Chapter 5

<*X*, *R*> models of Multicriteria Decision Making and Their Analysis

In this chapter, we present an introduction to preference modeling realized in terms of binary fuzzy relations. In dealing with <*X*, *R*> models, which serve for multiattribute decision making, a fundamental question arises on how one can construct fuzzy preference relations. In practice, a DM maker can directly assess fuzzy preference relations. The corresponding techniques are considered. A natural and convincing approach to constructing fuzzy preference relations based on the ordering of fuzzy quantities is discussed as well. Since any involved expert or any included criterion may require different formats for representing preferences (five main types of preference formats are considered in this chapter), the questions of their convertion into fuzzy preference relations on the basis of so-called transformation functions are considered. We discuss methods used to analyze problems of multicriteria evaluation, comparison, choice, prioritization, and/or ordering of alternatives. There exist two types of situations which give rise to these problems. The first one is related to the direct statement of multiattribute decision-making problems when the consequences associated with solutions to problems cannot be estimated with a single criterion. The second class is related to problems that may be solved on the basis of a single criterion or several criteria; however, if the uncertainty of information does not permit a unique solution to be obtained, it is possible to include additional criteria and thereby convert these problems into multiattribute tasks. Diverse techniques of multiattribute analysis in a fuzzy environment are discussed. Although these techniques are directly related to individual decision making, they can be applied to procedures of group decision making (diverse aspects of their application in group decision making are discussed, for instance, in (Ekel *et al*., 2009; Parreiras *et al*., 2010; Pedrycz, Ekel, and Parreiras, 2011; Parreiras, Ekel, and Bernandes, 2012; Parreiras, Ekel, and Morais, 2012). The use of the presented results is illustrated by solving practical problems coming from different areas.

5.1 Introduction to Preference Modeling with Binary Fuzzy Relations

As it was stated in Chapter 2, the binary fuzzy relation consists of a fuzzy set with a bidimensional membership function $R: X \times X \to [0,\ 1]$. In essence, such a relation associates each ordered pair of elements (X_k, X_l), where $X_k, X_l \in X$, with an entry $R(X_k, X_l)$ coming from the unit interval that reflects the degree to which elements X_k and X_l are in the relation R.

In the case of dealing with discrete (finite) sets of alternatives in preference modeling, the binary fuzzy relations can be represented (Pedrycz, Ekel, and Parreiras, 2011) in two ways:

▪ a square matrix R of dimension equal to the number of elements of X, where each entry R_{kl} corresponds to $R(X_k, X_l)$;
▪ a weighted graph where each element from X corresponds to a node and the relations between the elements are represented as arcs, in such a way that $R(X_k, X_l)$ corresponds to an arc oriented from X_k towards X_l.

Example 5.1. Consider a set $X = \{X_1, X_2, X_3, X_4\}$, where all possible pairs of elements are interrelated as follows: $R(X_1, X_1) = R(X_2, X_2) = R(X_3, X_3) = R(X_4, X_4) = 1$, $R(X_1, X_2) = 0.6$, $R(X_1, X_3) = 0.4$, $R(X_1, X_4) = 0.4$, $R(X_2, X_1) = 0.8$, $R(X_2, X_3) = 1$, $R(X_2, X_4) = 0.7$, $R(X_3, X_1) = 0.7$, $R(X_3, X_2) = 0.5$, $R(X_3, X_4) = 0.5$, $R(X_4, X_1) = 0.2$, $R(X_4, X_2) = 0$, and $R(X_4, X_3) = 0.6$. These dependencies can be represented as the following fuzzy binary relation

$$R = \begin{bmatrix} 1 & 0.6 & 0.6 & 0.4 \\ 0.8 & 1 & 1 & 0.7 \\ 0.7 & 0.5 & 1 & 0.5 \\ 0.2 & 0 & 0.6 & 1 \end{bmatrix} \tag{5.1}$$

or as the weighted graph as follows:

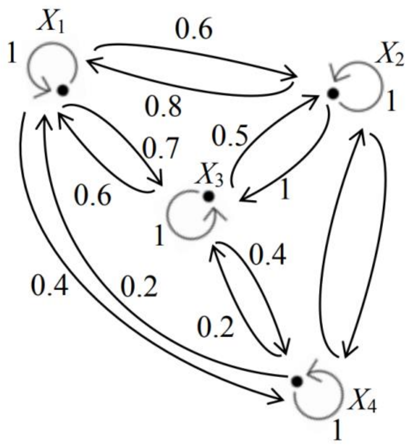


Figure 5.1 Relation *R* of Example 5.1 represented as a graph.

