Pr. H2.

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$$F(x)=\int_{-\infty}^{x} f(x)\,dx \Leftrightarrow F'(x)=f(x)$$

$(x<0)$ $$F'(x)=\left(\frac{1}{2}(1+x^2)^{-1}\right)\frac{d}{dx}=\frac{1}{2}(-1)(1+x^2)^{-2}2x$$

$$=\frac{-x}{(1+x^2)^2}$$

$(x\geq 0)$ $$F'(x)=\frac{4x\,2(1+x^2)-(1+2x^2)4x}{4(1+x^2)^2}=\frac{8x^3+8x-8x^3-4x}{4(1+x^2)^2}$$

$$=\frac{x}{(1+x^2)^2}$$

$$\boxed{f(x)=\frac{|x|}{(1+x^2)^2}.}$$

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X v.a. $m\in\mathbb{R}$ es mediana si

$P(X\leq m)\geq 1/2$, $P(X\leq m)\geq 1/2$.

a) Sea F una función de distribución,

$P(X\leq m)=F(m)$, $P(X\geq m)=1-\lim_{x\uparrow m} F(x)$

Sea $m=\min\{x: F(x)\geq 1/2\}$

existe, F cont. por la derecha

Como F es creciente, $F(m^-) \leq \frac{1}{2} \leq F(m)$

$\Rightarrow P(X \geq m) = 1 - F(m^-) \geq 1/2$.

---

tj.: $F(x) = \begin{cases} 0 & \text{si } x < 0 \\ 1/2 & \text{si } x \in [0,1] \\ 1 & \text{si } x > 1 \end{cases}$ $\forall x \in [0,1)$ x mediana.

---

b) F continua

Sea m mediana $\Rightarrow$ $\left.\begin{array}{l} F(m) \geq 1/2, \\ 1 - F(m^-) = 1 - F(m) \geq 1/2 \end{array}\right\} F(m) = 1/2.$

Sea $F(m) = 1/2 \Rightarrow F(m) \geq 1/2,\ 1 - F(m^-) = 1/2 \geq 1/2$

$\Rightarrow m$ mediana.

(c) $f(x) = \lambda e^{-\lambda x}, \ x \geq 0$

$F(x) = \int_0^x \lambda e^{-\lambda t}\, dt = 1 - e^{-\lambda x}$ $\rightarrow$ Continua $(0, \infty)$

Sea m mediana $\Leftrightarrow F(m) = 1/2 = 1 - e^{-\lambda m}$

$\Leftrightarrow e^{-\lambda m} = \frac{1}{2} \Leftrightarrow 2 = e^{\lambda m}$

$\Leftrightarrow m = \frac{\ln 2}{\lambda}$

2.6.

$$F(x) = \begin{cases} 0 & x \le 0 \\ \dfrac{x}{1+x} & x \geq 0 \end{cases} \qquad X \equiv \text{voltaje.}$$

$$\left.\begin{array}{l} \text{Si } x_0 \le 0, \quad \lim_{x \to x_0} F(x) = F(x_0) = 0 \\ \text{Si } x_0 \geq 0, \quad \lim_{x \to x_0} F(x) = F(x_0) = \dfrac{x_0}{1+x_0} \end{array}\right\} F \text{ continua}$$

Sea $x \geq y$:

$$\begin{cases} \text{Caso 1: } & y \le x \le 0 \Rightarrow F(x) = F(y) = 0 \\ \text{Caso 2 } & y \le 0 \le x \Rightarrow F(y) = 0 \le \dfrac{x}{1+x} = F(x) \\ \text{Caso 3 } & 0 \le y \le x \Rightarrow F(y) = \dfrac{y}{1+y} \le \dfrac{x}{1+x} = F(x) \\ & \Leftrightarrow y + xy \le x + xy \quad ✓ \end{cases}$$

F no decreciente.

$$\lim_{x \downarrow -\infty} F(x) = 0, \quad \lim_{x \uparrow \infty} F(x) = 1$$

