HOJA DE EJERCICIOS 9
Análisis Matemático (Grupo 130)
CURSO 2021–2022.

Problema 1. Para cada uno de los siguientes campos de velocidades en el plano, halla los caminos integrales y las transformaciones de flujo. Dibuja el retrato de fase.

$$(1,y)\quad,\quad(1,x)\quad,\quad(x,x^2)\;,$$
$$(x,y)\quad,\quad(x,-y)\quad,\quad(y,-x)\;.$$

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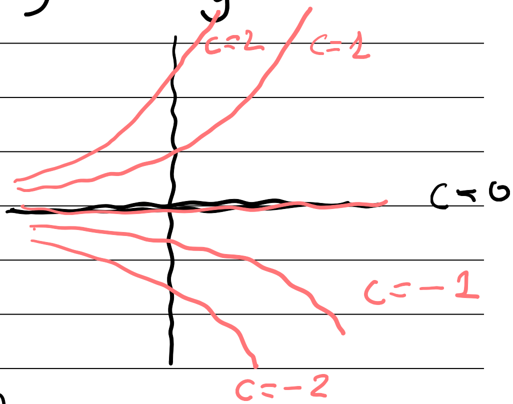
a) $V_1(x,y)=(1,y)$; $\alpha(t)=(x(t),y(t))$ es un camino integral para V_1 si

$$\begin{cases} \frac{dx}{dt}=1 \\ \frac{dy}{dt}=y \end{cases} \Rightarrow \frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}=y/1=y \Rightarrow \frac{dy}{y}=dx$$

$\Rightarrow \ln|y|=x+C \Rightarrow y=Ce^x$



Transformación de flujo:

$\varphi:\mathbb{R}\times\mathbb{R}^2\longrightarrow\mathbb{R}^2$

$\varphi(t,x,y)=(\varphi_1,\varphi_2)$

tal que

$$\begin{cases} \frac{\partial}{\partial t}\varphi(t,x,y)=V_1(\varphi_1,\varphi_2)=(1,\varphi_2) \\ \varphi(0,x,y)=(x,y) \end{cases}$$

Es decir

$$\begin{cases} \frac{\partial}{\partial t}\varphi_1=1 \\ \varphi_1(0,x,y)=x \end{cases} \Rightarrow \varphi_1(t,x,y)=t+C_1(x,y) \text{ con } x=\varphi_1(0,x,y)=C_1(x,y) \Rightarrow \varphi_1(t,x,y)=t+x$$

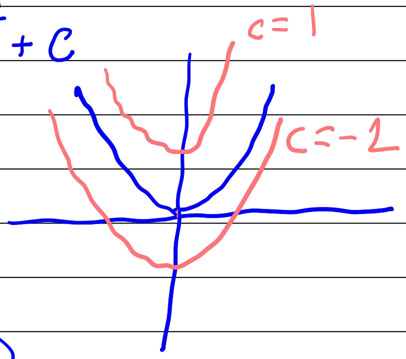
$$\begin{cases} \frac{\partial}{\partial t}\varphi_2=\varphi_2 \\ \varphi_2(0,x,y)=y \end{cases} \Rightarrow \varphi_2(t,x,y)=C_2(x,y)e^t \text{ con } y=\varphi_2(0,x,y)=C_2(x,y) \Rightarrow \varphi_2(t,x,y)=ye^t$$

Conclusión: $\varphi(t,x,y)=(t+x,\; ye^t)$

(c) $V_3(x,y)=(x,x^2)$. Camino integral $\alpha(t)=(x(t),y(t))$ tal que

$$\left\{\begin{array}{l} \dfrac{dx}{dt} = x \\ \dfrac{dy}{dt} = x^2 \end{array}\right\} \Rightarrow \frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt} \Rightarrow \frac{dy}{dx} = \frac{x^2}{x} = x \Rightarrow$$

$$dy = x\,dx \Rightarrow y = \frac{x^2}{2} + C$$



Transformación del flujo

$$\varphi : \mathbb{R} \times \mathbb{R}^2 \longrightarrow \mathbb{R}^2$$

$$\varphi(t,x,y) = (\varphi_1, \varphi_2)$$

tal que

$$\left\{\begin{array}{l} \dfrac{\partial \varphi}{\partial t}(t,x,y) = V_2(\varphi_1,\varphi_2) = (\varphi_1, \varphi_1^2) \\ \varphi(0,x,y) = (x,y) \end{array}\right\}$$

