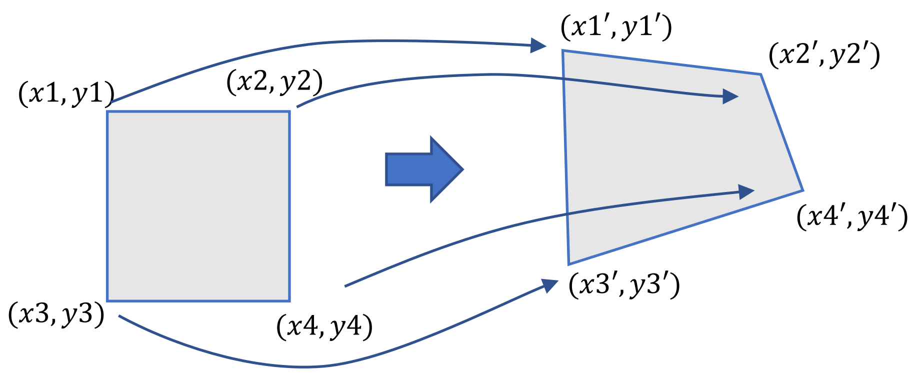
Perspective Transformation

we can transform rectangular shape to a different 3d orientation



for every (xi, yi) coordinate in the square we can find the transformed location (xi', yi')

by multiplying the point with the perspective transformation matrix

$$\begin{bmatrix} a11 & a12 & a13 \\ a21 & a22 & a23 \\ a31 & a32 & a33 \end{bmatrix} \begin{bmatrix} xi \\ yi \\ 1 \end{bmatrix} = \begin{bmatrix} xi' \\ yi' \\ 1 \end{bmatrix} \cdot \frac{1}{w}$$

$$w = xi \cdot a31 + yi \cdot a32 + a33$$

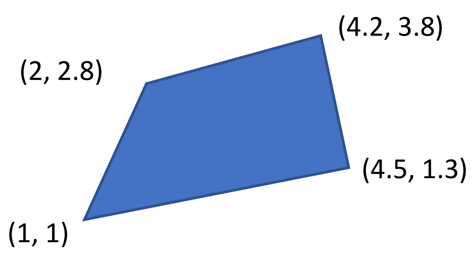
can also be writing like this

$$xi' = \frac{xi \cdot a11 + yi \cdot a12 + a13}{xi \cdot a31 + yi \cdot a32 + a33}$$

$$yi' = \frac{xi \cdot a21 + yi \cdot a22 + a23}{xi \cdot a31 + yi \cdot a32 + a33}$$

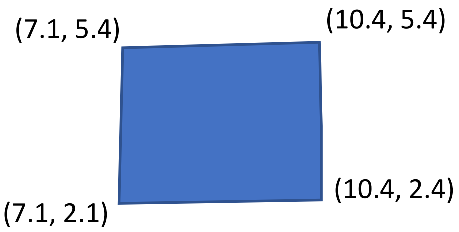
for example:

suppose we have the following rectangular shape and matrix



$$\begin{bmatrix} \frac{11571141}{2525110} & \frac{-1951189}{505022} & \frac{7800907}{1262555} \\ \frac{4169007}{5050220} & \frac{323397}{1010044} & \frac{1129257}{1262555} \\ \frac{356991}{1010044} & \frac{-385795}{1010044} & 1 \end{bmatrix}$$

The result of multiplying every point with the matrix in the way mentioned above is the following rectangle



now for a given set of 4 points and their corresponding transformed destinations

in order to find the matrix that can translate every point to its counterpart, we need to find the solution to the following set of equations

$$xi' = \frac{xi \cdot a11 + yi \cdot a12 + a13}{xi \cdot a31 + yi \cdot a32 + a33}$$

$$yi' = \frac{xi \cdot a21 + yi \cdot a22 + a23}{xi \cdot a31 + yi \cdot a32 + a33}$$

$$for\ i = 1, 2, 3, 4$$

moving things around…

$$xi \cdot a11 + yi \cdot a12 + a13 - xi' \cdot xi \cdot a31 - xi' \cdot yi \cdot a32 - xi' \cdot a33 = 0$$

$$xi \cdot a21 + yi \cdot a22 + a23 - yi' \cdot xi \cdot a31 - yi' \cdot yi \cdot a32 - yi' \cdot a33 = 0$$

