PROBLEM SET 1. Propositional Logic. EDyL.

EXERCISE 1.

$$\Delta = \{ B \Leftrightarrow [A \Leftrightarrow (\neg B \wedge A \wedge \neg C)],\ C \Leftrightarrow (A \vee B) \}.$$

(i) Truth table:

	A	B	C	B	$\Leftrightarrow$ [3]	A	$\Leftrightarrow$ [2]	$(\neg B \wedge A \wedge \neg C)$ [1]	C	$\Leftrightarrow$ [5]	$(A \vee B)$ [4]	Δ
I_1:	1	1	1	1	0	1	0	0	1	1	1	–
I_2:	1	1	0	1	0	1	0	0	0	0	1	–
I_3:	1	0	1	0	1	1	0	0	1	1	1	Model
I_4:	1	0	0	0	0	1	1	1	0	0	1	–
I_5:	0	1	1	1	1	0	1	0	1	1	1	Model
I_6:	0	1	0	1	1	0	1	0	0	0	1	–
I_7:	0	0	1	0	0	0	1	0	1	0	0	–
I_8:	0	0	0	0	0	0	1	0	0	1	0	–

(ii) The knowledge base is (SAT) satisfiable, but not a tautology, because there are two interpretations which are models (I_3 and I_5 have truth value "TRUE" for both WFF that form the knowledge base.)

A WFF is a logical consequence of a knowledge base when the models of said knowledge base are also models of the WFF.

Therefore, (iii) A is not a logical consequence of Δ: truth values of A for models of Δ: I_3 "1", I_5 "0"; $\Delta \not\models A$.

(iv) $\Delta \not\models \neg A$, truth values of $\neg A$ for the models of Δ: I_3: "0", I_5: "1".

(v) $\Delta \not\models B$, truth values of B for the models of Δ: I_3: "0", I_5 "1".

(vi) $\Delta \not\models \neg B$, truth values of $\neg B$ for the models of Δ: I_3: "1", I_5: "0".

(vii) $\Delta \models C$, truth values of C for the models of Δ: I_3: "1", I_5: "1".

(viii) $\Delta \not\models \neg C$, truth values of $\neg C$ for the models of Δ: I_3: "0", I_5: "0".

EXERCISE 2

(i) w_1, w_2 and w are WFF which simultaneously fulfill:

$\{w_1, w_2, \neg w\}$ is UNSAT , $\{w_1, w_2, w\}$ is SAT.

This is one example very similar to proof by contradiction, where we have the knowledge base $\Delta = \{w_1, w_2\}$* and, to know if w is a logical consequence of Δ, we proof that $\alpha = \{w_1, w_2, \neg w\}$ is UNSAT.

Therefore it is true that $\Delta \models w$, so $\{w_1, w_2\} \models w$.

* NOTE: we know that Δ is SAT because if $\{w_1, w_2, w\}$ is SAT, then $\{w_1, w_2\}$ must be also SAT.

(a) $\{w_1, w_2\} \not\models w$ is incorrect, because w is, in fact, a logical consequence of $\{w_1, w_2\}$.

(b) $\{w_1, w_2\} \not\models \neg w$ is correct, because $\{w_1, w_2\} \models w$, so, for all models of $\{w_1, w_2\}$ the truth value of $\neg w$ is "FALSE".

(c) "w is SAT" is correct, because if $\{w_1, w_2, w\}$ is SAT, then w_1, w_2 and w have truth value "TRUE" all in the same interpretation at least once, and that means that w is "TRUE" for at least one interpretation and is, therefore SAT.

(d) "$w_1 \wedge w_2$ is SAT" is also correct, just like in (c), there must be at least one interpretation where w_1, w_2, w are all three "TRUE", so there is at least one interpretation where w_1 and w_2 are both true, so there are models of $(w_1 \wedge w_2)$ and it is SAT.

(e) It is not possible to determine whether $w_1 \wedge w_2 \wedge w$ is a tautology, because there may be interpretations that are not models, but it is still SAT. There is not information enough to know.

(EXERCISE 2) (ii) $\Delta = \{A \vee B,\ (A \Rightarrow B) \vee (A \Leftrightarrow \neg C),\ \neg C \Leftrightarrow (\neg A \vee \neg B)\}$.

