&|az+b|^2 = |cz+d|^2 \\
&(az+b)\overline{(az+b)} = (cz+d)\overline{(cz+d)} \\
&a\bar{a}z\bar{z} + a\bar{b}z + \bar{a}b\bar{z} + b\bar{b} = c\bar{c}z\bar{z} + c\bar{d}z + \bar{c}d\bar{z} + d\bar{d} \\
&z\bar{z}(a\bar{a} - c\bar{c}) + b\bar{b} - d\bar{d} + z(a\bar{b} - c\bar{d}) + \bar{z}(\bar{a}b - \bar{c}d) = 0 \\
&|z|^2\left(|a|^2 - |c|^2\right) + |b|^2 - |d|^2 + z(a\bar{b} - c\bar{d}) + \bar{z}(\bar{a}b - \bar{c}d) = 0
\end{aligned}$$

Thus if $z = 1$ and $z = -1$, we may add the equations and get $|a|^2 + |b|^2 = |c|^2 + |d|^2$. Then we have that $a\bar{b} - c\bar{d} = 0$ and $\bar{a}b - \bar{c}d = 0$ which are equivalent under complex conjugation. Thus these are sufficient conditions.

We have $T(z) = e^{i\theta}z$ and $T(z) = 1/z$ both take the unit circle to the unit circle. Thus (a, b, c, d) is $(e^{i\theta}, 0, 0, 1)$ and $(0, 1, 1, 0)$. We let $a = e\bar{d}$. Thus we have $e\bar{d}\bar{b} = c\bar{d}$. Therefore $e\bar{b} = c$. Therefore we have that

$$\begin{aligned}
|a|^2 + |b|^2 &= \left|e\bar{d}\right|^2 + \left|\frac{\bar{c}}{\bar{e}}\right|^2 \\
&= |e|^2|d|^2 + |c|^2/|e|^2 \\
&= |d|^2 + |c|^2
\end{aligned}$$

Therefore $|e|^2 = 1$ Hence $\bar{e}e = 1$ or $e = e^{i\theta}$. Therefore we have that

$$\begin{aligned} T(z) &= \frac{az+b}{cz+d} \\ &= \frac{az+b}{e\bar{b}z + \bar{a}/\bar{e}} \\ &= \frac{az+b}{e(\bar{b}z + \bar{a}/(e\bar{e})} \\ &= \frac{az+b}{e(\bar{b}z+\bar{a})} \\ &= e^{-i\theta}\frac{az+b}{\bar{b}z+\bar{a}} \end{aligned}$$

#10 Let $D = \{z : |z| < 1\}$ and find all Mobius transformations T such that $T(D) = D$.

We have that $T(0) \in D$. Then $T(0) = \omega \in D$. But since 0 and ∞ are symmetric with respect to the unit circle then so to must their images. Therefore $T(\infty) = \omega * = 1/\bar{\omega}$. This is true since $\omega \notin \partial D$. Therefore $T(z) = C\frac{z+\omega}{\bar{\omega}z+1}$. Now since the interior maps to the interior and the outside maps to the outside by the symmetric property then the boundary must map to the boundary. Therefore $T(e^{i\theta}) \in \partial D$. Therefore $\left|T(e^{i\theta})\right| = 1$.

$$\begin{aligned} 1^2 &= \left|T(e^{i\theta})\right|^2 \\ &= \left|C\frac{e^{i\theta}+\omega}{\bar{\omega}e^{i\theta}+1}\right|^2 \\ &= |C|^2\frac{\left|e^{i\theta}+\omega\right|^2}{\left|\bar{\omega}e^{i\theta}+1\right|^2} \\ &= |C|^2\frac{\left|e^{i\theta}+\omega\right|^2}{\left|e^{i\theta}\right|^2\left|\bar{\omega}+e^{-i\theta}\right|^2} \\ &= |C|^2\frac{(e^{i\theta}+\omega)(e^{-i\theta}+\bar{\omega})}{(\bar{\omega}+e^{-i\theta})(\omega+e^{i\theta})} \\ &= |C|^2 \end{aligned}$$

Therefore we have $C = e^{i\theta}$ and $T(z) = e^{i\theta}\frac{z+\omega}{\bar{\omega}z+1}$ for $\theta \in (-\pi, \pi]$ and $\omega \in D$.

