Ejercicios

- ej. 4, hoja 4

a) (1) $M x_{k+1} = N x_k + b$ t.q. $\boxed{x_k \longrightarrow \underline{x}}$

$\Rightarrow A \underline{x} = b$

- $A x_{k+1} = M x_{k+1} - N x_{k+1} = N(\underline{x_k - x_{k+1}}) + b$

- $t_k = x_k - x_{k+1} \Rightarrow t_k \longrightarrow 0$

$\| x_k - x_{k+1} \| = \| x_k - \underline{x} + \underline{x} - x_{k+1} \| \leq \| x_k - \underline{x} \| + \| \underline{x} - x_{k+1} \|$

- $f_N(y) = Ny$ es continua $\Rightarrow f_N(t_k) \longrightarrow f_N(0) = 0$

$\Rightarrow A x_{k+1} \longrightarrow b$

- $f_A(y) = Ay$ es continua $\Rightarrow \underbrace{\lim_{k \to \infty} f_A(x_k)}_{= b} = f_A(\underbrace{\lim_{k \to \infty} x_k}_{= \underline{x}})$

$\Rightarrow A \underline{x} = b$

$\| f_A(y_1) - f_A(y_2) \| = \| A(y_1 - y_2) \| \leq \| A \| \, \| y_1 - y_2 \|$

b) $B_1 = M^{-1} N$ matriz de iteración de (1)

$B_2 = N^{-1} M$ " (2)

$M x_{k+1} = N x_k + b \Leftrightarrow x_{k+1} = \underbrace{M^{-1} N}_{B_1} x_k + M^{-1} b$

- $B_2 = B_1^{-1}$. sea v_λ autovector de B_1 con λ

$B_1 v_\lambda = \lambda v_\lambda \Leftrightarrow B_1^{-1} v_\lambda = \frac{1}{\lambda} v_\lambda = B_2 v_\lambda$

↳ intuición: relación entre $\rho(B_1)$ y $\frac{1}{\rho(B_2)}$

- $\rho(B_1) = \max\{|\lambda| : \lambda \text{ e.v. } B_1\} = \max \dfrac{1}{\{\frac{1}{|\lambda|} : \lambda \text{ e.v. } B_1\}}$

$$= \frac{1}{\min\{\frac{1}{|\lambda|} : \lambda \text{ e.v. } B_1\}} = \frac{1}{\min\{|\mu| : \mu \text{ e.v. } B_2\}}$$

$$\geq \frac{1}{\max\{|\mu| : \mu \text{ e.v. } B_2\}} = \frac{1}{\rho(B_2)}$$

$\Rightarrow \quad \rho(B_1)\,\rho(B_2) \geq 1 \quad$: no pueden ser ambos < 1

¿ pueden ser ambos divergentes?

$\hookrightarrow$ ¿ $\exists A : \rho(B_1) > 1$ y $\rho(B_2) > 1$?

si $B_1 = \begin{pmatrix} 2 & 0 \\ 0 & 1/2 \end{pmatrix}$, $B_2 = B_1^{-1} = \begin{pmatrix} 1/2 & 0 \\ 0 & 2 \end{pmatrix} \Rightarrow \rho(B_1) = \rho(B_2) = 2$

$\Rightarrow \quad M = I, \quad N = B_1, \quad A = M - N = \begin{pmatrix} -1 & 0 \\ 0 & 1/2 \end{pmatrix}$, por ejemplo

c) $A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$, $M = \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}$, $N = \begin{pmatrix} -a & 0 \\ 0 & -a \end{pmatrix}$

$$B_1 = \begin{pmatrix} 0 & -1/b \\ 1/b & 0 \end{pmatrix}\begin{pmatrix} -a & 0 \\ 0 & -a \end{pmatrix} = \begin{pmatrix} 0 & a/b \\ -a/b & 0 \end{pmatrix} \Rightarrow \rho(B_1) = \left|\frac{a}{b}\right|$$

$$B_2 = \begin{pmatrix} 0 & -b/a \\ b/a & 0 \end{pmatrix} \Rightarrow \rho(B_2) = \left|\frac{b}{a}\right|$$

d) $E_k = x_k - \underline{x} = B_1^k (x_0 - \underline{x}) = B_1^k E_0$

- $B_1 = \begin{pmatrix} 0 & a/b \\ -a/b & 0 \end{pmatrix} = \underbrace{\frac{a}{b}}_{\gamma} \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{J}$

$$B_1^k = \gamma^k J^k \quad , \quad J^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\hookrightarrow J^2 = -I \quad , \quad J^3 = J \cdot J^2 = -J$$