It is possible to put in correspondence to a generic binary fuzzy relation R, its inverse (or transpose) relation R^{-1}, its complementary relation R^c, and its dual relation R^d, which are defined, respectively, as follows (Fodor and Roubens, 1994b; Pedrycz, Ekel, and Parreiras, 2011):

$$R^{-1}(X_k, X_l) = R(X_l, X_k) \tag{5.2}$$
$$R^c(X_k, X_l) = 1 - R(X_k, X_l) \tag{5.3}$$
$$R^d(X_k, X_l) = 1 - R(X_l, X_k) = (R^{-1}(X_k, X_l))^c \tag{5.4}$$

Example 5.2. The application of (5.2)-(5.4) to matrix R of Example 5.1 generates the following matrices:

$$R^{-1} = \begin{bmatrix} 1 & 0.8 & 0.7 & 0.2 \\ 0.6 & 1 & 0.5 & 0 \\ 0.6 & 1 & 1 & 0.6 \\ 0.4 & 0.7 & 0.5 & 1 \end{bmatrix} \tag{5.5}$$

$$R^{c} = \begin{bmatrix} 0 & 0.4 & 0.6 & 0.6 \\ 0.2 & 0 & 0 & 0.3 \\ 0.3 & 0.5 & 0 & 0.5 \\ 0.8 & 1 & 0.4 & 0 \end{bmatrix} \tag{5.6}$$

$$R^{d} = \begin{bmatrix} 0 & 0.2 & 0.3 & 0.8 \\ 0.4 & 0 & 0.5 & 1 \\ 0.4 & 0 & 0 & 0.4 \\ 0.6 & 0.3 & 0.5 & 0 \end{bmatrix} \tag{5.7}$$

The operations of intersection, union, and complement of fuzzy relations were considered in Chapter 2.

The properties characterizing the binary fuzzy relations, which are of interest from the point of view of preference modeling, are considered in (Öztürk, Tsoukiàs, and Vincke, 2005; Pedrycz, Ekel, and Parreiras, 2011) as well as in Chapter 2.

Among these properties, it is necessary to distinguish the properties related to the family of *T*-transitivities. In particular, as it is indicated in Chapter 2, the basic idea of transitivity is that the strength of the direct relationship between two elements should not be weaker than their indirect relationship involving other elements. In the case of preference modeling, transitivity can be adopted as a consistency condition related to the following consideration (Pedrycz, Ekel, and Parreiras, 2011): if someone says that X_k is better than X_j and that X_j is better than X_l, then it is expected that this person prefers X_k to X_l at least at a minimum strength and not that they prefer X_l to X_k.

From the practical point of view, depending on the selected transitivity property, the corresponding consistency condition may be more rigorous or more relaxed. The family of *T*-transitivities includes the following conditions (Pedrycz, Ekel, and Parreiras, 2011):

▪ the min-transitivity condition for fuzzy preference relations

$$R(X_k, X_l) \geq \min(R(X_k, X_j), R(X_j, X_l)) \tag{5.8}$$

▪ the product-transitivity condition

$$R(X_k, X_l) \geq R(X_k, X_j) \bullet R(X_j, X_l) \tag{5.9}$$

▪ the Lukasiewicz-transitivity condition

$$R(X_k, X_l) \geq \max(R(X_k, X_j) + R(X_j, X_l) - 1,\ 0) \tag{5.10}$$

Example 5.3. Let us consider $R(X_1, X_3) = 0.4$ and $R(X_3, X_4) = 0.5$ from Example 5.1. Suppose that the element $R(X_1, X_4)$ is missing. It can be estimated, for instance, by the min-transitivity condition as follows: $R(X_1, X_4) \geq 0.4$.

Let us discuss questions of preference modeling on the basis of using binary fuzzy preference relations.

If a DM is asked to compare two alternatives $X_k, X_l \in X$ and determine which one of these alternatives he/she prefers, it is possible to expect one of the following answers:

- X_k and X_l are indifferent;
- X_k is strictly better than X_l;
- X_l is strictly better than X_k;
- X_k and X_l are incomparable (a DM is not able to compare the alternatives).

Taking this into account, three types of judgments can be distinguished: indifference, strict preference, and incomparability. They can be modeled with the use of binary fuzzy relations in such a way that the membership function of each binary fuzzy relation reflects the credibility (or intensity) of the judgment with quantifying into the interval [0, 1] (Pedrycz, Ekel, and Parreiras, 2011). The coherence between the model and the corresponding judgment is provided by some basic properties that each binary fuzzy relation has to have in accordance with the judgment that it is to reflect.

Let us consider the indifference, strict preference, and incomparability judgments as binary fuzzy relations (Pedrycz, Ekel, and Parreiras, 2011).

The judgment of indifference is used in situations where a DM believes that both alternatives satisfy equally his/here interests. Indifference can be modeled as a binary fuzzy relation I with the following properties:

- reflexivity: a DM is always indifferent to X_k and X_k;
- symmetry: a statement " X_k is indifferent to X_l " is equivalent to a statement " X_l is indifferent to X_k ".

