PROBLEM SET 4

EXERCISE 1: Consider the sequence $X_1 X_2 X_3 \cdots X_n$ where X_i is any decimal digit and $n \in \mathbb{N}$

(a) A sequence in which no digit is repeated must have $n \leq 10$.

(or)

$T = T_1 + T_2 + \cdots + T_{10}$ where T_i is the task that corresponds to selecting i ordered digits without repetition

$$|T_i| = \binom{10}{i} \cdot i! = \frac{10!}{(10-i)!} \quad ; \begin{cases} \binom{10}{i} = (\text{ways of selecting } i \text{ digits out of } 10 \text{ possible digits}) \\ i! \text{ ways of ordering each set of } i \text{ digits} \end{cases}$$

Therefore $|T| = \sum_{i=1}^{10} |T_i| = \frac{10!}{9!} + \frac{10!}{8!} + \cdots + \frac{10!}{0!}$

$$\boxed{|T| = 10 + 10 \cdot 9 + 10 \cdot 9 \cdot 8 + \cdots + 10!}$$

(b) How many sequences: $X_1 X_2 \ldots X_n$ where $\forall i \in \{1, 2 \ldots n-1\}$ $X_i \leq X_{i+1}$?

$$\underbrace{* * * \cdots *}_{x_0} | \underbrace{* * * * \cdots}_{x_1} | * \cdots * | \cdots | \underbrace{* \cdots *}_{x_9} \quad \begin{cases} n \ * \\ + \\ 9 \ | \end{cases}$$

Where x_i is the number of digits "i" selected $x_i \in \{0, \ldots, n\}$

$$\binom{n+9}{9} = \frac{(n+9)!}{9! \, (n+9-9)!} = \frac{(n+9)!}{9! \, n!}$$

(c) Each digit can either appear (its possition will be the same) or not.

As there are 10 possible digits, there are 2^{10} possible combinations

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EXERCISE 3 A group of 24 students Divided in 6 groups of 4 students

How many combinations are there?

Out of 24, we choose 4; out of 20, we choose 4 ...

... and out of 4 remaining, we choose 4.

$$|T| = |T_1| \cdot |T_2| \cdots |T_6| = \binom{24}{4}\binom{20}{4}\binom{16}{4}\binom{12}{4}\binom{8}{4}\binom{4}{4} =$$

$$= \frac{24!}{(4!)^6}$$

EXERCISE 4 3 distinguishable dolls, 7 different dresses, 5 different hats.

(a) In how many ways can we dress the dolls?

$T = T_1$ and T_2 and T_3 where T_i is dressing doll i.

(dressing doll 1:)

T_1: 7 dresses and 5 hats : $7 \cdot 5 = |T_1|$

T_2: 6 dresses and 4 hats $|T_2| = 6 \cdot 4$

T_3: 5 dresses and 3 hats $|T_3| = 5 \cdot 3$

$$\boxed{|T| = |T_1| \cdot |T_2| \, |T_3| = 7 \cdot 6 \cdot 5 \cdot 5 \cdot 4 \cdot 3}$$

(b) Which is the answer if the dolls are indistinguishable?

We'd have to divide $|T|$ by $3!$ (ways of ordering the tree dolls).

Then answer would be: $\dfrac{7 \cdot 6 \cdot 5 \cdot 5 \cdot 4 \cdot 3}{3!} = \underline{\underline{7 \cdot 6 \cdot 5 \cdot 5 \cdot 2}}$

EXERCISE 5 $x_1 + x_2 + x_3 = 10$

(a) Possible solutions if $x_1, x_2, x_3 \geq 0$ $\rightarrow \binom{12}{2}$.

$\longrightarrow$ $\underbrace{x\,x\,x}_{x_1} \,|\, \underbrace{x\,x\,x\,x}_{x_2} \,|\, \underbrace{x\,x\,x}_{x_3}$ $\rightsquigarrow \binom{12}{2}$.

