lema: sea $\ell^{(k)} = (0 \dots 0 \ \ell^{(k)}_{k+1} \dots \ell^{(k)}_{n})^t$, por $k = 1 \dots n-1$

decimos $L_k = I + \ell^{(k)} \otimes e_k \qquad [(e_k)_j = \delta_{k,j}]$

$\Rightarrow$ i) $L_k^{-1} = I - \ell^{(k)} \otimes e_k$

ii) $L_k L_{k+1} = I + \ell^{(k)} \otimes e_k + \ell^{(k+1)} \otimes e_k$

demo:

i) $(I + \ell^{(k)} \otimes e_k)(I - \ell^{(k)} \otimes e_k)$

$$= I + \cancel{\ell^{(k)} \otimes e_k} - \cancel{\ell^{(k)} \otimes e_k} - \underbrace{(\ell^{(k)} \otimes e_k)^2}$$

ejercicio: demostrar que esta matriz es 0

ii) $(I + \ell^{(k)} \otimes e_k)(I + \ell^{(k+1)} \otimes e_{k+1})$

$$= I + \ell^{(k)} \otimes e_k + \ell^{(k+1)} \otimes e_{k+1} + T$$

$$T = \underbrace{(\ell^{(k)} \otimes e_k)}_{M} \underbrace{(\ell^{(k+1)} \otimes e_{k+1})}_{N}, \qquad t_{ij} = \sum_q m_{iq} n_{qj}$$

$$t_{ij} = \sum_{\cancel{q}\, k} \ell^{(k)}_i \underbrace{(e_k)_q}_{\delta_{k,q}} \ell^{(k+1)}_q (e_{k+1})_j$$

$$= \ell^{(k)}_i \underbrace{\ell^{(k+1)}_k}_{0} \delta_{j,k+1} = 0 \quad \forall\, i,j = 1 \dots n$$

$\#$

alternativa: $T = \begin{pmatrix} | \\ \ell^{(k)} \\ | \end{pmatrix} \underbrace{(- e_k -) \begin{pmatrix} | \\ \ell^{(k+1)} \\ | \end{pmatrix}} (- e_{k+1} -)$

$$\langle e_k, \ell^{(k+1)} \rangle = \ell^{(k+1)}_k = 0$$

demostración (teorema LU)

- $$U = U^{(n-1)} = L_{n-1}^{-1} U^{(n-2)} = \dots = L_{n-1}^{-1} L_{n-2}^{-1} \cdots L_1^{-1} \underbrace{U^{(0)}}_{A}$$

$$\Rightarrow \quad A = \underbrace{L_1 L_2 \cdots L_{n-1}}_{\text{ii) lema: } = I + \sum_{k=1}^{n-1} \ell^{(k)} \otimes e_k} U$$

- tenemos que demostrar que U triangular alta

$$U^{(k)} = L_k^{-1} U^{(k-1)}$$

$$u_{ij}^{(k)} \overset{\text{i) lema}}{=} \sum_q \left(I - \ell^{(k)} \otimes e_k\right)_{iq} u_{qj}^{(k-1)}$$

$$= u_{ij}^{(k-1)} - \sum_q \ell_i^{(k)} \delta_{qk} u_{q,j}^{(k-1)}$$

$$= u_{ij}^{(k-1)} - \ell_i^{(k)} u_{k,j}^{(k-1)}$$

$$\Rightarrow \quad u_{ij}^{(k)} = \begin{cases} u_{ij}^{(k-1)}, & i = 1 \dots k \\ u_{ij}^{(k-1)} - \dfrac{u_{ik}^{(k-1)}}{u_{kk}^{(k-1)}} u_{kj}^{(k-1)}, & i = k+1 \dots n \end{cases}$$

las primeras k filas no cambian

ver luego lo que se hace en el algoritmo

$\boxed{\text{si } j = k}$ $$u_{ik}^{(k)} = \begin{cases} u_{ik}^{(k-1)}, & i = 1 \dots k \\ 0, & i = k+1 \dots n \end{cases}$$

$\downarrow$

en la columna k de la k-ésima iteración, los elementos de las filas $k+1 \dots n$ son 0

$$\begin{pmatrix} \text{———} & & & \\ 0 & \text{———} & & \\ 0 & 0 & \text{——} & \\ 0 & 0 & 0 & \ddots \\ 0 & 0 & 0 & \end{pmatrix} \Rightarrow \text{la matriz } U^{(n-1)} \text{ es triangular alta}$$

\#

Algoritmo: input A matriz $n \times n$
outputs L, U "

```
U = A; L = I;
for k = 1 : n-1
    for i = k+1 : n
        L(i,k) = U(i,k) / U(k,k);
        for j = k : n
            U(i,j) = U(i,j) - L(i,k) U(k,j);
        end
    end
end
```

⚠ no hay control sobre dividir por 0

pregunta: qué pasa si ponemos j = k+1 : n ?