The judgment of strict preference is utilized when a DM can define which one is the better one of two alternatives. The strict preference can be modeled as a binary fuzzy relation P, which is to satisfy the following conditions:

- irreflexivity: a DM cannot strictly prefer X_k to X_k;
- asymmetry: a DM cannot strictly prefer X_k to X_l and X_l to X_k, at the same time.

Incomparability is used when a DM cannot express his/her opinion and in situations where a DM is asked about his/her preference and the answer is "I do not know," due to missing or uncertain information or as a consequence of the existence of conflicting information. The judgment of incomparability is reflected by means of a binary fuzzy relation J with the following properties:

▪ irreflexivity: a DM cannot say that X_k is incomparable to X_k;
▪ symmetry: a statement "X_k is incomparable to X_l" is equivalent to a statement "X_l is incomparable to X_k."

In the solution of decision making problems based on fuzzy preference modeling, it is also important to consider a fuzzy nonstrict preference relation R (also called a fuzzy weak preference relation in literature). This relation is a starting point in analyzing models associated with binary fuzzy relations (Orlovsky, 1978; Fodor and Roubens, 1994a; Ekel, 2002).

The relation $R(X_k,\ X_l)$ represents the degree to which X_k weakly dominates X_l (X_k is at least as good X_l or X_k is not worse than X_l). In a somewhat loose sense, $R(X_k, X_l)$ also represents the degree of truth of the statement "X_k is preferred over X_l" or the statement "X_k is at least as good as X_l" (Kulshreshtha and Shekar, 2000).

The relation R is a reflective one (Pedrycz, Ekel, and Parreiras, 2011) and can be defined as the union of strict preference and indifference as follows:

$$R = P \cup I \tag{5.11}$$

Equivalently, in a less intuitive way, (5.11) (Pedrycz, Ekel, and Parreiras, 2011) can be stated in the following form:

$$R^d = P \cup J \tag{5.12}$$

With the use of the operations on fuzzy sets it is possible to define P, I, and J exclusively in terms of R. The corresponding results are given, for instance, in (Fodor and Roubens, 1993; Pedrycz, Ekel, and Parreiras, 2011).

There are several ways for building the fuzzy nonstrict preference relations (some of them are discussed below). Considering this, it is necessary to indicate that regardless of the method applied to construct the fuzzy nonstrict preference relation, the fuzzy strict preference relation as well as the fuzzy indifference relation can be obtained from that relation on the basis of applying several ways. The most popular way which, probably, is one of the first introduced for this goal, is presented in (Orlovsky, 1978; Orlovsky, 1981). In particular, the fuzzy strict preference relation can be presented in the following form:

$$P(X_k, X_l) = \max\left(R(X_k, X_l) - R(X_l, X_k),\ 0\right) \tag{5.13}$$

At the same time, the fuzzy indifference relation can be represented as follows:

$$I(X_k, X_l) = \min\,(R(X_k, X_l),\ R(X_l, X_k)) \tag{5.14}$$

Example 5.4. Let us consider the fuzzy nonstrict preference relation

$$R = \begin{bmatrix} 1 & 1 & 1 & 0.8 \\ 0.8 & 1 & 1 & 0.7 \\ 0.7 & 0.5 & 1 & 0.6 \\ 1 & 1 & 1 & 1 \end{bmatrix} \tag{5.15}$$

Applying (5.13) to (5.15), it is possible to obtain

$$P = \begin{bmatrix} 0 & 0.2 & 0.3 & 0 \\ 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0 \\ 0.2 & 0.3 & 0.4 & 0 \end{bmatrix} \tag{5.16}$$

The use of (5.14) provides

$$I = \begin{bmatrix} 1 & 0.8 & 0.7 & 0.8 \\ 0.8 & 1 & 0.5 & 0.7 \\ 0.7 & 0.5 & 1 & 0.6 \\ 0.8 & 0.7 & 0.6 & 1 \end{bmatrix} \tag{5.17}$$

The way of constructing fuzzy strict preference relations and indifference relations on the basis of (5.13) and (5.14), respectively, is not the only one. Several researches proposed other methods. Among the most important results in this field, it is possible to refer to Ovchinnikov (1981), Roubens (1989), and Ovchinnikov and Roubens (1991), for instance. As an example, in Ovchinnikov (1981), it is proposed that

$$P(X_k, X_l) = \begin{cases} R(X_k, X_l) & \text{if } R(X_k, X_l) > R(X_l, X_k) \\ 0 & \text{if } R(X_k, X_l) \le R(X_l, X_k) \end{cases} \tag{5.18}$$

and the indifference relation I the same as in (Orlovsky, 1978; Orlovsky, 1981).

