(8.9) $F(x) = \begin{cases} 0 & \text{si } x<0 \\ x & \text{si } x\in[0,1) \\ 1 & \text{si } x\geq 1 \end{cases}$

a) $dF \ll m$

b) Encontrar la derivada de Radon-Nikodym de dF.

---

b) $dF([a,b)) = F(b)-F(a) \overset{(*)}{=} \int_a^b f(x)dx =: \nu_f([a,b))$

$(**)$ con $f(x) = \left.\begin{cases} 0 & \text{si } x\leq 0 \\ 1 & \text{si } 0<x<1 \\ 0 & \text{si } 1\leq x \end{cases}\right\} = \chi_{(0,1)}(x)$.

$(*)$ • Caso $(a<b<0)$ ó $(0<a<b<1)$ ó $(1<a<b)$

$\Rightarrow$ F derivable en (a,b) y $F'=f$ en (a,b), $f\in C(a,b)$

Se sigue por TFC.

• Caso $a<0<b<1$

$\Rightarrow F(b)-F(a) = b-0=b$

$\int_a^b f(x)dx = m(\chi_{(0,1)\cap(a,b)}) = m((0,b)) = b$

Casos $(0<a<1<b)$, $(a<0<1<b)$ } análogos

---

dF, ν_f son premedidas en $\mathcal{A} = \{$uniones finitas de intervalos $[a,b)$, $[a,\infty)$, $(-\infty,b)\}$

Como premedidas coinciden.

Extendemos dF (Caratheodory) a una medida dF en $\mathcal{B}_{\mathbb{R}}$.

. dF finita. $dF|_{\mathcal{A}} = \nu_f|_{\mathcal{A}}$.

con $\nu_f(E) = \int_E f\,dm$ en $E\in\mathcal{L}_{\mathbb{R}} = \overline{\mathcal{B}_{\mathbb{R}}}$

. Por unicidad de la extensión de Caratheodory, $dF \equiv \nu_f$ en $\mathcal{L}_{\mathbb{R}}$

a) $E\in\mathcal{L}_{\mathbb{R}}$ $m(E)=0 \Rightarrow dF(E) = \int_E f\,dm = 0$.

b) f es la derivada de Radon-Nikodym (es única salvo med. cero).

8.1

(c) $\mu(A) = \text{card}(A \cap \mathbb{Q} \cap [0,1])$. Probar $dF \perp m$.

$E = \mathbb{Q} \cap [0,1]$, $P = \mathbb{R} \setminus (\mathbb{Q} \cap [0,1])$. $E \cap P = \phi$, $E \cup P = \mathbb{R}$.

Con μ nula en P, dF nula en E.

(8.1) ν medi. con signo en $(X, \mathcal{A})$.

a) $\{E_j\}_{j \in \mathbb{N}} \subset \mathcal{A}$, $E_1 \subset E_2 \subset \cdots$.

~~Es claro que~~ $E_n = \bigcup_{k=1}^{n} E'_k$. Con $\begin{cases} E'_k = E_k \setminus \left(\bigcup_{j=1}^{k-1} E_j\right) \\ \forall k > 1. \\ E'_1 = E_1 \end{cases}$

$\nu(E_n) = \sum_{k=1}^{n} \nu(E'_k)$.

De hecho, $E = \bigcup E_j$. $\nu(E) = \sum_{j=1}^{\infty} \nu(E'_j) =$

$E = \bigcup E'_j$ ← disj. 2 a 2

$= \lim_{n \to \infty} \underbrace{\sum_{j=1}^{n} \nu(E'_j)}_{\nu(E_n)} = \lim_{n \to \infty} \nu(E_n)$.

---

b) $\{E_j\}_{\mathbb{N}} \subset \mathcal{A}$, $E_1 \supseteq E_2 \supseteq \cdots$, con $|\nu(E_1)| < \infty$

$E = \bigcap_{j=1}^{\infty} E_j$. $\nu(E_1) = \nu(E_1 \cap E) + \nu(E_1 \setminus E)$

$\nu(E_1 \setminus E) = \nu(E_1 \cap (\cap E_j)^c) = \nu\left(\bigcup (E_1 \cap E_j^c)\right) =$

$E_1 \supseteq E_2 \supseteq \cdots \Rightarrow E_1^c \subseteq E_2^c \subseteq \cdots$

$\Rightarrow E_1 \cap E_2^c \subseteq E_1 \cap E_3^c \subseteq \cdots$

$= \lim_{j \to \infty} \underbrace{\nu(E_1 \cap E_j^c)}_{\left(\nu(E_1) - \nu(\underbrace{E_1 \cap E_j}_{E_j})\right)}$ $= \nu(E_1) - \nu(E_1 \cap E)$

$\Rightarrow \boxed{\nu(E) = \lim \nu(E_j)}$.

