(7.1) $X = Y = \mathbb{N}$, $\mathcal{M} = \mathcal{N} = \mathcal{P}(\mathbb{N})$. $\mu = \nu$ = medida de Contar

Probar $\mu \otimes \nu$ = medida de contar en $\mathbb{N}^2$.

$(m, n) \in \mathbb{N}^2 \Rightarrow \{(m, n)\} = \{m\} \times \{n\} \in \mathcal{M} \otimes \mathcal{N}$

Por ser un rectángulo

Por tanto, si $A \subset \mathbb{N} \times \mathbb{N}$, $A = \bigcup_{(m,n) \in A} \{(m,n)\}$ unión numerable de rectángulos

$\longrightarrow A \in \mathcal{M} \otimes \mathcal{M}$.

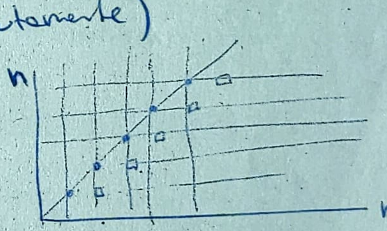
$\therefore \mathcal{M} \otimes \mathcal{M} = \mathcal{P}(\mathbb{N}^2)$.

$\mu \otimes \nu(\{(m,n)\}) = 1 = 1 \cdot 1 = \mu(m) \cdot \nu(n)$.

$\Rightarrow \mu \otimes \nu(A) = \# A$ $\square$

Si $f : \mathbb{N}^2 \to \mathbb{C}$

$$\int_{\mathbb{N} \times \mathbb{N}} f \, d(\mu \otimes \nu) = \sum_{m,n \in \mathbb{N}} f(m,n)$$ (es integrable ssi la serie converge absolutamente)



Con $f(m,n) = \begin{cases} 1 & \text{si } m = n \\ -1 & \text{si } m = n+1 \\ 0 & \text{otro caso.} \end{cases}$

- $\int |f| \, d(\mu \times \nu) = \infty$.
- $\underset{n \text{ fijo}}{\int f \, d\mu(m) =} \underset{m=n}{1} - \underset{m=n+1}{1} = 0 \Rightarrow \int \left(\int f d\mu \right) d\nu = 0$.
- $\underset{m \text{ fijo}}{\int f \, d\nu(n) =} \begin{cases} 1 - 1 & \text{si } m > 1 \\ 1 & \text{si } m = 1 \end{cases} \Rightarrow \int \left(\int f d\nu \right) d\mu = 1$.

No se puede usar Tonelli: f toma valores negativos.

No se puede usar Fubini: f no integrable.

7.1

(7.2.) $(X, \mathcal{M}, \mu) \cdot (Y, \mathcal{N}, \nu)$ σ-finitos

- $f: X \to \mathbb{R}$ medible. $g: Y \to \mathbb{R}$ medible.

$\Rightarrow h: X \times Y \to \mathbb{R}$ con $h(x,y) = f(x)h(y)$ medible $\mathcal{M} \otimes \mathcal{N}$.

$\circledast$ Sean f, g simples: $f = \sum_{i=1}^{p} a_i \chi_{A_i}$, $A_i \in \mathcal{M}$.

$g = \sum_{j=1}^{q} b_j \chi_{B_j}$, $B_j \in \mathcal{N}$.

$\Rightarrow h = \sum_{j=1}^{q} \sum_{i=1}^{p} a_i b_j \chi_{A_i \times B_j}$ función simple y por tanto medible.
medible porque $A_i \times B_j \in \mathcal{M} \otimes \mathcal{N}$.

- Sean f, g no necesariamente simples,

$\exists \{s_n\}, \{t_n\}$ funciones simples (med.) / $s_n \to f$, $t_n \to g$ conv. puntual

$\Rightarrow \underbrace{s_n(x)\, t_n(y)}_{\text{medible por } \circledast} \xrightarrow{n \to \infty} f(x)\, g(y) = h(x,y)$ (lím. puntual).