An overview of works indicated above as well as other similar studies is presented in (De Baets and Fodor, 1997) with conclusions that these works correspond to independent efforts to define fuzzy preference relations. A more formal class of results can be found, for instance, in (Alsina, 1985; Ovchinnikov and Roubens, 1992; Fodor and Roubens, 1994b; Bufardi, 1998; Llamazares, 2003; Fodor and Rudas, 2006; Fodor and De Baets, 2008). These results are based on applying axiomatic methods to extending the classical (Boolean) preference models to the fuzzy environment, in an attempt to derive the fuzzy strict preference, indifference, and incomparability relations from a fuzzy nonstrict preference relation, without losing the fuzzy counterparts of the Boolean preference structures (Pedrycz, Ekel, and Parreiras, 2011). One of the important practical conclusions of works indicated above and, in particular, of (Fodor and Roubens, 1994b), is the validity of the results of (Orlovsky, 1978; Orlovsky, 1981) on constructing fuzzy strict and indifference relations on the basis of (5.13) and (5.14). Finally, taking into account the results of (Fodor and Roubens, 1994a; Fodor and Roubens, 1994b), it is possible to define the fuzzy incomparability relation in the following form:

$$J(X_k, X_l) = \min\,(1 - R(X_k, X_l),\ 1 - R(X_l, X_k)) \tag{5.19}$$

5.2 Construction of Fuzzy Preference Relations

As it was indicated above, a fuzzy nonstrict preference relation $R(X_k, X_l)$ reflects the degree to which the alternative X_k is at least as good as X_l by mans of its membership function $\mu_R(X_k, X_l) : X \times X \to [0, 1]$.

When fuzzy preference relations are constructed by direct assessment, it is supposed that a DM is capable to indicate to what extend X_k is better (for instance, more or less, more intelligent, more attractive, etc.) than X_l by providing a subjective value from the unit interval. Different encoding schemes can be utilized to reflect the preference strength of one alternative over another. Following (Pedrycz, Ekel, and Parreiras, 2011), we discuss below two schemes. One of them is directed at the construction of a nonreciprocal fuzzy preference relation NR (Orlovsky,1978; Fodor and Roubens, 1994b; Ekel, 2002), which has a correspondence with the notion of fuzzy nonstrict preference relation discussed in the previous section. Besides, a natural and convincing approach to constructing fuzzy preference relations based on the ordering of fuzzy quantities is discussed below is also generates nonreciprocal fuzzy preference relations. Another encoding scheme helps one to build an additive reciprocal fuzzy preference relation RR, which is a fuzzy preference relation satisfying (Tanino, 1984; Chiclana, Herrera, and Herrera-Viedma 1998; Chiclana, Herrera, and Herrera-Viedma, 2001) the following property of the additive reciprocity:

$$RR(X_k, X_l) + RR(X_l, X_k) = 1 \quad \forall X_k, X_l \in X \tag{5.20}$$

The encoding scheme for constructing the additive reciprocal fuzzy preference relation can be presented (Pedrycz, Ekel, and Parreiras, 2011) in the following form:

- $RR(X_k, X_l) = 0.5$ means that X_k is indifferent to X_l;
- $0.5 < RR(X_k, X_l) \leq 1$ means that X_k is preferred to X_l;
- $0 \leq RR(X_k, X_l) < 0.5$ means that X_l is preferred to X_k;
- the entries of the main diagonal are filled with 0.5, since each element is equal to itself and, as a result, indifferent to itself.

It is natural, when a DM provides a value $RR(X_k, X_l)$, the value of $RR(X_l, X_k)$ can be obtained, taking into account (5.20), as $RR(X_l, X_k) = 1 - RR(X_k, X_l)$. The authors of (Pedrycz, Ekel, and Parreiras, 2011) claim, there are no preference eliciting procedures to assist a DM to directly define additive reciprocal fuzzy preference relations. Thus, a DM has to articulate preferences based on his/her own intuition and capabilities of quantifying coherently preference strengths with the rules presented above.

Besides, the additive reciprocal fuzzy preference relations accommodate intransitivity (Tanino, 1984; Chiclana, Herrera-Viedma, and Herrera, 2004). However, it is desirable to collect consistent fuzzy preference relations, since, in general, the methods for the analysis of such relations also require them to be consistent, in order to guarantee high-quality outcomes (Pedrycz, Ekel, and Parreiras, 2011). The additive transitivity property which is one of the most intuitively appealing conditions for attesting the consistency of additive reciprocal fuzzy preference relations is given in (Tanino, 1984; Herrera-Viedma *et al*., 2004) in the following form:

$$RR(X_k, X_j) - 0.5 = (RR(X_k, X_l) - 0.5) + (RR(X_l, X_j) - 0.5) \quad \forall k, j, l \in \{1, 2, ..., n\} \tag{5.21}$$

The elicitation process for constructing additive reciprocal fuzzy preference relations requires $n(n-1)/2$ pairwise comparisons. However, by enforcing additive transitivity, it is also possible to collect only $(n-1)$ pairwise comparisons and estimate the missing comparisons with the use of certain techniques such as the one proposed by in (Herrera-Viedma *et al*., 2004). If a DM provides all pairwise comparisons and they do not satisfy (5.21), it is possible to identify which pairwise comparisons are to be reviewed by a DM. Another way consists of using an automated method to repair (enhance) the provided judgments (without the need to ask a DM to review the corresponding judgments), by modifying them as slightly as possible, just to guarantee an acceptable level of consistency (Pedrycz, Ekel, and Parreiras, 2011).

