Cosmology Group Project
PROJECT DOCUMENT

February 12, 2024

PHYS 364 Group Project (2023/24) v2.0 (12/2)
Dark Energy: Evidence and Determination

Overview

The accelerating expansion of the Universe is one of the biggest mysteries in science today. This can be explained by adding an exotic component to the energy density known as "dark energy". In this project you will show why dark energy is necessary, find out how we hope to understand its nature in the future by observing the expansion history of the Universe via standard length scales and standard candles, and investigate a class of dark energy models based on scalar fields. You will also investigate an alternative explanation for the accelerating expansion, based on modifying gravity rather than adding a dark energy component.

* Throughout this project it will be assumed that the Universe is flat. This is expected from inflation, which is necessary to explain why the observed Universe is even close to a flat Universe today (as observed), and which shows that the most natural value for the total energy density at present corresponds to a flat Universe with $\Omega = 1$.

The project consists of three sections:

<u>(I) Direct Observational Evidence for Dark Energy:</u> Explaining why we need dark energy based on the *directly observed* age and content of the Universe.

<u>(II) Future Probes of Dark Energy:</u> Studying future satellite missions and how they will determine the nature of dark energy from:

(i): The observation of the Baryon Acoustic Oscillation (BAO) feature in galaxy surveys.

(ii): The determination of the luminosity distance versus redshift relation by using the peak magnitudes of Type Ia supernovae.

<u>(III) Modified Gravity as an Alternative to Dark Energy:</u> Investigating a Modified Gravity model and its predictions for future observations.

The project requires:

• Explanations of background physics.
• Theoretical calculations.
• Computational analyses.
• Summaries of the future Euclid (ESA) and Nancy Grace Roman [previously known as WFIRST] (NASA) space telescope missions.

Summary of the Project and Report Objectives

Part I:

(a) ΛCDM model:

• Brief explanation of the use of globular clusters for age bounds.

• Clear explanation of the derivation of Eq.1 showing t_0 as a function of $\Omega_{\Lambda,0}$ for the ΛCDM model.

• A figure showing t_0 vs. $\Omega_{\Lambda,\,0}$ for the ΛCDM model, with 1-σ and 2-σ observed age ranges from globular clusters shown on the plot.

• Statement of the range of $\Omega_{\Lambda,\,0}$ values consistent with the 1-σ and 2-σ age bounds from globular clusters, respectively.

• A plot of the age t in Gyr versus redshift z for the case of ΛCDM with $\Omega_{\Lambda,\,0} = 0.7$, for z in the range 0 to 2.

(b) General w_X dark energy model:

• A figure showing plots of t_0 versus $\Omega_{m,\,0}$ for equation of state parameters $w_X = -0.3, -0.5, -0.7, -0.9$ for dark energy with a constant equation of state w_X.

• Determination (with method explained) of the range of w_X values consistent with:

(i) the 1-σ observational bounds on $\Omega_{m,0}$ and t_0

(ii) the 2-σ observational bounds on $\Omega_{m,0}$ and t_0.

Part II, preliminary calculation:

• Explanation of the form of the $d(t_i)$ integral.

• Derivation of the $d(z)$ integral in terms of $F(a)$ and statement of the function $F(a)$.

Part II(i), Euclid space telescope/BAO project:

• Description of the Euclid telescope and project. This should include a statement of the predicted sensitivity of Euclid to the CPL w_p and w_a parameters.

• Explanation of the physics of BAO and the BAO length.

• Derivation of the integral for $d(z)$ for the case of the CPL equation of state.

• Explanation of the code you have developed to determine the mean values of $(\overline{w}_p, \overline{w}_a)$ and the chi-squared error ellipses.

• A figure showing plots of the 1-σ and 2-σ error ellipses on the (w_a, w_p) plane and statement of the values of $(\overline{w}_p, \overline{w}_a)$ for your Euclid simulated data.

Comment on:

• whether the simulated dark energy can be distinguished from a cosmological constant

• whether the time-dependence of its equation of state can be established.

Part II(ii), Roman (WFIRST) space telescope/Luminosity Distance project:

• Description of the Roman telescope and project.

• Explanation of how d_L is related to the measured light flux $\mathcal{F}$ from a Type Ia supernova and its known absolute luminosity $\mathcal{L}$, and why $d_L(z) = (1+z)d(z)$.

• Explanation of the code you have developed to determine $d_L(z)$ for a general dark energy model with equation of state parameter $w_X(a)$.

• Application of the general dark energy code to obtain a figure showing plots of $d_L(z)$ for ΛCDM and the Test SFQ model over the range $0 < z < 2$. (d_L should be expressed in Mpc.)

• A figure showing the plot of $\Delta d_L(z)/d_L(z)$ for the Test SFQ model.

• Determination of whether the Test SFQ model can be distinguished from ΛCDM by Roman.

• [Main Objective]: Plot of the values of a_t as a function of τ (for τ in the range 0.1 to 1) above which $|\Delta d_L/d_L|$ is (i) generally larger than the 2-σ bound expected from Roman and (ii) general smaller, over the range $0 < z < 2$. This will show the range of SFQ models that can and cannot be distinguished from ΛCDM by Roman.

Part III, Modified Gravity

• Qualitative description of the physics of the $\alpha = 1$ DGP model.

• Explanation of the derivation of r_c and $H(a)$ for the $\alpha = 1$ model. Statement of the value of r_c in Mpc.

• Derivation of the general expression for the deceleration parameter q in terms of the function g, with the function g stated.

• Derivation of q as a function of a for the $\alpha = 1$ DGP model, and statement of the present value of q for the DGP model.

- A figure showing a plot of $q(z)$ for the $\alpha = 1$ model together with a plot of $q(z)$ for the ΛCDM model.

- A figure showing a plot of $d_L(z)$ for the $\alpha = 1$ model, together with a plot of $d_L(z)$ for the ΛCDM model. (d_L should be expressed in Mpc.)

- A figure showing a plot of $|\Delta d_L(z)/d_L|$ for the $\alpha = 1$ model.

- Comment on the observability of the $\alpha = 1$ DGP model by Roman.

- Description of a code to analyse the case of a general value of α.

Project Details

Part (I): Evidence for dark energy from the directly observed age and matter content of the Universe.

In this section, the need for dark energy and limits on its nature will be established by comparing the theoretical age of the Universe with its *directly observed* age, t_0, and *directly observed* present matter content, $\Omega_{m,\,0}$.

• This has recently become important due to a discrepancy between the directly observed expansion rate from Type Ia supernova observations and the value theoretically inferred from the CMB observations using a ΛCDM dark energy model. It was found that the value from direct observation is larger than the value determined from CMB, with a $>$ 3-σ discrepancy that is too large to be purely due to observational errors. This is known as the 'Hubble tension' or the H_0 tension. As a result, it is important to determine as much about the Universe by direct observation (independently of CMB data), in order to understand the origin of this tension. (This is a major problem in modern cosmology.)