8.2

(8.2) ν med. con signo sobre $(X, \mathcal{A})$. $E \in \mathcal{A}$ positivo $\Leftrightarrow$ ~~$\forall F \subset E, F \in \mathcal{A} \Rightarrow \nu(F)$~~

$\Rightarrow \{\forall G \subset E,\ G \in \mathcal{A} \Rightarrow G$ positivo$\}$ Es trivial

---

Unión numerable de conjuntos positivos es positivo:

$\{E_i\}_{i=1}^{\infty} \subset \mathcal{A}$ positivos. Sea $A \subset \bigcup E_i$, $A \in \mathcal{A}$

$\Rightarrow$ $\cancel{A = \bigcup_{i=1}^{\infty} (A \cap E_i)}$ $A = \bigcup_{i=1}^{\infty} G_i$

$G_1 = E_1 \cap A$, $G_k = (E_k \cap A) \setminus \bigcup_{i=1}^{k-1} G_i \quad \forall k \in \mathbb{N} \setminus \{1\}$

Son disjuntos dos a dos y $G_k \subset E_k$

$\Rightarrow \nu(G_k) \geq 0 \quad \forall k \in \mathbb{N} \setminus \{1\}$

$\Rightarrow \nu(A) = \sum_{i=1}^{\infty} \nu(G_i) \geq 0$ □

(8.5) $X = [0,1]$, $\mathcal{A} = \mathcal{B}_{[0,1]}$, m (med. de Lebesgue)

μ (med. de contar en $\mathcal{A}$)

(a) $m \ll \mu$: $[\mu(E) = 0 \Rightarrow E = \emptyset \Rightarrow m(E) = 0.$ □$]$

Pero $dm \neq f\,d\mu \ \forall f$. Sup. que existe:

$\forall x \in X \quad m(\{x\}) = 0 \Rightarrow \int_{\{x\}} f\,d\mu = f(x) = 0$

$\Rightarrow f \equiv 0 \Rightarrow m \equiv 0$ ✗

(b) μ no tiene descomp. de Lebesgue respecto de m.

Sup $\mu = \lambda + \rho$ con $\lambda \perp m$, $\rho \ll m$.

$\Rightarrow 1 = \mu(\{x_0\}) = \lambda(\{x_0\}) + \underbrace{\rho(\{x_0\})}_{=0} = \lambda(\{x_0\}) = 1 \quad \forall x_0 \in [0,1]$

$\therefore \lambda \equiv \mu$, $\rho \equiv 0 \Rightarrow \lambda = \mu \perp m$ contradicción

porque $m \ll \mu$ y no son $\equiv 0$.

~~Ana~~ 8.3

(8.6) $(X, \mathcal{A}, \mu)$ EdM ~~σ~~-finito. $\mathcal{B} \subset \mathcal{A}$ una ~~subálgebra~~ (sub σ-álgebra)

ν la restricción de μ a $\mathcal{B}$: $\nu = \mu|_{\mathcal{B}}$.

$f \in \mathcal{L}^1(\mu) \longrightarrow \exists\, g \in \mathcal{L}^1(\nu)$ tal que

$$\int_E f\,d\mu = \int_E g\,d\nu \quad \forall E \in \mathcal{B}.$$

Además, g única salvo conjuntos ν-nulos.

(A g se le llama esperanza ~~de f~~ condicionada de f con respecto a $\mathcal{B}$: $E(f|\mathcal{B})$)

---

dem: definimos en $(X, \mathcal{B})$ la medida $\rho(E) = \underbrace{\int_E f\,d\mu}_{= \int_E f\,d\nu}$.

$\rho \ll \nu$: si $E \in \mathcal{B}$, $\nu(E) = 0$,
entonces $\rho(E) = \int_E f\,d\nu = 0$.

ν, ρ finitas porque μ lo es.

Con σ-finitas no se asegura nada

$\overset{\text{L-R-N}}{\Longrightarrow}$ $\exists\, g \in \mathcal{L}^1(d\nu)$ / $\rho(E) = \int_E g\,d\nu$
y g única salvo ν-medida cero. □

medida σ-finita en $(X, \mathcal{A}) \not\Rightarrow \mu|_{\mathcal{B}}$ σ-finita en $(X, \mathcal{B} \subseteq \mathcal{A})$

{Contraejemplo: $\mathcal{B} = \{\emptyset, X\}$, $\mu(X) = \infty$.

