Ejercicio: sea $p, q \in \mathbb{R}$ y $A = \begin{pmatrix} 1 & p & 0 \\ 1 & 1 & q \\ 0 & 1 & 1 \end{pmatrix}$

estudiar la convergencia de las iteraciones de Jacobi y de Gauss-Seidel

JACOBI: $B_J(A) = -D_A^{-1}(L_A + U_A) = -\begin{pmatrix} 0 & p & 0 \\ 1 & 0 & q \\ 0 & 1 & 0 \end{pmatrix}$

$$0 = \det \underbrace{\begin{pmatrix} \lambda & p & 0 \\ 1 & \lambda & q \\ 0 & 1 & \lambda \end{pmatrix}}_{= -B_J(A) + \lambda I} = \lambda(\lambda^2 - q) - \lambda p = \lambda^3 - \lambda(p+q)$$

$\lambda = 0$

$\lambda^2 = p+q$

$\Rightarrow \rho(B_J(A)) = \sqrt{|p+q|}$ : J converge $\Leftrightarrow |p+q| < 1$

GAUSS-SEIDEL: $B_{GS}(A) = -(D_A + L_A)^{-1} U_A$

$$= -\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 & p & 0 \\ 0 & 0 & q \\ 0 & 0 & 0 \end{pmatrix} = -\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 0 & p & 0 \\ 0 & 0 & q \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -p & 0 \\ 0 & p & -q \\ 0 & -p & q \end{pmatrix}$$

$$0 = \det \begin{pmatrix} -\lambda & -p & 0 \\ 0 & p-\lambda & -q \\ 0 & -p & q-\lambda \end{pmatrix} = -\lambda \det \begin{pmatrix} p-\lambda & -q \\ -p & q-\lambda \end{pmatrix} = -\lambda((p-\lambda)(q-\lambda) - pq)$$

$$= -\lambda(\lambda^2 - \lambda(p+q)) = -\lambda^2(\lambda - (p+q))$$

$\lambda = 0$

$\lambda = p+q$

$\Rightarrow \rho(B_{GS}(A)) = |p+q|$ : G-S converge $\Leftrightarrow |p+q| < 1$

observación: $\rho(B_{GS}(A)) = \rho(B_J(A))^2$

$\hookrightarrow$ G-S converge 2 veces más rápido

(si $r = \sqrt{|p+q|}$ $\quad \|x_k - \underline{x}\| \lesssim r^k \|x_0 - \underline{x}\|$ Jacobi

$\|x_k - \underline{x}\| \lesssim r^{2k} \|x_0 - \underline{x}\|$ G-S )

**Ejemplo:** (de que no siempre G-S es mejor que J)

$$A = \begin{pmatrix} 1 & 1 & 2 \\ \frac{5+\sqrt{21}}{2} & 1 & 1 \\ -\frac{5}{2} & \frac{2}{5+\sqrt{21}} & 1 \end{pmatrix}$$

usamos $\beta = \frac{5+\sqrt{21}}{2}$ $\left(\beta + \frac{1}{\beta} = 5\right)$

$$B_J(A) = -D_A^{-1}(L_A + U_A) = -\begin{pmatrix} 0 & 1 & 2 \\ \beta & 0 & 1 \\ -\frac{5}{2} & \frac{1}{\beta} & 0 \end{pmatrix}$$

$$0 = \det\begin{pmatrix} \lambda & 1 & 2 \\ \beta & \lambda & 1 \\ -\frac{5}{2} & \frac{1}{\beta} & \lambda \end{pmatrix} = \lambda\left(\lambda^2 - \frac{1}{\beta}\right) - \left(\beta\lambda + \frac{5}{2}\right) + 2\left(1 + \frac{5}{2}\lambda\right)$$

$$= \lambda^3 - \left(\beta + \frac{1}{\beta}\right)\lambda - \frac{5}{2} + 2 + 5\lambda = \lambda^3 - \frac{1}{2}$$

$$\Rightarrow \rho(B_J(A)) = \left(\frac{1}{2}\right)^{1/3} \simeq 0.79 < 1$$

$$B_{GS}(A) = -(D_A + L_A)^{-1} U_A = -\begin{pmatrix} 1 & 0 & 0 \\ \beta & 1 & 0 \\ -5/2 & 1/\beta & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= -\begin{pmatrix} 1 & 0 & 0 \\ -\beta & 1 & 0 \\ 7/2 & -\frac{1}{\beta} & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = -\begin{pmatrix} 0 & 1 & 2 \\ 0 & -\beta & 1-2\beta \\ 0 & 7/2 & 7 - 1/\beta \end{pmatrix}$$

$$0 = \det\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda - \beta & 1-2\beta \\ 0 & 7/2 & \lambda - \frac{1}{\beta} + 7 \end{pmatrix} = \lambda\left((\lambda-\beta)\left(\lambda - \frac{1}{\beta} + 7\right) - \frac{7}{2}(1-2\beta)\right)$$

$$= \lambda\left(\lambda^2 + 2\lambda - 5/2\right) \qquad \lambda = 0, \quad \lambda = -1 + \sqrt{1 + \frac{5}{2}}, \quad \lambda = -1 - \sqrt{1 + \frac{5}{2}}$$

$$\Rightarrow \rho(B_{GS}(A)) = 1 + \sqrt{1 + \frac{5}{2}} > 1$$

$\hookrightarrow$ para esta matriz J converge y G-S no converge

**proposición:** sea $A \in \mathbb{K}^{n\times n}$

si $\sum_{j=1, j\neq i}^{n} |a_{ij}| < |a_{ii}| \quad \forall i \in \{1..n\}$

MATRIZ DIAGONAL DOMINANTE ESTRICTA POR FILAS

$\Rightarrow$ la iteración de Jacobi es convergente.

**demostración:**

$$B_J(A) = -D_A^{-1}(L_A + U_A)$$

$$(B_J(A))_{ij} = -\sum_{k=1}^{n} (D_A^{-1})_{ik} (L_A + U_A)_{kj}$$

$$= -\sum_{k=1}^{n} \frac{1}{a_{ii}} \delta_{i,k} \, a_{kj} (1 - \delta_{kj}) \qquad \delta_{ik} = \begin{cases} 1, i=k \\ 0, i\neq k \end{cases}$$

$$= \frac{a_{ij}}{a_{ii}} (\delta_{ij} - 1)$$

$$|||B_J(A)|||_\infty = \max_{i\in\{1..n\}} \sum_{j=1}^{n} |B_J(A)_{ij}|$$

$$= \max_{i\in\{1..n\}} \frac{1}{|a_{ii}|} \sum_{j=1, j\neq i}^{n} |a_{ij}| < 1$$

$\sum_{j=1, j\neq i}^{n} |a_{ij}| < |a_{ii}| \quad \forall i \in \{1..n\}$

$$\Rightarrow \rho(B_J(A)) \leq |||B_J(A)|||_\infty < 1 \qquad \#$$

**observación:** verificar si $A$ es diagonal dominante (estricta) requiere $O(n^2)$ operaciones