Example 5.5. The four apartments (namely, X_1, X_2, X_3, and X_4) are considered for renting with taking into account their infrastructure convenience. A DM believes that X_1 and X_4 are the best alternatives, considering the transport accessibility, proximity of schools, grocery stores, restaurants, pharmacy stores, and cinemas. Besides, these alternatives are located in quiet streets. X_2 is characterized by the lack of nearby school. Taking into account that the family has two children 9 and 11 years old, this is a significant drawback. X_3 is characterized by the absence of nearby cinemas which is much smaller flaw.

Taking these considerations into account, the DM provided the following comparisons between the pairs of alternatives, based on the infrastructure convenience:

- $RR(X_1, X_2) = 1$ and $RR(X_2, X_1) = 0$, since X_1 is extremely better than X_2;
- $RR(X_1, X_3) = 0.8$ and $RR(X_3, X_1) = 0.2$, since X_1 is strongly better than X_3;
- $RR(X_1, X_4) = 0.5$ and $RR(X_4, X_1) = 0.5$, since X_1 is as good as X_4;
- $RR(X_2, X_3) = 0.4$ and $RR(X_3, X_2) = 0.6$, since X_3 is to moderate extent better than X_2;
- $RR(X_2, X_4) = 0$ and $RR(X_4, X_2) = 1$, since X_4 is extremely better than X_2;
- $RR(X_3, X_4) = 0.2$ and $RR(X_4, X_3) = 0.8$, since X_4 is strongly better than X_3.

The results of these comparisons are reflected by the following additive reciprocal fuzzy preference relation:

$$RR = \begin{bmatrix} 0.5 & 1 & 0.8 & 0.5 \\ 0 & 0.5 & 0.4 & 0 \\ 0.2 & 0.6 & 0.5 & 0.2 \\ 0.5 & 1 & 0.8 & 0.5 \end{bmatrix} \tag{5.22}$$

Although (5.22) does not fully satisfy the condition (5.21), its quality is enough good (the reader is invited to verify this).

The encoding scheme for constructing the nonreciprocal fuzzy preference relation is associated (Pedrycz, Ekel, and Parreiras, 2011) with the following conditions:

▪ if $NR(X_k, X_l) = 1$ and $NR(X_l, X_k) = 1$, then X_k is indifferent to X_l;
▪ if $NR(X_k, X_l) = 1$ and $NR(X_l, X_k) = 0$, then X_k is strictly preferred to X_l;
▪ if $NR(X_k, X_l) = 0$ and $NR(X_l, X_k) = 1$, then X_l is strictly preferred to X_k;
▪ if $NR(X_k, X_l) = 0$ and $NR(X_l, X_k) = 0$, then X_l is strictly preferred to X_k;
▪ the entries of the main diagonal are filled with 1, as each element is equal to itself and, as a result, indifferent to itself.

Intermediate judgments are allowed as well. Unlike (Pedrycz, Ekel, and Parreiras, 2011), they can be presented as follows:

▪ if $NR(X_k, X_l) = 1$ and $0 \le NR(X_l, X_k) < 1$, then X_k is weakly preferred to X_l;
▪ if $0 \le NR(X_k, X_l) < 1$ and $NR(X_l, X_k) = 1$, then X_l is weakly preferred to X_k.

Example 5.6. Taking into account the considerations related to the four apartments, which are the subject of Example 5.5, it is possible to provide (to construct the nonreciprocal fuzzy preference relation) the following comparisons between the pairs of alternatives:

▪ $NR(X_1, X_2) = 1$ and $NR(X_2, X_1) = 0$, since X_1 is extremely better than X_2;
▪ $NR(X_1, X_3) = 1$ and $NR(X_3, X_1) = 0.3$, since X_1 is strongly better than X_3;
▪ $NR(X_1, X_4) = 1$ and $NR(X_4, X_1) = 1$, since X_1 is as good as X_4;
▪ $NR(X_2, X_3) = 0.8$ and $NR(X_3, X_2) = 1$, since X_3 is to moderate extent better than X_2;
▪ $NR(X_2, X_4) = 0$ and $NR(X_4, X_2) = 1$, since X_4 is extremely better than X_2;
▪ $NR(X_3, X_4) = 0.3$ and $NR(X_4, X_3) = 1$, since X_4 is strongly better than X_3.