Hoja nº

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(8.7) λ medida en $\mathbb{R}$ / $\underbrace{\exists \{x_i\}_{i=1}^{\infty} \;/\; \lambda(\mathbb{R} \setminus \overbrace{\{x_i\}_{i=1}^{\infty}}^{K}) = 0}$

i.e. λ tiene soporte en $\{x_i\}_{i=1}^{\infty}$ (diferentes dos a dos)

Demostrar que $\exists c_i \geq 0 \;/\; \lambda(E) = \sum c_i \chi_E(x_i) \;\; \forall E \in \mathcal{A}$.

Es decir, $\lambda = \sum c_i \delta_{x_i}$ con $\delta_{x_i}(E) = \begin{cases} 1 & \text{si } x_i \in E \\ 0 & \text{si } x_i \notin E. \end{cases}$

$\lambda(E) = \underbrace{\lambda(E \cap K)}_{\substack{0 \\ \text{porque } \lambda(K)=0}} + \sum_{i=1}^{\infty} \lambda(\underbrace{E \cap \{x_i\}}) =$

$E \cap \{x_i\} = \begin{cases} \{x_i\} & \text{si } x_i \in E \\ \emptyset & \text{si } x_i \notin E \end{cases}$

$= \sum_{i=1}^{\infty} \lambda(\{x_i\}) \delta_{x_i}$ $\qquad \lambda(E \cap \{x_i\}) = \lambda(\{x_i\}) \delta_{x_i}$

$\therefore c_i = \lambda(\{x_i\})$ $\square$

(8.3) a) $\nu \ll \mu$

$\Rightarrow$ Sea E con $\mu(E) = 0$

$\Rightarrow \begin{cases} \mu(E \cap P) = 0 \Rightarrow \nu^+(E) = \nu(E \cap P) = 0 \\ \mu(E \cap N) = 0 \Rightarrow \nu^-(E) = \nu(E \cap N) = 0 \end{cases}$

$\Rightarrow \nu^+, \nu^- \ll \mu$.

$\nu^+, \nu^- \ll \mu \Rightarrow \underset{\substack{\| \\ \nu^+ + \nu^-}}{|\nu|} \ll \mu$ (obvio)

$|\nu| \ll \mu \Rightarrow \nu^+ + \nu^- \ll \mu$

Sea $E \,/\, \mu(E) = 0 \Rightarrow \nu^+(E) + \nu^-(E) = 0$

medidas (≥ 0) $\Rightarrow \nu^+(E) = 0, \; \nu^-(E) = 0$

8.5 $\Rightarrow \nu(E) = (\nu^+ - \nu^-)(E) = 0$ $\square$

(b) $\nu \ll \mu$, $\nu \perp \mu$

. Entonces, sean $E, F \in \mathcal{A}$ / $E \cap F = \emptyset$, $E \cup F = X$,

F nulo para ν, E nulo para μ.

Sea $A \in \mathcal{A} \Rightarrow \nu(A) \overset{\downarrow}{=} \nu(A \cap F) = 0$.

Por $A \cap F \subset E \Rightarrow \mu(A \cap F) = 0$

$\Rightarrow \nu(A \cap F) = 0$

$\nu \ll \mu$

$\therefore \nu \equiv 0$

(8.4) $X = \mathbb{N}$. $\mathcal{A} = \mathcal{P}(\mathbb{N})$. ν med. de contar en $\mathbb{N}$.

$\mu(E) = \sum_{n \in E} (1/2^n)$

- $\nu \ll \mu$: $\mu(E) = 0 \Rightarrow E = \emptyset \Rightarrow \nu(E) = 0$.
- ~~Dado~~ ε $\exists \varepsilon > 0$ / $\forall \delta > 0$ $\exists E \in \mathcal{A}$: $\mu(E) < \delta$, $\nu(E) \geq \varepsilon$

$\varepsilon = 1$, Tomo n / $\delta > 1/2^n$

$E = \{1/2^n\}$ se tiene $\mu(E) = 1/2^n < \delta$,

$\nu(E) = 1 \geq \varepsilon$

(8.8) $F:\mathbb{R}\to\mathbb{R}$ abs. continua si

@ $\forall \varepsilon>0 \ \exists \delta>0 /$

$\{a_k\}_k, \{b_k\}_k \subset \mathbb{R}$ con $a_k<b_k \ \forall k$

$$\sum_{k\geq 1}|b_k-a_k|<\delta \Longrightarrow \sum_{k\geq 1}|F(b_k)-F(a_k)|<\varepsilon$$