$\Longrightarrow$ h medible. (límite de medibles)

- f, g como antes, $f \in L^1(\mu)$, $g \in L^1(\nu) \Rightarrow h \in L^1(\mu \times \nu)$

y además:

$\circledast\circledast$ $\int_{X \times Y} h \, d(\mu \times \nu) = \left(\int_X f d\mu\right)\left(\int_Y g \, d\nu\right)$

$\int_{X \times Y} |h| \, d(\mu \times \nu) \underset{\uparrow}{=} \left(\int_X |f| \, d\mu\right)\left(\int_Y |g| \, d\nu\right) < \infty$.

Tonelli: las secciones de $|h|$ son $|f|$ y $|g|$

$\circledast\circledast$ Por Fubini.

7.2

(7.3) $f: X \to [0,\infty)$ med. $(X, \mathcal{M}, \mu)$

$A_f = \{(x,y) \in X \times \mathbb{R} \,/\, 0 \leq y < f(x)\}$.

(a) $A_f \in \mathcal{M} \otimes \mathcal{B}_{\mathbb{R}}$

- f simple: $f = \sum_{i=1}^{p} c_i \chi_{C_i}$ $\quad c_i > 0$, $C_i \in \mathcal{M}$.

$\Rightarrow A_f = \bigsqcup_{i=1}^{p} C_i \times [0, c_i) \in \mathcal{M} \otimes \mathcal{B}_{\mathbb{R}}$.

(b) Y la medida producto de A_f es: $\underbrace{(\mu \times m)(A_f) = \sum_{i=1}^{p} c_i \mu(C_i)}$

$$\Rightarrow \boxed{\mu \times m (A_f) = \int_X f \, d\mu}$$

(a) - f cualquiera: $\{s_n\}$ simples, ≥ 0, med, crecientes / $\{s_n\} \to f$

$A_f = \{(x,y) \in X \times \mathbb{R} \,/\, 0 \leq y < f(x)\} = \bigcup_{n=1}^{\infty} A_{s_n} \in \mathcal{M} \otimes \mathcal{B}_{\mathbb{R}}$

Unión creciente.

$$\boxed{\begin{array}{l} A_{s_n} \subset A_{s_{n+1}} \ \forall n \in \mathbb{N} \\ \text{Y si } y < f(x) \Rightarrow \exists n_0 \,/\, \forall n \geq n_0 \ \ s_n(x) > y \end{array}}$$

(b) Por tanto, $(\mu \times m) A_f = \lim_{n\to\infty} (\mu \times m)(A_{s_n}) =$

$= \lim_{n\to\infty} \int_X s_n \, d\mu \underset{\text{TCM}}{=} \int_X f \, d\mu$ $\square$.

~~(b)~~

7.3

(7.6)

$$f(x,y)=\begin{cases}\dfrac{xy}{(x^2+y^2)^2} & \text{si } -1\le x\le 1,\ -1\le y\le 1 \quad (x,y)\ne(0,0)\\[2ex] 0 & \text{si } x=y=0\end{cases}$$

x fijo: $x=0 \Rightarrow f(0,y)\equiv 0$

$x\neq 0 \Rightarrow$ $\underbrace{f(x,y) \text{ continua y acotada (Riemann = Lebesgue) (y es impar)}}$

$$\Downarrow$$

$$\int_{[-1,1]} f(x,y)\,dm(y)=0 \quad \forall x\in[-1,1]$$

También (por simetría) $\int_{[-1,1]} f(x,y)\,dm(y)=0 \quad \forall y\in[-1,1]$

Pero f no es integrable:

$$\iint_{[-1,1]\times[-1,1]} |f|\,dm^2 \underset{\uparrow}{=} 4\int_0^1\int_0^1 \frac{xy}{(x^2+y^2)^2}\,dx\,dy \underset{\text{Tonelli}}{=} 4\int_0^1 y\left(\int_0^1 \frac{x}{(x^2+y^2)^2}\,dx\right)dy=$$

$$=4\int_0^1 y\left[\frac{1}{x^2+y^2}\left(\frac{-1}{2}\right)\right]_0^1 dy=-2\int_0^1 y\left(\frac{1}{1+y^2}-\frac{1}{y^2}\right)dy=$$

$$=+2\int_0^1 \underbrace{y\left(\frac{1}{y^2}-\frac{1}{1+y^2}\right)}_{y\left(\frac{1+y^2-y^2}{y^2(1+y^2)}\right)=\frac{1}{y(1+y^2)}}dy \underset{1+y^2\le 2}{\geq} 2\int_0^1\frac{1}{y2}\,dy=\int_0^1\frac{1}{y}\,dy=\infty$$

