Experimental Methods in Particle Physics - Homework 1

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1.

The Future Circular Collider, a 100 km long synchrotron accelerator at CERN is under planning by the High Energy Physics Community with proton-proton, electron-proton, electron-electron as well as heavy ion collision modes.

- How strong magnets are needed to keep 50 TeV proton beams on orbit at this facility?
- How high energy Pb^{82+} ion could such magnets keep on orbit?
- How large would be the center-of-mass energy of an electron – proton collider that accelerates protons to 50 TeV and electrons to 0.2 TeV?

ANSWER

Estimation of the necessary strength of magnet in synchrotron

Using the relativistic dispersion relation for the proton beam energy one can write $E_p^2 = (p_p c)^2 + (m_p c^2)^2$, where p_p denotes the magnitude of the spatial momentum for the proton, m_p is the proton's rest mass and c is the speed of light. Let us assume that the $m_p c^2 \approx 1$ GeV $\ll 50$ TeV rest energy of the proton is negligible in this case, hence we can use the ultrarelativistic limit: $E_p \approx p_p c$.

In order to estimate the necessary strength of the magnetic field, one can equate the relativistic centripetal force and the Lorentz force:

$$F_{\mathrm{rcp}} = F_{\mathrm{Lorentz}} \qquad \Longrightarrow \qquad \frac{\gamma m_p v^2}{R} = qvB \ , \tag{1}$$

where v and q are the velocity and the electric charge of the proton, R is the radius of the synchrotron and B is the magnetic field. The γ symbol stands for $1/\sqrt{1-({}^{v}\!/_{c})^2}$. One can notice that the magnitude of the spatial momentum appears in (1), hence the magnetic field can be expressed and estimated in the following way:

$$\gamma m_p v = p_p \approx \frac{E_p}{c} \approx qBR \qquad \Longrightarrow \qquad B \approx \frac{E_p}{qRc} \approx \frac{50\ \mathrm{TeV} \cdot 2\pi}{1.6 \cdot 10^{-19}\ \mathrm{C} \cdot 100\ \mathrm{km} \cdot 3 \cdot 10^{8}\ {}^{\mathrm{m}}\!/_{\mathrm{s}}} \approx 10.49\ \mathrm{T}\ . \tag{2}$$

Estimation of possible lead ion beam energy

For the Pb^{82+} lead ion we do not know what beam energy to expect, so we shall perform the calculation without the ultrarelativistic approximation. The rest energy of the lead is $m_{\text{Pb}}c^2 \approx 193$ GeV and its electric charge is 82 times the elementary charge, thus $q \approx 82 \cdot 1.6 \cdot 10^{-19}$ C. If one does not neglect the rest energy of the lead ion, the magnitude of the spatial momenta can be written as

$$p_{\text{Pb}} = qBR = \frac{\sqrt{E_{\text{Pb}}^2 - m_{\text{Pb}}^2 c^4}}{c} \; . \tag{3}$$

Thus the possible lead ion beam energy is[1]

$$\begin{aligned} E_{\text{Pb}} &= \sqrt{(qBRc)^2 + m_{\text{Pb}}^2 c^4} \\ &\approx \sqrt{\left(82 \cdot 1.6 \cdot 10^{-19}\ \text{C} \cdot 10.49\ \text{T} \cdot \frac{100\ \text{km}}{2\pi} \cdot 3 \cdot 10^8\ \text{m/s}\right)^2 + (193\ \text{GeV})^2} \approx 4.1\ \text{PeV} \; . \end{aligned} \tag{4}$$

Estimation of centre of mass energy in p - e collision

The beam energies for the proton and electron beams are $E_p = 50$ TeV and $E_e = 0.2$ TeV respectively in the lab frame. In order to calculate the centre of mass (CM) energy one can calculate the s Mandelstam variable (since $\sqrt{s} = E^{\text{CM}}$). By definition $s = c^2(p_p + p_e)^2$, where p_p and p_e are the four-momenta of the colliding proton and electron respectively. One can expand the expression for s in the following way:

$$s = c^2(p_p + p_e)^2 = m_p^2 c^4 + m_e^2 c^4 + 2c^2 p_p \cdot p_e \; . \tag{5}$$

Let us neglect the rest energies in this calculation as well.

$$s \approx 2c^2 p_p \cdot p_e = 2c^2(p_p^0 p_e^0 - \mathbf{p}_p \mathbf{p}_e) = 2E_p E_e - 2c^2 |\mathbf{p}_p||\mathbf{p}_e| \cos\vartheta \; . \tag{6}$$

Let us assume that the angle $\vartheta = \pi$ at collision, thus $\cos\vartheta = -1$. Using the relativistic dispersion relation (and neglecting the rest energies again)

$$s \approx 2E_p E_c + 2c^2 |\mathbf{p}_p||\mathbf{p}_e| = 2E_p E_c + 2\sqrt{(E_p^2 - m_p^2 c^4)(E_e^2 - m_e^2 c^4)} \approx 4E_p E_c \; . \tag{7}$$

Hence the centre of mass energy is $E^{\text{CM}} = \sqrt{s} \approx \sqrt{4 \cdot 50 \cdot 0.2}$ TeV ≈ 6.32 TeV.

[1] One might repeat the calculation with neglected rest energy and would get nearly the same result, hence the lead ion can be also considered ultrarelativistic.

2.

Estimate the energy stored in the LHC beams using the accelerator parameters given in: `http://pdg.lbl.gov/2018/reviews/rpp2018-rev-hep-collider-params.pdf`.

- How heavy should be a high-speed train that travels with 200 km/h to have the same energy?
- How much water could be boiled at room temperature with the same energy?

