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Math 8230 Homework 3

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#12 Let $I(r) = \int_\gamma \frac{e^{iz}}{z} dz$ where $\gamma : [0, \pi] \to \mathbb{C}$ is defined by $\gamma(t) = re^{it}$. Show that $\lim_{r\to\infty} I(r) = 0$.

We have that

$$\begin{aligned} I(r) &= \int_\gamma \frac{e^{iz}}{z} dz \\ &= \int_0^\pi \frac{e^{ire^{it}}}{re^{it}} ire^{it} dt \\ &= \int_0^\pi ie^{ire^{it}} dt \end{aligned}$$

We then have

$$\begin{aligned} |I(r)| &= \left| \int_0^\pi ie^{ire^{it}} dt \right| \\ &\le \int_0^\pi \left| ie^{ire^{it}} \right| dt \\ &= \int_0^\pi \left| e^{ire^{it}} \right| dt \\ &= \int_0^\pi \left| e^{ir(\cos(t)+i\sin(t))} \right| dt \\ &= \int_0^\pi \sqrt{e^{ir\cos(t)-r\sin(t)}\overline{e^{ir\cos(t)-r\sin(t)}}} dt \\ &= \int_0^\pi \sqrt{e^{-2r\sin(t)}} dt \\ &= \int_0^\pi e^{-r\sin(t)} dt \end{aligned}$$

Now we have that $\phi(r, t) = e^{-r\sin(t)}$ is continuous and infinitely differentiable and bounded by 1 for $t \in [0, \pi]$. Therefore we have that $\lim_{r\to\infty} \int_0^\pi e^{-r\sin(t)} = \int_0^\pi \lim_{r\to\infty} e^{-r\sin(t)} dt = \int_0^\pi 0 dt = 0$ by dominated convergence theorem. Hence we have that $\lim_{r\to\infty} |I(r)| \le \lim_{r\to\infty} \int_0^\pi e^{-r\sin(t)} dt = 0$. Hence we have that $\lim_{r\to\infty} I(r) = 0$.

#13 Find $\int_\gamma z^{-\frac{1}{2}} dz$ where:

(a) γ is the upper half of the unit circle from $+1$ to -1:

We have that $e^{-\frac{1}{2}\log z} = z^{-\frac{1}{2}}$. Then we have that $\gamma : [0, \pi] \to \mathbb{C}$ with $\gamma(t) = e^{it}$ parameterizes the upper half unit circle from $+1$ to -1. Also $\gamma_\epsilon : [0, \pi - \epsilon] \to \mathbb{C}$. We need to avoid the negative real numbers of the principal branch cut of log. Then we have:

$$\begin{aligned}
\left|\int_\gamma z^{-\frac{1}{2}}dz - \int_{\gamma_\epsilon} z^{-\frac{1}{2}}dz\right| &= \left|\int_{\{\gamma\}\setminus\{\gamma_\epsilon\}} z^{-\frac{1}{2}}dz\right| \\
&\leq \int_{\{\gamma\}\setminus\{\gamma_\epsilon\}} |z|^{-\frac{1}{2}}\,|dz| \\
&= \int_{\{\gamma\}\setminus\{\gamma_\epsilon\}} |dz| \\
&= \epsilon
\end{aligned}$$

Therefore we have $\lim_{\epsilon\to 0}\int_{\gamma_\epsilon} z^{-\frac{1}{2}}dz = \int_\gamma z^{-\frac{1}{2}}dz$.