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(7.5) $$f(x,y) = \begin{cases} \dfrac{x^2-y^2}{(x^2+y^2)^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

T. Fubini falla porque f no integrable ($\mathbb{R}^2$, $\mathcal{L}^2$, m^2 ó $m \times m$)

· Hay que ver $\displaystyle\int_{[0,1]^2} |f(x,y)|\, dm^2 = \infty$

· Lo podemos hacer con un cambio de variable: $|f| \geq 0$ y medible.
con T. Tonelli ($|f| \geq 0$, medible)

$$\int_{B(\vec{0},1)\cap\{x,y\geq 0\}} |f(x,y)|\, dm^2 \geq \int_0^{\pi/4}\int_0^1 \frac{r^2 \cos 2\theta}{r^4}\, r \, dr\, d\theta =$$

Jacobiano

$x^2 \geq y^2$

Coord. polares
$x = r\cos\theta$
$y = r\sin\theta$ $\quad 0<r<1,\ \theta \in (0,\pi/2)$
$x^2-y^2 = r^2(\cos 2\theta)$

$$= \underbrace{\int_0^1 \frac{1}{r}\, dr}_{\infty} \underbrace{\int_0^{\pi/4} \cos(2\theta)\, d\theta}_{} = \underbrace{-\log r\Big]_0^1}_{\infty} \cdot \underbrace{\frac{\operatorname{sen}(2\theta)}{2}\Big]_0^{\pi/4}}_{\frac{1}{2}-0} = \infty \quad \square$$

---

a) $\displaystyle\underbrace{\int_0^1 dx \int_0^1 f(x,y)\, dy}_{\text{lo mismo que}} = \frac{\pi}{4}$ $\quad$ Obs: $f(x,y) = -f(y,x)$

$\displaystyle\int_0^1 \left(\int_0^1 f(x,y)\, dy\right) dx = \pi/4$

· $\displaystyle\int_0^1 \frac{x^2-y^2}{(x^2+y^2)^2}\, dy =$

$t = \dfrac{y^2}{x^2}$
$dt = \dfrac{2y}{x^2}\, dy$

$$\frac{x^2-y^2}{(x^2+y^2)^2} = \frac{A}{(x^2+y^2)^2} + \frac{B}{x^2+y^2}$$

7.5

UNICAMENTE PARA USO ESCOLAR (IMPRESO EN PAPEL RECICLADO)

(7.3) $f: X \to \mathbb{R}$ $\mathcal{M}$-medible, $f \geq 0$. $A_f = \{(x,y) \in X \times \mathbb{R} / 0 \leq y < f(x)\}$

a) $A_f \in \mathcal{M} \otimes \mathcal{B}_{\mathbb{R}}$

· f simple: $f = \sum_1^m a_i \chi_{E_i}$ con $E_i \in \mathcal{M}$ disjuntos dos a dos.

$\Rightarrow A_f = \biguplus_{i=1}^m E_i \times [0, a_i) \in \mathcal{M} \otimes \mathcal{B}_{\mathbb{R}}$ por ser unión finita de rectángulos: $E_i \in \mathcal{M}$, $[0,a_i) \in \mathcal{B}_{\mathbb{R}}$.

(· $f = \chi_E \Rightarrow A_f = \{(x,y) \in X \times \mathbb{R} / 0 \leq y < \chi_E(x)\} =$

$= \{(x,y) \in E \times \mathbb{R} / 0 \leq y < 1\} = E \times [0,1) \in \mathcal{M} \otimes \mathcal{B}_{\mathbb{R}}$

$f = a\chi_E \Rightarrow A_f = E \times [0, a) \in \mathcal{M} \otimes \mathcal{B}_{\mathbb{R}}$)

· $f \geq 0$, medible. $\exists \{s_n\}$ simples medibles $0 \leq s_1 \leq s_2 \leq \ldots$ tales que

$s_n \longrightarrow f$ punto a punto.