Por tanto

$$\left\{\begin{array}{l} \dfrac{\partial \varphi_1}{\partial t} = \varphi_1 \\ \varphi_1(0,x,y) = x \end{array}\right\} \Rightarrow \varphi_1(t,x,y) = C_1(x,y)\,e^t \quad \text{con } x = \varphi_1(0,x,y)$$

$$= C_1(x,y) \Rightarrow \varphi_1(t,x,y) = x\,e^t$$

$$\left\{\begin{array}{l} \dfrac{\partial \varphi_2}{\partial t} = \varphi_1^2 \\ \varphi_2(0,x,y) = y \end{array}\right\} \quad \frac{\partial \varphi_2}{\partial t} = x^2 e^{2t} \Rightarrow \varphi_2(t,x,y) = x^2 \frac{e^{2t}}{2} + C_2(x,y)$$

$$\text{con } y = \varphi_2(0,x,y) = \frac{x^2}{2} + C_2(x,y) \Rightarrow$$

$$C_2(x,y) = y - \frac{x^2}{2}$$

Conclusión: $\varphi(t,x,y) = \left(x e^t,\ x^2 \dfrac{e^{2t}}{2} + y - \dfrac{x^2}{2}\right)$

Problema 2. Determina el valor de la constante c para que el campo sea un gradiente. Con ese valor de c, halla un potencial escalar para el campo.

$$(cxz\,,\,w\,,\,x^2\,,\,y) \qquad \text{en} \qquad \mathbb{R}^4_{xyzw}\,,$$

$$(2xye^z + xz\,,\,e^z x^2\,,\,x^2ye^z + cx^2) \qquad \text{en} \qquad \mathbb{R}^3_{xyz}\,,$$

$$(e^{yz} + z\,,\,xze^{yz} + y^2\,,\,xye^{yz} + cx) \qquad \text{en} \qquad \mathbb{R}^3_{xyz}\,.$$

(b) $V_2(x,y,z) = (2xye^z + xz,\ e^z x^2,\ x^2 y e^z + cx^2)$

Para que $V_2 = (F_1, F_2, F_2)$ sea un campo gradiente debe cumplirse que

$$\frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i} \qquad \forall\, i,j = 1,2,3$$

equivalente a que exista $f: \mathbb{R}^3 \to \mathbb{R}$ tal que $\nabla f = V_2$
(f se llama potencial escalar)

Debe cumplirse $\dfrac{\partial}{\partial x}(x^2 y e^z + cx^2) = \dfrac{\partial}{\partial z}(2xye^z + xz)$

$$2xye^z + 2cx = 2xye^z + x \Rightarrow c = 1/2$$

El potencial escalar $f(x,y,z) \in \mathbb{R}$ debe satisfacer

(1) $\dfrac{\partial f}{\partial x} = 2xye^z + xz$ (2) $\dfrac{\partial f}{\partial y} = e^z x^2$ (3) $\dfrac{\partial f}{\partial z} = x^2 y e^z + \frac{1}{2}x^2$

Usando (1)

$$\frac{\partial f}{\partial x} = 2xye^z + xz \Rightarrow f(x,y,z) = x^2 y e^z + \frac{x^2}{2}z + \varphi_1(y,z)$$

Usando (2)

$$e^z x^2 = \frac{\partial f}{\partial y} = x^2 e^z + \frac{\partial \varphi_1}{\partial y} \Rightarrow \frac{\partial \varphi_1}{\partial y} = 0 \Rightarrow \varphi_1(\ ,y,z) = \varphi_2(z)$$

$$f(x,y,z) = x^2 y e^z + \frac{x^2}{2}z + \varphi_2(z)$$

Usando (3)

$$x^2 y e^z + \frac{1}{2}x^2 = \frac{\partial f}{\partial z} = x^2 y e^z + \frac{x^2}{2} + \varphi_2'(z) \Rightarrow \varphi_2'(z) = 0$$

$$f(x,y,z) = x^2 y e^z + \frac{x^2}{2}z$$

(C) La solución es $c = 1$ y

$$f(x,y,z) = x e^{yz} + xz + \frac{y^3}{3}$$

Problema 3. Determina el valor de la constante c para que el siguiente campo en $\mathbb{R}^3$ sea un rotacional. Con ese valor de c, halla un potencial vector.