Part I(a): The age problem and its solution via a cosmological constant

In this part you will focus on the standard flat universe ΛCDM model (CDM = Cold Dark Matter), where dark energy is assumed to be a cosmological constant Λ. [For a flat Universe, $\Omega_{m\,,0}+\Omega_{\Lambda,\,0} = 1$.] The goal is to determine the 1-σ and 2-σ range of $\Omega_{\Lambda,\,0}$ required to account for the observed age of the Universe. From this you should be able to conclude that a pure matter Universe ($\Omega_{m\,,0} = 1$, $\Omega_{\Lambda,\,0} = 0$) is observationally excluded (this is the original age problem) and to show that it can be solved by including a cosmological constant.

• The age of the Universe can be determined via a range of observations and methods. One method is globular cluster ages. The project will use a quite recent estimation of the age of globular clusters, which is discussed in 1902.07081.

Tasks:

(1) [Update] Explanation of how the age is determined by using globular clusters. The report should review the methods used to obtain the age and dark matter content bounds discussed in 1902.07081. It should include an explanation of what a globular cluster is and why globular clusters are used for age determination. It should also discuss the developments in observation and data that have greatly improved age and content bounds in recent years, making the age of the Universe an increasingly competitive method for testing dark energy. It should also give short review of the Hubble constant problem (aka Hubble tension) and why direct observational bounds on the age and content of the Universe are important in this regard. The paper from which the data in Part I are obtained, 1902.07081, provides a good starting point. This replaces Task (1) in the original Project Document. In particular, the method used to obtain the bounds used in the project is not white dwarf ages but Main

Sequence Turn Off (MSTO) of globular clusters, which should be explained.

2.) The Age of the Universe in the ΛCDM Dark Energy Model:

The project requires a derivation of the theoretical formula for the age of a flat Universe made of pressureless matter and a cosmological constant (the simplest model of dark energy):

$$t_0 = \frac{2}{3}\frac{H_0^{-1}}{\sqrt{\Omega_{\Lambda,\,0}}} \ln\left[\frac{1+\sqrt{\Omega_{\Lambda,\,0}}}{\sqrt{1-\Omega_{\Lambda,\,0}}}\right] \ , \tag{1}$$

where H_0 is the present value of the Hubble constant. $\Omega_{m,\,0}$ is the present density parameter of pressureless matter (CDM + baryons) and $\Omega_{\Lambda,\,0}$ is the present density parameter of the cosmological constant, with $\Omega_{m\,,0} + \Omega_{\Lambda,\,0} = 1$ for a flat Universe. In general, $\Omega_{i,\,0} = \rho_{i,\,0}/\rho_0$ will denote the present fractional density of matter type i, where ρ_0 is the total energy density at present (= the present critical density $\rho_{c,\,0}$ in the case of a flat Universe).

(i) Start with the Friedmann equation for the case of a flat Universe with present total mass density $\rho_0 = \rho_{m\,,0} + \rho_\Lambda$, where $\rho_{m\,,0}$ and ρ_Λ are the present mass densities of matter and a cosmological constant respectively. You should be able to derive an integral relating the scale factor to time of the form

$$\int_0^a f(a')da' = \int_0^t \left(\frac{8\pi G}{3}\right)^{1/2} dt' \ , \tag{2}$$

where $f(a')$ is for you to derive. (Here we use a' and t' to distinguish the integration variables from the a and t.)

[Use the notation a_0 for the scale factor at the present time t_0. You may assume that $a(t) = 0$ at $t = 0$.]

(ii) Calculate the integral on the LHS. You may use standard integral results once you have manipulated the integral into the following simple form by making a suitable change of variable:

$$\int \frac{ds}{(1+s^2)^{1/2}}$$

This standard integral has two expressions: in terms of an inverse hyperbolic sine and in terms of a natural log. You should use the latter.

(iii) Obtain the final result for the Age of the Universe, Eq.(1), by evaluating your result at the present time, $t = t_0$.

(iv) It is also interesting to write down a formula for the Age of the Universe t at a redshift z. This tells us the age when an object with redshift z is observed. To do this, use your expression for the age t obtained from Eq.(2), but with a now set to the scale factor at redshift z rather than a_0. Hence state a formula for t at z for a given $\Omega_{\Lambda,0}$.

3). Results

Plot the theoretical age t_0 as a function of $\Omega_{\Lambda,\,0}$. (Age should be in Gyr.) On the same plot you should show the 1-σ and 2-σ observational age bounds. (The input data for Part I is given at the end of this section.) Hence determine (and show on the plot) the ranges of $\Omega_{\Lambda,\,0}$ for which the theoretical age is compatible with the 1-σ and 2-σ observationally allowed age ranges.

• In the report, state the range of $\Omega_{\Lambda,\,0}$ compatible with the 1-σ age range and the 2-σ age range, respectively.

You should be able to conclude that a pure matter Universe ($\Omega_{\Lambda,\,0} = 0$, $\Omega_{m,\,0} = 1$) is ruled out and that a component of dark energy in the form of a cosmological constant can make the observed age of the Universe consistent with the theoretical age.

[As a check, make sure that your numerical value for the theoretical age of the Universe in the case $\Omega_{\Lambda,\,0} = 0$, $\Omega_{m,\,0} = 1$ is consistent with the expected value for a purely matter-dominated Universe, $t_0 = (2/3)H_0^{-1}$.]

• For comparison with astrophysics (and for science popularisation purposes), it is also useful to show the age of the Universe t as a function of redshift z for a typical example of the ΛCDM model.

For the case of ΛCDM with $\Omega_{\Lambda,\,0} = 0.7$, plot a figure of t in Gyr versus z, with z in the range 0 to 10. Indicate on the plot the redshift and age at which the Universe became dominated by the cosmological constant.

Part I(b): Bounds on w_X from the observed age and the matter content.

The project requires a determination of how different the dark energy X could be from a cosmological constant Λ and still satisfy the observational bounds on the age of the Universe and on $\Omega_{m,\,0}$.

To do this, you will compute the theoretical age of the Universe as a function of $\Omega_{m,\,0}$ for different *constant* values of the equation of state parameter (also called barotropic parameter) $w_X \equiv p_X/\rho_X c^2$, where p_X is the pressure of the dark energy. For a constant barotropic parameter, density of dark energy varies with scale factor according to:

$$\rho_X(a) = \rho_{X,\,0} \left(\frac{a_0}{a}\right)^{3(1+w_X)}$$

You may assume that $\rho_{X,\,0} = \rho_\Lambda$ i.e. $\Omega_{X,\,0} = \Omega_{\Lambda,\,0}$.

Tasks:

1). Numerically integrate Equation (2) for the case where ρ_Λ is replaced by $\rho_X(a)$, where the a dependence of $\rho_X(a)$ is determined by w_X.