$$\begin{aligned}
\int_\gamma z^{-\frac{1}{2}}dz &= \lim_{\epsilon\to 0}\int_{\gamma_\epsilon} z^{-\frac{1}{2}}dz \\
&= \lim_{\epsilon\to 0}\int_{\gamma_\epsilon} e^{-\frac{1}{2}\log z}dz \\
&= \lim_{\epsilon\to 0}\int_0^{\pi-\epsilon} e^{-\frac{1}{2}\log e^{it}}ie^{it}dt \\
&= \lim_{\epsilon\to 0} i\int_0^{\pi-\epsilon} e^{-\frac{it}{2}+it}dt \\
&= \lim_{\epsilon\to 0} i\int_0^{\pi-\epsilon} e^{\frac{it}{2}}dt \\
&= \lim_{\epsilon\to 0} 2\left(e^{it/2}\right]_0^{\pi-\epsilon} \quad \text{by FTC} \\
&= \lim_{\epsilon\to 0} 2(e^{i(\pi-\epsilon)/2}-1) \\
&= 2(i-1)
\end{aligned}$$

(b) γ is the lower half of the unit circle from $+1$ to -1.

We have that $e^{-\frac{1}{2}\log z} = z^{-\frac{1}{2}}$. Then we have that $\gamma : [0,\pi] \to \mathbb{C}$ with $\gamma(t) = e^{-it}$ parameterizes the lower half unit circle from $+1$ to -1. Also $\gamma_\epsilon : [0, \pi-\epsilon] \to \mathbb{C}$ again to avoid the principal branch cut. Then we have:

$$\begin{aligned}
\left|\int_\gamma z^{-\frac{1}{2}}dz - \int_{\gamma_\epsilon} z^{-\frac{1}{2}}dz\right| &= \left|\int_{\{\gamma\}\setminus\{\gamma_\epsilon\}} z^{-\frac{1}{2}}dz\right| \\
&\leq \int_{\{\gamma\}\setminus\{\gamma_\epsilon\}} |z|^{-\frac{1}{2}}\,|dz| \\
&= \int_{\{\gamma\}\setminus\{\gamma_\epsilon\}} |dz| \\
&= \epsilon
\end{aligned}$$

Therefore we have $\lim_{\epsilon\to 0}\int_{\gamma_\epsilon} z^{-\frac{1}{2}}dz = \int_\gamma z^{-\frac{1}{2}}dz$.

$$\begin{aligned}
\int_\gamma z^{-\frac{1}{2}}dz &= \lim_{\epsilon\to 0}\int_{\gamma_\epsilon} z^{-\frac{1}{2}}dz\\
&= \lim_{\epsilon\to 0}\int_{\gamma_\epsilon} e^{-\frac{1}{2}\log z}dz\\
&= \lim_{\epsilon\to 0} -\int_0^{\pi-\epsilon} e^{-\frac{1}{2}\log e^{-it}} ie^{-it}dt\\
&= \lim_{\epsilon\to 0} -i\int_0^{\pi-\epsilon} e^{\frac{it}{2}-it}dt\\
&= \lim_{\epsilon\to 0} -i\int_0^{\pi-\epsilon} e^{-\frac{it}{2}}dt\\
&= \lim_{\epsilon\to 0} 2\left(e^{-it/2}\right]_0^{\pi-\epsilon} \text{ by FTC}\\
&= \lim_{\epsilon\to 0} 2(e^{-i(\pi-\epsilon)/2}-1)\\
&= 2(-i-1)
\end{aligned}$$

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#7 Use the results of this section to evaluate the following integrals:

(a) $\int_\gamma \frac{e^{iz}}{z^2}dz$, $\gamma(t)=e^{it}$, $0\le t\le 2\pi$;

We have that $f^{(1)}(0)=\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{(z-0)^{1+1}}dz$ by Generalized Cauchy's Integral Formula for disks. As e^{iz} is entire then we have $f(z)=e^{iz}$ Thus $f'(z)=ie^{iz}$. Thus $i=\frac{1}{2\pi i}\int_\gamma \frac{e^{iz}}{z^2}dz$. Therefore $\int_\gamma \frac{e^{iz}}{z^2}dz=-2\pi$.

(d) $\int_\gamma \frac{\log z}{z^n}dz$, $\gamma(t)=1+\frac{1}{2}e^{it}$, $0\le t\le 2\pi$ and $n\ge 0$.