F continua

$$P(X \in (3,5)) = P(3 < X < 5) \underline{\underline{\quad}}$$

$$= F(5) - F(3) = \frac{5}{6} - \frac{3}{4} = \underline{\underline{\frac{1}{12}}}$$

2.7. $Y = F(X)$: $Y \in (0,1)$

$F_Y(y) = 0$ si $y < 0$,

$F_Y(y) = 1$ si $y > 0$.

si $y \in (0,1) \Rightarrow \exists x_0$ tq $y = F(x_0)$

$$\Rightarrow F_Y(y) = P(F(X) \leq y) =$$

$$= P(F(X) \leq F(x_0)) \underset{\substack{\uparrow \\ F \text{ no decrec.}}}{=} P(X \leq x_0) = F(x_0) = y$$

$$F_Y(y) = \begin{cases} 0 & \text{si } y \leq 0 \\ y & \text{si } y \in (0,1) \\ 1 & \text{si } y \geq 1 \end{cases} \Rightarrow Y \sim U(0,1)$$

2.8. $X = F^{-1}(U)$.

F estrictam. creciente $\Rightarrow F: \mathbb{R} \xrightarrow{\circledast} [0,1]$ bijective.

$\Rightarrow F^{-1}$ estm. creciente.

$$F_X(x) = P(X \leq x) = P(F^{-1}(U) \leq x) = P(U \leq F(x)) =$$

$$= F_U(F(x)) = F(x).$$

$\circledast$ $F(x) \in [0,1]$

$\hookrightarrow F(x) = 1 - e^{-\lambda x}. \Rightarrow 1 - F(x) = e^{-\lambda x}$

$\Rightarrow -\log(1 - F(x)) = \lambda x \Rightarrow F^{-1}(y) = \dfrac{-\log(1-y)}{\lambda}.$

Tomando un $y \in [0,1]$, tomado con prob. uniforme $(0,1)$ se obtiene: $F^{-1}(y) = x \in [0, \infty)$. Con distrib. exp.

2.9. $p(k) = P(X = k) = \dfrac{c}{k(k+1)}, \quad k \in \mathbb{N} \setminus \{0\}$

$1 = \sum_{k \in \mathbb{N}} \dfrac{c}{k(k+1)} \Rightarrow \dfrac{1}{c} = \sum_{k=1}^{\infty} \dfrac{1}{k(k+1)}$

Valores	n
1/2	1
2/3	2
3/4	3
4/5	n
	5

$\sum_{k=1}^{n} \dfrac{1}{k(k+1)} = 1 - \dfrac{1}{n+1}$

Caso base

$\sum_{k=1}^{n+1} \dfrac{1}{k(k+1)} = \dfrac{1}{(n+1)(n+2)} + 1 - \dfrac{1}{n+1}$

$= 1 + \dfrac{1-n-2}{(n+1)(n+2)} = 1 + \dfrac{-n-1}{(n+1)(n+2)} = 1 - \dfrac{1}{n+2}$ ⊔

$n \to \infty$

$\Rightarrow \dfrac{1}{c} = \sum_{k=1}^{\infty} \dfrac{1}{k(k+1)} = 1 \Rightarrow \boxed{c = 1}$

2.10 $p(k) = c\,k^{\alpha}, \quad k \in \mathbb{N} \setminus \{0\}$

$$1 = \sum_{k=1}^{\infty} c\,k^{\alpha}$$

2.11. $X \equiv$ periódicos en un día.

(a) $$p(x) = \begin{cases} cx, & x \in \{1 \dots 50\} \\ c(100-x), & x \in \{51 \dots 100\} \\ 0, & \text{otro caso.} \end{cases}$$

$$1 = \sum_{x=1}^{\infty} p(x) = \sum_{x=1}^{50} cx + \sum_{x=51}^{100} c(100-x)$$

$$= c\left(\frac{51 \cdot 50}{2} + \sum_{x=0}^{49} x \right) = c\,\frac{50}{2}(51+49)$$

$$= 25c \cdot 100 \Rightarrow c = \frac{1}{2500}$$

(b) i) $P(A) = P(X > 50) =$

$$= c\sum_{x=51}^{100}(100-x) = c\sum_{x=0}^{49} x = \frac{50 \cdot 49}{2 \cdot 2500} = \underline{\underline{0{,}49}}.$$