2). Plot on a single graph the values of t_0 as a function of $\Omega_{m,\,0}$ for $w_X = -0.3, -0.5, -0.7, -0.9$.

On the same graph, plot the 1-σ and 2-σ observational limits on t_0 and $\Omega_{m,\,0}$.

3). By adjusting w_X, determine the ranges of w_X which are compatible with the 1-σ and 2-σ limits on the observed age and matter content of the Universe. [You may wish to plot a second graph of t_0 as a function of $\Omega_{m,\,0}$ to do this.]

- In the report, state clearly the 1-σ and 2-σ compatible ranges of w_X.

Input Data for Part I:

The project will use the values given in 1902.07081.

The 1-σ range of the Age of the Universe, t_0, from globular cluster ages is

$$t_0 = 13.2 \pm 0.4\,\mathrm{Gyr}$$

[See p.7 of 1902.07081, where t_0 is denoted as t_U^{GC}.]

The direct observational bound on $\Omega_{m,\,0}$ from galaxy cluster dynamics gives a 1-σ range [see p.7 of 1902.07081]

$$\Omega_{m,\,0} = 0.30 \pm 0.02$$

In Part I you will use the directly observed Hubble Space Telescope value for H_0 [Riess et al (2018), arXiv:1801.01120]:

$$H_0 = 73.48 \text{ km s}^{-1} \text{ Mpc}^{-1}$$

The error on H_0 from HST is small and we will neglect it in this part of the project.

NB: This is the value of the expansion rate H_0 *directly* determined by observation of Type Ia supernovae. This value is independent of the dark energy model. It is larger than the value used in Part II, which uses the value determined *indirectly* by fitting the CMB predictions of an assumed dark energy model (the ΛCDM model) to the observed CMB temperature fluctuations.

• This difference is presently a mystery and it is of great interest to current research, known as the "H_0 problem" (or "H_0 tension"). [See Riess et al, arXiv:1804.10655.]

Part (II) Future space telescope determination of the nature of dark energy from BAO and Type Ia supernova observations.

In this section, you will investigate how the precise nature of the dark energy equation of state parameter can be determined by future space telescope observations of structures of known length scale at different redshifts and of objects of known brightness.

There are two distinct parts to this:

Part II(i) Future Euclid space telescope observations of the BAO feature in galaxy surveys.

Part II(ii) Future Roman space telescope determination of the Luminosity-Distance vs. Redshift of Type Ia (SNIa) supernovae.

[The Roman space telescope (more precisely, the Nancy Grace Roman space telescope) was previously known as WFIRST (before May 2020).]

Both Part II(i) and Part II(ii) (and some of Part III) are built on the calculation of the present physical distance $d(z)$ to an object which is observed to have a redshift z. This will need to be done for a time-dependent (and so scale factor dependent) dark energy barotropic parameter, $w_X(a)$.

NB: Part II will use different input values from those used in Part I (next page).

Part II Input Data for the BAO and Type Ia supernova analysis:

• The errors on all of these quantities are small and will be neglected in this project.

Cosmological parameters: Use the Planck CMB best fit results: $\Omega_{m,\,0} = 0.308$, $\Omega_{X,\,0} = 0.692$, $H_0 = 67.81\,\mathrm{km\ s^{-1}\ Mpc^{-1}}$.

[From "Planck 2015 results. XIII. Cosmological Parameters", arXiv.1502.01589]

• When considering dark energy X, you will assume that the present dark energy $\Omega_{X,\,0}$ is the same as the Planck value for the cosmological constant, $\Omega_{X,\,0} \equiv \Omega_{\Lambda,\,0} = 0.692$.

Speed of light: $2.998 \times 10^8\,\mathrm{m\,s^{-1}}$

BAO length: You will use the best-fit value from Planck: $l_{BAO} = 147.60$ Mpc.

(In the Planck results, l_{BAO} is equal to the "comoving sound horizon at the baryon drag epoch", r_{drag}. In the ESA Euclid report [arXiv:1110.3193], it is denoted as s.)

Part II Preliminary calculation: The present physical distance $d(z)$ to an object as a function of the redshift at which is its observed, z.

When we observe an object, we see the photons that have left the object at an initial time t_i and travelled to the present time t_0. These photons will be redshifted.

The present physical distance to an object observed at the present time t_0 with redshift z can be calculated by considering a photon emitted by the object at the initial time t_i and summing the distance travelled by the photon until it reaches the observer at t_0, taking into account the effect of the expansion of the Universe on the distance travelled during each increment dt of the photon's journey.

(i) Show that the total physical distance $d(t_i)$ to the object which emits a photon at t_i is

$$d(t_i) = a(t_0) \int_{t_i}^{t_0} \frac{cdt'}{a(t')}$$

(The prime on the t is to distinguish the integration variable from t.)

NB: Explain the reasoning behind your derivation fully.

(ii) To compute the physical distance to an object with observed redshift z, $d(z)$, you will first need to change variable in the integral from t to the scale factor $a(t)$, and derive an integral of the form

$$d(z) = \int_{a(z)}^{a_0} F(a', H(a'))da' \qquad (1)$$

where you need to determine $F(a', H(a'))$. Here $a_0 = a(t_0)$, and $a(z) = a(t_i)$, where z is the observed redshift of the object from which the light was emitted at t_i. (We write a' to distinguish the integration variable from a.)

• The integral obtained for $d(z)$ at this stage can be used in the study of the Modified Gravity model in Part III, by substituting $H(a')$ for the Modified Gravity model into Equation (1).

(iii) You will then need to insert $H(a')$ for a flat Universe which is composed of pressureless matter Ω_m (dark matter and baryons) and dark energy Ω_X (with $\Omega_m + \Omega_X = 1$). [You can neglect radiation and other minor components of the energy density.]

To find $H(a')$, you will first need to derive how the dark energy density varies with scale factor a' when w_X is a function of a'. In general, the density of each independent matter type ρ_i with pressure p_i obeys the Continuity (or Fluid) Equation:

$$\frac{d\rho_i}{dt} + 3H\left(\rho_i + \frac{p_i}{c^2}\right) = 0$$

Using this equation, show that

$$\ln\left(\frac{\rho_X(a')}{\rho_{X,\,0}}\right) = -\int_{a_0}^{a'} \frac{G[w_X(a'')]}{a''} da'' \qquad (2)$$

where $w_X(a'')$ is a function of scale factor a'' and $G[w_X(a'')]$ is a function for you to determine. (We write a'' to distinguish the integration variable from a'.)

This integral will be used in Part II(i) (the BAO analysis with the CPL form of $w_X(a)$) and in Part II(ii) (the computation of luminosity distance for a specific dark energy model).

[End of Preliminary calculation.]

Part II(i) Future Euclid observations of the BAO length

The European Space Agency (ESA) approved the Euclid space telescope in 2011, as part of its Cosmic Vision 2015-2025 programme. [See "Euclid Definition Study Report" arXiv:1110.3193 .] It launched on 1st July 2023.