Probar: F abs. continua y creciente, entonces la medida de L-S dF es $dF \ll m$ en $\mathbb{R}$

Sup $E\subset \mathcal{L}$, $m(E)=0$, ~~dF(E)...~~

$$dF(E)=\inf\left\{\sum_{j=1}^{\infty} dF((a_j,b_j]) \,\middle|\, \underbrace{\{(a_j,b_j]\}}_{a_j<b_j}\subset \mathcal{A},\ E\subset \bigcup (a_j,b_j]\right\}$$

Dado $\varepsilon>0$ Toma su δ correspondiente

~~Entonces tomo $\{(a_j,b_j]\}_{j\in\mathbb{N}} / \sum [F(b_j)-F(a_j)]$~~

$$m(E)=\inf\left\{\sum_{j=1}^{\infty} m((a_j,b_j]) \,\middle|\, \begin{array}{l} a_j<b_j\in\mathbb{R}\ \forall j\in\mathbb{N}, \\ E\subset \bigcup (a_j,b_j] \end{array}\right\}=0$$

Tomo $\{(a_j,b_j]\}_{j\in\mathbb{N}}\subset \mathcal{A}$ tales que

$$\sum_{j\in\mathbb{N}} m((a_j,b_j])<\delta$$

$$\Rightarrow \sum_{j\in\mathbb{N}} dF((a_j,b_j])<\varepsilon$$

Podemos encontrar $\{(a_j,b_j]\} / \sum dF((a_j,b_j])<\varepsilon$ $\forall\varepsilon>0$ $\therefore$ $dF(E)=0$ □

8.6

(8.10) $(X, \mathcal{A})$ espacio medible

ν. medida con signo σ-finita en $\mathcal{A}$

λ, μ. medidas σ-finitas en $\mathcal{A}$

con $\nu << \mu$, $\mu << \lambda$

(a) $g \in L^1(X, \mathcal{A}, \nu) \Rightarrow g \left(\frac{d\nu}{d\mu}\right) \in L^1(X, \mathcal{A}, \mu)$

y $\int_X g \, d\nu = \int_X g \left(\frac{d\nu}{d\mu}\right) d\mu$.

$\nu = \sigma + \nu$ con $\sigma \perp \mu$, $\nu << \mu$, $\nu(E) = \int_E \left(\frac{d\nu}{d\mu}\right) d\mu$

$\cancel{\theta = \alpha + \beta \text{ con } \alpha \perp \mu, \ \beta << \mu}$

$\cancel{\beta(E) = \int_E \left(\frac{d\nu}{d\mu}\right) d\mu}$

$$\int_X \left| \left(g \frac{d\nu}{d\mu} \right) \right| d\mu$$

$$\int_X |g| \, d\nu = \underbrace{\int_P |g| \, d\nu^+}_{\hat{\infty}} - \underbrace{\int_N |g| \, d\nu^-}_{\hat{\infty}}$$

$$\int_X g \, d\nu = \int_X g \left(\frac{d\nu}{d\mu}\right) d\mu$$

Sup $\nu \geq 0$: • $g = \chi_E$, $E \in \mathcal{A} \Rightarrow$ cierto

• g simple $\Rightarrow$ cierto por linealidad de $\int \cdot \, d\nu$

• g medible $\rightarrow$ cierto por TCM (primero $g \geq 0$, luego $g^+, g^- \rightarrow g$)

Caso $\nu \neq 0$ se hace:

$$P = \left\{ \frac{d\nu}{d\mu} \geq 0 \right\},\quad N = \left\{ \frac{d\nu}{d\mu} < 0 \right\}$$

$$\int_X g\, d\nu = \int_P g\, d\nu^+ - \int_N g\, d\nu^- =$$

$$= \int_P g \frac{d\nu}{d\mu}\, d\mu - \int_N g\left(-\frac{d\nu}{d\mu}\right) d\mu$$

$$= \int_X \left(g \frac{d\nu}{d\mu}\right) d\mu .$$

Para ver $g\frac{d\nu}{d\mu}$ ~~med~~ integrable.