iii) $P(C) = p(50) = \dfrac{50}{2500} = \underline{\underline{0{,}02}}.$

ii) $P(B) = P(X < 50) = 1 - P(X > 50) - P(X = 50)$

$$= 1 - 0{,}49 - 0{,}02 = \underline{\underline{0{,}49}}$$

iv) $P(D) = P(25 \le X \le 75)$

$$= c\left(\underbrace{\sum_{x=25}^{50} x}_{\sum_{x=1}^{50} x - \sum_{x=1}^{24} x = \frac{50\cdot 51 - 24\cdot 25}{2} = 975} + \sum_{x=50}^{75} (100-x) \right) =$$

$$= c\left(975 + \sum_{x=25}^{50} x \right) = \frac{2\cdot 975}{2500} = \underline{\underline{0{,}78}}$$

$$P(25 \le X \le 75)$$

$$P(25 < X < 75) = \underline{\underline{0{,}76}}$$

(v) $P(E) = P(X \text{ par}) =$

$$= c\left(\sum_{y=0}^{25} 2y + \sum_{y=26}^{50} (100-2y) \right) \quad 2\sum_{y=26}^{50} (50-y)$$

$$= \cancel{2}c\left(\frac{25\cdot 26}{\cancel{2}} + \frac{24\cdot 25}{\cancel{2}} \right) = \frac{1}{2}. \Rightarrow P(E) = \frac{1}{2}.$$

2.12. $P(X=k) = p(1-p)^{k-1}$, $k=1,2,\dots$

$$\Updownarrow$$

$$P(X>m) = (1-p)^m \quad \forall m \in \mathbb{N}$$

i.e. $P(X>m) = P(X>1)^m$.

($\Leftarrow$) Sup. $P(X>m) = (1-p)^m \quad \forall m \in \mathbb{N}$

$\Rightarrow$ $P(X=1) = 1 - P(X>1)$

$= p$

Base

$P(X=2) = 1 - P(X>2) - P(X=1)$

$= 1-(1-p)^2 - p$

$= 1 - 1 + 2p - p^2 - p$

$= p(1-p)$

Ind sup. $P(X=n) = p(1-p)^{n-1}$

$\Rightarrow P(X=n+1) = 1 - P(X>n) - P(X<n)$

$= 1 - (1-p)^n - \sum_{i=1}^{n} p(1-p)^{i-1}$

$$= 1 - (1-p)^n - p \sum_{i=0}^{n-1} (1-p)^i$$

$$= 1 - (1-p)^n - p \frac{(1-p)^{n+1} - 1}{1-p-1}$$

$$= 1 - (1-p)^n + (1-p)^{n+1} - 1$$

$$= (1-p)^n (-1 + 1 + p) = p(1-p)^n \quad \square$$

$$(\Rightarrow) \quad P(X = k) = p(1-p)^{k-1}, \quad k = 1, 2, \ldots$$

$$\Rightarrow P(X > k) = 1 - P(X \le k)$$

$$= 1 - \sum_{i=1}^{k} p(1-p)^{i-1} = 1 - p \sum_{i=0}^{k-1} (1-p)^i$$

$$= 1 - p \frac{(1-p)^k - 1}{1-p-1} = 1 + (1-p)^k - 1$$

$$= (1-p)^k \quad \square$$

(b) Sea X geométrica:

$(\Rightarrow)$ $P(X>m+n \mid X>m) = \dfrac{P(X>m+n,\ X>m)}{P(X>m)} =$

$\overset{*}{=} \dfrac{P(X>m+n)}{P(X>m)} = \dfrac{(1-p)^{m+n}}{(1-p)^{m}} = (1-p)^{n}$

