(4.1) $f: [0,1] \longrightarrow \mathbb{R}^+$. $f(x) = 0$ si x racional

$f(x) = n$ si n es el número de ceros después del punto decimal en la representación de x en escala decimal

Calcula $\int f(x)\, dm$

$$g_1(x) = \begin{cases} 0 & \text{si } x \geq 0.1 \\ 1 & \text{si } 0 \leq x < 0.1 \end{cases} \qquad g_2(x) = \begin{cases} 0 & \text{si } x \geq 0.1 \\ 1 & \text{si } 0.01 \leq x < 0.1 \\ 2 & \text{si } \cancel{0} < x < 0.01 \end{cases}$$

$$\ldots \quad g_n(x) = \begin{cases} 0 & \text{si } x \geq 0.1 \\ k & \text{si } \frac{1}{10^{k+1}} \leq x < \frac{1}{10^k} \quad \text{para } k \in \{1,2,\ldots,n-1\} \\ n & \text{si } 0 \leq x < \frac{1}{10^n} \end{cases}$$

Converge $\forall x \in [0,1]$

$g \xleftarrow[n\to\infty]{} g_n$ simples, medibles, crecientes: $g_1 \leq g_2 \leq \ldots$ $\forall$ punto $[0,1]$

g

$\|$

f ctpx

En todo $[0,1] \setminus \mathbb{Q}$ medida cero

$$\int g_n\, dm = \sum_{k=1}^{n-1} \underbrace{\left(\frac{1}{10^k} - \frac{1}{10^{k+1}}\right)}_{\frac{9}{10^{k+1}} = \frac{9}{10}\left(\frac{1}{10}\right)^k} k + n\frac{1}{10^n}$$

$$= \frac{9}{10^2} \underbrace{\sum_{k=1}^{n-1} k \left(\frac{1}{10}\right)^{k-1}}_{} + \left(\frac{n}{10^n}\right)$$

$$\frac{d}{dx}\left(\sum_{k=1}^{n-1} x^k\right) = \frac{d}{dx}\left(\frac{x^n - x}{x-1}\right) = \frac{(nx^{n-1}-1)(x-1) - x^n + x}{(x-1)^2}$$

$$= \frac{nx^n - nx^{n-1} - x + 1 - x^n + x}{(x-1)^2} = \frac{(n-1)x^n - nx^{n-1} + 1}{(x-1)^2}$$

$$= \frac{9}{10^2} \cdot \frac{(n-1)\left(\frac{1}{10}\right)^n - n\left(\frac{1}{10}\right)^{n-1} + 1}{(9/10)^2} + n\frac{1}{10^n}$$

$$\xrightarrow[n \to \infty]{} \frac{9}{10^2} \cdot \left(\frac{10}{9}\right)^2 = \frac{1}{9}$$

Por tanto, $\int f\, dm \overset{1}{=} \int g\, dm \underset{TCM}{=} \lim_{n\to\infty} \int g_n\, dm = \boxed{1/9}$

Ej 4.1

(4.2) $f(x) = 0$ si $x \in \mathcal{C}$ (conjunto de Cantor en $[0,1]$)
$f(x) = p$ en cada intervalo del complementario de longitud $\frac{1}{3^p}$. Demostrar f medible y calcular $\int f\,dm$.

$$\mathcal{B}_{\mathbb{R}} \ni \begin{cases} C_0 = [0,1] \\ C_1 = [0, 1/3] \cup [2/3, 1] \\ C_2 = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1] \\ \vdots \end{cases}$$

$$f_1(x) = \begin{cases} 0 & \text{en otro caso} \\ 1 & \text{en } (1/3, 2/3) \end{cases} = \chi_{C_0 \setminus C_1}(x)$$

$$f_2(x) = \begin{cases} 0 & \text{otro caso} \\ 1 & \text{en } [1/3, 2/3] \\ 2 & \text{en } [1/9, 2/9] \cup [7/9, 8/9] \end{cases} = \cancel{\chi_{C_1}(x) + 2\chi_{C_2 \setminus C_1}(x)}$$

$$= \chi_{C_0 \setminus C_1}(x) + 2\chi_{C_1 \setminus C_2}(x)$$

$\vdots$

$$f_n = \sum_{k=1}^{n} k\,\chi_{C_{k-1} \setminus C_k} \xrightarrow{n \to \infty} f$$

y $\exists$ límite $\forall x$

$\rightarrow C_{k-1} \setminus C_k \in \mathcal{B}_{\mathbb{R}} \rightarrow f_n$ medibles, monótonas, simples

