3. Worst, best and average cases for basic alg.

23) $S_{LSearch} = \{ \underbrace{1, \dots, N}_{(K \in T)}, \underbrace{T[i] < K < T[i+1]}_{①}, \underbrace{K < T[1]}_{②}, \underbrace{K > T[N]}_{③} \} \Rightarrow |S_{LSearch}| = 2N+1$

↳ cost i ↳ for every $i \in \{1, \dots, N-1\}$

① i, ② 1, ③ N

$$\Rightarrow A_{LSearch} = \left(\sum_{i=1}^{N} i + \sum_{i=1}^{N-1} i + 1 + N \right) \overbrace{\frac{1}{2N+1}}^{\text{Probability of every case}}$$

$$= \left(\frac{N(N+1)}{2} + \frac{N(N-1)}{2} + N+1 \right) \frac{1}{2N+1}$$

$$= \left(\frac{N^2+N+N^2-N+2N+2}{2} \right) \frac{1}{2N+1} = \frac{2N^2+2N+2}{2(2N+1)}$$

$$= \frac{N^2+N+1}{2N+1}.$$

24) $P(K = T[i]) = \frac{1}{C_N} \frac{\log(i)}{i}$, $C_N = \sum_1^N \frac{\log i}{i}$

(We do not consider K not being in T) $\Rightarrow \sum_{i=1}^{N} P(K = T[i]) = \frac{\sum_1^N \frac{\log i}{i}}{C_N} = 1$.

If $K = T[i] \Rightarrow n_{LSearch}(K) = i$

$$\rightarrow A_{LSearch}(N) = \sum_{i=1}^{N} n_{LSearch}(K = T[i]) \cdot p(K = T[i])$$

$$= \sum_{i=1}^{N} i \frac{\log i}{i} \cdot \frac{1}{C_N} = \frac{1}{C_N} \sum_{i=1}^{N} \log i$$

C_N is decreasing and $C_N = \sum_{i=1}^{N} \frac{\log i}{i} = \sum_{i=2}^{N} \frac{\log i}{i}$

$$\Rightarrow \int_1^N \frac{\log x}{x} dx \geq C_N \geq \int_2^{N+1} \frac{\log x}{x} dx. \quad \int \frac{\log x}{x} dx = \underset{\uparrow}{\int} y\, dy = \log^2 x \cdot \frac{1}{2} \ (+C)$$

$[y = \log x]$

$$\Rightarrow \frac{\log^2 x}{2}\Bigg|_1^N \geq C_N \geq \frac{\log^2 x}{2}\Bigg|_2^{N+1}$$

$$\Rightarrow \frac{\log^2 N}{2} \geq C_N \geq \frac{\log^2(N+1) - \log^2(2)}{2}$$

$$\Rightarrow 1 \geq \frac{C_N}{\left(\frac{\log^2 N}{2}\right)} \geq \frac{\log^2(N+1) - \log^2(2)}{\log^2(N)} \xrightarrow[N\to\infty]{} 1$$

$$1 \xrightarrow{N\to\infty} 1$$

$$\Rightarrow \lim_{N\to\infty} \frac{C_N}{\left(\frac{\log^2 N}{2}\right)} = 1 \Rightarrow C_N \sim \frac{\log^2 N}{2}$$

$$\Rightarrow A_{LSearch}(N) = \frac{1}{C_N} \sum_{i=1}^{N} \log i \approx \frac{1}{C_N} N \log N \approx \frac{N \log N}{\log^2 N} \cdot 2$$

$$\approx \frac{2N}{\log N}$$

27) $\sum_1^N i^2 = \frac{N(N+1)(2N+1)}{6}$. For $N=1$: ✓ $\frac{1(2)(3)}{6} = 1$

Assume this is true $\forall k \leq N$

$$\sum_1^{N+1} i^2 = \sum_1^N i^2 + (N+1)^2 = \frac{N(N+1)(2N+1)}{6} + (N+1)^2$$

$$= \frac{1}{6}((N^2+N)(2N+1) + 6N^2 + 12N + 6) = \frac{1}{6}(2N^3 + 2N^2 + N^2 + N + 6N^2 + 12N + 6)$$

$$= \frac{1}{6}(2N^3 + 9N^2 + 13N + 6) = \frac{1}{6}(N+1)(N+2)(2N+3) \quad \square$$

$((N+1)+1)$ $(2(N+1)+1)$

$\Rightarrow$ Also true for $K = N+1$.

Escaneado con CamScanner

27, b) $\sum_1^N i^3 = \left(\frac{N(N+1)}{2}\right)^2$. True for 1: $\left(\frac{1(2)}{2}\right)^2 = 1$ ✓.

