11.2 $F:\mathbb{R}^3 \to \mathbb{R}^3$, $F(\vec{x}) = M\vec{x}$, $M = \begin{pmatrix} 2 & 0 & 5 \\ -1 & 3 & 5 \\ -2 & 1 & -2 \end{pmatrix}$

(a) λ autovalor $\leftrightarrow$ $\begin{vmatrix} 2-\lambda & 0 & 5 \\ -1 & 3-\lambda & 5 \\ -2 & 1 & -2-\lambda \end{vmatrix} = \begin{vmatrix} 2-\lambda & 0 & 5 \\ -3+\lambda & 3-\lambda & 0 \\ -2 & 1 & -2-\lambda \end{vmatrix} = \begin{vmatrix} 3-\lambda & -3+\lambda & 0 \\ -1 & 3-\lambda & 5 \\ -2 & 1 & -2-\lambda \end{vmatrix} = \begin{vmatrix} 3-\lambda & 0 & 0 \\ -1 & 2-\lambda & 5 \\ -2 & -1 & -2-\lambda \end{vmatrix} = 0$

$\leftrightarrow (3-\lambda)(-4+\lambda^2+5) = 0 \leftrightarrow (3-\lambda)(\lambda^2+1) = 0$

$\leftrightarrow \lambda = 3, \ \lambda = \pm i$ ← Autovalores ($\lambda = 3$ es el único real).

(b) Tomando $\lambda = -i$: $\ker(M+iI) = E_1(-i)$

$\rightarrow \left(\begin{array}{ccc|c} 2+i & 0 & 5 & 0 \\ -1 & 3+i & 5 & 0 \\ -2 & 1 & -2+i & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & -1 & -2+i & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} (-2+i)\gamma \\ (-2+i)\gamma \\ \gamma \end{pmatrix} = \begin{pmatrix} 5 \\ 5 \\ -2-i \end{pmatrix}\gamma$

$(i-2)\cdot c = 5 \rightarrow c = \frac{5}{i-2}\cdot\frac{i+2}{i+2} = \frac{5i+10}{-5} = \cdots = -i$

Autovector de la forma $\begin{pmatrix} 5 \\ \alpha \\ \beta \end{pmatrix} \longrightarrow \begin{pmatrix} 5 \\ +5 \\ -2-i \end{pmatrix}$ (*) ; $-5, -6-3i \in \mathbb{C}$

(c) Tomando $\lambda = 3$, $E_1(3) = \ker(M-3I)$: $\left(\begin{array}{ccc|c} -1 & 0 & 5 & 0 \\ -1 & 0 & 5 & 0 \\ -2 & 1 & -5 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & -5 & 0 \\ 0 & 1 & -15 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

Autovector: $\begin{pmatrix} 5 \\ 15 \\ 1 \end{pmatrix} = \vec{u_3}$, $M\vec{u_3} = 3\vec{u_3}$

Por el autovector encontrado en (*), podemos tomar la base:

$\mathcal{B} = \left\{ \vec{u_1} = \begin{pmatrix} 5 \\ +5 \\ -2 \end{pmatrix}, \vec{u_2} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}, \vec{u_3} = \begin{pmatrix} 5 \\ 15 \\ 1 \end{pmatrix} \right\} \Rightarrow M\vec{u_1} = \vec{u_2}, \ M\vec{u_2} = -\vec{u_1}$

$\Rightarrow \boxed{M(F;\mathcal{B}) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix}}$

$\boxed{11.3}$ $f_1 : \mathbb{R}^2 \to \mathbb{R}^2$. $f_1(\vec{x}) = \begin{bmatrix} 3 & 1 \\ -1 & 5 \end{bmatrix} \vec{x}$. λ autovalor $\Leftrightarrow$

$$\begin{vmatrix} 3-\lambda & 1 \\ -1 & 5-\lambda \end{vmatrix} = 15 + \lambda^2 - 8\lambda = \lambda^2 - 8\lambda + 16 = 0 \Leftrightarrow \cancel{\lambda = \frac{8 \pm \sqrt{64-}}{}}$$

$$\Leftrightarrow (\lambda - 4)^2 = 0 \Leftrightarrow \lambda = 4 \text{ (doble)}$$

$E_1(4) = \text{Ker}(f_1 - 4I)$ $\left(\begin{array}{cc|c} -1 & 1 & 0 \\ -1 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{cc|c} -1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right) \to \vec{v_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ autovector.