The results of the comparisons given above are reflected by the following nonreciprocal fuzzy preference relation:

$$NR = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0.8 & 0 \\ 0.3 & 1 & 1 & 0.3 \\ 1 & 1 & 1 & 1 \end{bmatrix} \tag{5.23}$$

Example 5.7. The application of (5.13) to (5.23) leads to the construction of the following fuzzy strict preference relations:

$$P = \begin{bmatrix} 0 & 1 & 0.7 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0.2 & 0 & 0 \\ 0 & 1 & 0.7 & 0 \end{bmatrix} \tag{5.24}$$

At the same time, applying (5.14) to (5.23), we can build the following fuzzy indifference preference relation:

$$I = \begin{bmatrix} 1 & 0 & 0.3 & 1 \\ 0 & 1 & 0.8 & 0 \\ 0.3 & 0.8 & 1 & 0.3 \\ 1 & 0 & 0.3 & 1 \end{bmatrix} \tag{5.25}$$

With respect to the transitivity of nonreciprocal fuzzy preference relations, it should be mentioned that the min-transitivity, estimated by (5.8), is one of the most common consistency conditions. Among the most utilized consistency conditions, it can also be indicated the weak-transitivity condition (Pedrycz, Ekel, and Parreiras, 2011) given as follows:

$$\begin{aligned} &\text{If } RR(X_k, X_j) \geq RR(X_j, X_k) \text{ and } RR(X_j, X_l) \geq RR(X_l, X_j) - 0.5), \\ &\text{then } RR(X_k, X_l) \geq RR(X_l, X_k) \quad \forall X_k X_j, X_l \in X \end{aligned} \tag{5.26}$$

A natural and rational (from the fundamental as well as psychological points of view) approach for deriving fuzzy preference relations is the use of fuzzy estimates provided by a DM to evaluate each alternative. In essence, the use of this approach is associated with the need to compare or rank fuzzy numbers to choose the best (largest or smallest) or worst (smallest or largest) among them on the basis of applying the corresponding techniques (Pedrycz, Ekel, and Parreiras, 2011).

Various works can be indicated that are dedicated to the techniques for comparing or ranking fuzzy numbers. For instance, it is possible to distinguish, (Jain, 1976; Baas and H. Kwakernaak, 1977; Baldwin and Guild, 1979; Orlovsky, 1981; Yager, 1981; Dubois and Prade, 1983; Lee and Li, 1988; Tseng and Klein, 1989; Chen and Hwang, 1992; Fortemps and Roubens, 1996; Cheng, 1998; Horiuchi and Tamura, 1998; Raj and Cumar, 1999), Modares and Sadi-Nezhad, 2001; Chu and Tsao, 2002; Facchinetti, 2002; Liu and Han, 2005; Abbasbandy and Asady, 2006;Wang and Lee, 2008; Abbasbandy and Hajjari, 2009; Chen and Chen, 2009; Wang and Liuo, 2009; Saadi-Nezhad and Shahnazari-Shahrezaei, 2013; Destercke and Couso, 2015).

The authors of (Chen and Hwang, 1992) have proposed a way for classifying the groups of techniques related to the ordering of fuzzy quantities. In particular, the following groups of techniques have been classified:

▪ preference relations;
▪ use of fuzzy mean and spread characteristics;
▪ fuzzy scoring techniques;
▪ linguistic methods.

Among these classes, the authors of (Horiuchi and Tamura, 1998) consider the construction of fuzzy preference relations by means of pairwise comparisons as being the most practical and justified approach. Considering this, it is worth distinguishing the fuzzy number ranking index proposed by Orlovsky (Orlovsky, 1981). It is based on the concept of a membership function of a generalized preference relation.

In particular, if $F(X_k)$ and $F(X_l)$ are fuzzy sets reflecting evaluations of the objective function F or the attribute F for alternatives X_k and X_l, respectively, the quantity $\eta\{\mu[F(X_k)],\mu[F(X_l)]\}$ is the degree of preference $\mu[F(X_k)] \succcurlyeq \mu[F(X_l)]$, while $\eta\{\mu[F(X_l)],\mu[F(X_k)]\}$ is the degree of preference $\mu[F(X_l)] \succcurlyeq \mu[F(X_k)]$. Then, the membership functions of the generalized preference relations $\eta\{\mu[F(X_k)],\mu[F(X_l)]\}$ and $\eta\{\mu[F(X_l)],\mu[F(X_k)]\}$ take the following forms:

$$\eta\{\mu[F(X_k)],\mu[F(X_l)]\} = \sup_{F(X_k),F(X_l)\in F} \min\{\mu[F(X_k)],\ \mu[F(X_l)],\mu_R[F(X_k),F(X_l)]\} \tag{5.27}$$

$$\eta\{\mu[F(X_l)],\mu[F(X_k)]\} = \sup_{F(X_k),F(X_l)\in F} \min\{\mu[F(X_l)],\ \mu[F(X_k)],\mu_R[F(X_l),F(X_k)]\} \tag{5.28}$$

respectively.

In (5.27) and (5.28), $\mu_R[F(X_k),F(X_l)]$ and $\mu_R[F(X_l),F(X_k)]$ are the membership functions of the corresponding fuzzy preference relations which, respectively, reflect the essence of the preferences of X_k over X_l and X_l over X_k (for instance, "more attractive", "more flexible", etc.).