(b) Possible solutions if: $x_1 \geq 0$, $x_2 \geq -2$, $x_3 \geq 4$

$$\left.\begin{aligned} x_2 &= -2 + x_2' \\ x_3 &= 4 + x_3' \end{aligned}\right\} \quad \begin{aligned} &x_1 + x_2 + x_3 = x_1 + x_2' + x_3' + 4 - 2 = 10 \\ &\Leftrightarrow \underline{x_1 + x_2' + x_3' = 8} \end{aligned}$$

$\rightarrow \binom{10}{2}$.

EXERCISE 6 In how many ways can numbers 1, 2, 4, 5, 6, 8, 9, 10, 12 and 15 be arranged on a row such that any number must appear before its double?

1 before 2 before 4 before 8 — A, A, A, A

5 before 10 — B, B

6 before 12 — C, C

9 — D

15 — E

As numbers B (5 and 10) are always ordered between them in the same way, we'll treat them as if they were not distinguishable, the same with the rest.

4 A's, 2 B's, 2 C's, 1 D, 1 E → Total of 10

_ _ _ _ _ _ _ _ _ _

We place the A's then the B ... and then the E

$$|T| = \binom{10}{4}\cdot\binom{6}{2}\cdot\binom{4}{2}\binom{2}{1}\binom{1}{1} = \frac{10!}{4!\,\cancel{6!}}\cdot\frac{\cancel{6!}}{2!\,\cancel{4!}}\cdot\frac{\cancel{4!}}{2!\,\cancel{2!}}\cdot\frac{\cancel{2!}}{\cancel{1!}\,\cancel{1!}}\cdot\frac{\cancel{1!}}{1!}$$

$$= \boxed{\frac{10!}{4!\cdot 2!\cdot 2!}}$$

EXERCISE 7 3 strawberry candies, 3 lemon candies, 2 orange candies.

(a) In how many different ways can we give one candy to each of 8 different kids.

$T = T_s$ and T_l and T_o

T_s ↳ strawberry, T_l ↳ lemon, T_o ↳ orange candies

$$|T_s| = \binom{8}{3}, \quad |T_l| = \binom{8-3}{3} = \binom{5}{3}, \quad |T_o| = \binom{2}{2} = 1$$

$$|T| = \binom{8}{3}\cdot\binom{5}{3} = \boxed{\frac{8!}{3!\,3!\,2!}}$$

(b) If we add another candy (chocolate), but still give one to each of 8 different kids?

$K_1, K_2, K_3, K_4, K_5, K_6, K_7, K_8$, (To no kid) : 9! ways of ordering the candies

We divide by 3!, 3!, 2!, 1!, because candies of

$$\text{So } |T| = \boxed{\frac{9!}{3!\,3!\,2!}}$$

EXERCISE 8

(a) Ana, Beatriz, Carlos and David have each read a book from a list of 10 books. How many different cases are there? ⊛

Total number of cases: 10^4 (each person can read 1 of the 10 books)

Cases in which they all read the same book: 10 (one for each book)

$$⊛ \# \text{cases} = 10^4 - 10 = 10(10^3 - 1) = 10(999) = \boxed{9990}.$$

(b) Juan has lego bricks of 8 different colours. After having made 56 different groups where all have the same number of bricks and all bricks in a group are of different colours, he discovers that it's impossible to make a different group. Is it possible to know how many bricks are there in each group?

Let's say there are n bricks per group.

$$\underbrace{- - - \cdots -}_{n \leq 8}$$

We have to choose n bricks out of 8 every time.

Cases: $\binom{8}{n} = \frac{8!}{n!(8-n)!} = 56$

$\binom{8}{1} = 8$, $\binom{8}{2} = \frac{8!}{2!6!} = \frac{8 \cdot 7}{2} = 4 \cdot 7 = 28.$

$\binom{8}{3} = \frac{8 \cdot 7 \cdot \cancel{6}}{\cancel{3!}} = 56$

$\binom{8}{5} = \binom{8}{3} = 56$

There are 3 or 5 bricks per group.