$$m(C_k) = \left(\frac{2}{3}\right)^k, \quad C_k \subset C_{k-1} \qquad \underset{\substack{\shortparallel \\ C_k}}{C_k \cap C_{k-1}}$$

$$\Rightarrow m(C_{k-1}) = (2/3)^{k-1} = m(C_{k-1} \setminus C_k) + m(C_k)$$

$$\rightarrow m(C_{k-1} \setminus C_k) = (2/3)^{k-1} - (2/3)^k$$

$$\int f_n\,dm = \sum_{k=1}^{n} k\left[(2/3)^{k-1} - (2/3)^k\right] = \sum_{k=0}^{n-1} (k+1)(2/3)^k - \sum_{k=1}^{n} k(2/3)^k$$

$$= (2/3)^0 + \sum_{k=1}^{n-1} (k+1)(2/3)^k - \sum_{k=1}^{n-1} k(2/3)^k - n(2/3)^n$$

$$= 1 - n(2/3)^n + \sum_{k=1}^{n-1} (2/3)^k = 1 - n\left(\frac{2}{3}\right)^n + \frac{(2/3)^n - (2/3)}{2/3 - 1} \longrightarrow$$

$$\xrightarrow{n\to\infty} 1 + \frac{2/3}{1-2/3} = 1 + \frac{2/3}{1/3} = 1+2=3$$

TCM
$$\int f\,dm = \lim_{n\to\infty} \int f_n\,dm = \boxed{3}$$

(4.3)
$$f_n(x) = \begin{cases} 0 & \text{si } \left[\frac{1}{x}\right] > n \\ k & \text{si } k-1 < \left[\frac{1}{x}\right] \end{cases} = \begin{cases} [1/x] & \text{si } [1/x] \leq n \\ 0 & \text{en otro caso.} \end{cases}$$

$$= \begin{cases} 1 & \text{si } \frac{1}{2} < x \leq 1 \Longleftrightarrow 2 > \frac{1}{x} \geq 1 \Leftrightarrow [1/x] = 1 \\ 2 & \text{si } \frac{1}{3} < x \leq \frac{1}{2} \longrightarrow 3 > \frac{1}{x} \geq 2 \leftrightarrow [1/x] = 2 \\ \vdots & \\ n & \text{si } \frac{1}{n+1} < x \leq \frac{1}{n} \longleftarrow [1/x] = n \end{cases}$$

$$= \sum_{k=1}^{n} \chi_{\left(\frac{1}{n+1}, \frac{1}{n}\right]}(x) \cdot k \qquad f_1(x) \leq f_2(x) \leq \dots \ \forall x \in (0,1]$$

simples, medibles, no negativas

Para cada $x \in [0,1]$, $\exists n_0 \,/\, f_n(x) = f_{n_0}(x) \ \forall n \geq n_0$
$$\Rightarrow \exists \lim_{n\to\infty} f_n(x).$$

medible por ser sup f_n

$$g(x) = \begin{cases} k & \text{si } \frac{1}{k+1} < x \leq \frac{1}{k} \quad \forall k \in \mathbb{N} \\ 0 & \text{si } x = 0 \end{cases} \xleftarrow[n\to\infty]{} f_n(x)$$

$$\int f_n\,dm = \sum_{k=1}^{n} k\left(\frac{1}{k} + \frac{1}{k+1}\right) = \cancel{\sum_{k=1}^{n} k\frac{1}{k} - \sum_{k=1}^{n} k\frac{1}{k+1}}$$

$$\cancel{1 + \sum^{n}}$$

$$= \sum_{k=1}^{n} k\left(\frac{k+1-k}{k(k+1)}\right) = \sum_{k=1}^{n} \frac{1}{k+1} = \sum_{k=2}^{n+1} \frac{1}{k} \xrightarrow[n\to\infty]{} \infty$$

La $f(x)$ descrita es igual a $g(x)$ en $(0,1) \setminus \underbrace{\mathbb{Q}}_{\text{medida } 0}$.

$$\Rightarrow \int f\,dm = \int g\,dm \underset{\text{TCM}}{=} \lim_{n\to\infty} \int f_n\,dm = \underline{\underline{\infty}}.$$

hoja 4.2

$d_i : (0,1) \longrightarrow \{0,1,\dots,9\}$.