#14 Suppose that one circle is contained inside another and that they are tangent at the point a. Let G be the region between the two circles and map G conformally onto the open unit disk. (Hint: first try $(z-a)^{-1}$.)

We may take $T(z) = z - a$ then the two circles are tangent at the origin. Let L be the line through the centers and the origin. Then we may take $R(z) = e^{i\theta}z$ such that L is along the imaginary axis. Then $I(z) = 1/z$ takes the region to between two parallel lines. Now after another shift and rescaling S we can get the region between $\{0 < Imz < \pi\}$. Then the exponential map takes to the upper half plane and the Cayley transformation, $C(z) = \frac{z-i}{z+i}$ takes it to the unit disk.

Thus we have $C(e^{(S(I(R(T(G)))))}) = \{z : |z| < 1\}$

#15 Can you map the open unit disk conformally onto $\{z : 0 < |z| < 1\}$?

We have that $T(z) = (z, -i, -1, 1) = i\frac{z+1}{-z+1}$ maps the open unit disk to the upper half plane. Let $D = \{z : |z| < 1\}$ then $T(D) = \{z : Imz > 0\} = H$. Then e^z maps the left half plane onto the open punctured disk $D - \{0\}$ by part 1. Hence $S(z) = iz$ rotates the upper half plane to the left half plane. Thus the composition of their maps is $f(z) = e^{i^2 \frac{z+1}{-z+1}} = e^{\frac{z+1}{z-1}}$.

Now we check conformal. Clearly the function is analytic as it is a composition of the exponential and two Mobius transformations. The derivative is $f'(z) = e^{\frac{z+1}{z-1}} \frac{-2}{(z-1)^2}$ and $f'(z) \neq 0$ for $z \in D$. Hence by Theorem 3.4.

#16 Map $G = \mathbf{C} - \{z : -1 \leq z \leq 1\}$ onto the open unit disk by an analytic function f. Can f be one-one?

We have $T(z) = (z, 0, -1, 1) = \frac{1+z}{1-z}$. Then $T(G) = \mathbf{C} - [0, \infty)$. Then we may take $R(z) = e^{i\pi}z$ such that $R(T(G)) = \mathbf{C} - (-\infty, 0]$. Then $S(z) = z^2$. Hence we have $S(R(T(G))) = \{z : Re(z) > 0\}$. Then using $P(z) = \frac{z-1}{z+1}$ we map the open right half plane the the open unit disk. Thus we have a composition of functions that are analytic. The function is then $\frac{2z}{1+z^2}$. This is not one-one as $(-2 \pm \sqrt{5})$ maps to $i/2$.

We have that f cannot be one to one as not simply connected maps to not simply connected for f analytic.

#17 Let G be a region and suppose that $f : G \to \mathbf{C}$ is analytic such that $f(G)$ is a subset of a circle. Show that f is constant.

First assume that Γ is a circle centered at the origin and $f(G) = \Gamma$. Then $|f(z)| = r$ for all $z \in G$. Then $f(z)\overline{f(z)} = r^2$. Then since f is analytic we have that $f'(z)\overline{f(z)} + f(z)\overline{f'(z)} = 0$. Hence $f'(z)\overline{f(z)} = -\overline{f'(z)\overline{f(z)}}$. Thus the only number that is the negative of its conjugate is 0. Therefore $f'(z)\overline{f(z)} = 0$. Since $|f(z)| = r$ then $f(z) \neq 0$ for all $z \in G$. Hence $f'(z) = 0$ for all $z \in G$ and f is then a constant function.