For $n\ne 1$, then $F(z)=\frac{z^{1-n}\log z}{1-n}-\frac{z^{1-n}}{n^2-2n+1}$. Then $F'(z)=\frac{(1-n)z^{-n}\log z+z^{-n}}{1-n}-\frac{(1-n)z^{-n}}{(n-1)^2}=z^{-n}\log z$. Clearly we have that z^{-n} and $\log z$ are analytic on the disk $B(1;\frac{1}{2})$ with $F(z)$ is primitive. Thus we can use Cauchy's Theorem on a disk and $\int_\gamma \frac{\log z}{z^n}dz=0$.

When $n=1$, then $\int_\gamma \frac{\log z}{z}dz$. Let $G(z)=\frac{\log^2(z)}{2}$. Then $G'(z)=\log(z)z^{-1}$. Hence $G(z)$ is primitive and $G(z)$ is analytic over $B(1;\frac{1}{2})$. Therefore $\int_\gamma \frac{\log z}{z}dz=0$ by Cauchy's theorem.

#10 Evaluate $\int_\gamma \frac{z^2+1}{z(z^2+4)}dz$ where $\gamma(t)=re^{it}$, $0\le t\le 2\pi$, for all possible values of r, $0<r<2$ and $2<r<\infty$.

$$\begin{aligned}
\int_\gamma \frac{z^2+1}{z(z^2+4)}dz &= \int_\gamma\left(\frac{\frac{1}{4}}{z}+\frac{\frac{3}{8}}{z+2i}+\frac{\frac{3}{8}}{z-2i}\right)dz \text{ by partial fraction decomposition}\\
&= \frac{1}{4}\int_\gamma \frac{1}{z}dz+\frac{3}{8}\int_\gamma \frac{1}{z-(-2i)}dz+\frac{3}{8}\int_\gamma\frac{1}{z-2i}dz\\
&= \begin{cases}\frac{1}{4}\int_\gamma \frac{1}{z}dz+\frac{3}{8}\int_\gamma \frac{1}{z-(-2i)}dz+\frac{3}{8}\int_\gamma\frac{1}{z-2i}dz & \text{if } 0<r<2\\ \frac{1}{4}\int_\gamma \frac{1}{z}dz+\frac{3}{8}\int_\gamma \frac{1}{z-(-2i)}dz+\frac{3}{8}\int_\gamma\frac{1}{z-2i}dz & \text{if } 2<r<\infty\end{cases}
\end{aligned}$$

We apply the Complete Cauchy's Integral Formula on disks as constant functions are entire.

$$\int_\gamma \frac{z^2+1}{z(z^2+4)}dz = \begin{cases} \frac{1}{4}2\pi i + \frac{3}{8}\int_\gamma \frac{1}{z-(-2i)}dz + \frac{3}{8}\int_\gamma \frac{1}{z-2i}dz & \text{if } 0 < r < 2 \\ \frac{1}{4}2\pi i + \frac{3}{8}2\pi i + \frac{3}{8}2\pi i & \text{if } 2 < r < \infty \end{cases}$$
$$= \begin{cases} \frac{1}{2}\pi i + 0 + 0 & \text{if } 0 < r < 2 \\ 2\pi i & \text{if } 2 < r < \infty \end{cases}$$

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#1 Let f be an entire function and suppose there is a constant M, an $R > 0$, and an integer $n \geq 1$ such that $|f(z)| \leq M|z|^n$ for $|z| > R$. Show that f is a polynomial of degree $\leq n$.