$d_1 = 0\chi_{[0,1/10)} + 1\chi_{[1/10,2/10)} + \dots + 9\chi_{[9/10,1)}$

$d_2 = 0\chi_{[0,1/100)\cup[1/10,11/100)\cup\dots\cup[9/10,91/100)} + \dots +$

$+ 9\chi$

etc.

d_i son funciones simples, no negativas y medibles

$$\int d_i \, dm = \sum_{k=0}^{9} k \chi_{E_k}, \quad E_k = \{\text{números con } k \text{ en la } i\text{-ésima posición decimal}\}$$

$$m(E_k) = 1/10$$

$$\| $$

$$\frac{1}{10}\sum_{k=0}^{9} k = \frac{1}{10}\frac{9(9+1)}{2} = 9/2$$

$f_n(x) = \sum_{i=1}^{n} \frac{d_i(x)}{2^i}$ simple, medible, no negativa.

Converge $\forall x$ por ser creciente y acotada $f_n(x) \leq \sum_{i=1}^{n} \frac{9}{2^i} \leq 9 \sum_{i=1}^{\infty} \frac{1}{2^i} = 9$

$f = \sup_{n\in\mathbb{N}} f_n$ medible y TCM: $\int_0^1 f\,dm = \lim_{n\to\infty} \int_0^1 f_n\,dm =$

$$= \lim_{n\to\infty} \int \left(\sum_{i=1}^{n} \frac{d_i(x)}{2^i}\right) dm = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{2^i} \underbrace{\left(\frac{9}{2}\right)}_{\int d_i} = \boxed{9/2 = \int_0^1 f}$$

$g_n = (-1)^{d_i(x)} = -1\chi_{E_{imp}} + 1\chi_{E_{par}}$. $(g_n^+ = \chi_{E_{par}},\ g_n^- = \chi_{E_{imp}})$

$E_{imp} = \bigcup_{k \text{ impar}} E_k$ medibles y $m(E_{imp}) = m(E_{par}) = 1/2$

$E_{par} = \bigcup_{k \text{ par}} E_k$

$$\sum_{i=1}^{\infty} \int \left|\frac{(-1)^{d_i(x)}}{2^i}\right| = \sum_{i=1}^{\infty} \frac{1}{2^i} \int \left(\chi_{E_{imp}} + \chi_{E_{par}}\right) = \sum_{i=1}^{\infty} \frac{1}{2^i} = 1 < \infty$$

$\Longrightarrow g(x)$ converge ctp x (converge $\forall x$, es fácil de ver)

Además $\int \left(\sum_{i=1}^{\infty} \frac{(-1)^{d_i(x)}}{2^i}\right) = \sum_{i=1}^{\infty} \underbrace{\int \frac{(-1)^{d_i(x)}}{2^i}\,dm}_{0} = 0$

$\int g_n = \int g_n^+ - \int g_n^- = \frac{1}{2} - \frac{1}{2} = 0 \longrightarrow 0$

(4.5) desig. de Fatou: $\int (\liminf f_n) \le \liminf \int f_n$,

si $\{f_n\}$ medibles, no neg.

Para todo $n \in \mathbb{N}$, $f_{2n-1} = \chi_{[0,1]}$

$f_{2n} = \chi_{[1,2]}$

$\liminf \left(\int f_n\right) = \liminf 1 = 1$ ✓

$\int (\liminf f_n) = \int 0 = 0$

(4.7) $f_n \ge 0$ medible, $\lim f_n = f$, $f_n \le f$ $\forall n$ (*)

Lema de Fatou: $\int \liminf f_n = \int f \underset{(**)}{\le} \liminf \int f_n$

~~Existe por~~ $\longrightarrow \; \| \; \lim \int f_n$

$\int f_n \, d\mu \overset{(*)}{\le} \int f \, d\mu \Rightarrow$

$\Rightarrow \sup_{n \ge m} \int f_n \le \int f \, d\mu \Rightarrow \lim_{m\to\infty} \sup \int f_n \, d\mu \le \int f \, d\mu$

$\Rightarrow \limsup \int f_n \le \int f \underset{(**)}{\le} \liminf f_n \le \limsup f_n$

$\longrightarrow$ ~~$\int f = \lim \int f_n = \dots$~~ $\boxed{\int f \, d\mu = \lim_{n\to\infty} \int f_n \, d\mu}$

(4.8) $f_n = \min(f, n)$, $f \ge 0$ medible

$\longrightarrow 0 \le f_n = \min(f, n) \le f_{n+1} = \min(f, n+1) \le f$

$f_1 \le f_2 \le \dots$ $\forall x$ medibles no neg

Además, dado $x \in X$, $\exists n_0 = \lfloor f(x) \rfloor + 1 / \forall n \ge n_0$ $f_n(x) = \min\{f(x), n\}$

$f(x) \le \lfloor f(x) \rfloor + 1 \le n_0 \le n$

$\Rightarrow f_n(x) = f(x)$

i.e. $f_n(x) \xrightarrow[n\to\infty]{} f(x)$ $\forall x \in X$

dim. ej. 4.3 $\Rightarrow \int f_n \le \int f$, $\int f_1 \le \int f_2 \le \dots \le \int f$, $\int f \overset{TCM}{=} \lim_{n\to\infty} \int f_n$.