$= P(X>n).$ $\square$

$(\Leftarrow)$ $P(X>m+n \mid X>m) = P(X>n)$

$\overset{*}{\Longrightarrow} P(X>m+n) = P(X>m)\,P(X>n)$

Sea $P(X>1) = p$

$\Rightarrow P(X>m) = P(X>(m-1)+1)$

$= P(X>m-1)\,P(X>1)$

$= \cdots = P(X>1)^{m}$ $\square$

2.13. $X \equiv$ nº de tirades hasta que salen 4 6's.

$$\underbrace{\underline{a_1}\ \ \underline{a_2}\ \ \underline{a_3}\ \ \ldots\ \ \underline{\ \ }}_{k-1,\ 3 \text{ seises}}\ \ \underline{6}$$

$$P(X=k) = \binom{k-1}{3}\left(\frac{1}{6}\right)^4\left(\frac{5}{6}\right)^{k-4}$$

(a) $$P(X=10) = \binom{9}{3}\left(\frac{1}{6}\right)^4\left(\frac{5}{6}\right)^6$$

$$= \frac{9!}{3!\,6!}\ \frac{5^6}{6^{10}}$$

$$= \frac{9\cdot 8\cdot 7}{3\cdot 2}\cdot\frac{5^6}{6^{10}} = \underline{\underline{0{,}02170635}}$$

(b) $$\underbrace{\underline{\ \ }\ \ \underline{\ \ }\ \ \ldots\ \ \underline{\ \ }}_{8 \text{ tirades},\ 2 \text{ seises}}\ \ \underline{6}\ \ \underline{6}$$

$S_9 \equiv$ Sexta tirada es un 6

$$\rightsquigarrow P(X=10, S_9) =$$

$$= \binom{8}{2}\left(\frac{1}{6}\right)^4\left(\frac{5}{6}\right)^6 = \frac{8\cdot 7}{2}\cdot\frac{5^6}{6^{9}}$$

$$= \underline{\underline{0{,}0072354 5\ldots}}$$

2.14. $p = 0{,}03$. (prob. bombilla defectuosa)

$$X \sim B(n=100,\ p=0{,}03)$$

$X \equiv$ nº de bomb. defectuosas de 100

$$P(X=k) = \binom{100}{k} p^k q^{100-k}$$

Para $k = 0, 1 \ldots 4$.

Aproximación a Poisson

Si $n \to \infty$, $\binom{n}{k} p^k (1-p)^{n-k}$

$$= \frac{n!}{k!\,(n-k)!}\, p^k (1-p)^{n-k}$$

$$= \frac{n(n-1)\cdots(n-k-1)}{k!} \cdot \frac{\lambda^k}{n^k} \left(1-\frac{\lambda}{n}\right)^{n-k}$$

$np = \lambda$

$$= \frac{\lambda^k}{k!} \underbrace{\left(\frac{n}{n} \cdot \frac{n-1}{n} \cdots \frac{n-k+1}{n}\right)}_{\downarrow\, n\to\infty,\ (k \text{ fijo})\ \ 1} \left(1-\frac{\lambda}{n}\right)^{n-k}$$

$$\xrightarrow{n\to\infty} \frac{\lambda^k}{k!} e^{-\lambda} \approx P(X=k)$$

$$\Rightarrow \quad np = \lambda = 3$$

$$\begin{cases} P(X=0) = \frac{3^0}{0!} e^{-3} = e^{-3} \\ P(X=1) = \frac{3}{1!} e^{-3} = 3e^{-3} \\ P(X=2) = \frac{9}{2!} e^{-3} = \frac{9}{2} e^{-3} \\ P(X=3) = \frac{27}{6} e^{-3} = \frac{9}{2} e^{-3} \\ P(X=4) = \frac{81}{4!} e^{-3} = \frac{27}{8} e^{-3} \end{cases}$$