We have that for any $r > R$, $\left|f^{(m)}(0)\right| = \left|\frac{m!}{2\pi i}\int_{|w|=r}\frac{f(w)}{w^{m+1}}dw\right|$ as f is analytic on $\mathbb{C}$. Thus we have that

$$\begin{aligned} \left|f^{(m)}(0)\right| &= \left|\frac{m!}{2\pi i}\int_{|w|=r}\frac{f(w)}{w^{m+1}}dw\right| \text{ by Generalized Cauchy's Integral Formula on disks} \\ &\leq \frac{m!}{2\pi i}\int_{|w|=r}\frac{|f(w)|}{|w|^{m+1}}|dw| \\ &\leq \frac{m!}{2\pi i}\int_{|w|=r}\frac{M|w|^n}{|w|^{m+1}}|dw| \text{ by apply the bound of } |f(z)| \\ &= \frac{m!}{2\pi i}\int_{|w|=r}\frac{Mr^n}{r^{m+1}}|dw| \\ &= \frac{m!}{2\pi i}Mr^{n-m-1}2\pi r \\ &= Cr^{n-m} \end{aligned}$$

Thus we have this is true for any $r > R$. Hence we let $r \to \infty$. Therefore when $m > n$. $\left|f^{(m)}(0)\right| = 0$. However since f is entire we have a unique power series representation $f(z) = \sum_{i=0}^{\infty} a_i z^i$ where $a_i = \frac{f^{(i)}(0)}{i!}$. Therefore $f(z) = \sum_{i=0}^{n} a_i z^i$.

#3 Find all entire functions f such that $f(x) = e^x$ for $x \in \mathbb{R}$.

Suppose $f(z)$ is an entire function such that $f(z) = e^x$. Let $g(z) = f(z) - e^z$. Then $g(x) = 0$ for $x \in \mathbb{R}$. Furthermore since f and e are entire then g is also entire. Let $g(z) = \sum_{i=0}^{\infty} a_i z^i = \sum_{i=0}^{\infty}\frac{g^{(i)}(0)}{i!}z^i$. Taking the derivate at $x \in \mathbb{R}$, we may choose any direction. Along the real axis or h is real we have $g^{(1)}(x) = \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} = \lim_{h\to 0}\frac{0}{h} = 0$. Then by induction we have our induction hypothesis that $g^{(i)}(x) = 0$. Therefore $g^{(i+1)}(x) = \lim_{h\to 0}\frac{g^{(i)}(x+h)-g^{(i)}(x)}{h} = \lim_{h\to 0}\frac{0}{h} = 0$. Thus we have shown $g^{(i)}(x) = 0$ for all $i > 0$, $x \in \mathbb{R}$ and clearly $g(x) = 0$. Therefore we have that $g(z) = \sum_{i=0}^{\infty}\frac{g^{(i)}(0)}{i!}(z-0)^i \equiv 0$. Therefore $f(z) = e^z$.

#8 Let G be a region and let f and g be analytic functions on G such that $f(z)g(z) = 0$ for all z in G. Show that either $f \equiv 0$ or $g \equiv 0$.

Suppose that $g \not\equiv 0$. Then there exists some $a \in G$, $g(a) \neq 0$. Then $f(a)g(a) = 0$ implies $f(a) = 0$. If f is constant with $f(a) = 0$ then $f \equiv 0$ and we are done. Otherwise suppose f is not constant. Then by Corollary 3.10 we have that there is an $R > 0$ such that $B(a;R) \subset G$ and $f(z) \neq 0$ for $0 < |z-a| < R$. Therefore for $z \in B(a;R)\backslash\{a\}$ we have that $f(z)g(z) = 0$ with $f(z) \neq 0$ implies

$g(z) = 0$. Therefore have $g(z) = 0$ for $z \in B(a;R)\backslash\{a\}$ and $g(a) \neq 0$. Therefore g is not continuous on $B(a;R)$ a contradiction that g is analytic on G. Therefore our original supposition is false and $g \equiv 0$.