(4.9) $f, g \geq 0$ medibles, $f \geq g$ $\int g < \infty$

$\Rightarrow$ $(f-g) \geq 0$ medible,

$$\int f = \int [\underbrace{g + (f-g)}_{\text{medibles}}] = \int g + \int (f-g)$$

$$\Rightarrow \int f - \underbrace{\int g}_{<\infty, \text{ se puede restar}} = \int (f-g).$$

(4.10) f_n medibles, acotadas, no neg. $f_n \downarrow f$.
Para algún k, $\int f_k < \infty$

$\underbrace{f_1 \geq f_2 \geq \cdots}$ $\forall x$, $\int f_1 \geq \int f_2 \geq \cdots \geq \int f_k \geq \cdots$

$0 \leq g_n = \underbrace{f_k}_{\text{integrables}} - \underbrace{f_{k+n}}$ sucesión creciente: $g_n = f_k - \underbrace{f_{k+n}}_{\substack{\vee\! | \\ f_{k+n+1}}} \leq \underbrace{f_k - f_{k+n+1}}_{\substack{\| \\ g_{n+1}}}$

g_n integrables $\quad$ integrables ($\infty > \int f_k \geq \int f_{k+n}$)

$0 \leq g_1 \leq g_2 \leq \cdots$. Como $f_n \downarrow f$,

$$\lim_{n\to\infty} f_n(x) = f(x) \Rightarrow \lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} (f_k(x) - f_{k+n}(x)) =$$

$$= f_k(x) - f(x).$$

TCM: $\int g_n \xrightarrow[n\to\infty]{} \int (f_k(x) - f(x)) = \int f_k(x) - \int f(x)$

$$\boxed{\lim_{n\to\infty} \int f_n = \lim_{n\to\infty} \int f_{k+n} = \lim_{n\to\infty} \int (f_k - g_n)} = \lim \left(\int f_k - \int g_n\right) =$$

(4.9)

$$= \int f_k - \lim \int g_n = \int f_k - \left[\int f_k - \int f(x)\right] = \underline{\int f}$$

(4.6) $f(x) = 1/x$. si $x \geq 1$ Tomemos la sucesión de funciones simples:

$$s_n(x) = \begin{cases} 2^n & \text{si } 1/x \geq 2^n \text{ (nunca)} \\ \frac{k}{2^n} & \text{si } \frac{k}{2^n} \leq \frac{1}{x} < \frac{k+1}{2^n} \quad \forall k \in \{0, 1, \dots, 2^n - 1\} \text{ ⊛} \\ 0 & \text{si } x < 1 \end{cases}$$

visto en clase,

$0 \leq s_1 \leq s_2 \leq \dots$

$s_n \leq f$, $s_n \xrightarrow[n\to\infty]{} f$

$$\Rightarrow s_n(x) = \left\{ k/2^n \text{ si } \frac{2^n}{k+1} < x \leq \frac{2^n}{k} \quad \forall k \in \{1, \dots, 2^n - 1\}\right.$$

o siguiendo caso

$1 < x \leq \frac{2^n}{2^n - 1}$ ⊛

$$= \sum_{k=0}^{2^n - 1} \frac{k}{2^n} \chi_{\underbrace{\left(2^n/k+1,\ 2^n/k\right]}}$$

medida: $2^n\left(\frac{1}{k} - \frac{1}{k+1}\right) = 2^n \frac{1}{k(k+1)}$

$$\int s_n \, dm = \sum_{k=0}^{2^n - 1} \frac{k}{2^n} \frac{2^n}{k(k+1)} = \sum_{k=0}^{n} \frac{1}{k+1} =$$

$$= \sum_{k=1}^{2^n} \frac{1}{k} \xrightarrow{n\to\infty} \infty$$

$$\xrightarrow{TCM} \int f = \lim \int s_n = \infty \quad \square$$

⊛ Como queremos $1/x < 1$ $\left(\int_1^\infty\right)$,
a lugar de $k \in \{0, 1, \dots, 2^{2n} - 1\}$, tomamos:
$k \in \{0, 1, \dots, 2^n - 1\}$ $\longrightarrow$ $1/x < \frac{2^n}{2^n} = 1$