ANSWER

Estimating the energy stored in the LHC beam

According to the *heavy ion collider* data the maximum beam energy in the LHC for Pb-Pb collisions is $E_{\text{beam},n} = 2.51$ TeV/nucleon, which means $E_{\text{beam}} = 208 \cdot E_{\text{beam},n} = 522.08$ TeV for the whole lead nucleus. There are $N_{\text{particle}} = 1.9 \cdot 10^8$ particles in a single bunch and there are $N_{\text{bunch}} \approx 518$ bunches in one ring (per species), which can be considered as the number of bunches in a beam. This way the total energy stored in a beam can be estimated as

$$E_{\text{total}} = N_{\text{bunch}} \cdot N_{\text{particle}} \cdot E_{\text{beam}} \approx 518 \cdot 1.9 \cdot 10^8 \cdot 522.08 \text{ TeV} \approx 5.138 \cdot 10^{13} \text{ TeV} \approx 8.232 \text{ MJ} \ . \tag{8}$$

Estimating the mass of the train

The classical formula for the kinetic energy is $E_{\text{kin}} = {}^1\!/_2\, mv^2$, from which the mass is

$$\frac{2E_{\text{kin}}}{v^2} \stackrel{!}{=} \frac{2E_{\text{total}}}{v^2} \approx \frac{2 \cdot 8.232 \text{ MJ}}{(200 \ {}^{\text{km}}\!/_{\text{h}})^2} \approx 5.334 \text{ t} \ . \tag{9}$$

Estimating the amount of water

The specific heat of water is $c \approx 4.18$ kJ kg^{-1} °C^{-1}, $\Delta T = 80$ °C is the temperature difference from 20 to 100 °C and the latent heat for boiling water is $L = 2256.37$ kJ kg^{-1}. The required energy to reach the boiling temperature of water of mass m is $Q = cm\Delta T$ and to convert all the water to vapor we need $Q_{\text{boiling}} = Lm$. This way one can write $E_{\text{total}} = Q + Q_{\text{boiling}} = m(c\Delta T + L)$, from which the mass is

$$m = \frac{Q + Q_{\text{boiling}}}{c\Delta T + L} \stackrel{!}{=} \frac{E_{\text{total}}}{c\Delta T + L} \approx \frac{8.232 \text{ MJ kg}}{4.18 \text{ kJ °C}^{-1} \cdot 80 \text{ °C} + 2256.37 \text{ kJ}} = 3.177 \text{ kg} \ . \tag{10}$$

3.

Deuterons, the nuclei of heavy hydrogen, are accelerated in a cyclotron. Determine the frequency of the voltage source, if the value of magnetic field strength in the cyclotron is 1.5 T. Determine the cyclotron radius for particles, which leave the cyclotron with a kinetic energy of 16 MeV. How many times does the deuteron cross between the "D" electrodes (also called "dees"), if the electrical potential difference between the two dees is 50 kV?

ANSWER

Calculation of the frequency of the voltage source

We know that the rest energy of the deuteron is $m_d c^2 \approx 1.875$ GeV and its electric charge is the same as the elementary charge: $q \approx 1.6 \cdot 10^{-19}$ C. Ideally (in cyclotron resonance) the frequency of the voltage source is equal to the cyclotron frequency, hence it can simply be calculated (using the previously used notations) as follows:

$$f_{\mathrm{RF}} \stackrel{!}{=} f_{\mathrm{cyc}} = \frac{qB}{2\pi m_d} \approx \frac{1.6 \cdot 10^{-19}\ \mathrm{C} \cdot 1.5\ \mathrm{T} \cdot (3 \cdot 10^8\ \mathrm{m/s})^2}{2\pi \cdot 1.875\ \mathrm{GeV}} \approx 11.44\ \mathrm{MHz}\ . \tag{11}$$

Determining the proper cyclotron radius at given kinetic energy

Let us do this calculation relativistically[2]. The kinetic energy can be written as the difference of the total and the rest energy: $E_{\mathrm{kin}} = E - m_d c^2 = \sqrt{(p_d c)^2 + (m_d c^2)^2} - m_d c^2 = 16$ MeV. From this formula and using the results from Exercise 1. one can write

$$p = \frac{\sqrt{(E_{\mathrm{kin}} + m_d c^2)^2 - (m_d c^2)^2}}{c} \stackrel{!}{=} qRB\ . \tag{12}$$

From this the cyclotron radius can be expressed and calculated as

$$R = \frac{\sqrt{(E_{\mathrm{kin}} + m_d c^2)^2 - (m_d c^2)^2}}{cqB} = \frac{\sqrt{(16\ \mathrm{MeV} + 1.875\ \mathrm{GeV})^2 - (1.875\ \mathrm{GeV})^2}}{3 \cdot 10^8\ \mathrm{m/s} \cdot 1.5\ \mathrm{T} \cdot 1.6 \cdot 10^{-19}\ \mathrm{C}} \approx 54.6\ \mathrm{cm}\ . \tag{13}$$

Calculation of number of "dee"-crosses

In every single cross the deuteron particle gets additional $\Delta E = q\Delta U = 1.6 \cdot 10^{-19}\ \mathrm{C} \cdot 50\ \mathrm{kV} = 50$ keV energy. This way, while getting closer and closer to the final $E = 16$ MeV energy, it crosses between the "dees" $N = E/\Delta E = 320$ times.

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[2] We could have done classically (as $E_{\mathrm{kin}} = 1/2\, m_d v^2$) as well, the difference is $\mathcal{O}(m^{-4})$.