#9 For $R > 0$, show that $\frac{1}{2\pi R}\int_{\partial B_R(0)} \frac{|dz|}{z-z_0} = \begin{cases} 0, & z_0 \in B_R(0) \\ -\frac{1}{z_0}, & z_0 \in \mathbb{C}\backslash\overline{B_R(0)} \end{cases}$

$$\begin{aligned} \frac{1}{2\pi R}\int_{\partial B_R(0)} \frac{|dz|}{z-z_0} &= \frac{1}{2\pi R}\int_0^{2\pi} \frac{\left|iRe^{it}\right| dt}{Re^{it}-z_0} \\ &= \frac{1}{2\pi}\int_0^{2\pi} \frac{dt}{Re^{it}-z_0} \end{aligned}$$

We have $\frac{d}{dt}\left(-\frac{t+i\log(Re^{it}-z_0)}{z_0}\right) = -\frac{1}{z_0} - \frac{i}{z_0}\frac{Rie^{it}}{Re^{it}-z_0} = \frac{1}{z_0}\frac{-Re^{it}+z_0+Re^{it}}{Re^{it}-z_0} = \frac{1}{Re^{it}-z_0}$.

$\log(Re^{it}-z_0)$ is analytic when $z_0 \notin \overline{B_R(0)}$ as $\log(Re^{it}-z_0)$ is far enough away from the origin to take a branch cut in the $\pi + \arg(z_0)$ direction.

Therefore for $z_0 \notin \overline{B_R(0)}$

$$\begin{aligned} \frac{1}{2\pi R}\int_{\partial B_R(0)} \frac{|dz|}{z-z_0} &= \frac{1}{2\pi}\left(-\frac{t+i\log(Re^{it}-z_0)}{z_0}\right]_0^{2\pi} \quad \text{by FTC} \\ &= \frac{1}{2\pi}\left(-\frac{2\pi}{z_0}\right) \\ &= -\frac{1}{z_0} \end{aligned}$$

Suppose $z_0 \in B_R(0)$. Let $z_0 = r_0 e^{i\theta_0}$. Then let $\partial B_R(0) = Re^{it}$ for $t \in [-\pi + \theta_0, \pi + \theta_0]$.

$$\begin{aligned}
\frac{1}{2\pi R}\int_{\partial B_R(0)} \frac{|dz|}{z - z_0} &= \frac{1}{2\pi R}\int_{-\pi+\theta_0}^{\pi+\theta_0} \frac{\left|iRe^{it}\right| dt}{Re^{it} - r_0 e^{i\theta_0}} \\
&= \frac{1}{2\pi R}\int_{-\pi+\theta_0}^{\pi+\theta_0} \frac{e^{-it}dt}{1 - \frac{r_0}{R}e^{i\theta_0 - it}} \\
&= \frac{1}{2\pi R}\int_{-\pi}^{\pi} \frac{e^{-i(s+\theta_0)}ds}{1 - \frac{r_0}{R}e^{-is}} \text{ with } s = t - \theta_0 \\
&= \frac{1}{2\pi R}\int_{-\pi}^{\pi} \frac{e^{-i\theta_0}ds}{e^{is} - \frac{r_0}{R}} \\
&= \frac{1}{2\pi R}\int_{-\pi}^{\pi} \frac{e^{-i\theta_0}(e^{-is} - \frac{r_0}{R})ds}{(e^{is} - \frac{r_0}{R})(e^{-is} - \frac{r_0}{R})} \\
&= \frac{e^{-i\theta_0}}{2\pi R}\int_{-\pi}^{\pi} \frac{(e^{-is} - \frac{r_0}{R})ds}{1 - 2\frac{r_0}{R}\cos(s) + \left(\frac{r_o}{R}\right)^2} \\
&= \frac{e^{-i\theta_0}}{2\pi R}\int_{-\pi}^{\pi} \frac{\cos(-s) + i\sin(-s) - \frac{r_0}{R}ds}{1 - 2\frac{r_0}{R}\cos(s) + \left(\frac{r_o}{R}\right)^2} \\
&= \frac{e^{-i\theta_0}}{2\pi R(1-\rho^2)}\int_{-\pi}^{\pi} \frac{(1-\rho^2)\cos(s) - (1-\rho^2)i\sin(s) - (1-\rho^2)\rho ds}{1 - 2\rho\cos(s) + \rho^2} \\
&= \frac{e^{-i\theta_0}}{2\pi R(1-\rho^2)}\int_{-\pi}^{\pi} P_\rho(s)\cos(s) - iP_\rho(s)\sin(s) - P_\rho(s)\rho ds \\
&= \frac{e^{-i\theta_0}}{2\pi R(1-\rho^2)}\int_{-\pi}^{\pi} P_\rho(s)\cos(s) - P_\rho(s)\rho ds \text{ since } \sin(s) \text{ is even} \\
&= \frac{e^{-i\theta_0}}{2\pi R(1-\rho^2)}(-2\pi\rho + 2\pi\rho) \text{ by Poisson Kernel Formulas for } \rho < 1 \\
&= 0
\end{aligned}$$