The primary science objective of Euclid is the determination of the nature of dark energy i.e. its equation of state parameter w_X, including possible time-dependence of w_X. It will do this by observing the Baryon Acoustic Oscillation (BAO) length scale, which appears as a statistical feature in the distribution of galaxies as a function of redshift. The BAO length scale is precisely known theoretically and therefore provides a 'standard ruler' with which to measure the expansion history of the Universe.

Euclid will measure the angular size of the BAO length scale at different redshifts. These observations can then be compared with the theoretical angular size for a given form of dark energy. This will allow us to extract the dark energy equation of state.

Tasks:

1). You should prepare a description of the Euclid mission (the telescope and the mission objectives) for the final report.

2). You should explain the physics of BAO and how the BAO length is obtained observationally, making clear the statistical nature of its observation at different redshifts.

3). [The observed angular size of the BAO length as a function of redshift.] The observed angular size θ_{BAO} of the BAO length is determined by the size of the BAO length *at present*, l_{BAO}, and the physical distance *at present* to the objects forming the observed BAO feature at redshift z, $d(z)$. For $\theta_{BAO} \ll 1$, use the small-angle approximation ($\sin\theta \approx \tan\theta \approx \theta$ if $|\theta| \ll 1$) to show that:

$$\theta_{BAO}(z) = \frac{l_{BAO}}{d(z)}$$

[We have expressed this as an exact equality for very small angles $\theta_{BAO} \ll 1$, although it is really only a very accurate approximation.]

4). [Theoretical $d(z)$ and $\theta_{BAO}(z)$ for the CPL parameterisation.] The Euclid space telescope will look for dark energy with an equation of state of the form

$$w_X(a) \equiv \frac{p_X}{\rho_X c^2} = w_p + \left(\frac{a_p}{a_0} - \frac{a}{a_0}\right) w_a$$

This is the CPL (Chevalier-Polarski-Linder) parameterisation of $w_X(a)$. It has two free parameters, w_p and w_a.

w_a parameterises a possible time dependence (i.e. scale factor dependence) of the dark energy equation of state. Measurement of w_p and w_a is the prime mission goal of the Euclid project.

Without loss of generality, we can set $a_p = a_0$. This is just a redefinition of w_p, so there is no loss of information. (The Euclid team sets the present value of the scale factor to one, $a_0 = 1$. We will keep a_0 explicit here.)

You will next theoretically derive the integral which gives the physical distance $d(z)$ as a function of redshift for a dark energy barotropic parameter w_X described by a CPL parameterisation:

Starting from integral (2) in the preliminary calculation, derive the function ρ_X in the form

$$\rho_X(w_p, w_a, \rho_{X,0}, a_0, a) \qquad (3)$$

where we have indicated the parameters in the function ρ_X.

By substituting ρ_X into equation (1) from the preliminary calculation, show that, for a flat Universe, $d(z)$ for the CPL parameterisation becomes

$$d(z) = \frac{c}{a_0^{1/2}} \frac{1}{H_0} \int_{a(z)}^{a_0} \frac{da'}{a'^{1/2} \left\{ \Omega_{m,\,0} + \Omega_{X,\,0} \left(\frac{a_0}{a'}\right)^{3(w_p+w_a)} \exp\left[3w_a \left(\frac{a'}{a_0} - 1\right)\right] \right\}^{1/2}} \qquad (4)$$

where we write a' to distinguish the integration variable from a. (You will use $\rho_X(w_p, w_a, \rho_{X,0}, a_0, a')$ in the integral.)

Development of a Code to Determine w_p and w_a from observed BAO angles

New experiments, such as telescopes, CMB satellites and particle detectors, require data analysis methods to be developed. To do this, *simulated data* is generated, to allow the project team to develop the theory and software necessary to extract the key information from the raw data.

Overview: You will consider a table of simulated Euclid BAO angles $\theta_{BAO}(z)$ which include observational errors of the magnitude expected for Euclid. (For simplicity, we will write $\theta_{BAO}(z)$ as $\theta(z)$.) You will develop your code to determine the best-fit values of w_a and w_p by using the chi-square goodness of fit method. To do this, you need to determine the minimum value of $\chi^2(w_a, w_p)$, defined by

$$\chi^2(w_p, w_a) = \sum_{i=1}^{N} \frac{(\theta(z_i) - \theta_{th}(z_i))^2}{\Delta\theta_i^2}$$

where $\theta(z_i)$ is the mean value observed at z_i, $\theta_{th}(z_i)$ is the theoretical value calculated for a pair of input (w_p, w_a) values, and $\Delta\theta_i$ is the error (standard deviation) in the value of the observed $\theta(z_i)$.

[For a discussion of the chi-square method, see for example:

http://astronomy.swin.edu.au/ cblake/StatsLecture3.pdf (This has a good summary of error ellipses)

http://physics.ucsc.edu/ drip/133/ch4.pdf]

By varying the input (w_p, w_a), you should be able to determine the best-fit values $(\overline{w}_p, \overline{w}_a)$ that give the minimum of χ^2.

The chi-square method can also determine the error on the best-fit values of w_p and w_a, by determining a 68% confidence level (1-σ) or 95% confidence level (2-σ) error ellipse in the (w_p, w_a) plane.

In the project you will first do this for a test case where the value of w_a is known to be exactly zero. You will determine $\overline{w}_p$ and the $1-\sigma$ and 2-σ errors on w_p. This will allow you to establish and check your method.

You will then analyse a realistic dataset with errors in both w_p and w_a. You will determine $(\overline{w}_p, \overline{w}_a)$ and plot the 1-σ and 2-σ error ellipse in the (w_p, w_a) plane.

You should be then be able to determine whether it is possible to:

(i) Distinguish the simulated dark energy from a cosmological constant at 1-σ and 2-σ;

(ii) Determine whether the simulated dark energy equation of state is time-dependent (i.e.

a dependent) at 1-σ and 2-σ accuracy.

The dataset tables for each research collaboration will be provided in a separate document before the research phase of the project, which begins in Week 3.

• When writing the BAO section of the report, don't forget that this is just *simulated* dark energy, used in order to develop a method for analysing the data from future real observations. Please don't write as if you have discovered dark energy!

Data Tables and Analysis:

Important: The BAO angle tables in the project will be not be given directly as $\theta_{BAO}(z)$. Instead, the data tables will given in terms of the observed BAO angle θ_{BAO} minus the theoretical BAO angle $\theta_{\Lambda CDM}$ obtained by computing $d(z)$ for the case of ΛCDM:

$$\delta\theta_{BAO}(z) = \theta_{BAO}(z) - \theta_{\Lambda CDM}(z)$$

[The reason for this is that the mean values of w_p and w_a extracted from analysing the observed $\theta_{BAO}(z)$ will have a dependence on the accuracy of the code used to analyse the data. Since there is a practical limit on how small the step size used in the code can be if we are to keep the run time reasonable, this means that different codes will produce different values for $(\overline{w}_p, \overline{w}_a)$ for a given data table.