Transformation to CNF:

[1] $(A \vee B)$

[2] $(A \Rightarrow B) \vee (A \Leftrightarrow \neg C) \overset{(\text{def} \Rightarrow, \Leftrightarrow)}{\equiv} (\neg A \vee B) \vee [(\neg A \wedge \neg\neg C) \vee (A \wedge \neg C)] \equiv$ [¬¬Elim], [Associative laws]

$\equiv \neg A \vee B \vee (\neg A \wedge C) \vee (A \wedge \neg C)$ [Distributive laws]

$\equiv (\neg A \vee B \vee \neg A \vee A) \wedge (\neg A \vee B \vee \neg A \vee \neg C) \wedge (\neg A \vee B \vee C \vee A) \wedge (\neg A \vee B \vee C \vee \neg C)$

[Idempotency, excluded middle laws.]

$\equiv \top \wedge (\neg A \vee B \vee \neg C) \wedge \top \wedge \top$ [Neutral element]

$\equiv (\neg A \vee B \vee \neg C)$ [2]

[3] $\neg C \Leftrightarrow (\neg A \vee \neg B) \overset{(\text{def} \Leftrightarrow), (\text{def} \Rightarrow)}{\equiv} (\neg\neg C \vee (\neg A \vee \neg B)) \wedge (\neg(\neg A \vee \neg B) \vee \neg C)$ [DM] [¬¬ elim.]

$\equiv (C \vee (\neg A \vee \neg B)) \wedge ((A \wedge B) \vee \neg C)$ [Associative and Distributive laws]

$\equiv (C \vee \neg A \vee \neg B) \wedge (A \vee \neg C) \wedge (B \vee \neg C)$ [3]. [1], [2], [3] $\vdash_{\wedge \text{intro}}$

$$\boxed{\Delta: \{\underbrace{(A \vee B)}_{[1]} \wedge \underbrace{(\neg A \vee B \vee \neg C)}_{[2]} \wedge \underbrace{(C \vee \neg A \vee \neg B)}_{[3.1]} \wedge \underbrace{(A \vee \neg C)}_{[3.2]} \wedge \underbrace{(B \vee \neg C)}_{[3.3]}\}}$$

$\vdash_{\wedge \text{elim}}$

[1] $A \vee B$
[2] $\neg A \vee B \vee \neg C$ $\vdash_{RES_A}$ $B \vee B \vee \neg C \overset{(\text{Idempotency})}{\equiv} B \vee \neg C$

[3.1] $C \vee \neg A \vee \neg B$
[3.2] $A \vee \neg C$ $\vdash_{Res_C}$ $A \vee \neg A \vee \neg B \overset{(\text{excluded middle law})}{\equiv} \top$

[3.3] $\longrightarrow (B \vee \neg C)$

$\vdash_{\wedge \text{INTRO}}$ $(B \vee \neg C) \wedge \top \wedge (B \vee \neg C) \equiv$ [Neutral element, Idempotency]

$\equiv (B \vee \neg C)$ is alredy part of Δ, while neither C nor $\neg C$ can be infered by Resolution with the clauses from Δ, so we can say that:

Neither C nor $\neg C$ is a logical consequence of Δ.

EXERCISE 3

When two individuals of the same species speak to each other, they always tell the truth. When two individuals of different species speak to each others, they always lie.

Atoms: A: "A is Akritian". ($\neg A$: "A is Denobulan")

B: "B is Akritian". ($\neg B$: "B is Denobulan")

C: $\underbrace{\text{"C is Akritian"}}_{\text{Denotation.}}$ ($\neg C$: "C is Denobulan").

(i) Formalize the knowledge base as WFFs in propositional logic.

A to B: $\underbrace{\text{You are Akritian if and only if C is Akritian}}$

$B \Leftrightarrow C$ is true if and only if A and B are of the same species: $A \Leftrightarrow B \equiv (A \wedge B) \vee (\neg A \wedge \neg B)$

The same goes for the two others, so:

Sentence in natural language	WFFs
• A and B are of the same species if and only if B and C are of the same species :	[1] $(A \Leftrightarrow B) \Leftrightarrow (B \Leftrightarrow C)$
• B and C are of the same species if and only if either C is Denobulan or A is Akritian (or both)	[2] $(B \Leftrightarrow C) \Leftrightarrow (A \vee \neg C)$
• A and C are of the same species if and only if B and A are Akritians :	[3] $(A \Leftrightarrow C) \Leftrightarrow (A \wedge B)$

(ii) $\Delta: \{(A \Leftrightarrow B) \Leftrightarrow (B \Leftrightarrow C), (B \Leftrightarrow C) \Leftrightarrow (A \vee \neg C), (A \Leftrightarrow C) \Leftrightarrow (A \wedge B)\}$

[1] $(A \Leftrightarrow B) \Leftrightarrow (B \Leftrightarrow C)$ [def $\Leftrightarrow$]