$$for\ i = 1, 2, 3, 4$$

we can write the solution as the following matrix

$$\begin{bmatrix} x1 & y1 & 1 & 0 & 0 & 0 & -x1' \cdot x1 & -x1' \cdot y1 & -x1' \\ 0 & 0 & 0 & x1 & y1 & 1 & -y1' \cdot x1 & -y1' \cdot y1 & -y1' \\ x2 & y2 & 1 & 0 & 0 & 0 & -x2' \cdot x2 & -x2' \cdot y2 & -x2' \\ 0 & 0 & 0 & x2 & y2 & 1 & -y2' \cdot x2 & -y2' \cdot y2 & -y2' \\ x3 & y3 & 1 & 0 & 0 & 0 & -x3' \cdot x3 & -x3' \cdot y3 & -x3' \\ 0 & 0 & 0 & x3 & y3 & 1 & -y3' \cdot x3 & -y3' \cdot y3 & -y3' \\ x4 & y4 & 1 & 0 & 0 & 0 & -x4' \cdot x4 & -x4' \cdot y4 & -x4' \\ 0 & 0 & 0 & x4 & y4 & 1 & -y4' \cdot x4 & -y4' \cdot y4 & -y4' \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a11 \\ a12 \\ a13 \\ a21 \\ a22 \\ a23 \\ a31 \\ a32 \\ a33 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

one way to solve the matrix is to LU decompose it, then solving using gaussian elimination.

for example:

suppose we have the following set of points and their transformed counterpart:

$$(x, y) \rightarrow (x', y')$$

$$(1, 1) \rightarrow (7.1, 2.1)$$

$$(2, 2.8) \rightarrow (7.1, 5.4)$$

$$(4.2, 3.8) \to (10.4, 5.4)$$

$$(4.5, 1.3) \to (10.4, 2.4)$$

placing the points in the matrix

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & -7.1\cdot 1 & -7.1\cdot 1 & -7.1 \\ 0 & 0 & 0 & 1 & 1 & 1 & -2.1\cdot 1 & -2.1\cdot 1 & -2.1 \\ 2 & 2.8 & 1 & 0 & 0 & 0 & -7.1\cdot 2 & -7.1\cdot 2.8 & -7.1 \\ 0 & 0 & 0 & 2 & 2.8 & 1 & -5.4\cdot 2 & -5.4\cdot 2.8 & -5.4 \\ 4.2 & 3.8 & 1 & 0 & 0 & 0 & -10.4\cdot 4.2 & -10.4\cdot 3.8 & -10.4 \\ 0 & 0 & 0 & 4.2 & 3.8 & 1 & -5.4\cdot 4.2 & -5.4\cdot 3.8 & -5.4 \\ 4.5 & 1.3 & 1 & 0 & 0 & 0 & -10.4\cdot 4.5 & -10.4\cdot 1.3 & -10.4 \\ 0 & 0 & 0 & 4.5 & 1.3 & 1 & -2.4\cdot 4.5 & -2.4\cdot 1.3 & -2.4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a11 \\ a12 \\ a13 \\ a21 \\ a22 \\ a23 \\ a31 \\ a32 \\ a33 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

the solution is

$$\begin{bmatrix} a11 \\ a12 \\ a13 \\ a21 \\ a22 \\ a23 \\ a31 \\ a32 \\ a33 \end{bmatrix} = \begin{bmatrix} \frac{11571141}{2525110} \\ \frac{-1951189}{505022} \\ \frac{7800907}{1262555} \\ \frac{4169007}{5050220} \\ \frac{323397}{1010044} \\ \frac{1129257}{1262555} \\ \frac{356991}{1010044} \\ \frac{-385795}{1010044} \\ 1 \end{bmatrix}$$

and in the matrix form

$$\begin{bmatrix} a11 & a12 & a13 \\ a21 & a22 & a23 \\ a31 & a32 & a33 \end{bmatrix} = \begin{bmatrix} \frac{11571141}{2525110} & \frac{-1951189}{505022} & \frac{7800907}{1262555} \\ \frac{4169007}{5050220} & \frac{323397}{1010044} & \frac{1129257}{1262555} \\ \frac{356991}{1010044} & \frac{-385795}{1010044} & 1 \end{bmatrix}$$

using LU decomposition and gaussian elimination to solve a matrix

$$Mx = b \rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{bmatrix} \begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

manipulating matrix M to find the LU decomposition of one of its forms

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{bmatrix}$$

use elementary row operations to get to a row echelon form, only by swapping rows and $Ri = Ri - (k \cdot Rj)$ operation allowed.