EXERCISE 9 | I have planted 5 seeds randomly. (Pomelo, Lemon, Orange)

(a) What are the possible different combinations if order does not matter?

$$* \ * \mid * \ * \mid * \rightsquigarrow \overset{(5)}{*\ \text{Seeds}}, \overset{}{\mid (2)}$$

$5 \geq n_P, n_L, n_O \geq 0$

$\rightarrow \# = \binom{7}{2} = \frac{7!}{2!5!} = 21.$

(b) And if order does matter?

3 cases: $(5,0,0) \longrightarrow 3$

3 cases $*2 \rightarrow \begin{cases} (4,1,0) \longrightarrow 3 \cdot 2\binom{5}{1} = 6 \cdot 5 = 30 \\ (3,2,0) \longrightarrow 3 \cdot 2\binom{5}{2} = 6 \cdot 10 = 60 \end{cases}$

3 cases $(3,1,1) \longrightarrow 3 \cdot \frac{5!}{3!} = 5 \cdot 4 \cdot 3 = 60$

3 cases $(1,2,2) \longrightarrow 3 \cdot \frac{5!}{2!2!} = \frac{3 \cdot 5 \cdot \cancel{4} \cdot 3 \cdot 2}{\cancel{2!2!}} = 90$

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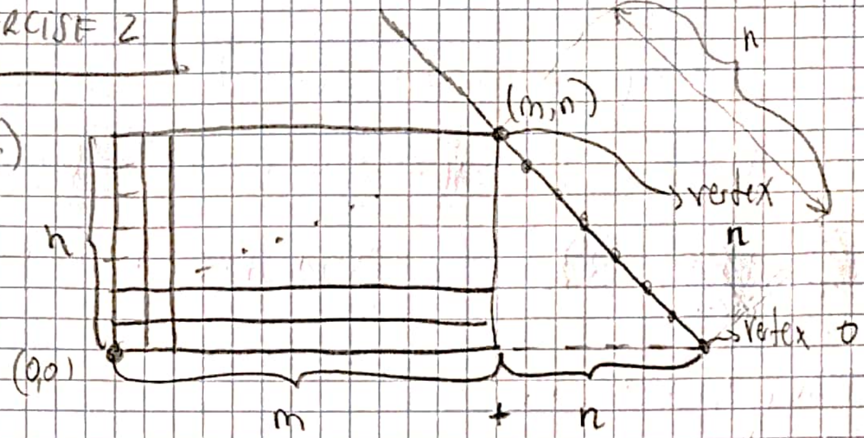
(21 cases.)

$3 + 30 + 60 + 60 + 90 = \underline{\underline{243}}$

EXERCISE 2

(a)



We can reduce this by seeing the point (m,n) as a number of the form $\binom{a}{b}$ from Pascal's triangle.

Where a is the row in which it appears and b, the position it has in said row.

We can see that, if $(0,0)$ is the uppermost point of the triangle, then (m,n) is in row $n+m$, with position n.

So there will be $\binom{n+m}{n}$ different paths from $(0,0)$ to (m,n).

(b) How many different paths are there from $(0,0)$ to (m,n) that do not pass through (i,j) where $0<i<m$, $0<j<n$?

$\longrightarrow$ # paths from $(0,0)$ to (m,n)

$-$ (# paths from (i,j) to (m,n) · # paths from $(0,0)$ to (i,j)).

$$= \boxed{\binom{m+n}{n} - \binom{m-i+n-j}{n-j}\cdot\binom{i+j}{j}}$$

(*)

(*) (# paths from (i,j) to (m,n) =

$=$ # path from $(i-i, j-j)=(0,0)$ to $(m-i, n-j)$

$$= \binom{m-i+n-j}{n-j}\Bigg)$$