$\equiv ((A \Leftrightarrow B) \wedge (B \Leftrightarrow C)) \vee (\neg (A \Leftrightarrow B) \wedge \neg (B \Leftrightarrow C))$ [def $\Leftrightarrow$], [Associative]

$\equiv [(\neg A \vee B) \wedge (\neg B \vee A) \wedge (\neg B \vee C) \wedge (\neg C \vee B)] \vee [\neg ((A \wedge B) \vee (\neg A \wedge \neg B)) \wedge \neg ((B \wedge C) \vee (\neg B \wedge \neg C))]$

[DM, $\neg\neg$ Elimin, Associative laws]

$\equiv [(\neg A \vee B) \wedge (\neg B \vee A) \wedge (\neg B \vee C) \wedge (\neg C \vee B)] \vee [(\neg A \vee \neg B) \wedge (A \vee B) \wedge (\neg B \vee \neg C) \wedge (B \vee C)]$

[Distributive laws, Idempotency, excluded middle laws - $(\neg A \vee A) \equiv T$]

$\equiv T \wedge T \wedge T \wedge (\neg A \vee B \vee C) \wedge T \wedge T \wedge (A \vee \neg B \vee \neg C) \wedge T \wedge (\neg A \vee \neg B \vee C) \wedge T \wedge T \wedge T \wedge T \wedge (A \vee B \vee \neg C) \wedge T \wedge T$. [Neutral element.]

$\equiv (\neg A \vee B \vee C) \wedge (A \vee \neg B \vee \neg C) \wedge (\neg A \vee \neg B \vee C) \wedge (A \vee B \vee \neg C)$ [1].

[2] $(B \Leftrightarrow C) \Leftrightarrow (A \vee \neg C)$ [def $\Leftrightarrow$]

$\equiv ((B \Leftrightarrow C) \wedge (A \vee \neg C)) \vee (\neg(B \Leftrightarrow C) \wedge \neg(A \vee \neg C))$ [def $\Leftrightarrow$, $\neg\neg$ elimin, DM]

$\equiv [(\neg B \vee C) \wedge (B \vee \neg C) \wedge (A \vee \neg C)] \vee [\neg((B \wedge C) \vee (\neg B \wedge \neg C)) \wedge (\neg A \wedge C)]$ [Associative law, DM, $\neg\neg$ Elimin]

$\equiv [(\neg B \vee C) \wedge (B \vee \neg C) \wedge (A \vee \neg C)] \vee [(\neg B \vee \neg C) \wedge (B \vee C) \wedge \neg A \wedge C]$ [Absorption, Commutative law: $(B \vee C) \wedge C \equiv C$.]

$\equiv [(\neg B \vee C) \wedge (B \vee \neg C) \wedge (A \vee \neg C)] \vee [(\neg B \vee \neg C) \wedge \neg A \wedge C]$ [Distributive laws, Idempotency, excluded middle laws]

$\equiv T \wedge (\neg A \vee \neg B \vee C) \wedge (\neg B \vee C) \wedge T \wedge (\neg A \vee B \vee \neg C) \wedge T \wedge (A \vee \neg B \vee \neg C) \wedge T \wedge T$
[Neutral element]

$\equiv (\neg A \vee \neg B \vee C) \wedge (\neg B \vee C) \wedge (\neg A \vee B \vee \neg C) \wedge (A \vee \neg B \vee \neg C)$ [2]

[3] $(A \Leftrightarrow C) \Leftrightarrow (A \wedge B)$ [def $\Leftrightarrow$], [Associative laws]

$\equiv [(A \Leftrightarrow C) \wedge A \wedge B] \vee [\neg(A \Leftrightarrow C) \wedge \neg(A \wedge B)]$ [DM, def $\Leftrightarrow$

$\equiv [(\neg A \vee C) \wedge (\neg C \vee A) \wedge A \wedge B] \vee [\neg((\neg A \wedge \neg C) \vee (A \wedge C)) \wedge (\neg A \vee \neg B)]$ [Associative + Commutative laws, DM, $\neg\neg$ elimin., Absorption: $(\neg C \vee A) \wedge A \equiv A$]

$\equiv [(\neg A \vee C) \wedge A \wedge B] \vee [(A \vee C) \wedge (\neg A \vee \neg C) \wedge (\neg A \vee \neg B)]$
[Distributive laws, Idempotency, excluded middle laws]

$\equiv T \wedge T \wedge (\neg A \vee \neg B \vee C) \wedge (A \vee C) \wedge T \wedge T \wedge (A \vee B \vee C) \wedge (\neg A \vee B \vee \neg C) \wedge T$
{Neutral element]