$E_2(4) = \text{Ker}(f_1 - 4I)^2 \to \left(\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right) \to E_2(4) = \mathbb{R}^2$

Tomamos $\vec{u_2} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in E_2(4) - E_1(4) \Rightarrow \vec{u_1} = (f_1 - 4I)\vec{u_2} = \begin{pmatrix} -1 \\ -1 \end{pmatrix} \in E_1(4)$

y $f(\vec{u_2}) = \vec{u_1} + 4\vec{u_2}$

$$\Rightarrow \boxed{\mathcal{B} = \left\{ \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}, \quad J = \begin{pmatrix} 4 & 1 \\ 0 & 4 \end{pmatrix}}$$

$f_2 : \mathbb{R}^3 \to \mathbb{R}^3$. $f_2 = \begin{pmatrix} 0 & 2 & 1 \\ -4 & 6 & 1 \\ 8 & 4 & -2 \end{pmatrix}$

$$|f_2 - \lambda I| = \begin{vmatrix} -\lambda & 2 & 1 \\ -4 & 6-\lambda & 1 \\ 8 & 4 & -2-\lambda \end{vmatrix} = \begin{vmatrix} 4-\lambda & -4+\lambda & 0 \\ -4 & 6-\lambda & 1 \\ 8 & 4 & -2-\lambda \end{vmatrix} = \begin{vmatrix} 4-\lambda & 0 & 0 \\ -4 & 2-\lambda & 1 \\ 8 & 12 & -2-\lambda \end{vmatrix} = 0$$

$\Leftrightarrow (4-\lambda)(-4 + \lambda^2) = 0 \Leftrightarrow \lambda = 4$ ó $\lambda = \pm 4$.

$\to$ Autovalores: $\lambda = 4$ (doble). $\lambda = -4$.

$E_1(-4)$: $\left(\begin{array}{ccc|c} 4 & 2 & 1 & 0 \\ -4 & 10 & 1 & 0 \\ 8 & 4 & 2 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 4 & 2 & 1 & 0 \\ 0 & 6 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \to \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 1 \\ -6 \end{pmatrix}$. $E_1(-4) = \left\langle \begin{pmatrix} 1 \\ 1 \\ -6 \end{pmatrix} \right\rangle$.

$E_1(4)$: $\left(\begin{array}{ccc|c} -4 & 2 & 1 & 0 \\ -4 & 2 & 1 & 0 \\ 8 & 4 & -6 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} -4 & 2 & 1 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \to \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$, $E_1(4) = \left\langle \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \right\rangle$

$E_2(4)$: $\left(\begin{array}{ccc|c} 16 & 0 & -8 & 0 \\ 16 & 0 & -8 & 0 \\ -96 & 0 & 48 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 2 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \to E_2(4) = \left\langle \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \right\rangle$

$\vec{u_3} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \in E_2(4) - E_1(4)$, $\vec{u_2} = (f_2 - \lambda I)\vec{u_3} = \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix}$ $\vec{u_1} = \begin{pmatrix} 1 \\ 1 \\ -6 \end{pmatrix}$

$$\Rightarrow \boxed{\mathcal{B} = \{\vec{u_1}, \vec{u_2}, \vec{u_3}\} \Rightarrow J = \begin{pmatrix} -4 & 0 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{pmatrix}}$$

$$f_3 = \begin{bmatrix} 4 & -3 & 1 \\ 1 & 0 & 1 \\ 2 & 6 & 5 \end{bmatrix} \text{ en } \mathbb{R} = \mathbb{K}$$

$$\det(f_3 - \lambda I) = \begin{vmatrix} 4-\lambda & -3 & 1 \\ 1 & -\lambda & 1 \\ 2 & 6 & 5-\lambda \end{vmatrix} = \begin{vmatrix} 3-\lambda & -3 & 1 \\ 0 & -\lambda & 1 \\ -3+\lambda & 6 & 5-\lambda \end{vmatrix} = \begin{vmatrix} 3-\lambda & -3 & 1 \\ 0 & -\lambda & 1 \\ 0 & 3 & 6-\lambda \end{vmatrix} = 0$$

$$\Leftrightarrow (3-\lambda)(\lambda^2 - 6\lambda - 3) = 0 \Leftrightarrow \lambda = 3 \text{ ó } \lambda = \frac{6 \pm \sqrt{48}}{2} = 3 \pm 2\sqrt{3}$$