2.15. Para cada $S > 0$

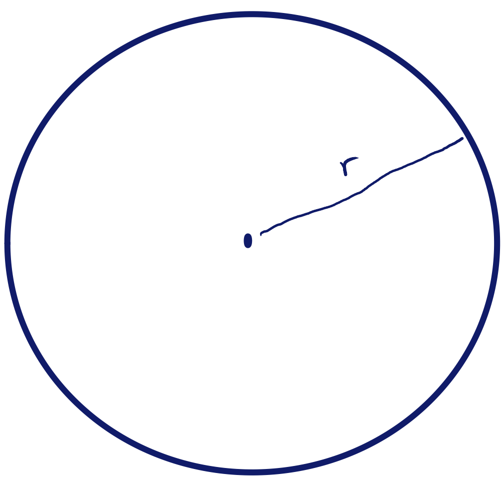
$X_S \equiv$ nº de árboles en zona de extensión S. $X \sim P(\lambda S)$

$$P(X_S = k) = e^{-\lambda S} \frac{(\lambda S)^k}{k!}$$

Donde $\lambda > 0$, constante.

$$P(X_S > 0) = 1 - P(X_S = 0)$$
$$= 1 - e^{-\lambda S}$$

Sea $Y \equiv$ dist. de un punto al árbol más cercano



$\rightarrow S = \pi r^2$

$$P(Y < r) = P(X_S > 0) = P(X_{\pi r^2} < 0)$$
$$= \left(1 - e^{-\lambda \pi r^2}\right).$$

2.16.

a) $f(x) = c\, e^{-|x|}$

$$\int_{-\infty}^{\infty} c\, e^{-|x|}\, dx = 1 = \int_{0}^{\infty} 2c\, e^{-x}\, dx =$$

$$= 2c \left(-e^{-x}\right)\Big|_0^{\infty} = 2c \Rightarrow c = 1/2.$$

b) $f(x) = c \exp(-x - e^{-x})$

$$= c\, e^{-x-e^{-x}} = c\, e^{-x} \cdot \underbrace{e^{-e^{-x}}}_{g(x)}$$

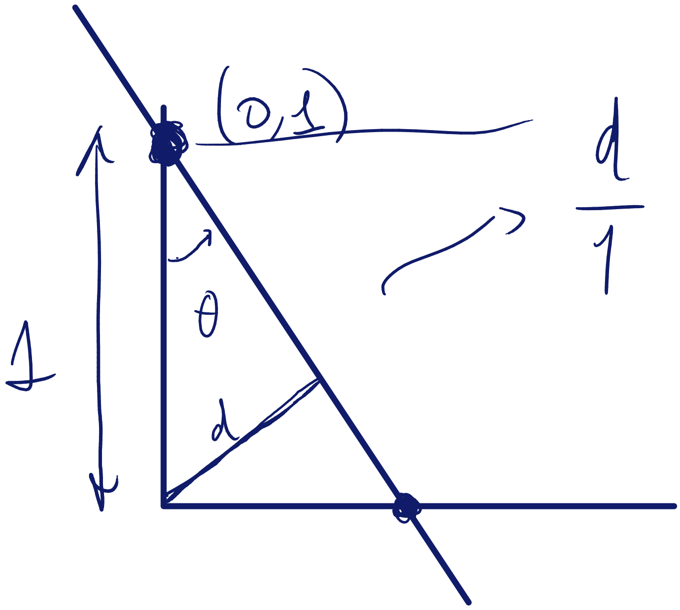
$$g'(x) = +e^{-e^{-x}} \cdot e^{-x}$$

$$\Rightarrow \frac{f(x)}{g'(x)} = \frac{c\, e^{-x} e^{-e^{-x}}}{e^{-x}\, e^{-e^{-x}}} = c.$$

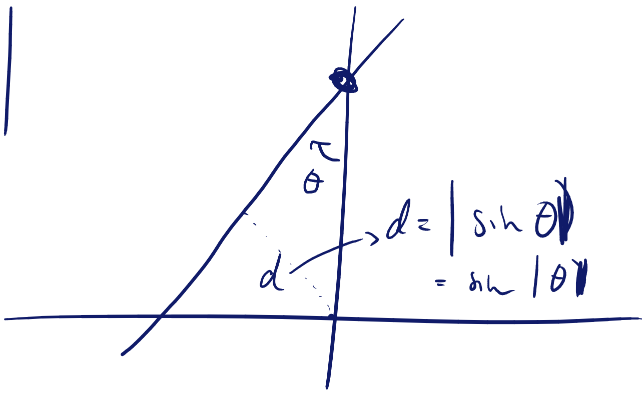
$$\int_{-\infty}^{\infty} f(x)\, dx = \int_{-\infty}^{\infty} c \cdot g'(x)\, dx = c\, g(x)\Big|_{-\infty}^{\infty} = c(1-0) = c = 1$$