However, this problem does not arise if we analyse instead the difference $\theta_{BAO}(z) - \theta_{\Lambda CDM}(z)$, because the leading-order step size dependences in theoretically calculating $\theta_{BAO}(z)$ and $\theta_{\Lambda CDM}(z)$ cancel out, leaving a negligible error due to step size.]

To find the best fit, you will perform a chi-square analysis by writing a code to do this. You will need to determine the minimum value of $\chi^2(w_a, w_p)$, defined by

$$\chi^2(w_p, w_a) = \sum_{i=1}^{N} \frac{(\delta\theta_{BAO}(z_i) - \delta\theta_{th}(z_i))^2}{\Delta\theta_i^2}$$

where $\delta\theta_{BAO}(z_i)$ is the mean observed value of $\theta_{BAO} - \theta_{\Lambda CDM}$ at redshift z_i as given in the tables of simulated Euclid data, $\delta\theta_{th}(z_i)$ is the theoretical value of $\delta\theta_{BAO}(z_i)$ calculated for a pair of input (w_p, w_a) values by calculating $d(z)$ for these inputs, and $\Delta\theta_i$ is the error (standard deviation) in the observed value of $\delta\theta_{BAO}(z_i)$ ($\Delta\theta_i$ will be given in the Euclid data tables).

The chi-square method has the following steps:

(i) Suppose we have N data points. We compute the value of χ^2 for a given input (w_p, w_a).

(ii) To get the best-fit values of (w_p, w_a), we then minimize χ^2 by varying w_p and w_a to obtain χ^2_{min}.

You should calculate and state χ^2_{min}.

You should also calculate and state the minimum "reduced χ- square" value, given by χ^2_{min}/ν, where ν is the number of degrees of freedom, and compare this to 1. If we are testing a model with d free parameters (in our case, $d = 2$, corresponding to (w_p, w_a)), then $\nu = N - d$. As a rough rule-of-thumb, the best-fit (w_p, w_a) will be a good fit to data if $\chi^2_{min}/\nu \sim 1$.

(iii) To determine the error in the best-fit, you need to search for values of (w_p, w_a) that are shifted from their best-fit values such that χ^2 is given by

$$\chi^2(w_p, w_a) = \chi^2_{min} + \Delta\chi^2$$

The limits of (w_p, w_a) will correspond to the values at which $\Delta\chi^2$ is equal to a particular value, called the critical value, for a given probability or significance level. This depends on the number of free parameters in the model. For the dark energy model there are 2 free parameters, w_p and w_a. In general:

For 1-σ error and 1 free parameter, critical $\Delta\chi^2 = 1.01$

For 2-σ error and 1 free parameter, critical $\Delta\chi^2 = 4.00$

For 1-σ error and 2 free parameters, critical $\Delta\chi^2 = 2.30$

For 2-σ error and 2 free parameters, critical $\Delta\chi^2 = 6.17$

Tasks:

• The simulated Euclid BAO data tables with $\delta\theta_{BAO}(z)$ will be released on the first project day, Friday 2nd February.

(1) Test Case (single unknown, w_p): You will first consider a dataset for which the value of $w_a = 0.0$. (This corresponds to the case of a time-independent dark energy equation of state.) Therefore there is only one free parameter, w_p.

The result is known for this case, so you can use this to test your method.

• You should determine the best-fit value of w_p, which we denote by $\overline{w}_p$, to three decimal places and the value χ^2_{min} and of χ^2_{min}/ν.

• You should then determine the 1-σ and 2-σ ranges of the values of w_p.

The expected results are $\overline{w}_p = -0.936$, with the 1-σ range $w_p = -0.937$ to -0.934 and 2-σ range $w_p = -0.938$ to -0.933. $\chi^2_{min} = 17.73$ and $\chi^2_{min}/\nu = 1.04$. (Your results may differ slightly from these due to code dependence.)

(2) Simulated Euclid data (Two unknowns, w_p and w_a): You will next consider a dataset which has errors in both w_p and w_a.

• You should determine the best-fit values of w_p and w_a. $\overline{w}_p$ should be determined to three decimal places and $\overline{w}_a$ to two decimal places.

(In all simulated data sets the best-fit w_p is in the range -1.0 to -0.95, and the best-fit w_a is in the range 0.0 to 0.3.)

• You should state the values of χ^2_{min} and χ^2_{min}/ν.

• You should then determine the 1-σ and 2-σ error ellipses in the (w_a, w_p) plane,

• You can then determine whether the model can be observationally distinguished from a cosmological constant and whether it is possible to observationally determine if the model has a time-dependent equation of state, to 1-σ and 2-σ accuracy.

NB: When numerically calculating the integral for $d(z)$ in your code, you should use a step size in the scale factor a which is at least as small as 0.001. This value is sufficient for an accurate calculation. You can go to smaller step sizes, but your code will take longer to scan the (w_p, w_a) parameter space. If you find that the runtime for your code is too long, you can use a smaller step size in a, 0.01, but this will cause some loss in accuracy.

Part II(ii) Future Roman observations of Type Ia Supernova Luminosity Distances

One of the methods for determining the dark energy equation of state is to compare the theoretical *luminosity distance* for a given $w_X(a)$ to the observed luminosity distances of Type Ia supernovae (obtained from the observed light flux of the supernovae) as a function of redshift. Determination of the luminosity distance-redshift relation is a prime objective of NASA's Roman space telescope project.

In this section you will develop a general code that can produce the theoretical luminosity distance versus redshift for *any* dark energy model with an equation of state $w_X(a)$. You will then use your code to calculate the theoretical luminosity distance versus redshift for a class of dark energy models known as Scaling Freezing Quintessence (SFQ). You will compare the SFQ results with the predictions for a cosmological constant and determine whether future observations by Roman will be able to distinguish the SFQ models from a cosmological constant.

A. Luminosity distance versus redshift

Tasks:

1). You should give a description of the Roman telescope (launch no later than 2027, latest estimate) and the mission objectives in your report. [See "WFIRST-AFTA Final Report", Spergel at al, arXiv:1305.5422]

2). [Luminosity distance:] You should explain what is meant by the luminosity distance $d_L(z)$ and how it is related to the observed light flux from an object $\mathcal{F}$ and the absolute luminosity of the object $\mathcal{L}$.

• You should also explain its relation to the physical distance $d(z)$ i.e. why $d_L(z) = (1+z)d(z)$ in an expanding Universe.

3). You will next write a computer code that can determine the luminosity distance versus redshift relation as a function of the equation of state parameter $w_X(a)$ for any dark energy model.