$$\lambda = 3 \to \begin{pmatrix} 1 & -3 & 1 \\ 1 & -3 & 1 \\ 2 & 6 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & -3 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \left\langle \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\rangle = \operatorname{Ker}(f_3 - 3I)$$

$$\lambda = 3+2\sqrt{3} \begin{pmatrix} 1-2\sqrt{3} & -3 & 1 \\ 1 & -3-2\sqrt{3} & 1 \\ 2 & 6 & 2-2\sqrt{3} \end{pmatrix} \sim \begin{pmatrix} 1 & -3-2\sqrt{3} & 1 \\ -2\sqrt{3} & +2\sqrt{3} & 0 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 0 & -2-2\sqrt{3} & 1 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\operatorname{Ker}(f_3 - (3+2\sqrt{3})I) = \left\langle \begin{pmatrix} 1 \\ 1 \\ 2+2\sqrt{3} \end{pmatrix} \right\rangle$$

$$\lambda = 3-2\sqrt{3} \begin{pmatrix} 1+2\sqrt{3} & -3 & 1 \\ 1 & -3+2\sqrt{3} & 1 \\ \cancel{2} & \cancel{6} & \cancel{2+2\sqrt{3}} \end{pmatrix} \begin{pmatrix} 1 & -3+2\sqrt{3} & 1 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -2+2\sqrt{3} & 1 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\operatorname{Ker}(f_3 - (3-2\sqrt{3})I) = \left\langle \begin{pmatrix} 1 \\ 1 \\ 2-2\sqrt{3} \end{pmatrix} \right\rangle$$

$$\Rightarrow J = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3+2\sqrt{3} & 0 \\ 0 & 0 & 3-2\sqrt{3} \end{pmatrix} \text{ con } \mathcal{B} = \left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2+2\sqrt{3} \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2-2\sqrt{3} \end{pmatrix} \right\}.$$

$$f_4 = \begin{bmatrix} 5 & -2 & -1 \\ 3 & 0 & 0 \\ 1 & -1 & 4 \end{bmatrix}, \quad \begin{vmatrix} 5-\lambda & -2 & -1 \\ 3 & -\lambda & 0 \\ 1 & -1 & 4-\lambda \end{vmatrix} = \begin{vmatrix} 3-\lambda & -2 & -1 \\ 3-\lambda & -\lambda & 0 \\ 0 & -1 & 4-\lambda \end{vmatrix} = \begin{vmatrix} 3-\lambda & -2 & -1 \\ 0 & 2-\lambda & 1 \\ 0 & -1 & 4-\lambda \end{vmatrix} =$$

$$= (3-\lambda)(\lambda^2 - 6\lambda + 9) = (3-\lambda)^3 = 0 \Leftrightarrow \lambda = 3 \text{ triple}$$

$$E_1(3) = \operatorname{Ker}(f_4 - 3I): \begin{pmatrix} 2 & -2 & -1 \\ 3 & -3 & 0 \\ 1 & -1 & 1 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \to \left\langle \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\rangle = \operatorname{Ker}(f_4 - 3I)$$

$$E_2(3) = \operatorname{Ker}(f_4 - 3I)^2: \begin{pmatrix} -3 & 3 & -3 \\ -3 & 3 & -3 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} -1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \to E_2(3) = \left\langle \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\rangle$$

$E_3(3) = \mathbb{R}^3$. tomo $\vec{u_3} = (1, 0, 0) \in \mathbb{R}^3 - E_2(3)$

$\Rightarrow \vec{u_2} = (f_4 - 3I)\vec{u_3} = (2, 3, 1) \in E_2 - E_1(3)$

$\Rightarrow \vec{u_1} = (f_4 - 3I)\vec{u_2} = (-3, -3, 0)$

$\beta = \{\vec{u_1}, \vec{u_2}, \vec{u_3}\}$ t.q.
$$\begin{aligned} f_4(\vec{u_1}) &= 3\vec{u_1} \\ f_4(\vec{u_2}) &= \vec{u_1} + 3\vec{u_2} \\ f_4(\vec{u_3}) &= \vec{u_2} + 3\vec{u_3} \end{aligned} \to J = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix}$$