$$J^4 = J^2 \cdot J^2 = I \quad , \quad J^5 = J , \ldots$$

- $J \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ -x_1 \end{pmatrix} \Rightarrow \| J x \|_p = \| x \|_p \ , \ \forall x \ , \ \forall p$

$$\| J^k x \|_p = \| x \|_p \ , \quad \forall x , \forall p , \forall k$$

- $\| E_k \|_p = \| B_1^k E_0 \|_p = \| \gamma^k J^k E_0 \|_p = |\gamma|^k \| J^k E_0 \|_p$

$$= \left| \frac{a}{b} \right|^k \| E_0 \|_p \qquad \forall \ p \in [1, \infty)$$

$$\hookrightarrow \rho = \rho(B_1) .$$

$\cdot \sim \cdot$

- ej. 2 hoja 3

a) • $\| x \|_\infty \leq \| x \|_2$

- $\| x \|_\infty = \max_i |x_i|$, y sea $k = \arg\max_i |x_i|$ $(= k(x))$

$$\| x \|_2^2 = \sum_{i=1}^{n} |x_i|^2 = \underbrace{|x_k|^2}_{\| x \|_\infty^2} + \underbrace{\sum_{\substack{i=1 \\ i \neq k}}^{n} |x_i|^2}_{\geq 0}$$

- si $x = e_j \Rightarrow \| x \|_2 = \| x \|_\infty = 1$

• $\| x \|_2 \leq \sqrt{n} \ \| x \|_\infty$

- $\| x \|_2^2 = \sum_{i=1}^{n} |x_i|^2 \leq \| x \|_\infty \sum_{i=1}^{n} 1 = n \ \| x \|_\infty$

- si $x = \begin{pmatrix} 1 \\ -1 \\ \vdots \\ 1 \end{pmatrix} \Rightarrow \sum_{i=1}^{n} |x_i|^2 = \sum_{i=1}^{n} 1 = n \ , \quad \| x \|_\infty = 1$

b) $$|||A|||_\infty = \sup_{x \neq 0} \frac{\|Ax\|_\infty}{\|x\|_\infty}$$

- $$\frac{\|Ax\|_\infty}{\|x\|_\infty} \leqslant \frac{\|Ax\|_2}{\frac{1}{\sqrt{n}}\|x\|_2} = \sqrt{n}\,\frac{\|Ax\|_2}{\|x\|_2} \Rightarrow |||A|||_\infty \leq \sqrt{n}\,|||A|||_2$$

$$\hookrightarrow \quad A = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ & & 0 & \end{pmatrix} \quad : \quad |||A|||_\infty = n$$

$$Ax = \begin{pmatrix} \sum_i x_i \\ 0 \\ \vdots \\ 0 \end{pmatrix} \Rightarrow \|Ax\|_2 = \left|\sum_i x_i\right| = \left|\sum_i 1 \cdot x_i\right| \leq \underbrace{\left(\sum x_i^2\right)^{1/2}}_{\|x\|_2} \sqrt{n}$$

$$\Rightarrow |||A|||_2 = \sqrt{n} = \frac{1}{\sqrt{n}}\,|||A|||_\infty$$

$$\left(\text{si } x = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \Rightarrow \|Ax\|_2 = \|x\|_2\sqrt{n} = n\right)$$

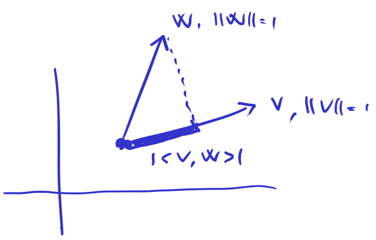
- $$\frac{\|Ax\|_\infty}{\|x\|_\infty} \geqslant \frac{\frac{1}{\sqrt{n}}\|Ax\|_2}{\|x\|_2} \Rightarrow |||A|||_\infty \geq \frac{1}{\sqrt{n}}\,|||A|||_2$$

$$A = \begin{pmatrix} 1 & \\ 1 & \\ \vdots & 0 \\ 1 & \end{pmatrix}, \quad |||A|||_\infty = 1$$

$$A^* A = \begin{pmatrix} n & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & 0 & \\ 0 & & & \end{pmatrix} \Rightarrow |||A|||_2 = \sqrt{n} = \sqrt{n}\,|||A|||_\infty$$

desigualdad de Cauchy-Schwartz :

$$v, w \in \mathbb{R}^n \Rightarrow |\langle v, w\rangle| \leq \|v\|_2 \, \|w\|_2$$



$$\| \qquad\qquad\qquad$$

$$\sum_i v_i w_i \leq \left(\sum_i v_i^2\right)^{1/2}\left(\sum_i w_i^2\right)^{1/2}$$

$$= \iff v \,/\!/\, w$$