proposición: sea $A \in \mathbb{K}^{n \times n}$ y sean $\{r_i\}_{i=1}^{n} \in \mathbb{R}_+$:

$$\begin{cases} r_1 = \dfrac{1}{|a_{11}|} \displaystyle\sum_{j=2}^{n} |a_{1j}| \\ r_i = \dfrac{1}{|a_{ii}|} \left(\displaystyle\sum_{j=1}^{i-1} |a_{ij}| r_j + \sum_{j=i+1}^{n} |a_{ij}| \right) \end{cases} \qquad \xrightarrow{i = 2, \dots, n}$$

si $\max\limits_{i \in \{1 \dots n\}} r_i < 1 \Rightarrow$ la iteración de GS converge $\forall x_0$

demostración: queremos mostrar que $|||B_{GS}(A)|||_\infty < 1$

- $B_{GS}(A)\, x = \underline{-(D_A + L_A)^{-1} U_A\, x = y}$

  $\hookrightarrow$ queremos ver que $\|y\|_\infty < 1$ para $\|x\|_\infty = 1$

- recordemos la forma explícita de G-S

  $\hookrightarrow$ $y_i = -\displaystyle\sum_{j=1}^{i-1} \frac{a_{ij}}{a_{ii}} y_j - \sum_{j=i+1}^{n} \frac{a_{ij}}{a_{ii}} x_j$

$$\Rightarrow |y_i| \leq \frac{1}{|a_{ii}|} \left(\sum_{j=1}^{i-1} |a_{ij}| \cdot |y_j| + \sum_{j=i+1}^{n} |a_{ij}| \cdot \overbrace{|x_j|}^{\overset{1}{\shortparallel}} \right)$$

$$\leq \frac{1}{|a_{ii}|} \left(\sum_{j=1}^{i-1} |a_{ij}| \cdot |y_j| + \sum_{j=i+1}^{n} |a_{ij}| \right)$$

- $i = 1 \qquad |y_1| \leq r_1$

  $i = 2 \qquad |y_2| \leq \dfrac{1}{|a_{22}|} \left(\displaystyle\sum_{j=1}^{2-1} |a_{2j}| \cdot \overset{r_1}{\overset{\shortparallel}{|y_j|}} + \sum_{j=2+1}^{n} |a_{2j}| \right)$

  $\leq r_2$

  $\hookrightarrow$ $|y_i| \leq r_i \quad \forall\, i \in \{1, \dots, n\}$

- $\|y\|_\infty \leq \max\limits_{i=1..n} r_i < 1$. #

corolario : sea $A \in \mathbb{K}^{n\times n}$

si $\sum_{j=1, j\neq i}^{n} |a_{ij}| < |a_{ii}| \quad \forall i \in \{1 \dots n\}$

DIAGONAL DOMINANTE ESTRICTA POR FILAS

$\Rightarrow$ G-S converge $\forall x_0$

demostración

- $r_1 = \frac{1}{|a_{11}|} \sum_{j=2}^{n} |a_{1j}| < 1$ por hipotesis

- $r_i = \frac{1}{|a_{ii}|} \left(\sum_{j=1}^{i-1} |a_{ij}| r_j + \sum_{j=i+1}^{n} |a_{ij}| \right)$

$$\leq \frac{1}{|a_{ii}|} \sum_{j=1, j\neq i}^{n} |a_{ij}| < 1 \text{ por hipotesis.} \quad \#$$

Ejemplos :

- $A = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 3 & 1/2 & 0 \\ 0 & 1/2 & 4 & 1 \\ 1 & 0 & -1 & 7 \end{pmatrix}$ es diagonal dominante estricta por filas

- $A = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$ no es diagonal dominante estricta por filas : en las filas 2 y 3 $a_{ii} = 2$, $\sum_{j=1, j\neq i}^{n} |a_{ij}| = 2$

pero : $r_1 = \frac{1}{2}, \quad < 1$

$r_2 = \frac{1}{2}\left(\frac{1}{2} + 1\right) = \frac{3}{4} \quad < 1$

$r_3 = \frac{1}{2}\left(\frac{3}{4} + 1\right) = \frac{7}{8} \quad < 1$

$r_4 = \frac{1}{2} \cdot \frac{7}{8} = \frac{7}{16} \quad < 1 .$

teorema: sea $A \in \mathbb{K}^{n\times n}$, $A = A^*$, $A > 0$ $(\langle Ax, x\rangle > 0 \ \forall x)$

$\Rightarrow$ G-S converge $\forall x_0$.