(4.11) $1 = a_1 \geq a_2 \geq \dots \geq a_n \geq \dots$ $\lim a_n = 0$

$f_n(x) = \frac{a_n}{x}$. $x > a > 0$

$f_1 \geq f_2 \geq \dots$. Además, dado $\varepsilon > 0$ $\exists$ n_0 tal que $\frac{a_{n_0}}{a} < \varepsilon$

$\Rightarrow \forall n \geq n_0$, $a_n \leq a_{n_0} < \varepsilon$

$\Rightarrow \forall n \geq n_0$ $f_n(x) = \frac{a_n}{x} < \frac{a_{n_0}}{a} < \varepsilon$. Pero $\int f_n = \int_a^\infty \left(\frac{a_n}{x}\right) dm$

$x > a$

$\frac{1}{x} < a$

$\overset{(4.6)}{=} \infty$

fin. ej. 4.4

(4.12) $f_n : [0,1] \to [0,\infty)$

$f_n(x) = \begin{cases} n & \text{si } 0 \leq x \leq 1/n \\ 0 & \text{otro caso} \end{cases}$

~~Dado $\varepsilon > 0$~~ $\forall x \in (0,1]$ $\exists n_0 /$ $x > \frac{1}{n_0}$ i.e. $\forall n \geq n_0$ $x > \frac{1}{n}$

i.e. $f_n(x) = 0$ $\forall n \geq n_0$

$\Rightarrow$ $f_n(x) \xrightarrow[n\to\infty]{} 0$ puntualmente.

$\int f_n = n m(\chi_{[0,1/n)}) = n \cdot \frac{1}{n} = 1$

(4.13) $g : (X, \mathcal{A}, \mu) \longrightarrow (\overline{\mathbb{R}}, \mathcal{B}_{\overline{\mathbb{R}}})$ integrable.

$\{E_n\}$ sucesión decreciente de conjuntos / $\underbrace{\bigcap_{n=1}^{\infty} E_n = \phi}$

$\lim_{n\to\infty} \mu(E_n) = 0 = \mu(\phi)$

Probar que $\lim_{n\to\infty} \underbrace{\int_{E_n} g \, d\mu}_{} = 0$

$$\text{Afirmo } ①, ② \quad \int_{E_n} g^+ - \int_{E_n} g^-$$

$\int_{E_n} g^+ \overset{①}{=} 0$, $\int_{E_n} g^- \overset{②}{=} 0$

Basta verlo si $g \geq 0$.

$g(x) \leq \sup_{x\in X} g(x) =: G$. Con $G \in [0,\infty]$

$\Rightarrow 0 \leq \int_{E_n} g \, d\mu \leq \int_{E_n} G \, d\mu = \int_X G \chi_{E_n} d\mu = G\, m(E_n)$

$\lim_{n\to\infty} G\, m(E_n) = G \cdot 0 = 0 \Rightarrow \lim_{n\to\infty} \int_{E_n} g \, d\mu = 0$ $\square$

(4.14) $f : \mathbb{R} \to [0,\infty)$ medible. $f \in L^1(m)$ (integrable)

$F : \mathbb{R} \to \mathbb{R}$. $F(x) = \int_{-\infty}^{x} f(t)\, dm$.

- Probar que F cont.
  - Probar que dados $x_1 < x_2 < \dots$ reales de forma
$$\sum_k |F(x_{k+1}) - F(x_k)| < \int |f|\, dm.$$

$$0 \leq F(x_{k+1}) - F(x_k) = \int \left[f(t)\chi_{[-\infty, x_{k+1}]} - f(t)\chi_{(-\infty, x_k]} \right) dm$$

$$= \int_{x_k}^{x_{k+1}} f(t)\, dm$$

$$\geq$$

$$\chi_{(-\infty, x_{k+1}]} - \chi_{(-\infty, x_k]} = \chi_{(x_k, x_{k+1}]}$$

$$\sum_{k=1}^{\infty} |F(x_{k+1}) - F(x_k)| = \lim_{N\to\infty} \sum_{k=1}^{N} \int f(t)\chi_{(x_k, x_{k+1}]}$$

$$= \lim_{N\to\infty} \int f(t) \underbrace{\sum_{k=1}^{N} \chi_{(x_k, x_{k+1}]}}_{\chi_{(x_1, x_N]}} = \lim_{N\to\infty} \int \underbrace{f(t)\chi_{(x_1, x_N]}}_{\leq f(t)} \leq \int f$$

Creciente y acotada, $\leq \int f$

converge

Dado $\varepsilon > 0$ $\exists \delta > 0$ / $|x-y| < \delta \Rightarrow |F(x) - F(y)| < \varepsilon$

$\delta = \varepsilon / (\int f)$

$$|F(x) - F(y)| = \int f(t)\, \chi_{\substack{(x,y)\\ o\,(y,x)}} \leq \left(\int_X f\right) \cdot |x-y| < \left(\int_X f\right) \cdot \delta = \varepsilon \quad \square$$

~~si $x \leq y$~~

(4.15) $\mu(X) < \infty$ $\{f_n\} \subset L^1(\mu)$ con $f_n \longrightarrow f$ unif.