To do this, you will need to write a program that can calculate $\rho_X(a)$ by integrating Equation (2) of the preliminary calculation for each value of a:

$$\ln\left(\frac{\rho_X(a')}{\rho_{X,\,0}}\right) = -\int_{a_0}^{a'} \frac{G[w_X(a'')]}{a''}da''$$

(using a'' for the integration variable and computing ρ_X at a scale factor a')

and then use $\rho_X(a')$ in the integral for $d(z)$, Equation (1):

$$d(z) = \int_{a(z)}^{a_0} F(a')da' \qquad (1)$$

This will require a "nested loop" in your program to obtain $\rho_X(a')$ for each value a' in the integral of Equation (1).

• To test the program, you should use the case of a pure cosmological constant Universe, with $\Omega_{X,\,0} = 1$ and $w_X = -1$. For this case the integral for $d(z)$ can be done analytically (i.e. it can be calculated on paper) and a formula for $d(z)$ and hence $d_L(z)$ obtained. Then by setting $\Omega_{X,\,0} = 1$, $w_X = -1$ and $\Omega_{m,\,0} = 0$ in your program, you should be able to reproduce the same results using your $d_L(z)$ program as obtained using the formula.

Complete this test before proceeding. Confirm in your report that this test has been done successfully.

NB: $d_L(z)$ should always be expressed in Mpc.

4). [The SFQ model:] The SFQ model is described in Section 3.2 of "Quintessence: A Review" S.Tsujikawa, arXiv:1304.1961. It is a scalar field model of dark energy. (Scalar field dark energy models are generally known as *quintessence models*).

At early times the SFQ model behaves as a *scaling* quintessence model. This means that the equation of state of the scalar field solution for the dark energy becomes equal to the equation of state of the dominant form of matter. So, for example, during matter domination, $w_X = w_m = 0$. The ratio of the dark energy density to the dark matter density is then a constant. (This may help us to understand why the present dark matter and dark energy densities are similar.)

However, in order to behave like dark energy today, the dark energy must break away from the scaling solution and *transition* to a form of matter with equation of state close to that of a cosmological constant, $w_X \approx -1$.

SFQ models have a time-dependent equation of state given by

$$w_X(a) = w_f + \frac{w_{past} - w_f}{1 + \left(\frac{a}{a_t}\right)^{1/\tau}}$$

Here w_{past} is the past value of w_X (as $a \to 0$) and w_f is the future value (as $a \to \infty$). a_t is the scale factor at the transition where the scaling behaviour ends and τ is called the transition width. (1304.1961 writes w_{past} as w_p. We use w_{past} to avoid confusion with the w_p parameter of the CPL parameterisation.)

• You should try to explain the physics of the Scaling Freezing Quintessence (SFQ) model as best you can. [You will not study scalar fields until the 4th year, but you should be able to qualitatively explain the concepts behind the SFQ models.]

5). Using your (now tested) $d_L(z)$ program, plot the predictions for $d_L(z)$ for two models of dark energy: a cosmological constant and a specific test case of the SFQ model.

A): A cosmological constant Λ, $w_X(a) = -1$.

This is the standard ΛCDM model, consisting of a cosmological constant plus matter. Other dark energy models are compared to this, to see if they can be distinguished from a cosmological contant.

B): Scaling Freezing Quintessence (SFQ) Test Model.

(This is the model given in Figure 2(a) of 1304.1961.)

The SFQ model replaces the cosmological constant in the ΛCDM model with scalar field dark energy. The model parameters for the test model are $a_t = 0.23$ and $\tau = 0.33$, $w_{past} = 0$ and $w_f = -1$.

Roman (WFIRST) will observe > 2700 SNIa in the redshift range $0.2 < z < 1.7$.

• Plot w_X as a function of z for the ΛCDM and the test SFQ model, in the range $0 < z < 2$.

• Plot $d_L(z)$ for the two dark energy models in the range $0 < z < 2$. [You should find that the plots are very close to each other.]

6). Roman has a predicted 1-σ fractional accuracy for the luminosity distance in the range $0.2 < z < 1.7$ of 0.18 %. [In the WFIRST-AFTA report (arXiv:1305.5422, Appendix C-4), this is called the "aggregate precision", which is expressed as a percentage.]

• This means that if the fractional deviation is less than 0.36% over the whole range $0.2 < z < 1.7$ then the model is definitely unobservable at 2-σ. If it is larger then 0.36% over at least part of the range $0.2 < z < 1.7$, then it may be observable.

This fractional accuracy can then be compared to the fractional deviation of the luminosity distance for the dark energy models from the luminosity distance for the ΛCDM model, by calculating

$$\left|\frac{\Delta d_L}{d_L}\right| = \left|\frac{d_L(z) - d_{L\,\Lambda cdm}(z)}{d_L(z)}\right|$$

where $d_L(z)$ is the luminosity distance for the dark energy model and $d_{L\,\Lambda cdm}(z)$ is the luminosity distance for the ΛCDM model.

• Plot $|\Delta d_L/d_L|$ as a function of z (for $0 < z < 2$) for the test SFQ models.

• By comparing the values of $|\Delta d_L/d_L|$ with the predicted accuracy of Roman over the range $0.2 < z < 1.7$, you should be able to show that the test SFQ Model definitely cannot be distinguished from the ΛCDM model at 2-σ accuracy.

7). [The general SFQ model.]

• This is the prime objective of the SFQ analysis. The goal is to determine the region of the parameter space of the SFQ model (a_t, τ) where the model cannot be distinguished at 2-σ accuracy from ΛCDM by Roman.

(We will restrict to values of τ between 0 and 1.)

Determine the smallest value of a_t, for $\tau = 0.1, 0.3, 0.5, 0.7$ and 1, for which $|\Delta d_L/d_L|$ is smaller than the 2-σ Roman accuracy over the whole range $0.2 < z < 1.7$

Plot these values of a_t as a function of τ. Hence indicate on the plot the region for which the deviation definitely cannot be detected at 2-σ accuracy by Roman. For points outside of this region, the model may be observable, with the likelihood of being observed increasing the further the value of $(a_t,\ \tau)$ is from the boundary.

(III) An alternative to dark energy: Modified Gravity models.

In this section you will explore the predictions of a model of accelerating expansion that is not based on dark energy. The Universe in this case consists only of conventional matter and radiation.

The idea behind this model is that gravity is modified such that when the mean density of the Universe is small enough, conventional gravity (i.e. General Relativity) no longer describes the expansion. The Friedmann equation of the modified gravity theory is:

$$H^2 - \frac{H^\alpha}{(r_c/c)^{2-\alpha}} = \frac{8\pi G}{3}\rho_m \quad (1)$$

where ρ_m is the density of conventional matter, r_c is called the cross-over scale (which is a distance) and α is a model-dependent parameter. The modification by the second term on the LHS means that there are solutions to this equation corresponding to an accelerating expansion. Therefore accelerating expansion can be achieved without the need for dark energy.