for every column pivot the highest absolute number.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{bmatrix} \xrightarrow{R1 \leftrightarrow R3} \begin{bmatrix} 7 & 8 & 10 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{bmatrix} \xrightarrow[R3 = R3 - \left(\frac{1}{7} \cdot R1\right)]{R2 = R2 - \left(\frac{4}{7} \cdot R1\right)} \begin{bmatrix} 7 & 8 & 10 \\ 0 & \frac{3}{7} & \frac{2}{7} \\ 0 & \frac{6}{7} & \frac{11}{7} \end{bmatrix} \xrightarrow{R2 \leftrightarrow R3}$$

$$\begin{bmatrix} 7 & 8 & 10 \\ 0 & \frac{6}{7} & \frac{11}{7} \\ 0 & \frac{3}{7} & \frac{2}{7} \end{bmatrix} \xrightarrow{R3 = R3 - (\frac{1}{2} \cdot R2)} \begin{bmatrix} 7 & 8 & 10 \\ 0 & \frac{6}{7} & \frac{11}{7} \\ 0 & 0 & -\frac{1}{2} \end{bmatrix} = U$$

to find L, start with an empty matrix and replay all operations

and for every operation $Ri = Ri - (k \cdot Rj)$ taking the k value and placing it in row i and column j.

at the end add I to the result

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R1 \leftrightarrow R3} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[R3 = R3 - \left(\frac{1}{7} \cdot R1\right)]{R2 = R2 - \left(\frac{4}{7} \cdot R1\right)} \begin{bmatrix} 0 & 0 & 0 \\ \frac{4}{7} & 0 & 0 \\ \frac{1}{7} & 0 & 0 \end{bmatrix} \xrightarrow{R2 \leftrightarrow R3}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ \frac{1}{7} & 0 & 0 \\ \frac{4}{7} & 0 & 0 \end{bmatrix} \xrightarrow{R3 = R3 - (\frac{1}{2} \cdot R2)} \begin{bmatrix} 0 & 0 & 0 \\ \frac{1}{7} & 0 & 0 \\ \frac{4}{7} & \frac{1}{2} & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{7} & 1 & 0 \\ \frac{4}{7} & \frac{1}{2} & 1 \end{bmatrix} = L$$

the result

$$L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{7} & 1 & 0 \\ \frac{4}{7} & \frac{1}{2} & 1 \end{bmatrix}, \qquad U = \begin{bmatrix} 7 & 8 & 10 \\ 0 & \frac{6}{7} & \frac{11}{7} \\ 0 & 0 & -\frac{1}{2} \end{bmatrix}$$

$$L \cdot U = \begin{bmatrix} 7 & 8 & 10 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

going back to the matrix M we need to <u>reverse</u> all row swapping operations

$$\begin{bmatrix} 7 & 8 & 10 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \xrightarrow[\substack{R2 \leftrightarrow R3 \\ R1 \leftrightarrow R3}]{} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{bmatrix}$$

it should be done also on the result b

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 10 \end{bmatrix} \begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \xrightarrow[\substack{R2 \leftrightarrow R3 \\ R1 \leftrightarrow R3}]{}$$

$$\begin{bmatrix} 7 & 8 & 10 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

now we know how to LU decompose the matrix in this equation

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{7} & 1 & 0 \\ \frac{4}{7} & \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 7 & 8 & 10 \\ 0 & \frac{6}{7} & \frac{11}{7} \\ 0 & 0 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

given $\mathrm{L} \cdot \mathrm{U} \cdot \mathrm{x} = \mathrm{b}$ suppose we have a solution to the following equation

$$L \cdot y = b$$

then with the result vector y solving

$$U \cdot x = y$$

to find x.

in our example first let's solve $L \cdot y = b$

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{7} & 1 & 0 \\ \frac{4}{7} & \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} y1 \\ y2 \\ y3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ \frac{1}{7} & 1 & 0 & 1 \\ \frac{4}{7} & \frac{1}{2} & 1 & 2 \end{array}\right]$$

$$y1 = 3$$

$$y2 + 3 \cdot \frac{1}{7} = 1 \rightarrow y2 = \frac{4}{7}$$

$$y3 + 3 \cdot \frac{4}{7} + \frac{4}{7} \cdot \frac{1}{2} = 2 \ \rightarrow y3 = 0$$

and now $U \cdot x = y$

$$\begin{bmatrix} 7 & 8 & 10 \\ 0 & \frac{6}{7} & \frac{11}{7} \\ 0 & 0 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix} = \begin{bmatrix} 3 \\ \frac{4}{7} \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 7 & 8 & 10 \\ 0 & \frac{6}{7} & \frac{11}{7} \\ 0 & 0 & -\frac{1}{2} \end{bmatrix} \ \begin{bmatrix} 3 \\ \frac{4}{7} \\ 0 \end{bmatrix}$$

$$-\frac{1}{2} \cdot x3 = 0 \ \rightarrow x3 = 0$$

$$\frac{6}{7} \cdot x2 + 0 \cdot \frac{11}{7} = \frac{4}{7} \ \rightarrow x2 = \frac{2}{3}$$

$$7 \cdot x1 + \frac{2}{3} \cdot 8 + 0 \cdot 10 = 3 \ \rightarrow x1 = -\frac{1}{3}$$