Demostrar $f \in L^1(\mu)$, $\int f_n \, d\mu \longrightarrow \int f \, d\mu$.

$\varepsilon_n(x) = f_n(x) - f(x) \longrightarrow 0$ unif

f medible por ser $\left(\liminf_n f_n \right)$

$\forall x \in X$ Dado $\varepsilon > 0$ $\exists n_0 \,/\, \forall n \geq n_0$: $|f_n(x) - f(x)| < \varepsilon / \mu(X)$

$$\left| \int f - \int f_n \right| = \left| \int (f - f_n) \right| \leq \int \underbrace{|f_n - f|}_{\wedge \text{ en } \frac{\varepsilon}{\mu(X)}} \leq$$

$$\leq \int_X \frac{\varepsilon}{\mu(X)} = \varepsilon \xrightarrow{\varepsilon \to 0} 0$$

$\Rightarrow \boxed{\lim_{n \to \infty} \int f_n = \int f}$. Tomo $\varepsilon = 1$, n_1

$$\left| \int f \right| \leq \left| \int f_{n_1}(x) - \int (f_{n_1}(x) - f(x)) \right| \leq$$

$$\leq \underbrace{\left| \int f_{n_1}(x) \right|}_{\wedge \, \infty} + \underbrace{\left| \int (f_{n_1} - f) \right|}_{\substack{\wedge \\ \underbrace{\int |f_{n_1} - f|}_{\wedge \, 1/\mu(X)}}} < \infty \Rightarrow f \in L^1(\mu)$$

$P_n = \{0, 1/n, 2/n, \dots, 1\}$

(4.16) $A = [0,1] \cap \mathbb{Q} = \{q_1, q_2, \dots\}$ $f_n : [0,1] \to \mathbb{R}$: $f_n(x) = \begin{cases} 1 & \text{si } x \in \{q_1 \dots q_n\} \\ 0 & \text{otro caso} \end{cases}$

Fijo $n \Rightarrow$ $\mathcal{U}(f_n, P_m) - \mathcal{L}(f_n, P_m) = \frac{1}{m} \sum_{i=1}^{m} \underbrace{(M_i - m_i)}_{} = \frac{n}{m} \xrightarrow{m \to \infty} 0$

Habrá n M_i's $= 1$ el resto, 0, $m_i = 0$ $\forall i$

$\Rightarrow f_n$ integrable Riemann.

Pero f no! $\underbrace{\mathcal{U}(f, P_m)}_{1} - \underbrace{\mathcal{L}(f_m, P_m)}_{0 \text{ siempre.}} = \frac{1}{m} \sum_{i=1}^{m} 1 = 1$!!

(4.13) $\int \underbrace{g\chi_{E_n}}\, d\mu \longrightarrow 0$

$\{f_n\}$ funciones ~~...~~ acotadas ($|f_n| < |g|$ con $\int |g| < \infty$

Además $f_n \to 0$ punt ($\bigcap E_n = \phi$, $E_1 \supset E_2 \supset \cdots$)

$\xrightarrow{TCD} \lim_{n\to\infty} \int \underbrace{g\chi_{E_n}}_{f_n} d\mu = \int (\lim_{n\to\infty} f_n) d\mu = \int 0\, d\mu = 0$.

Obs: Para usar TCD necesitamos ~~...~~ $g: X \to \mathbb{R}$, no $\overline{\mathbb{R}}$.

Como $\int |g| < \infty$: $\mu(\{x \in X / g(x) = \pm\infty\}) = 0$

$\Rightarrow$ definimos una $\hat{g}$ igual a g en todo punto, excepto en $\{x \in X / g(x) = \pm\infty\}$.