Tasks:

1). The modified Friedmann equation model is described in "Dark Energy as a Modification of the Friedmann Equation", Dvali and Turner, astro-ph/0301510. You should try to qualitatively summarise the ideas behind the original DGP model, which has $\alpha = 1$, as best you can. [The technical details involve General Relativity and concepts from extra-dimensional physics which are beyond your present knowledge, but you should be able to qualitatively understand and summarise the broad ideas and concepts of the model.]

2). You will first consider the DGP model with $\alpha = 1$. The present matter density is parameterised by $\Omega_{m,\,0}$, where

$$\Omega_{m,\,0} = \frac{\rho_{m,\,0}}{\rho_{c,\,0}}$$

and where, by definition

$$\rho_{c,\,0} = \frac{3H_0^2}{8\pi G}$$

where H_0 is the present expansion rate. Using this and Eq.(1) with $\alpha = 1$, you should be able to show that in order to have a given present matter density $\Omega_{m,\,0}$, r_c/c must equal

$$r_c/c = (1 - \Omega_{m,\,0})^{-1}\, H_0^{-1}$$

• Using the mean values of the input parameters for Part II, calculate the value of r_c in Mpc.

You should then be able to solve Eq.(1) with $\alpha = 1$ to express $H(a)$ as a function of H_0, $\Omega_{m,\,0}$, a_0 and a.

3). The deceleration parameter, q, is defined by

$$q(z) = -\frac{1}{H^2}\frac{\ddot{a}}{a}$$

This is defined so that $q > 0$ means the Universe expansion is decelerating, $\ddot{a} < 0$.

Starting from $H = \dot{a}/a$, you should derive a general expression for q for a given $H(a)$ in the form

$$q = -(1 + g(a, H, dH/da))$$

where you should determine the function g.

4) Using this expression for q and $H(a)$ for the $\alpha = 1$ model, derive q as a function of a for the $\alpha = 1$ modified gravity model.

Hence show that the Universe expansion is accelerating at present in the modified gravity model i.e. $q < 0$ when $a = a_0$.

[You should assume that $\Omega_{m,\,0} = 0.308$ and $H_0 = 67.81\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$.]

- In the report, state the value of q at present, q_0, for the $\alpha = 1$ model.

5) Plot the predictions for $q(z)$ for the $\alpha = 1$ modified gravity model for $0 < z < 5$. Plot the predictions for the standard ΛCDM dark energy model on the same figure.

6) Plot the predictions for the luminosity distance $d_L(z)$ for the $\alpha = 1$ modified gravity model for $0 < z < 2$. To do this, you will have to compute $d(z)$ by using the Preliminary Calculation in Part II, substituting the $H(a)$ for the modified gravity model into the integral for $d(z)$ given in Equation (1) of the Preliminary Calculation and computing $d_L(z)$ numerically.

Plot the $d_L(z)$ predictions for the standard ΛCDM dark energy model on the same figure.

7). Plot $|\Delta d_L/d_L|$ [defined in Part II(ii)] as a function of z (for $0 < z < 2$) for the $\alpha = 1$ model.

Hence determine whether Roman observation of $d_L(z)$ will be able to distinguish the $\alpha = 1$ modified gravity model from the ΛCDM model at 2-σ by comparing the predictions for $|\Delta d_L/d_L|$ to the expected accuracy of Roman at 2-σ in the range $0.2 < z < 1.7$. [See Part II(ii).]

Generalising to $\alpha \neq 1$ models

In this part you will discuss how to analyse the modified Friedmann equation a general value of α. This requires a code to be written in order to compute $H(a)$.

NB: You only need to explain how a code can be constructed. You don't need to write a code.

(a) Discuss how changing α (increasing or decreasing α) could bring the Modified Gravity model predictions closer to those of ΛCDM.

(b) Determine an expression for r_c for a general value of α in terms of α, $\Omega_{m,0}$ and H_0.

(c) Describe how a code could be designed to determine $d_L(z)$ for any value of α. This should explain:

(i) How $H(a)$ would be numerically determined for each a.

(ii) How $d_L(z)$ would then be determined.

Your plan should be sufficiently detailed that a coder could easily put it into practice.

References and Materials

List of papers. (These are a starting point.)

(1) “The Age of the Universe”, B.Chaboyer, astro-ph/9808200.
(2) “Euclid 2011 Definition Study Report”, arXiv:1110.3193
(3) “Baryon Oscillations as a Cosmological Probe”, E.Linder, astro-ph/0304001
(4) “Quintessence: A Review”, S.Tsujikawa, arXiv:1304.1961.
(5) “Dark Energy as a Modification of the Friedmann Equation”, G.Dvali and M.S.Turner, astro-ph/0301510
(6) “The Local and Distant Universe: Stellar Ages and H_0,”, Jimenez et al. arXiv:1902.07081
(7) “Wide-Field Infra-Red Survey Telescope-Astrophysics Focused Telescope Assets (WFIRST-AFTA) Final Report”, D.Spergel et al, arXiv:1305.5422

The Planck input values for $\Omega_{m,\,0}$, $\Omega_{X,\,0}$ and the BAO length are from:
“Planck 2015 results, XIII Cosmological parameters”, arXiv:1502.01589

Other information can be obtained from Google searches and websites such as ESA, NASA etc. Wikipedia is also very useful, <u>but the original references must be cited</u>.

NB: Do not cite Wikipedia as a reference!

The INSPIRE and arXiv databases are the main professional tools for finding relevant papers, review articles. You will be given a short tutorial on their use.

Websites: `http://inspirehep.net/` and `http://www.arxiv.org`

Group Project Generalities

Group Organisation

You will be formed into groups of 4 or 5. You will then need to form an organisational structure and come up with a research plan.

Groups need to have a Coordinator (overall manager) and an Administrator (responsible for organising meetings and agendas and keeping minutes of meetings). There should be a weekly formal meeting with agenda to discuss progress and tasks.

The agendas and minutes of the meetings and the research plan should be included in an appendix to your report. [This does not count towards the report page count.]

The project requires various tasks: e.g. background physics explanation and literature search, theory calculations, computer programming, figure generation, results presentation and report coordination and editing. You will need to decide who does what and when; you may choose to rotate tasks or have one or two people assigned to one task throughout.

• In Week 3 you will present your research plan and organisation. Note that this is not a review of the physics to be investigated. It is simply a description of the organisation of the group, its strategy, the resources to be used, its timeline and milestones etc.

Report

• You will produce a single report for the whole group. The maximum length is 25 pages. This does not include Appendices. Note that Appendices will not count directly towards the report's assessment (although they will be taken into account), therefore all important material must be discussed in the main text.

• Title, Abstract, Table of Contents, References and Appendices do not count towards the total page count. The minimum font size is 11pt and margins must be ≥ 1 inch.

NB: Do not exceed the upper limit. There is a one letter grade penalty for doing so.