When F can be measured on a numerical scale and if the essence of preference behind relation R is coherent with the natural order (≤) along the axis of measured values of F, then (5.27) and (5.28), respectively, are reduced to the following expressions:

$$\eta\{\mu[F(X_k)],\mu[F(X_l)]\} = \sup_{\substack{F(X_k),F(X_l)\in F \\ F(X_k)\le F(X_l)}} \min\{\mu[F(X_k)],\ \mu[F(X_l)]\} \tag{5.29}$$

$$\eta\{\mu[F(X_l)],\mu[F(X_k)]\} = \sup_{\substack{F(X_k),F(X_l)\in F \\ F(X_l)\le F(X_k)}} \min\{\mu[F(X_k)],\ \mu[F(X_l)]\} \tag{5.30}$$

when F is a maximization criterion or attribute.

If F is a maximization criterion or attribute, then the relationships (5.29) and (5.30) are to be modified. In particular, (5.29) is to be written for $F(X_k) \geq F(X_l)$ and (5.30) is to be written for $F(X_l) \geq F(X_k)$.
The expressions (5.29) and (5.30) are consistent with the Baas-Kwakernaak (Baas and Kwakernaak, 1977), Baldwin-Guild (Baldwin and Guild, 1979), and one of the Dubois-Prade (Dubois and Prade, 1983) fuzzy number ranking indices.

Example 5.8. Assume we are given the alternatives X_1 and X_2 with their respective fuzzy values $F(X_1)$ and $F(X_2)$ of the objective function, as illustrated in Figure 5.2. With the use of (5.29) and (5.30), we evaluate the degrees of the preferences of X_1 over X_2 ($X_1 \succcurlyeq X_2$) and of X_2 over X_1 ($X_2 \succcurlyeq X_1$), in order to select the smallest value between $F(X_1)$ and $F(X_2)$. For illustrative purposes, Table 5.1 shows the corresponding membership functions, given just for some selected points distributed in the universe of discourse.

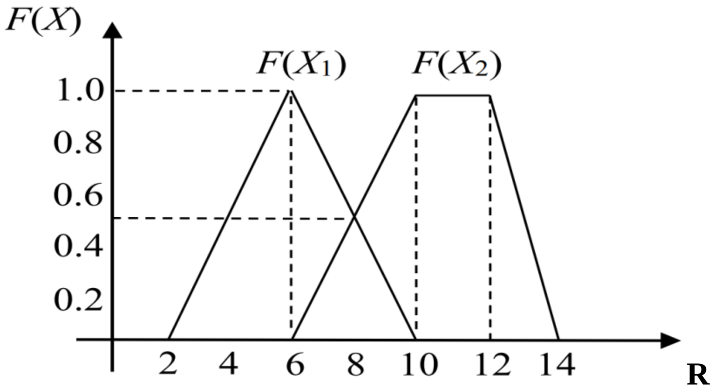


Figure 5.2 Fuzzy values of the objective function $F(X)$.

The formal application of (5.29) and (5.30) is associated with the construction of the Cartesian product of $F(X_1)$ and $F(X_2)$, as presented in Table 5.2.

Table 5.1 Membership functions of $F(X_1)$ and $F(X_2)$

R	2	4	6	8	10	12	14
μ[$F(X_1)$]	0	0.50	1	0.50	0	0	0
μ[$F(X_2)$]	0	0	0	0.50	1	1	0

Table 5.2 Cartesian product of $F(X_1)$ and $F(X_2)$

$F(X_1) \rightarrow$ / $F(X_2) \downarrow$	0/2	0.50/4	1/6	0.50/8	0/10	0/12	0/14
0/2	**0**	0	0	0	0	0	0
0/4	0	**0**	0	0	0	0	0
0/6	0	0	**0**	0	0	0	0
0.50/8	0	0.50	0.50	**0.50**	0	0	0
1/10	0	0.50	1	0.50	**0**	0	0
1/12	0	0.50	1	0.50	0	**0**	0.35

0/14	0	0	0	0	0	0	**0**

The entries located on the main diagonal (marked in bold) and below it are related to the area with $F(X_1) \leq F(X_2)$. Taking this into account, according to (5.29), it is possible to find $\eta\{\mu[F(X_1)],\mu[F(X_2)]\} = 1$. On the other hand, the entries located on the main diagonal and above it are associated with $F(X_2) \leq F(X_1)$. By applying (5.30), we obtain $\eta\{\mu[F(X_2)],\mu[F(X_1)]\} = 0.50$. Therefore, we have that X_1 is equally good (small) or better (smaller) than X_2. In particular, it is possible to interpret the obtained results as follows: X_2 is equivalent to X_1 with the degree $\eta\{\mu[F(X_2)],\mu[F(X_1)]\} = 0.50$. At the same time, X_1 is strictly better than X_2 with the degree $\eta\{\mu[F(X_1)],\mu[F(X_2)]\} - \eta\{\mu[F(X_2)],\mu[F(X_1)]\} = 0.50$.