$$\left(y\,\mathrm{sen}(yz)\,,\, x^2y+z\,,\, 3y^2+cx^2z \right).$$

Si existiera $\vec{F}=(F_1,F_2,F_3)$ en $\mathbb{R}^3$ tal que $\mathrm{rot}(F)=$ $=V(x,y,z)=(y\,\mathrm{sen}(yz),\ x^2y+z,\ 3y^2+cx^2z)$ se debería tener

$$\mathrm{div}(V)=\mathrm{div}(\mathrm{rot}(F))=0$$

Es decir $\quad 0+x^2+cx^2=0 \quad \Rightarrow \quad c=-1$

Tomar $F_3=0$ (la solución no es única). Entonces

$$V(x,y,z)=\mathrm{rot}(\vec{F})=\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & 0 \end{vmatrix}=$$

$$=\left(-\frac{\partial F_2}{\partial z},\ \frac{\partial F_1}{\partial z},\ \frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)$$

Debe cumplirse

$$(1)\ -\frac{\partial F_2}{\partial z}=y\,\mathrm{sen}(yz) \qquad (2)\ \frac{\partial F_1}{\partial z}=x^2y+z$$

$$(3)\ \frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}=3y^2-zx^2$$

Usando (1)

$$\frac{\partial F_2}{\partial z}=-y\,\mathrm{sen}(yz) \Rightarrow F_2(x,y,z)=\cos(yz)+\varphi_2(x,y)$$

Usando (2)

$$x^2y+z=\frac{\partial F_1}{\partial z} \Rightarrow F_1(x,y,z)=x^2yz+\frac{z^2}{2}+\varphi_1(x,y)$$

Usando (3)

$$3y^2-zx^2=\frac{\partial \varphi_2}{\partial x}-x^2z-\frac{\partial \varphi_1}{\partial y} \Rightarrow \frac{\partial \varphi_2}{\partial x}-\frac{\partial \varphi_1}{\partial y}=3y^2$$

Por ejemplo, $\varphi_2 = 0$, $\varphi_1 = -y^3$

Conclusión $F = (F_1, F_2, F_3)$ con

$$F_1 = x^2yz + \frac{z^2}{2} - y^3, \quad F_2 = \cos(yz), \quad F_3 = 0$$

Problema 4. Halla un potencial vector para cada uno de los campos siguientes.

$$(2\,,\,x-e^x\,,\,3x^2y-2y)\quad,\quad(yz\,,\,xz\,,\,xy)\,.$$

Sol: Para el primero

$F_1(x,y,z)=(x-e^x)z+y^2,\quad F_2(x,y,z)=-2z+x^3y$

$F_3=0$

Para el segundo $\quad G=(G_1,G_2,G_3)$

$G_1=\frac{xz^2}{2}-\frac{xy^2}{2},\quad G_2(x,y,z)=-y\,z^2/2,\quad G_3(x,y,z)=0$

Problema 6. Sea $n \geq 2$. Consideramos el **radio esférico** $\rho \stackrel{\text{def}}{=} \sqrt{x_1^2 + \cdots + x_n^2}$. Halla una constante α tal que el siguiente campo en $\mathbb{R}^n \setminus \{\mathbf{0}\}$

$$\rho^\alpha \, \nabla\rho \,,$$

tenga divergencia idénticamente nula.

Sea $\vec{F}(x_1, .., x_n) = \rho^\alpha \nabla \rho = (x_1^2 + .. + x_n^2)^{\alpha/2} \nabla \rho$. Hay que hallar α para que

$$\operatorname{div}(\vec{F}) = 0 \quad \text{en } \mathbb{R}^n - \{0\}$$

$$\frac{\partial \rho}{\partial x_j} = \frac{\partial}{\partial x_j} (x_1^2 + .. + x_n^2)^{1/2} = \frac{x_j}{\rho}, \quad j = 1, 2, .., n$$