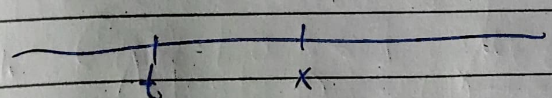
Mi 4.13 está mal

(4.14) F cont por sucesiones

Sea $x \in \mathbb{R}$. ~~Veamos~~ Sea $\{x_n\} \subset \mathbb{R}$ / $\{x_n\} \to x$. $\Rightarrow F(x_n) \to x$

demostración: $F(x_n) = \int_{-\infty}^{x_n} f(t)\,dt = \int_{\mathbb{R}} \underbrace{f\chi_{(-\infty, x_n]}}_{f_n \text{ medible}}$

$\cdot\, f_n \longrightarrow f\chi_{(-\infty, x]}$ (puntual) excepto quizás en x



$\forall t < x \ \exists n_0 / \forall n \geq n_0 \ |x - x_n| < (x-t)$

$\Rightarrow \chi_{(-\infty, x_n]}(t) = 1$

$\cdot\, |f_n| \leq |f|$, $\int_X |f| < \infty$

$\Rightarrow$ TCD: $\lim_{n\to\infty} \underbrace{\int_X f_n\, d\mu}_{F(x_n)} = \int f\chi_{(-\infty, x)} = F(x)$. $\square$

Ej. 4.6

... damos $A = \{\emptyset, X\}$ $X = \{1, 2\}$

(4.15) Otra forma de ver $\int f = \lim \int f_n$:

$\exists n_0 \quad \forall n \geq n_0 \quad |f_n| \leq \underbrace{|f| + 1}_{g}$ por convergencia uniforme ($\forall x$)

$\overset{TCD}{\Longrightarrow} \int f_n \, d\mu \xrightarrow{n\to\infty} \int f \, d\mu.$

(4.17) $\lim\limits_{n\to\infty} \int_0^\infty \frac{dx}{(1+\frac{x}{n})^n x^{\frac{1}{n}}}$

Pista $n \geq 2 \Rightarrow (1+\frac{x}{n})^n \geq \frac{x^2}{4}$

Sol: $f_n(x) = \frac{1}{(1+\frac{x}{n})^n x^{1/n}}$ $x > 0$

$f_n(x) \longrightarrow e^{-x} = f(x).$ $\boxed{\int_0^\infty f(x)\,dx = 1}$

$\odot$ $\int_0^\varepsilon x^{-\alpha} dx < \infty \iff \alpha < 1$

$\odot\odot$ $\int_\varepsilon^\infty x^{-\alpha} dx < \infty \iff \alpha > 1$

$|f_n| \leq g.$ con $g(x) = \begin{cases} x^{-1/2} & x \in (0,1) \\ 4x^{-2} & x \geq 1 \end{cases}$ integrable. $\odot\odot\odot$ $\overset{TCD}{\Longrightarrow}$ $\int_0^\infty f_n \to 1$

$f_n(x) \leq \frac{1}{(x^2/4)\, x^{1/n}} \leq \frac{4}{x^2}$ si $x \geq 1$

$(x^{1/n} \geq 1$ si $x \geq 1)$

$(x^{1/n} \geq x^{1/2}$ si $x \in (0,1)$, $n \geq 2)$

$x \in (0,1)$ $f_n(x) = \frac{1}{(1+\frac{x}{n})^n x^{1/n}} \leq \frac{1}{x^{1/2}}$

$(1+\frac{x}{n})^n \geq 1$

(4.18) $f_n(x) = \dfrac{nx - 1}{(x\log n + 1)(1 + nx^2 \log n)}$, $x \in (0,1]$

Comprobar $\lim\limits_{n\to\infty} f_n(x) \overset{①}{=} 0$ y $\lim\limits_{n\to\infty} \int_0^1 f_n(x)\,dx \overset{②}{=} 1/2$.

Sol: Con sugerencia es claro $f_n(x) \xrightarrow[n\to\infty]{} 0$ punto a punto.

$$\int_0^1 f_n(x)\,dx = \int_0^1 \frac{-1}{x\log n + 1}\,dx + \int_0^1 \frac{nx}{n\log(n)x^2 + 1}\,dx \xrightarrow[n\to\infty]{} 0$$

$$\cdot \int_0^1 \frac{-dx}{x\log n + 1} = -\frac{\log(x\log n + 1)}{\log n}\Big]_0^1 = \frac{-\log(\log n + 1)}{\log n} \xrightarrow[n\to\infty]{} 0$$

$$\cdot \int_0^1 \frac{nx}{n\log n\, x^2 + 1}\,dx = \frac{\log(n\log n\, x^2 + 1)}{2\log n}\Big|_0^1 \xrightarrow[n\to\infty]{} 0$$

(4.19) $\lim\limits_{n\to\infty} \int_a^\infty \dfrac{n}{1 + n^2x^2}\,dx$. Estudiando $a<0$, $a=0$, $a>0$.