Además, $A_{s_n} \subset A_{s_{n+1}} \cdots \subset A_f$

y $\bigcup_{n=1}^\infty \underset{\mathcal{M} \otimes \mathcal{B}_{\mathbb{R}}}{\underset{\cap}{A_{s_n}}} = A_f$ unión numerable de elementos de la σ-álgebra: $\underline{A_f \in \mathcal{M} \otimes \mathcal{B}_{\mathbb{R}}}$.

b) μ es σ-finita

· f simple: $f = \sum_1^m a_i \chi_{E_i}$ con $E_i \in \mathcal{M}$ disj. 2 a 2

$\left.\begin{array}{l} \int_X f \, d\mu = \sum_1^m a_i \mu(E_i). \\ \mu \times m(A_f) = \mu \times m\left(\bigcup_1^m E_i \times [0, a_i)\right) = \sum_1^m a_i \mu(E_i) \end{array}\right\} = .$

· $f \geq 0$, medible. $\mu \times m(A_{s_n}) \longrightarrow \mu \times m(A_f)$ (Unión creciente)

$\| $

$\int_X s_n \, d\mu \longrightarrow \int_X f \, d\mu$ (TCM)

(7.4) $X = Y = [0,1]$. $\mathcal{A}_1 = \mathcal{A}_2 = \mathcal{B}_{[0,1]}$. m = med. Lebesgue en $\mathcal{A}_1$, ν = med. contar en $\mathcal{A}_2$. $(X \times Y, \mathcal{A}_1 \otimes \mathcal{A}_2, m \times \nu)$.

$V = \{(x,y) : x = y\}$.

- Comprobar $V \in \mathcal{A}_1 \otimes \mathcal{A}_2$, pero integrales iteradas diferentes.

$V_n = \bigcup_{j=1}^{2^n} (I_n^j \times I_n^j)$ donde $I_n^j = [\frac{j-1}{2^n}, \frac{j}{2^n}]$ $\forall j \in \{1, \dots, 2^n\}$.

Además $V \subset V_n$ $\forall n \in \mathbb{N}$ y si $x \neq y$ $(x,y) \in [0,1]^2 \Rightarrow \exists n_0 / (x,y) \notin V_{n_0}$.

$V_1 \supset V_2 \supset \dots$ $V_n \in \mathcal{A}_1 \otimes \mathcal{A}_2$

y $V = \bigcap_{n=1}^{\infty} V_n \Longrightarrow V \in \mathcal{A}_1 \otimes \mathcal{A}_2$.

Otra forma: $f(x,y) = x - y$ continua $\Rightarrow f$ $\mathcal{B}_{[0,1]^2}$-medible

$\Rightarrow V = f^{-1}(\{0\}) \in \mathcal{B}_{[0,1]^2}$.

Falta ver que $\mathcal{B}_{[0,1]^2} \subset \mathcal{B}_{[0,1]} \otimes \mathcal{B}_{[0,1]}$. Esto probaría que f es $\mathcal{A}_1 \otimes \mathcal{A}_2$-medible. Pero

$\mathcal{B}_{[0,1]^2} = \mathcal{M}(\{\text{Abiertos en } [0,1]^2\}) \underset{\uparrow}{=} \mathcal{M}(\{U \times V : U, V \text{ abtos en } [0,1]\})$

$[0,1]^2$ es II axioma de numerabilidad.

$\Rightarrow \mathcal{B}_{[0,1]^2} \subset \mathcal{B}_{[0,1]} \otimes \mathcal{B}_{[0,1]}$. $\square$

- $\int_X \underbrace{\chi_V(x,y)} dm(x) = \int_{[0,1]} \chi_{\{y\}}(x) dm(x) = m(\{y\}) = 0 \Rightarrow \boxed{\int_Y \left(\int_X \chi_V dm\right) d\nu = 0}$

- $\int_Y \chi_V(x,y) d\nu(y) = \int_{[0,1]} \chi_{\{x\}}(y) d\nu(y) = \nu(\{x\}) = 1$

$\Rightarrow \boxed{\int_X \left(\int_Y \chi_V d\nu\right) dm = \int_{[0,1]} 1 \, dm = 1 m([0,1]) = 1}$

7.6

(7.7) $f, g \in L^1(\mathbb{R}, \mathcal{M}, m)$ con m = medida de Lebesgue

Demostrar que $f(x-y)\,g(y)$ integrable en y $\forall$ ctp $x \in X$.