Assume it is true ~~for~~ $\forall k \leq N$,

$$\sum_1^{N+1} i^3 = \sum_1^N i^3 + (N+1)^3 = \frac{(N^2+N)^2}{4} + N^3+3N^2+3N+1$$

$$= \frac{1}{4}\left(N^4+2N^3+N^2+4N^3+12N^2+12N+4\right)$$

$$= \frac{1}{4}\left(N^4+6N^3+13N^2+12N+4\right) = \frac{1}{4}(N+1)^2(N+2)^2 \quad \square$$

28) $$\sum_{n=1}^N n x^n = x \sum_{n=1}^N n x^{n-1} = x \frac{d}{dx}\left(\sum_{n=1}^N x^{n-1}\right)$$

$$= x\frac{d}{dx}\left(\sum_{n=0}^{N-1} x^n\right) = x\frac{d}{dx}\left(\frac{x^N-1}{x-1}\right)$$

① if $x=1 \Rightarrow \sum_{n=1}^N n x^n = \sum_{n=1}^N n = \frac{N(N+1)}{2}$.

② if $x \neq 1 \Rightarrow \sum_{n=1}^N n x^n = x\left(\frac{(Nx^{N-1})(x-1)-(x^N-1)}{(x-1)^2}\right) =$

$$= x\frac{(Nx^N - Nx^{N-1})-x^N+1}{(x-1)^2} = \frac{x\left((N-1)x^N - Nx^{N-1}+1\right)}{(x-1)^2}$$

$$= \frac{(N-1)x^{N+1} - Nx^N + x}{(x-1)^2}.$$

$$\sum_{n=1}^N n^2 x^n \quad \Big((n-1)^3 = n^3-3n^2+3n-1,$$

$$\sum_{n=1}^N \left(n^3-(n-1)^3\right)x^n = \sum_{n=1}^N n^3 x^n - \sum_{n=1}^N (n-1)^3 x^n =$$

$$= \sum_{n=1}^N n^3 x^n - \sum_{n=0}^{N-1} n^3 x^{n+1} \Big)$$

$$\int \underbrace{y^2}_{v} \underbrace{x^y dy}_{du} = y^2 \frac{x^y}{\log x} - \int 2y\frac{x^y}{\log x}dy = y^2\frac{x^y}{\log x} - \frac{2}{\log x}\left(y\frac{x^y}{\log x} - \overbrace{\int \frac{x^y}{\log x}dy}^{\frac{x^y}{\log^2 x}}\right)$$

$$\left(\begin{matrix} x^y = du \\ u = \frac{x^y}{\log x} \end{matrix} \rightarrow \frac{d}{dy}\left(\frac{x^y}{\log x}\right) = \frac{d}{dy}\left(\frac{e^{y\log x}}{\log x}\right) = \frac{(e^{y \log x})}{\log x}\log x = x^y \right)$$

$$= y^2 \frac{x^y}{\log x} - \frac{2y\, x^y}{\log^2 x} + \frac{2x^y}{\log^3 x} = f(y) \circledast$$

$$\Rightarrow S_N = \sum_{n=1}^{N} n^2 x^n \quad \text{~~cret~~ increasing}$$

$$\Rightarrow \int_0^N y^2 x^y \, dy \leq S_N \leq \int_1^{N+1} y^2 x^y \, dy$$

$$\circledast \Rightarrow f(y)\Big|_0^N \leq S_N \leq f(y)\Big|_1^{N+1}$$

$$\Rightarrow \left(x^N \left(\frac{N^2}{\log x} - \frac{2N}{\log^2 x} + \frac{2}{\log^3 x} \right) - \overbrace{\frac{2}{\log^3 x}}^{B} \right) \leq S_N \leq$$

$$\leq x^{N+1} \left(\frac{(N+1)^2}{\log x} - \frac{2(N+1)}{\log^2 x} + \frac{2}{\log^3 x} \right) - x \underbrace{\left(\frac{1}{\log x} - \frac{2}{\log^2 x} + \frac{2}{\log^3 x} \right)}_{A}$$

$$1 \leq \frac{S_N}{x^N \left(\frac{N^2}{\log x} - \frac{2N}{\log^2 x} + \frac{2}{\log^3 x} \right) - B}$$

$$\leq \frac{x \left(x^N \left(\frac{(N+1)^2}{\log x} - \frac{2(N+1)}{\log^2 x} + \frac{2}{\log^3 x} \right) - A \right)}{x^N \left(\frac{N^2}{\log x} - \frac{2N}{\log^2 x} + \frac{2}{\log^3 x} \right) - B} \longrightarrow x \neq 1.$$