demostración (teorema de regularización de Tikhonov)

0. recordamos algunas notaciones:

- $x \in \mathbb{R}^n \Rightarrow x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ vector columna
- si $u : \mathbb{R}^n \to \mathbb{R}$, decimos $\nabla u(x) = \left(\frac{\partial u}{\partial x_1}(x), \dots, \frac{\partial u}{\partial x_m}(x) \right)$ vector fila
- si $F : \mathbb{R}^n \to \mathbb{R}^x$, $F(x) = \begin{pmatrix} F_1(x) \\ \vdots \\ F_m(x) \end{pmatrix}$

  decimos $(J_F(x))_{ij} = \dfrac{\partial F_i(x)}{\partial x_j}$, $J_F(x) = \begin{pmatrix} — \nabla F_1(x) — \\ \vdots \\ — \nabla F_m(x) — \end{pmatrix}$

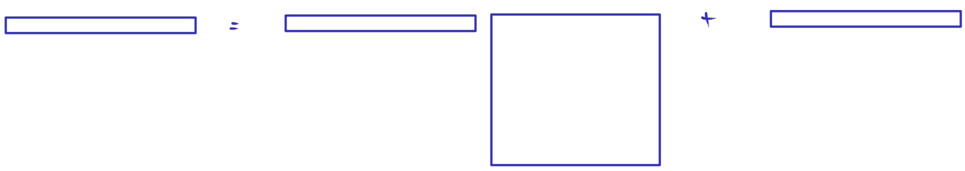
1. para encontrar $x_\alpha(b) = \underset{x \in \mathbb{R}^n}{\arg\min}\, f_\alpha(x)$

buscamos los puntos críticos de f_α : x t.q. $\nabla f_\alpha(x) = 0$

- $f_\alpha(x) = \|Ax - b\|^2 + \alpha \|x\|^2 = g(h(x)) + \alpha\, g(x)$
- $g(x) = \|x\|^2$, $h(x) = Ax - b$

  $g : \mathbb{R}^n \to \mathbb{R}$ $\qquad h : \mathbb{R}^n \to \mathbb{R}^n$

$\hookrightarrow \nabla f_\alpha(x) = \nabla g(h(x))\, J_h(x) + \alpha \nabla g(x)$ REGLA DE LA CADENA



- $\dfrac{\partial g}{\partial x_i}(x) = \dfrac{\partial}{\partial x_i} \sum_{k=1}^{m} x_k^2 = 2x_i \Rightarrow \boxed{\nabla g(x) = 2x^t}$

$\dfrac{\partial F_i}{\partial x_j}(x) = \dfrac{\partial}{\partial x_j} (Ax - b)_i = \dfrac{\partial}{\partial x_j} \sum_{k=1}^{m} A_{ik} x_k = A_{ij} \Rightarrow \boxed{J_h = A}$

$\hookrightarrow \nabla f_\alpha(x) = 2(Ax - b)^t A + \alpha\, 2x^t$

$\cdot\ \nabla f_\alpha(x) = 0 \iff \nabla f_\alpha(x)^t = 0$ ecuación traspuesta:

$$A^t(Ax-b) + \alpha x = 0$$

$\hookrightarrow\ \nabla f_\alpha(x) = 0 \iff (A^tA + \alpha I)\,x = A^t b$

2. sea $A = U\Sigma V^t$ una SVD de A

$\cdot\ A^tA + \alpha I = V(\Sigma^2 + \alpha I)V^t$: $\cdot\ A^tA = V\Sigma^2V^t$ $\cdot$ V unitaria

$\cdot\ V(\Sigma^2+\alpha I)V^t x = A^t b$

$\iff (\Sigma^2 + \alpha I)V^t x = V^t A^t b$

$\iff V^t x = (\Sigma^2+\alpha I)^{-1} V^t A^t b$

$\downarrow$

$V^tA^t = (AV)^t = (U\Sigma)^t = \Sigma U^t$

$\downarrow$

$\iff V^t x = (\Sigma^2+\alpha I)^{-1}\Sigma U^t b$

$\cdot\ (\Sigma^2+\alpha I)^{-1}\Sigma$ es la matriz diagonal cuyos elementos diagonales son $\dfrac{\sigma_k}{\sigma_k^2+\alpha}$: EJERCICIO

$\cdot$ escribiendo el sistema de ecuaciones obtenido componente por componente, observando que

$$V^t x = \begin{pmatrix} - V^{(1)} - \\ \vdots \\ - V^{(n)} - \end{pmatrix}\begin{pmatrix} | \\ x \\ | \end{pmatrix} = \begin{pmatrix} \langle x, V^{(1)}\rangle \\ \vdots \\ \langle x, V^{(n)}\rangle \end{pmatrix}$$

$$U^t b = \begin{pmatrix} - U^{(1)} - \\ \vdots \\ - U^{(n)} - \end{pmatrix}\begin{pmatrix} | \\ b \\ | \end{pmatrix} = \begin{pmatrix} \langle b, U^{(1)}\rangle \\ \vdots \\ \langle b, U^{(n)}\rangle \end{pmatrix}$$

tenemos $\langle x, V^{(k)}\rangle = \dfrac{\sigma_k}{\sigma_k^2+\alpha}\langle b, U^{(k)}\rangle$, $k = 1..n$

$\cdot$ como $\{V^{(k)}\}_{k=1}^n$ es una BON de $\mathbb{R}^n$

$$\Rightarrow x = \sum_{k=1}^{n}\langle x, V^{(k)}\rangle V^{(k)} = \sum_{k=1}^{n}\frac{\sigma_k}{\sigma_k^2+\alpha}\langle b, U^{(k)}\rangle V^{(k)}$$

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