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~~1) $x-y$ continua, f medible $\Rightarrow$ $f(x-y)$ medible~~

~~2) $f(x-y) \cdot g(y)$ prod. de medibles es medible~~

$f(x)$ medible, $g(y)$ también (en $\mathcal{M} \otimes \mathcal{M}$)

$T(x,y) = (x-y, y)$ lineal e invertible

$\Rightarrow T$ y T^{-1} mandan conjuntos medibles en conjuntos medibles.

$F(x,y) = f(x)\,g(y)$ medible (prod. de medibles)

$\Rightarrow F \circ T(x,y) = f(x-y)\,g(y)$ medible.

$$\int_{\mathbb{R}^2} |f(x-y)\,g(y)|\, d\, m \times m(x,y) \underset{\text{Tonelli}}{=} \int_{\mathbb{R}} |g(y)| \left(\int_{\mathbb{R}} |f(x-y)|\, dm(x) \right) dm(y)$$

$$= \int_{\mathbb{R}} |g(y)|\, \|f\|_1\, dm(y) = \|f\|_1 \cdot \|g\|_1 < \infty.$$

$\therefore f(x-y)\,g(y)$ integrable.

Fubini: $f(x-y)\,g(y)$ integrable en y ctp x.

$$h(x) = \int_{\mathbb{R}} f(x-y)\,g(y)\,dm(y)$$

$$\int_{\mathbb{R}} h(x)\,dm(x) \underset{\text{Fubini}}{=} \int_{\mathbb{R}} g(y) \int_{\mathbb{R}} f(x-y)\,dm(x)\,dm(y) = \left(\int g\,dy\right)\left(\int f\,dx\right)$$

$$\int_{\mathbb{R}} |h|\,dm(x) \leq \int_{\mathbb{R}} |g(y)| \int_{\mathbb{R}} |f(x-y)|\,dm(x)\,dm(y) = \|g\|_1 \|f\|_1 < \infty$$

(7.8) $d\vartheta$ medida en $\mathcal{P}(\mathbb{R}^2)$:

$$\vartheta(A) = \text{card}(A \cap \mathbb{Z}^2) \quad \forall A \subset \mathbb{R}^2.$$

$\phi: \mathbb{R}^2 \to \mathbb{R}$. $\phi(x,y) = e^{x^2+y^2}$: $d\vartheta_\phi$ la medida inducida por $d\vartheta$ y ϕ en $\mathbb{R}$.

$$\boxed{\vartheta_\phi(A) = \vartheta(\phi^{-1}(A))}$$

$$\boxed{\vartheta_\phi([1,e])} = \vartheta(\{(x,y)\in\mathbb{R}^2 : e^{x^2+y^2} \in [1,e]\}$$
$$= \vartheta(\{(x,y)\in\mathbb{R}^2 : 0 \le x^2+y^2 \le 1\} = \boxed{5}$$

$$\boxed{\vartheta_\phi((e^2,e^3]) = \vartheta(\{(x,y)\in\mathbb{R}^2 : 2 < x^2+y^2 \le 3\}) = 0}$$

x	y	x^2+y^2
0	0	0
±1	0	1
0	±1	1
±1	±1	2
±2	0	4
0	±2	4

(7.9) $X = [0,1]\times[0,1] \subset \mathbb{R}^2$. La medida de área de Lebesgue habitual: dm. $\varphi_1(x_1,x_2) = x_1+x_2$. $\varphi_2(x_1,x_2) = |x_1-x_2|$

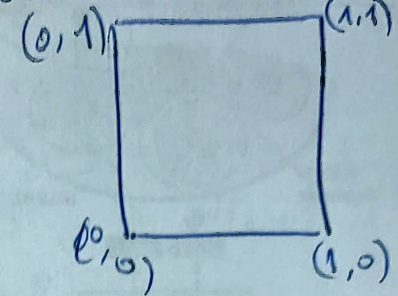
$\varphi_1(X) = [0,2]$. $\varphi_2(X) = [0,1]$