Note the Poisson Kernel Formulas give the previous result for $\rho > 1$.

#10 For $R > 0$, compute $\int_{\partial B_R(0)} z^\alpha dz$ for each $\alpha \in \mathbb{C}$. (For z^α, of course, use the principal branch.)

We first must look at $z^\alpha = e^{\alpha Log(z)}$ the principal branch that is analytic on $\mathbb{C}\backslash\{z \leq 0\}$. Let $\gamma(t) = Re^{it}$ for $t \in [-\pi, \pi]$. Then the trace $\{\gamma(t)\} = \partial B_R(0)$. We must avoid the principal branch cut therefore we must approximated the integral with another.

$$\begin{aligned}
\lim_{\epsilon \to 0}\left|\int_\gamma z^\alpha dz - \int_{\gamma_\epsilon} z^\alpha\right| &= \lim_{\epsilon \to 0}\left|\int_{\{\gamma\}\backslash\{\gamma_\epsilon\}} z^\alpha dz\right| \\
&\leq \lim_{\epsilon \to 0}\int_{\{\gamma\}\backslash\{\gamma_\epsilon\}} |z|^\alpha |dz| \\
&= \lim_{\epsilon \to 0} R^\alpha 2\epsilon R \\
&= 0
\end{aligned}$$

Thus we have convergence of the integral. Then we compute for $\alpha \neq -1$.

$$
\begin{aligned}
\int_\gamma z^\alpha dz &= \lim_{\epsilon\to 0}\int_{\gamma_\epsilon} z^\alpha dz \\
&= \lim_{\epsilon\to 0}\int_{\gamma_\epsilon} e^{\alpha\log z} dz \\
&= \lim_{\epsilon\to 0}\int_{-\pi+\epsilon}^{\pi-\epsilon} e^{\alpha\log Re^{it}} Rie^{it} dt \\
&= \lim_{\epsilon\to 0}\int_{-\pi+\epsilon}^{\pi-\epsilon} e^{\alpha(\log R+\log e^{it})} Rie^{it} dt \\
&= \lim_{\epsilon\to 0} iR^{\alpha+1}\int_{-\pi+\epsilon}^{\pi-\epsilon} e^{\alpha it}e^{it} dt \\
&= \lim_{\epsilon\to 0} iR^{\alpha+1}\int_{-\pi+\epsilon}^{\pi-\epsilon} e^{(\alpha+1)it} dt \\
&= \lim_{\epsilon\to 0} iR^{\alpha+1}\left(\frac{e^{(\alpha+1)it}}{i(\alpha+1)}\right]_{-\pi+\epsilon}^{\pi-\epsilon} \quad \text{by FTC} \\
&= \lim_{\epsilon\to 0} iR^{\alpha+1}\left(\frac{e^{(\alpha+1)i(\pi-\epsilon)} - e^{(\alpha+1)i(-\pi+\epsilon)}}{i(\alpha+1)}\right) \\
&= \frac{R^{\alpha+1}}{\alpha+1}\left(e^{(\alpha+1)i\pi} - e^{-(\alpha+1)i\pi}\right) \\
&= \frac{R^{\alpha+1}}{\alpha+1} 2i\sin(\pi(\alpha+1))
\end{aligned}
$$

Suppose $\alpha = -1$ then $\int_\gamma \frac{1}{z}dz = 2\pi i$ by the most important integral in complex analysis, Cauchy's integral formula.