#18 Let $-\infty < a < b < \infty$ and put $M(z) = \frac{z-ia}{z-ib}$. Define the lines $L_1 = \{z : Imz = b\}, L_2 = \{z : Imz = a\}$ and $L_3 = \{z : Rez = 0\}$. Determine which of the regions A, B, C, D, E, F in Figure 1, are mapped by M onto the regions U, V, W, X, Y, Z in Figure 2.

#28 Discuss the mapping properties of $(1-z)^i$.

We have that $(1-z)^i = \exp\{i\log(z)\} = \exp\{i\mathrm{Log}(z) - 2\pi k\} = \exp\{i\mathrm{Log}(z)\}e^{-2\pi k}$ for $k \in \mathbf{Z}$

#30 For $|z| < 1$ define $f(z)$ by

$$f(z) = \exp\{-i\log[i\left(\frac{1+z}{1-z}\right)]^{1/2}\}.$$

(a) Show that f maps $D = \{z : |z| < 1\}$ conformally onto an annulus G.

Clearly f is a composition of analytic functions. Furthermore

$$\begin{aligned}
f'(z) &= \exp\{-\frac{i}{2}\log[i\left(\frac{1+z}{1-z}\right)]\}\frac{-i}{2i\frac{1+z}{1-z}}\frac{2i}{(1-z)^2} \\
&= \exp\{-\frac{i}{2}\log[i\left(\frac{1+z}{1-z}\right)]\}\frac{-i}{(1+z)}\frac{1}{(1-z)} \\
&= \exp\{-\frac{i}{2}\log[i\left(\frac{1+z}{1-z}\right)]\}\frac{-i}{(1-z^2)}
\end{aligned}$$

Hence $f'(z) \neq 0$ as long as $1 - z^2 \neq 0$ or $z \neq \pm 1$ which is true for $z \in D$. Thus f is conformal by Theorem 3.4.

(b) Find all Mobius transformations $S(z)$ that map D onto D and such that $f(S(z)) = f(z)$ when $|z| < 1$.
From 10, we have $S(z) = e^{i\theta}\frac{z-w}{\bar{w}z-1}$. Furthermore if $f(S(z)) = f(z)$, then

$$\exp\{-\frac{i}{2}\log[i\left(\frac{1+S(z)}{1-S(z)}\right)]\} = \exp\{-\frac{i}{2}\log[i\left(\frac{1+z}{1-z}\right)]\}$$

$$\exp\{-\frac{i}{2}[\mathrm{Log}[i\left(\frac{1+S(z)}{1-S(z)}\right)] + i2\pi k]\} = \exp\{-\frac{i}{2}[\mathrm{Log}[i\left(\frac{1+z}{1-z}\right)]]\}$$

$$\exp\{-\frac{i}{2}[\log\left|i\left(\frac{1+S(z)}{1-S(z)}\right)\right| + i\arg(i\left(\frac{1+S(z)}{1-S(z)}\right)) + i2\pi k]\} =$$

$$\exp\{-\frac{i}{2}[\log\left|i\left(\frac{1+z}{1-z}\right)\right| + i\arg(i\left(\frac{1+z}{1-z}\right))]\}$$

Therefore $\log\left|i\frac{1+z}{1-z}\right| = \log\left|i\frac{1+S(z)}{1-S(z)}\right| + 2\pi j$ or $\left|\frac{1+z}{1-z}\right| = \left|\frac{1+S(z)}{1-S(z)}\right| e^{2\pi j}$

Hence we may take $e^{2\pi j} = \left|\frac{1+z}{1-z}\frac{1-S(z)}{1+S(z)}\right|$. Let $z = 0$ and $S(0) = e^{i\theta}w$. Then $e^{2\pi j} = \left|\frac{1-e^{i\theta}w}{1+e^{i\theta}w}\right|$. With this we can solve for θ and w.