$\boxed{11.4}$ $E = P_{\mathbb{R}}^{(1)}[x]$ $\quad L(p) = -2p(0) + (3+x)p - (1+x^2)p'$

$$\left.\begin{aligned} &\mathcal{B} = \{1, x\}. \quad L(1) = -2 + 3 + x = 1 + x \\ &L(x) = 3x + x^2 - 1 - x^2 = -1 + 3x \end{aligned}\right\} M(L, \mathcal{B}) = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} = M$$

$$|M - \lambda I| = 0 \Leftrightarrow \begin{vmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{vmatrix} = 1 + 3 + \lambda^2 - 4\lambda = (\lambda - 2)^2 = 0 \Leftrightarrow \lambda = 2 \text{ (doble)}$$

(a) $E_1(2) \Rightarrow \left(\begin{array}{cc|c} -1 & -1 & 0 \\ 1 & +1 & 0 \end{array}\right) \to E_1(2) = \left\langle \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\rangle = \langle 1 - x \rangle$

Tomando $p_2 = 1 \in E_2(2) - E_1(2)$, $(M - 2I)p_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} = -1 + x = p_1$

$\Rightarrow Mp_1 = 2p_1$, $\quad Mp_2 = p_1 + 2p_2$

En la base $\beta = \{-1 + x, 1\} \Rightarrow J = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$.

$$M(L,B) = A = C\ J\ C^{-1}$$

con $C = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} \rightarrow C^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & +1 \end{pmatrix}$

$$J^1 = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix},\quad J^2 = \begin{pmatrix} 4 & 4 \\ 0 & 4 \end{pmatrix},\quad J^3 = \begin{pmatrix} 8 & 12 \\ 0 & 8 \end{pmatrix}$$

$\rightarrow J^K = \begin{pmatrix} 2^K & g(K) \\ 0 & 2^K \end{pmatrix}$ $\quad g(K) = 2g(K-1) + 2^{K-1} = 2(g(K-1) + 2^{K-2})$

$g(1) = 1 \qquad g(5)$

$g(2) = 4 \qquad g(6)$

$g(3) = 12$

$g(4) = 32$

$$g(K) = 2(2(g(K-2) + 2^{K-3}) + 2^{K-2}) =$$

$$= (K-1)\,2^{K-1} + 2^{K-1} = K \cdot 2^{K-1}$$

Por inducción: Base: $J^1 = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$, $J^2 = \begin{pmatrix} 2^2 & 2 \cdot 2^1 \\ 0 & 2^2 \end{pmatrix} = \begin{pmatrix} 4 & 4 \\ 0 & 4 \end{pmatrix}$

H.I: $J^k = \begin{pmatrix} 2^k & k \cdot 2^{k-1} \\ 0 & 2^k \end{pmatrix}$

$$J^{k+1} = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 2^k & k \cdot 2^{k-1} \\ 0 & 2^k \end{pmatrix} = \begin{pmatrix} 2^{k+1} & 2^k + k\,2^k \\ 0 & 2^{k+1} \end{pmatrix} = \begin{pmatrix} 2^{k+1} & (k+1)\,2^k \\ 0 & 2^{k+1} \end{pmatrix}$$

$$\blacksquare \Rightarrow J^k = \begin{pmatrix} 2^k & k \cdot 2^{k-1} \\ 0 & 2^k \end{pmatrix}$$

$$\Rightarrow M^{13} = C J^{13} C^{-1} = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2^{13} & 13 \cdot 2^{12} \\ 0 & 2^{13} \end{pmatrix} C^{-1} = \begin{pmatrix} -2^{13} & -11 \cdot 2^{12} \\ 2^{13} & 13 \cdot 2^{12} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -11 \cdot 2^{12} & -13 \cdot 2^{12} \\ 13 \cdot 2^{12} & 15 \cdot 2^{12} \end{pmatrix}$$

$$M^{13} = 2^{12}\begin{pmatrix} -11 & -13 \\ 13 & 15 \end{pmatrix} \Rightarrow L^{13}(4-7x) = M^{13}\begin{pmatrix} 4 \\ -7 \end{pmatrix} = 2^{12}\begin{pmatrix} -11 & -13 \\ 13 & 15 \end{pmatrix}\begin{pmatrix} 4 \\ -7 \end{pmatrix} =$$

$$= 2^{12}\begin{pmatrix} 47 \\ -53 \end{pmatrix} \Rightarrow \boxed{L^{13}(4-7x) = 47 \cdot 2^{12} - 2^{12} \cdot 53\,x}$$

H11-3