Son finitas $\Rightarrow$ de Lebesgue-Stieltjes sobre $\mathbb{R}$.

$$F(x) = \begin{cases} -\mu((x,0]) & \text{si } x<0 \\ 0 & \text{si } x=0 \\ \mu((0,x]) & \end{cases}$$

7.7

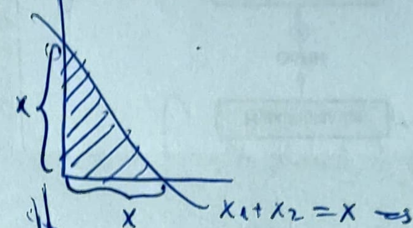
Como $\varphi_1(X), \varphi_2(X) \subset [0,\infty)$, es claro que si $x < 0$, entonces $m_\varphi((x,0]) \stackrel{\downarrow}{=} m(\underbrace{\varphi^{-1}(\{0\})}_{\text{En ambos casos mide } 0.}) = 0$



· Caso φ_1: Sea $x \in (0,2]$

$\varphi^{-1}((0,x]) =$

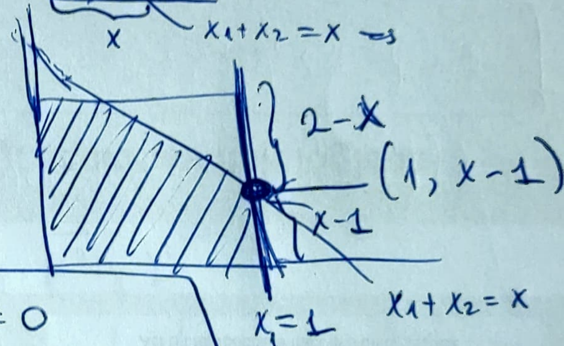
$= \{(x_1,x_2) \in X \mid 0 < x_1 + x_2 \leq x\}$



Sea $x \in (0,1]$:

$\Downarrow$

$m_\varphi((0,x]) = \frac{x^2}{2}$

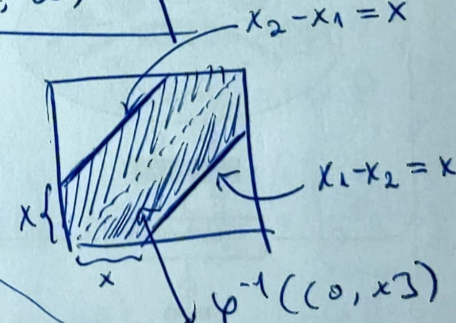


Sea $x \in [1,2]$

$m_\varphi((0,x]) = 1 - \frac{(2-x)^2}{2}$

$$F_1(x) = \begin{cases} 0 & \text{si } x \leq 0 \\ x^2/2 & \text{si } x \in (0,1] \\ 1-(2-x)^2/2 & \text{si } x \in (1,2] \\ 1 & \text{si } x \in [2,\infty) \end{cases}$$

· Caso φ_2: $x \in (0,1]$:



$m((0,x]) = \underbrace{1-(1-x)^2}$

$= 1 - x^2 - 1 + 2x$

$= 2x - x^2$

$$F_2(x) = \begin{cases} 0 & \text{si } x \leq 0 \\ 2x - x^2 & \text{si } 0 < x \leq 1 \\ 1 & \text{si } x > 1 \end{cases}$$

7-8

(7.6) $f(x,y) = \begin{cases} 0 & \text{si } x=y=0 \\ \dfrac{xy}{(x^2+y^2)^2} & \text{si } (x,y) \in [-1,1]^2 \setminus \{(0,0)\} \end{cases}$

$(y \neq 0)$ $\displaystyle\int_{[-1,1]} \underbrace{\frac{xy}{(x^2+y^2)^2}}_{\text{acotada, continua : Lebesgue = Riemann}} dm(x) = \int_{[-1,0]} \frac{xy}{(x^2+y^2)^2} dm(x) + \int_{[0,1]} \frac{xy}{(x^2+y^2)^2} dm(x)$

$= 0$

$y=0$, $f(x,0) \equiv 0 \Rightarrow \displaystyle\int_{[-1,1]} 0 \, dm(x) = 0$

Del mismo modo, $\displaystyle\int_{[-1,1]} \frac{xy}{(x^2+y^2)^2} dm(y) = 0$.