Sup $\nu \geq 0$ (medida)

- g simple: $\int_X \left| g \frac{d\nu}{d\mu} \right| d\mu = \int_X |g| \frac{d\nu}{d\mu}\, ~~d\mu~~$

$$= \int_X \sum_{i=1}^m |c_i| \chi_{C_i} \frac{d\nu}{d\mu}\, d\mu = \sum_1^m |c_i| \int_{C_i} \frac{d\nu}{d\mu} d\mu$$

$$= \sum_1^m |c_i|\, \nu(C_i) = \int_X |g|\, d\nu$$

- g no simple: s_n simples $0 \leq s_1 \leq s_2 \leq \ldots$ con $\nearrow s_n \longrightarrow |g|$ entonces: $s_n \frac{d\nu}{d\mu} \nearrow |g| \frac{d\nu}{d\mu}$ constr...

$$\int_X |g| \frac{d\nu}{d\mu}\, d\mu \overset{TCM}{=} \lim_{n\to\infty} \int_X s_n \frac{d\nu}{d\mu}\, d\mu \overset{\ominus}{=} \lim \int_X s_n\, d\nu \overset{TCM}{=} \int_X |g|\, d\nu < \infty$$

8.7

ν general (con signo)

$\Rightarrow \nu = \nu^+ - \nu^-$ con P, N descomp. de Hahn.
$\hookrightarrow P = \{\frac{d\nu}{d\mu} \geq 0\}, N = \{\frac{d\nu}{d\mu} < 0\}$

$$\int_X \left|g \frac{d\nu}{d\mu}\right| d\mu = \int_P |g| \frac{d\nu}{d\mu} d\mu - \int_N |g| \left(-\frac{d\nu}{d\mu}\right) d\mu =$$

$$\nu^+(E) = \int_{E\cap P} d\nu = \nu(E\cap P) = \int_{E\cap P} \underbrace{\left(\frac{d\nu}{d\mu}\right)}_{=\frac{d\nu^+}{d\mu}} d\mu$$ (restringida a P)

$$\nu^-(E) = \int_{E\cap N} \underbrace{\left(-\frac{d\nu}{d\mu}\right)}_{=\frac{d\nu^-}{d\mu}} d\mu$$ (restringida a N.)

$$= \int_P g \, d\nu^+ - \int_N g \, d\nu^- = \int_X g \, d\nu!$$

b) $\underline{\nu << \lambda}$ y $\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu} \frac{d\mu}{d\lambda}$ c.t.p. λ
q

Como: $\lambda(E) = 0 \overset{\mu << \lambda}{\Rightarrow} \mu(E) = 0 \overset{\nu << \mu}{\Rightarrow} \nu(E) = 0$

$$\nu(E) = \int_X \left(\frac{d\nu}{d\lambda}\right) d\lambda$$ porque $\nu = \underset{\perp\lambda}{0} + \underset{<<\lambda}{\nu}$ (descomp. de Lebesgue)

$$\mu(E) = \int_X \left(\frac{d\mu}{d\lambda}\right) d\lambda$$ por $\mu << \lambda$

Por (a), como $\frac{d\nu}{d\mu}$ ~~es~~ ~~μ integrable~~ μ-medible,

$$\int_X \left(\frac{d\nu}{d\mu}\right) d\mu =$$

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$X = \bigcup_{i=1}^{\infty} X_i$ con $\mu(X_i) < \infty$.

Como $\frac{d\vartheta}{d\mu}$ μ-integrable en $E \subset X_i$,

$$\vartheta(E) = \int_E \left(\frac{d\vartheta}{d\mu}\right) d\mu \overset{(a)}{=} \int_E \left(\frac{d\vartheta}{d\mu}\right)\left(\frac{d\mu}{d\lambda}\right) d\lambda$$

Si $E \in \mathcal{A}$, $E = \biguplus_{i=1}^{\infty} E_i$. $E_1 = E \cap X_1$,

$E_k = E \cap X_k \setminus \bigcup_{i=1}^{k-1} E_i$

$$\vartheta(E) = \sum_{i=1}^{\infty} \vartheta(E_i) = \sum_{i=1}^{\infty} \int_{E_i} \left(\frac{d\vartheta}{d\mu}\right)\left(\frac{d\mu}{d\lambda}\right) d\lambda =$$

$= \ldots ?$

TCM, $\Rightarrow$

198. Quizás lo hace para ϑ finita

$\mathbb{1}_E \frac{d\vartheta}{d\mu}$ μ-integrable

$$\Rightarrow \vartheta(E) = \int_E \frac{d\vartheta}{d\mu} d\mu \overset{(a)}{=} \int_E \left(\frac{d\vartheta}{d\mu}\frac{d\mu}{d\lambda}\right) d\lambda$$

$\square$ por T-L-R-N, unicidad salvo λ-medida cero de la derivada.

9.8