The report hand-in date and time will be announced later. (It will be during Week 20.) *There are significant penalties for handing in late.*

• The report will be marked according to the following criteria:

Presentation 20%
Project Motivation and Purpose 10%
Content 40%
Understanding 30%.

There will be a tutorial on report writing in Week 7.

Assessment

50% is for the group report. All group members will receive this mark.

15% is for your individual on-line logbook. This will be a OneNote logbook, which must be written up on a continuous basis during the project, with all entries dated.

20% is the individual contribution mark. This will be decided by all the group members via a system of peer-assessment.

[The remaining 15% will be awarded for your presentations at the PLACE mini-conference.]

Appendix A: A Brief Review of Relevant Big-Bang Cosmology

(i) The Cosmological Equations:

$$(i) \qquad \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3} - \frac{kc^2}{a^2} + \frac{\Lambda c^2}{3} \quad , \quad \text{FRIEDMANN EQ.}$$

$\rho =$ Mass Density.

The k term is the curvature term and Λ is the Cosmological Constant. The Cosmological Constant is equivalent to a constant density $\rho_\Lambda = \Lambda c^2/8\pi G$.

$k = 0 \Rightarrow$ Flat Universe, $k < 0 \Rightarrow$ Open Universe, $k > 0 \Rightarrow$ Closed Universe. We will assume $k = 0$ throughout.

$$(ii) \qquad \dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) = 0 \quad , \quad \text{FLUID (OR CONTINUITY) EQ.}$$

$$(iii) \qquad \frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) \quad , \quad \text{ACCELERATION EQ.}$$

(The Acceleration equation also known as the Raychaudhuri equation.)

$H = \frac{\dot{a}}{a}$ is the expansion rate or Hubble parameter.

With $k = 0$ ($\Rightarrow$ Flat Universe), the Friedmann equation implies that

$$H^2 = \frac{8\pi G\rho_c}{3}$$

$$\Rightarrow \rho_c = \frac{3H^2}{8\pi G} \quad , \quad \text{CRITICAL DENSITY} \quad (Definition)$$

The density parameter is defined as

$$\Omega_i = \frac{\rho_i}{\rho_c}$$

For a flat Universe $\Omega_{total} = 1$.

(ii) Redshift: If a_0 is the scale factor *at present*, then the redshift z at an earlier time t is defined by

$$1 + z = \frac{a_0}{a(t)}$$

(iii) Equation of State: This tells us the pressure p for a given density ρ, $p(\rho)$. Often p is proportional to ρ. We define the equation of state parameter (also known as the 'barotropic parameter') w by

$$w = \frac{p}{\rho c^2} \quad , \quad \text{EQUATION OF STATE PARAMETER}$$

Non-relativistic matter (essentially pressureless): $p = 0 \Rightarrow w = 0$.

Radiation (and other relativistic matter): $p = \frac{1}{3}\rho c^2 \Rightarrow w = \frac{1}{3}$.

(iv) ρ as a function of scale factor:

For matter of type i with constant equation of state w_i:

$$\frac{\rho_i}{\rho_{i\,0}} = \left(\frac{a_0}{a}\right)^{3(1+w_i)}$$

[Exercise: Prove this by solving the fluid equation, using $p_i/c^2 = w_i \rho_i$.]

For example:

$$\text{MATTER} - \text{DOMINATED UNIVERSE} \ : w = 0 \Rightarrow \frac{\rho}{\rho_0} = \left(\frac{a_0}{a}\right)^3$$

[NB: Here "matter" means non-relativistic matter.]

$$\text{RADIATION} - \text{DOMINATED UNIVERSE} \ : w = \frac{1}{3} \Rightarrow \frac{\rho}{\rho_0} = \left(\frac{a_0}{a}\right)^4$$

Appendix B: A Coding Method for a Simple Intergration Program

It is perfectly OK to use integration packages in your Python code, but sometimes it is difficult to get a package to do what you want.

The alternative is to write your own integration code. I will illustrate the core of the code using FORTRAN notation (my preferred language and simple to understand). It is straightforward to translate this into the equivalent Python commands.

Suppose we want to integrate x^2 from 0 to 1. The first step is to define the initial value of x, the step size dx, and the integral, which we will call I. So we would write

$x = 0.0$
$dx = 0.001$
$I = 0.0$
$dI = 0.0$

Here dI will be the change in the integral for each step in x.

Next we start the loop and do the integral. Note that there is no need to use complicated methods like Simpson's rule. For most purposes the integral can simply be done as a sum of the integrand values at each x (in this example, x^2) times the step size dx, since for small enough Δx:

$$\int_0^1 f(x)dx = \sum_i f(x_i)\Delta x$$

So the complete code would read (in FORTRAN) [my comments are in square brackets]:

[BEGIN CODE]

$x = 0.0$
$dx = 0.001$
$I = 0.0$
$dI = 0.0$

[START THE LOOP]

do while (x .le. 1.0)

[In FORTRAN this means "while $x \leq 1.0$"]

$dI = (x * *2.0) * dx$

$I = I + dI$
$x = x + dx$

[The last line moves x to the next value for the next loop]

[END THE LOOP]

end do

[END CODE]

This will sum x^2 times dx from 0 to 1, so the final value of I when the loop ends is the complete numerical value of the integral. So long as the step size is small enough, this will be a good approximation to the exact result, which in this case equals 1/3.

• In order to test the accuracy of the integration code, reduce dx by a factor of 2, in this case to 0.0005, and run the code again. If the result changes only slightly (more precisely, if the change is smaller than the level of precision you require) then the step size is good, otherwise reduce it to obtain the precision you require.

It is quite easy to write a nested loop code. For example, suppose I want to do the following integral:

$$I = \int_0^1 f(t)dt$$

where

$$f(t) = \int_0^t x^2 dx$$

In this case the complete code would be:

[BEGIN CODE]

[SET INITIAL CONDITIONS FOR THE OUTER $f(t)$ LOOP]

$t = 0.0$
$dt = 0.001$
$I = 0.0$
$dI = 0.0$

[START OUTER LOOP]

do while (t .le. 1.0)

[SET INITIAL CONDITIONS FOR THE INNER LOOP; this has to reset for each value of

t, so the x initial conditions are *inside* the t loop]

$x = 0.0$
$dx = 0.001$
$f = 0.0$
$df = 0.0$

[START INNER LOOP]

do while (x .le. t)

$df = (x**2.0)*dx$

$f = f + df$
$x = x + dx$

end do

[THIS ENDS THE INNER LOOP; this gives the value of $f(t)$ at the value of t reached by the outer loop]

$dI = f*dt$

$I = I + dI$
$t = t + dt$

[THESE LINES PERFORM THE t INTEGRAL]

end do

[THIS ENDS THE OUTER LOOP; the value of I will be the final value of the integral over t as required.]

[END CODE]