Pero $\displaystyle\int_{[-1,1]^2} \frac{|xy|}{(x^2+y^2)^2} dm^2 = 4 \int_{[0,1]^2} \frac{xy \, dm^2}{(x^2+y^2)^2} \geq$

$\displaystyle\geq 4 \int_{B(\vec{0},1) \cap [0,1]^2} \frac{xy}{(x^2+y^2)^2} dm^2(x,y) \underset{\left\{ \begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array} \right\} : G(r,\theta)}{=}$

$\displaystyle= 4 \int_0^1 \left(\int_0^{\pi/2} \frac{r^2 \cos\theta \sin\theta}{r^4} \cdot r \, d\theta \right) dr = 4 \int_0^1 \frac{1}{r} \underbrace{\left[\frac{-\cos(2\theta)}{4} \right]_0^{\pi/2}}_{\frac{-\cos\pi}{4} + \frac{\cos 0}{4} = \frac{1}{2}} d\theta =$

$\cos\theta \sin\theta = \dfrac{\sin(2\theta)}{2}$

$= 4 \cdot \frac{1}{2} \cdot \underbrace{\left[\log r \right]_0^1} = \infty$

7.9

(7.5)

$$f(x,y)=\begin{cases}0 & \text{si } (x,y)=(0,0)\\ \dfrac{x^2-y^2}{(x^2+y^2)^2} & \text{si } (x,y)\neq(0,0)\end{cases} \qquad . \quad X=[0,1]$$

· Si $y=0$, $\displaystyle\int_{[0,1]} \frac{x^2}{x^4}\,dx = \left(-\frac{1}{x}\right)\Big]_0^1 = \infty$

· Si $y\neq 0$ $\displaystyle\int_{[0,1]}\frac{x^2-y^2}{(x^2+y^2)^2}\,dx = \int_X\left(\frac{x^2+y^2}{(x^2+y^2)^2} - \frac{2y^2}{(x^2+y^2)^2}\right)dx$

· $\displaystyle\int_X \frac{dx}{(x^2+y^2)} = \int_X \frac{dx}{\left[1+\left(\frac{x}{y}\right)^2\right]y^2} = \frac{1}{y}\arctan\left(\frac{x}{y}\right)\Big]_0^1 =$

$= y\arctan(1/y)$

· $\displaystyle\int_X \frac{dx}{(x^2+y^2)^2}$

$\displaystyle\int_X \frac{2x^2}{(x^2+y^2)^2}\,dx - \int_X \frac{1}{(x^2+y^2)}\,dx$

· $\displaystyle\int_X \frac{x^2}{(x^2+y^2)^2}\,dx$.

$$\frac{d}{dx}\left(\frac{x}{(x^2+y^2)}\right) = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{x^2+y^2}$$

$$\therefore \int_{[0,1]}\frac{x^2-y^2}{(x^2+y^2)^2}\,dx = \left(\frac{-x}{x^2+y^2}\right)\Big]_0^1 = \frac{-1}{1+y^2}$$

$$\int_{[0,1]}\frac{-dy}{1+y^2} = -\arctan y\Big|_0^1 = -\frac{\pi}{4}$$

7.10

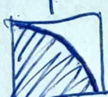
$$\rightarrow \boxed{\int_X \int_X f(x,y)\,dx\,dy = -\frac{\pi}{4}}$$

Como $-f(x,y) = f(y,x)$, se tiene

$$\boxed{\int_X \int_X f(x,y)\,dy\,dx = -\int_X \int_X f(y,x)\,dy\,dx = +\frac{\pi}{4}}$$

Para Fubini no se verifica la integrabilidad def.

$$\int_{X^2} |f|\,dm^2 \geq \int_0^1 \int_0^{\pi/2} \left(\frac{1}{r^3}\right) r^2 |\cos^2\theta - \sin^2\theta|\,d\theta\,dr \geq$$



$$\geq \int_0^1 \int_0^{\pi/4} (1/r) \underbrace{(\cos^2\theta - \sin^2\theta)}_{\cos 2\theta}\,d\theta\,dr = \infty .$$

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