$\equiv (\neg A \vee \neg B \vee C) \wedge (A \vee C) \wedge (A \vee B \vee C) \wedge (\neg A \vee B \vee \neg C)$ [3]

[1], [2], [3] $\vdash_{\wedge \text{INTRO}}$ [Associative laws, Idempotency, Commutative laws]

$$\boxed{\begin{aligned}\Delta: \{ & (A \vee B \vee C) \wedge (A \vee B \vee \neg C) \wedge (A \vee \neg B \vee \neg C) \wedge (\neg A \vee B \vee C) \wedge (\neg A \vee B \vee \neg C) \wedge (\neg A \vee \neg B \vee C) \\ & \wedge (A \vee C) \wedge (\neg B \vee C) \}. \quad \text{(CNF)}.\end{aligned}}$$

$\vdash_{\wedge \text{Elimin}}$

[a] $A \vee B \vee C$
[b] $A \vee B \vee \neg C$ $\vdash_{RES_A}$ $A \vee B$ [ab]

[c] $A \vee \neg B \vee \neg C$
[g] $A \vee C$ $\vdash_{RES_C}$ [cg] $A \vee \neg B$

[ab], [cg] $\vdash_{RES_B}$ A

[a] $A \vee B \vee C$
[h] $\neg B \vee C$ $\vdash_{RES_B}$ [ah] $A \vee C$

[d] $\neg A \vee B \vee C$
[h] $\neg B \vee C$ $\vdash_{RES_B}$ [dh] $\neg A \vee C$

[ah], [dh] $\vdash_{RES_A}$ C

[d] $\neg A \vee B \vee C$
[e] $\neg A \vee B \vee \neg C$ $\vdash_{RES_C}$ [de] $\neg A \vee B$

[ab] $A \vee B$

[de], [ab] $\vdash_{RES_A}$ B

A, B, C $\vdash_{\wedge \text{INTRO}}$ $(A \wedge B \wedge C)$

A, B and C are all Arabitan.

[f] $\neg A \vee \neg B \vee C$, [e] $A \vee B \vee \neg C$ $\vdash_{RES_C}$ $A \vee \neg A \vee B \vee \neg B \equiv T$ [Neutral element]

$$\boxed{\Delta \models A \wedge B \wedge C}$$

EXERCISE 4

If The lobster, The Gryphon or the Mock Turtle are sane, their beliefs are always true, otherwise they are false.

(a)

Atoms		Denotation
L	:	"The Lobster is sane"
G	:	"The Gryphon is sane"
M	:	"The Mock Turtle is sane"

(b)

	WFF	Meaning
Δ: [1]	$L \Longleftrightarrow (G \Leftrightarrow (G \wedge \neg L \wedge \neg M))$.	The lobster believes that the gryphon believes that he (the gryphon) is the only one of the three that is sane.
[2]	$M \Leftrightarrow (L \vee G)$.	The mock turtle believes that either the lobster or the gryphon (or both) is sane

(c) Transform the WFFs in Δ to CNF.

[1] $L \Leftrightarrow (G \Leftrightarrow (G \wedge \neg L \wedge \neg M))$ [def $\Leftrightarrow$]

$\equiv (L \wedge (G \Leftrightarrow (G \wedge \neg L \wedge \neg M)) \vee (\neg L \wedge \neg (G \Leftrightarrow (G \wedge \neg L \wedge \neg M)))$

[def $\Leftrightarrow$]

$\equiv (L \wedge ((\neg G \vee (G \wedge \neg L \wedge \neg M)) \wedge (G \vee \neg (G \wedge \neg L \wedge \neg M)))) \vee$

$\vee (\neg L \wedge \neg [(G \wedge (G \wedge \neg L \wedge \neg M)) \vee (\neg G \wedge \neg (G \wedge \neg L \wedge \neg M))])$ [Associative laws, DM, $\neg\neg$ Elimin., Distributive laws]

$\equiv (L \wedge \overbrace{(\neg G \vee G)}^{T} \wedge (\neg G \vee \neg L) \wedge (\neg G \vee \neg M) \wedge \overbrace{(G \vee \neg G \vee L \vee M)}^{T}) \vee$

$\vee (\neg L \wedge \neg [(G \wedge \neg L \wedge \neg M) \vee \overbrace{(\neg G \wedge (\neg G \vee L \vee M))}^{\neg G}]$ [Excluded middle laws, DM $\neg\neg$ Elimin, Associative laws, neutral element, Absorption]