2.17.

$$f(\theta) = \begin{cases} \frac{1}{2}\cos(\theta), & \theta \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right) \\ 0 & \text{otro caso} \end{cases}$$



$$\frac{d}{1} = \sin|\theta|$$



Variable: $\theta \equiv$ ángulo ...

$$D = \sin|\theta|$$

$$\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \Rightarrow D \in [0, 1) \boxed{\ni z}$$

$$P(D \leq z) = P(\sin|\theta| \leq z) =$$

$$= P(|\sin\theta| \leq z) = P(-z \leq \sin\theta \leq z)$$

$$= P\left(-\arcsin z \le \Theta \le \arcsin z\right)$$

$$\arcsin : [0,1] \longrightarrow \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$= \int_{-\arcsin z}^{\arcsin z} \frac{1}{2}\cos\theta \, d\theta = \frac{1}{2} \sin\theta \Big|_{-\arcsin z}^{\arcsin z} = z$$

$$\Rightarrow P(D \le z) = \begin{cases} z & \text{si } z \in [0,1) \\ 0 & \text{si } z < 0 \\ 1 & \text{si } z \ge 1 \end{cases}$$

2.18.

$$f(x) = \begin{cases} 0 & \text{si } x < 0 \\ c & \text{si } x \in [0,d] \\ 0 & \text{si } x > d \end{cases}$$

$d \equiv$ dist entre ciudades, $X \equiv$ dist a A.

$$\int_0^d c \, dx = 1 = dc \Rightarrow c = 1/d$$

$$Y \equiv \text{razón } \frac{\text{dist a } A}{\text{dist a } B}.$$

$$Y = \frac{X}{d-X} \longrightarrow Y \in (0, \infty)$$

a) Sea $r>0$,

$$P(Y > r) = P\left(\frac{X}{d-X} > r\right)$$

$(d > X)$

$$= P(X > r(d-X)) =$$

$$= P(X(1+r) > rd) = P\left(X > \frac{rd}{1+r}\right)$$

$$= 1 - \left(\frac{r}{1+r}\right), \text{ ya que } \frac{rd}{1+r} \in (0, d)$$

$$\boxed{P(Y=3) = 0}$$

$x \in (0, d)$

$$P(X < x) = \int_0^x \frac{1}{d}\,dt = \frac{x}{d}$$

$$P(X > x) = \int_x^d \frac{1}{d}\,dt = 1 - \frac{x}{d}$$

2.19. X = t. de servicio

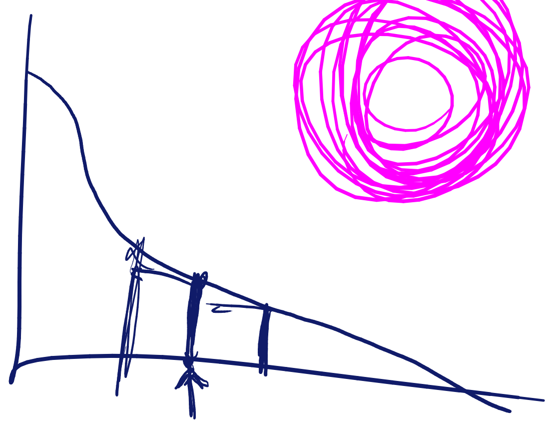
$$P(X < x) = \alpha e^{-\alpha x}$$ $\boxed{\begin{matrix}\alpha > 0 \\ x > 0\end{matrix}}$

Supongamos en días.

$$\Rightarrow \eta = \lfloor X \rfloor$$

$$F_\eta(y) = P(\eta \leq y) = P(\lfloor X \rfloor \leq y)$$

si $y \in \mathbb{N} \cup \{0\}$



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