Entonces

$$\vec{F}(x_1, .., x_n) = \rho^{\alpha - 1} (x_1, x_2, .., x_n)$$

$$0 = \operatorname{div}(\vec{F}) = \sum_{j=1}^{n} \frac{\partial}{\partial x_j} (x_j \rho^{\alpha - 1}) = \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \left(x_j (x_1^2 + .. + x_n^2)^{\frac{\alpha - 1}{2}}\right)$$

$$\frac{\partial}{\partial x_j} \left(x_j (x_1^2 + .. + x_n^2)^{\frac{\alpha - 1}{2}}\right) = \rho^{\alpha - 1} + x_j \rho^{\alpha - 3} \cdot x_j (\alpha - 1)$$

Entonces,

$$0 = \sum_{j=1}^{n} \rho^{\alpha - 1} + (\alpha - 1) x_j^2 \rho^{\alpha - 3} = n \rho^{\alpha - 1} + (\alpha - 1) \rho^{\alpha - 3} \rho^2$$

$$= n \rho^{\alpha - 1} + (\alpha - 1) \rho^{\alpha - 1} = (n + \alpha - 1) \rho^{\alpha - 1}$$

$$\Rightarrow \alpha = 1 - n$$

Problema 7. En $\mathbb{R}^3 \setminus \{\mathbf{0}\}$ consideramos el "campo gravitatorio" $F \equiv \rho^{-2}\,\nabla\rho$.

a) Haz un dibujo de los abiertos siguientes

$$U_1 \;=\; \mathbb{R}^3 \setminus \big(\{(0,0)\}\times[0,+\infty)\big) \quad , \quad U_2 \;=\; \mathbb{R}^3 \setminus \big(\{(0,0)\}\times(-\infty,0]\big)\,.$$

b) Para $j = 1, 2$, comprueba que el campo G_j está definido en U_j y es un potencial vector para $F|_{U_j}$:

$$G_1 \;=\; \frac{(\,y\,,\,-x\,,\,0\,)}{\rho\,(\rho - z)} \quad , \quad G_2 \;=\; \frac{(\,-y\,,\,x\,,\,0\,)}{\rho\,(\rho + z)}\,.$$

¿Coinciden G_1 y G_2 en $U_1 \cap U_2$?

(a) $\vec{F} = \rho^{-2}\nabla\rho$ con $\rho = \sqrt{x^2+y^2+z^2}$. Entonces

$$\vec{F} = \frac{1}{\rho^2}\left(\frac{x}{\rho}, \frac{y}{\rho}, \frac{z}{\rho}\right) = \frac{1}{\rho^3}(x, y, z)$$

Comprobar que

$$\mathrm{rot}(G_1) = \vec{F}$$

$$\mathrm{rot}(G_1) = \nabla\times G_1 = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{y}{\rho(\rho-z)} & \frac{-x}{\rho(\rho-z)} & 0 \end{vmatrix}$$

Primera componente: $\frac{\partial}{\partial z}\left(\frac{x}{\rho(\rho-z)}\right) =$

$$x\frac{\partial}{\partial z}\left((x^2+y^2+z^2)^{-1/2}\left((x^2+y^2+z^2)^{1/2}-z\right)^{-1}\right)$$

$$= x\left[-\frac{1}{2}\cdot 2z\,\frac{1}{\rho^3(\rho-z)} + \frac{1}{\rho}(-1)\frac{1}{(\rho-z)^2}\left(\frac{1}{2}2z\frac{1}{\rho}-1\right)\right] =$$

$$= x\left[\frac{-z}{\rho^3(\rho-z)} - \frac{1}{\rho(\rho-z)^2}\left(\frac{z}{\rho}-1\right)\right]$$

$$= x\left[\frac{-z}{\rho^3(\rho-z)} + \frac{1}{\rho^2(\rho-z)^2}(\rho-z)\right] =$$

$$= \frac{x}{\rho^2(\rho-z)}\left[\frac{-z}{\rho}+1\right] = \frac{x}{\rho^3}$$

Para la segunda componente $-\frac{\partial}{\partial z}\left(\frac{y}{\rho(\rho-z)}\right)=\frac{y}{\rho^3}$

Haciendo el cálculo con la tercera componente

$$\frac{\partial}{\partial x}\left(\frac{-x}{\rho(\rho-z)}\right)-\frac{\partial}{\partial y}\left(\frac{y}{\rho(\rho-z)}\right)=\frac{z}{\rho^3}$$