demostración: queremos mostrar $\rho(B_{GS}(A)) < 1$

$\Leftrightarrow \forall \lambda$ autovalor de $B_{GS}(A)$: $|\lambda| < 1$

. $A = A^* \Rightarrow U_A = L_A^*$

. $A > 0 \Rightarrow$ todos sus elementos diagonales son > 0:

$a_{ii} = \langle A e_i, e_i \rangle > 0$

$\Rightarrow$ en particular podemos definir

$$D_A^{-1} = \begin{pmatrix} 1/a_{11} & & 0 \\ & 1/a_{22} & \\ 0 & & \ddots \ \frac{1}{a_{nn}} \end{pmatrix}, \quad D_A^{1/2} = \begin{pmatrix} \sqrt{a_{11}} & & 0 \\ & \sqrt{a_{22}} & \\ 0 & & \ddots \ \sqrt{a_{nn}} \end{pmatrix}$$

. $\boxed{B_{GS}(A) = -(D_A + L_A)^{-1} L_A^* \text{ tiene los mismos autovalores de } T B_{GS}(A) T^{-1}, \ \forall T \in \mathbb{K}^{n\times n} \text{ invertible}}$

. sea $T = D_A^{1/2}$:

$$D_A^{1/2} B_{GS}(A) D_A^{-1/2} = -D_A^{1/2} (D_A + L_A)^{-1} \underbrace{}_{I = D_A^{1/2} D_A^{-1/2}} L_A^* D_A^{-1/2}$$

$$= -D_A^{1/2} (D_A + L_A)^{-1} D_A^{1/2} D_A^{-1/2} L_A^* D_A^{-1/2}$$

$$= -\Big(\underbrace{D_A^{-1/2} (D_A + L_A) D_A^{-1/2}}_{I + \underbrace{D_A^{-1/2} L_A D_A^{-1/2}}_{= L}}\Big)^{-1} \underbrace{D_A^{-1/2} L_A^* D_A^{-1/2}}_{= L^*}$$

$$= -(I + L)^{-1} L^*$$

- sea $v_\lambda \in \mathbb{C}^n$: $-(I+L)^{-1} L^* v_\lambda = \lambda v_\lambda$, $\|v_\lambda\|_2 = 1$

$$\hookrightarrow \quad -L^* v_\lambda = \lambda (I+L) v_\lambda$$

$$\Rightarrow \quad -\langle v_\lambda, L^* v_\lambda \rangle = \lambda \langle v_\lambda, (I+L) v_\lambda \rangle = \lambda (1 + \langle v_\lambda, L v_\lambda \rangle)$$

$$\Rightarrow \quad \lambda = - \frac{\langle v_\lambda, L^* v_\lambda \rangle}{1 + \langle v_\lambda, L v_\lambda \rangle}$$

$$z = \langle v_\lambda, L v_\lambda \rangle \in \mathbb{C}$$

$$= \langle L^* v_\lambda, v_\lambda \rangle$$

$$= \overline{\langle v_\lambda, L^* v_\lambda \rangle}$$

$$= - \frac{\bar{z}}{1+z}$$

- $z = a + ib$, $a = \mathrm{Re}(z) \in \mathbb{R}$, $b = \mathrm{Im}(z) \in \mathbb{R}$

$$|\lambda|^2 = \left| \frac{a - ib}{1 + a + ib} \right|^2 = \frac{a^2 + b^2}{1 + a^2 + 2a + b^2}$$

$$\hookrightarrow \quad \boxed{|\lambda| < 1} \iff a^2 + b^2 < 1 + 2a + a^2 + b^2 \iff \boxed{1 + 2a > 0}$$

- $1 + 2a = \|v_\lambda\|_2^2 + 2\,\mathrm{Re}\, z = \|v_\lambda\|_2^2 + z + \bar{z}$

$$= \|v_\lambda\|_2^2 + \langle v_\lambda, L v_\lambda \rangle + \langle v_\lambda, L^* v_\lambda \rangle$$

$$= \langle v_\lambda, (I + L + L^*) v_\lambda \rangle$$

$$D_A^{-1/2} D_A D_A^{-1/2} + D_A^{-1/2} L_A D_A^{-1/2} + D_A^{-1/2} L_A^* D_A^{-1/2}$$

$$= D_A^{-1/2} \underbrace{(D_A + L_A + U_A)}_{A} D_A^{-1/2}$$

$$= \langle \underbrace{D_A^{-1/2} v_\lambda}_{x}, A \underbrace{D_A^{-1/2} v_\lambda}_{x} \rangle = \langle Ax, x \rangle > 0$$

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