(a>0) $\left|\dfrac{n}{1+n^2x^2}\right| \le \dfrac{1}{nx^2} \underset{(n\ge 1)}{\le} \dfrac{1}{x^2} = g(x)$. $\int_a^\infty g(x)\,dx = -x^{-1}\Big|_a^\infty = 1/a < \infty$

$\lim\limits_{n\to\infty} \dfrac{n}{1+n^2x^2} = 0 \quad \forall x \neq 0$	$\lim\limits_{n\to\infty} \dfrac{n}{1+n^2x^2} = \infty$ para $x=0$	$f(x)=$

$f_n(x) \to 0$ c.t.p. x

Utilizando TCD, $\lim\limits_{n\to\infty} \int_a^\infty \dfrac{n\,dx}{1+n^2x^2} = \int_a^\infty 0\,dx = 0$.

(a=0) No aplicable TCD, pues ~~$\frac{n}{1+n^2x^2}$ no se puede acotar uniformemente cerca de $x=0$~~

~~Como $f_n(x) \xrightarrow{x\to 0} \infty$,~~

$$\int_0^\infty \frac{n}{1+n^2x^2}\,dx \underset{z=nx}{=} \int_0^\infty \frac{\cancel{n}}{1+z^2}\frac{dz}{\cancel{n}} = \arctan z\Big]_0^\infty = \frac{\pi}{2}.$$

~~$\dfrac{\partial}{\partial n}\left(\dfrac{n}{1+n^2x^2}\right) = \dfrac{1+n^2x^2 - 2n^2x^2}{(1+n^2x^2)^2} = \dfrac{1-n^2x^2}{(1+n^2x^2)^2} \underset{(x\le 1)}{\ge} \dfrac{1-n^2}{(1+n^2x^2)^2}$~~

~~Y el TCD solo podría concluir $\lim\limits_{n\to\infty} \int f_n = 0 = \int \lim f_n$~~ 0

tira ej. 4.7

(4.20) Calculer $\lim_{n\to\infty} \int_0^\infty \frac{1+nx^2}{(1+x^2)^n}dx$

$$\cancel{\frac{\partial}{\partial n}\frac{1+nx^2}{(1+x^2)^n} = \frac{x^2(1+x^2)^n - (1+nx^2)(1+x^2)^n\log(1+x^2)}{(1+x^2)^{2n}} =}$$

$$\cancel{\rightarrow e^{n\log(1+x^2)}}$$

$$\cancel{= \frac{x^2 - \log(1+x^2) - nx^2\log(1+x^2)}{(1+x^2)^n}}$$

$$\left(\frac{1+nx^2}{(1+x^2)^n}\right) = \frac{1}{(1+x^2)^n} + \frac{nx^2}{(1+x^2)^n} \leq \frac{1}{(1+x^2)} + \frac{1}{(1+x^2)}$$ intégrable

$$1 \leq \binom{n}{k}\frac{1}{n} = \frac{n(n-1)(n-2)!}{2!\,(n-2)!\,n}$$

$$\frac{nx^2}{(1+x^2)^n} = \frac{nx^2}{\sum_{k=0}^{n}\binom{n}{k}x^{2k}} = \frac{1}{\sum_{k=1}^{n}\binom{n}{k}\frac{x^{2k-2}}{n}} \leq_{\substack{k=2\\k=1}} \frac{1}{1+\binom{n}{2}\frac{x^2}{n}} \leq \frac{1}{1+x^2}$$

TCM, $\lim_{n\to\infty}\int_0^\infty \frac{1+nx^2}{(1+x^2)^n}dx = \int_0^\infty \underbrace{\left(\lim_{n\to\infty}\frac{1+nx^2}{(1+x^2)^n}\right)}_{0}dx = 0$

(4.19) $\frac{d}{dx}\arctan(nx) = \frac{n}{1+n^2x^2}$

$a=0$: $\int_0^\infty \left(\frac{n}{1+n^2x^2}\right)dx = \arctan(nx)\Big]_0^\infty = \pi/2$

$a<0$: $\int_a^0 \left(\frac{n}{1+n^2x^2}dx\right) + \frac{\pi}{2} = \frac{\pi}{2} + \arctan nx\Big|_a^0 = \frac{\pi}{2} - \arctan na$

$\xrightarrow[\text{ou } a<0]{n\to\infty} \frac{\pi}{2} + \frac{\pi}{2} = \pi.$

$$\rightarrow \boxed{\lim_{n\to\infty}\int_a^\infty \frac{n}{1+n^2x^2}dx = \begin{cases} 0 & \text{si } a>0 \\ \pi/2 & \text{si } a=0 \\ \pi & \text{si } a<0 \end{cases}}$$