$\equiv (L \wedge (\neg G \vee \neg L) \wedge (\neg G \vee \neg M)) \vee$

$\vee (\neg L \wedge (\neg G \vee L \vee M) \wedge G)$ [Distributive Laws, Commutative laws]

$\equiv (((L \wedge \neg G) \vee \underbrace{(L \vee \neg L)}_{T}) \wedge (\neg G \vee \neg M)) \vee (((\underbrace{\neg L \wedge L}_{T}) \vee (\neg L \wedge (\neg G \vee M))) \wedge G)$ [Excluded middle laws, neutral element, Associative laws]

$\equiv (L \wedge \neg G \wedge (\neg G \vee \neg M)) \vee (\neg L \wedge (\neg G \vee M) \wedge G)$ [Distributive laws]

$\equiv (L \wedge \neg G \wedge (\neg G \vee \neg M)) \vee (\neg L \wedge ((\neg G \wedge G) \vee (G \wedge M))$ [excluded middle laws]

$\equiv (L \wedge \neg G \wedge (\neg G \vee \neg M)) \vee (\neg L \wedge G \wedge M)$ [Absorption]

$\equiv (L \wedge \neg G) \vee (\neg L \wedge G \wedge M)$ [Distributive laws, excluded middle law]

$\equiv \top \wedge (L \vee G) \wedge (L \vee M) \wedge (\neg G \vee \neg L) \wedge \top \wedge (\neg G \vee M)$ [neutral element

[1] $\equiv \underline{(L \vee G) \wedge (L \vee M) \wedge (\neg G \vee \neg L) \wedge (\neg G \vee M)}$

[2] $M \Leftrightarrow L \vee G$ [def $\Leftrightarrow$

$\equiv (M \wedge (L \vee G)) \vee (\neg M \wedge \neg (L \vee G))$ [D.M, Associative law]

$\equiv (M \wedge (L \vee G)) \vee (\neg M \wedge \neg L \wedge \neg G)$ [Distributive law, excluded middle law]

$\equiv \top \wedge (M \vee \neg L) \wedge (M \vee \neg G) \wedge (L \vee G \vee \neg M) \wedge \top \wedge \top.$ [neutral element]

$\equiv \underline{(M \vee \neg L) \wedge (M \vee \neg G) \wedge (L \vee G \vee \neg M)}$

Δ: [1], [2] $\vdash_{\wedge INTRO}$ [Associative law, Idempotency]

$$\boxed{\Delta = \{(L \vee G) \wedge (L \vee M) \wedge (\neg L \vee \neg G) \wedge (M \vee \neg G) \wedge (M \vee \neg L) \wedge (L \vee G \vee \neg M)\}}$$

(d) $\vdash_{\wedge ELIMIN}$ $\begin{matrix} L \vee M \\ M \vee \neg L \end{matrix} \vdash_{RES_L} M \rightarrow \Delta \models M$ M is a logical consequence of Δ.
The Mock turtle is sane.

$\begin{matrix} L \vee G \\ M \vee \neg G \end{matrix} \vdash_{RES_G} L \vee M$ → Either the M.Turtle or the Lobster is sane, or both; but we alredy have this clause.

(e) If we also know that either the lobster, ~~but~~ or the M. Turtle is sane, but not both.

then we know: $M \Leftrightarrow \neg L \equiv (\neg M \vee \neg L) \wedge (M \vee L)$
[def $\Leftrightarrow$, $\neg\neg$ Elimin]

Our knowledge base becomes:

$\Delta = \{(L \vee G) \wedge (L \vee M) \wedge (\neg L \vee \neg G) \wedge (M \vee \neg G) \wedge (M \vee \neg L) \wedge (L \vee G \vee \neg M) \wedge (\neg M \vee \neg L) \wedge (M \vee L)\}$

$\vdash_{\wedge ELIMIN}$ $\begin{matrix} L \vee M \\ M \vee \neg L \end{matrix} \vdash_{RES_L} M$; $\begin{matrix} M \\ \neg M \vee \neg L \end{matrix} \vdash_{RES_M} \neg L$; $\begin{matrix} \neg L \\ L \vee G \end{matrix} \vdash_{RES_L} G$; $\left\{\begin{matrix} M \\ \neg L \\ G \end{matrix}\right. \vdash_{\wedge INTRO} M \wedge \neg L \wedge G$

[Now $\Delta \models M$, $\Delta \models \neg L$, $\Delta \models G$.]

If we know $M \Leftrightarrow \neg L$, then $(M \wedge \neg L \wedge G)$ is a logical consequence of Δ, so the Mock Turtle and the Gryphon are sane, and the Lobster is not.