EJEMPLO C Estudia la curva $2x^2-12xy-7y^2-16x-2y-3=0$, reduciéndola a su forma canónica, indicando el cambio de sistema de referencia y sus elementos geométricos

S/ $A=\begin{pmatrix}2 & -6\\ -6 & -7\end{pmatrix}$; $|A| = -14-36=-50$; Tipo hiperbólico

Autovalores

$$\begin{vmatrix}2-\lambda & -6\\ -6 & -7-\lambda\end{vmatrix} = (2-\lambda)(-7-\lambda)-36 = \lambda^2+5\lambda-50=0$$

$$\boxed{\lambda_1=5} \ , \ \boxed{\lambda_2=-10}$$

Autovectores:

$\boxed{\lambda_1=5}$ $\begin{pmatrix}-3 & -6\\ -6 & -12\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix} \Leftrightarrow -x-2y=0$; $u_1=\frac{1}{\sqrt5}\begin{pmatrix}2\\ -1\end{pmatrix}$

$\boxed{\lambda_2=-10}$ $\begin{pmatrix}12 & -6\\ -6 & 3\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix} \Leftrightarrow -6x+3y=0$; $\vec{u}_2=\frac{1}{\sqrt5}\begin{pmatrix}1\\ 2\end{pmatrix}$

Cambio de base:

$$\begin{pmatrix}x\\ y\end{pmatrix}=\frac{1}{\sqrt5}\begin{pmatrix}2 & 1\\ -1 & 2\end{pmatrix}\begin{pmatrix}x_1\\ y_1\end{pmatrix} \Leftrightarrow \left\{\begin{array}{l} x=\frac{1}{\sqrt5}(2x_1+y_1)\\ y=\frac{1}{\sqrt5}(-x_1+2y_1)\end{array}\right\} \quad (1)$$

Sustituimos en la curva:

$$5x_1^2-10y_1^2-16\left(\frac{1}{\sqrt5}(2x_1+y_1)\right)-\frac{2}{\sqrt5}(-x_1+2y_1)-3=0$$

$$5x_1^2-10y_1^2-\frac{30}{\sqrt5}x_1-\frac{20}{\sqrt5}y_1-3=0$$

$$5\left(x_1-\frac{3}{\sqrt5}\right)^2-10\left(y_1+\frac{1}{\sqrt5}\right)^2-9+2-3=0$$

$$5\left(x_1-\frac{3}{\sqrt5}\right)^2-10\left(y_1+\frac{1}{\sqrt5}\right)^2=10$$

Treslación

$$\left\{\begin{array}{l} x_2 = x_1 - \frac{3}{\sqrt{5}} \\ y_2 = y_1 + \frac{1}{\sqrt{5}} \end{array}\right\} \qquad (2)$$

$$\boxed{\frac{x_2^2}{2} - \frac{y_2^2}{1} = 1}$$

Forma canónica

~~Elipse~~ Hipérbola

$a = \sqrt{2}$, $b = 1$, $c = \sqrt{2+1} = \sqrt{3}$

El cambio de sistema de referencia está dado por (1) y (2):

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x_2 + 3/\sqrt{5} \\ y_2 - 1/\sqrt{5} \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} + \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 3/\sqrt{5} \\ -1/\sqrt{5} \end{pmatrix}$$

$$= \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} + \frac{1}{5} \begin{pmatrix} 5 \\ -5 \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad (3)$$

El centro de la hipérbola es $(x_2, y_2) = (0,0) \Rightarrow \boxed{\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}} = C$

Eje principal: $C + \langle \vec{u}_1 \rangle$ o bien $y_2 = 0$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x_2 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} 2x_2 \\ -x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{2}{\sqrt{5}} x_2 + 1 \\ -\frac{1}{\sqrt{5}} x_2 - 1 \end{pmatrix}$$

$$\left.\begin{array}{l} x - 1 = \frac{2}{\sqrt{2}} x_2 \\ y + 1 = -\frac{1}{\sqrt{2}} x_2 \end{array}\right\} \quad x - 1 = 2(-y-1); \quad \boxed{x + 2y + 1 = 0}$$

Eje secundario $C + \langle u_2 \rangle$, o bien $x_2 = 0$, o bien el perpendicular al eje principal que pasa por $C = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$:

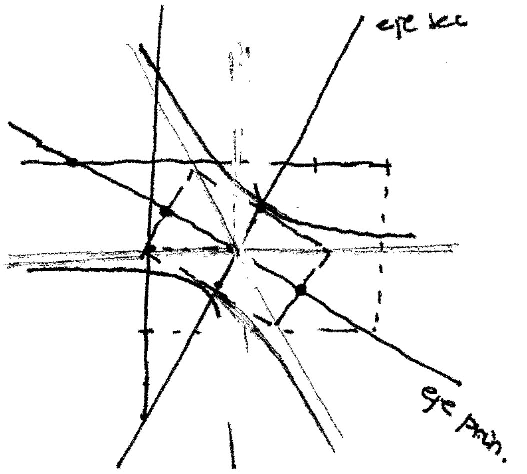
$$-2x + y = d \quad \text{con} \quad d = -2 - 1 = -3; \quad \boxed{-2x + y + 3 = 0}$$

Focos: $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} \sqrt{2} \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} \sqrt{2} \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} 2\sqrt{2} \\ -\sqrt{2} \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

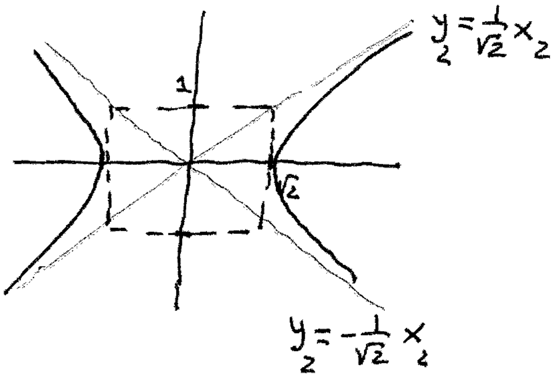
$$F_1 = \begin{pmatrix} \frac{2\sqrt{2}}{\sqrt{5}} + 1 \\ -\frac{\sqrt{2}}{\sqrt{5}} - 1 \end{pmatrix}$$

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{5}} + 1 \\ \frac{2}{\sqrt{5}} - 1 \end{pmatrix}.$$

Asíntotas



Forma canónica



$$\boxed{x_2 = \sqrt{2}\, y_2} \qquad \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} \sqrt{2}\, y_2 \\ y_2 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} 2\sqrt{2}+1 \\ -\sqrt{2}+2 \end{pmatrix} y_2 + \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} \left(\frac{2}{\sqrt{5}}\sqrt{2} + \frac{1}{\sqrt{5}}\right) y_2 + 1 \\ \left(-\frac{\sqrt{2}}{\sqrt{5}} + \frac{2}{\sqrt{5}}\right) y_2 - 1 \end{pmatrix}$$

$$y_2 = \frac{x-1}{\frac{2\sqrt{2}}{\sqrt{5}} + \frac{1}{\sqrt{5}}} = \frac{\sqrt{5}\,(x-1)}{2\sqrt{2}+1} \; ; \quad y_2 = \frac{y+1}{-\frac{\sqrt{2}}{\sqrt{5}} + \frac{2}{\sqrt{5}}} = \frac{\sqrt{5}\,(y+1)}{2-\sqrt{2}}$$

$$\boxed{\frac{x-1}{2\sqrt{2}+1} = \frac{y+1}{2-\sqrt{2}}} \; ;$$

$$\boxed{x_2 = -\sqrt{2}\, y_2} \qquad \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} -\sqrt{2}\, y_2 \\ y_2 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} -2\sqrt{2}\, y_2 + y_2 \\ -\sqrt{2}\, y_2 + 2y_2 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{\sqrt{5}}(-2\sqrt{2}+1)\, y_2 + 1 \\ \left(-\frac{\sqrt{2}}{\sqrt{5}} + \frac{2}{\sqrt{5}}\right) y_2 - 1 \end{pmatrix}$$

$$y_2 = \frac{x-1}{1-2\sqrt{2}}\sqrt{5} \; ; \quad y_2 = \frac{y+1}{2-\sqrt{2}}\sqrt{5}$$

$$\boxed{\frac{x-1}{1-2\sqrt{2}} = \frac{y+1}{2-\sqrt{2}}}$$