By the way, the fact that the level of the intersection of $F(X_1)$ and $F(X_2)$ corresponds to 0.50 (see Figure 5.2) permits one to define $\eta\{\mu[F(X_1)], \mu[F(X_2)]\}$ =1 and $\eta\{\mu[F(X_2)], \mu[F(X_1)]\}$ =0.50 without the need to construct the Cartesian product of $F(X_1)$ and $F(X_2)$. This way of determining $\eta\{\mu[F(X_k)], \mu[F(X_l)]\}$ and $\eta\{\mu[F(X_l)], \mu[F(X_k)]\}$ can be used in many practical situations.

Let us consider another example where membership functions of the compared fuzzy quantities have more than one intersection.

Example 5.9. We are given the alternatives X_1 and X_2 with fuzzy quantities of the objective function $F(X_1)$ and $F(X_2)$, which are presented in Figure 5.3. It is necessary to evaluate the degrees of the preferences X_1 over X_2 ($X_1 \succcurlyeq X_2$) and X_2 over X_1 ($X_2 \succcurlyeq X_1$) to select the largest value between $F(X_1)$ and $F(X_2)$. The corresponding membership functions are given in Table 5.3.

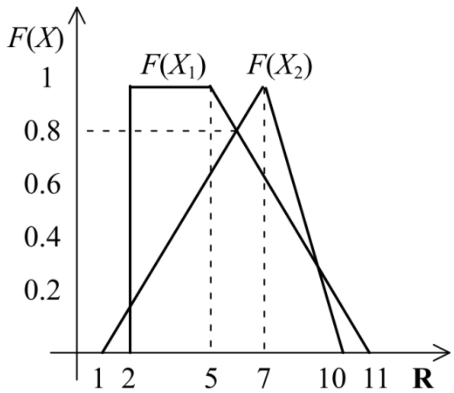


Figure 5.3 Comparison of alternatives with trapezoidal membership functions.

The Cartesian product of $F(X_1)$ and $F(X_2)$, constructed on the basis of (5.29) and (5.30) is presented in Table 5.4.

One can note that $\eta\{\mu[F(X_1)],\mu[F(X_2)]\}$ =0.83 and $\eta\{\mu[F(X_2)],\mu[F(X_1)]\}$ =1. Besides, as shown in Example 5.8, we can conclude that $F(X_1)$ is equivalent to $F(X_2)$ with the degree $\eta\{\mu[F(X_1)],\mu[F(X_2)]\}$ =0.83. At the same time, $F(X_2)$ is strictly larger (better) than $F(X_1)$ with the degree $\eta\{\mu[F(X_2)],\mu[F(X_1)]\}$ - $\eta\{\mu[F(X_1)],\mu[F(X_2)]\}$ =0.17.

Table 5.3 Membership functions of $F(X_1)$ and $F(X_2)$

R	1	2	3	4	5	6	7	8	9	10	11
$F(X_1)$	0	1	1	1	1	0.83	0.67	0.50	0.34	0.17	0
$F(X_2)$	0	0.17	0.34	0.50	0.67	0.83	1	0.67	0.34	0	0

Table 5.4 Cartesian product of $F(X_1)$ and $F(X_2)$

$F(X_1)\rightarrow$ / $F(X_2)\downarrow$	0/1	1/2	1/3	1/4	1/5	0.83/6	0.67/7	0.50/8	0.34/9	0.17/10	0/11
0/1	**0**	0	0	0	0	0	0	0	0	0	0
0.17/2	0	**0.17**	0.17	0.17	0.17	0.17	0.17	0.17	0.17	0.17	0
0.34/3	0	0.34	**0.34**	0.34	0.34	0.34	0.34	0.34	0.34	0.17	0
0.50/4	0	0.50	0.50	**0.50**	0.50	0.50	0.50	0.50	0.34	0.17	0
0.67/5	0	0.67	0.67	0.67	**0.67**	0.67	0.67	0.50	0.34	0.17	0
0.83/6	0	0.83	0.83	0.83	0.83	**0.83**	0.67	0.50	0.34	0.17	0
1/7	0	1	1	1	1	0.83	**0.67**	0.50	0.34	0.17	0
0.67/8	0	0.67	0.67	0.67	0.67	0.67	0.67	**0.50**	0.34	0.17	0
0.34/9	0	0.34	0.34	0.34	0.34	0.34	0.34	0.34	**0.34**	0.17	0
0/10	0	0	0	0	0	0	0	0	0	**0**	0
0/11	0	0	0	0	0	0	0	0	0	0	**0**

In such a manner, on the basis of the relations between (5.29) and (5.30) (or, generally, between (5.27) and (5.28)) , it is possible to express the degree of preference of any of the alternatives compared and, therefore, to construct nonreprocical fuzzy preference relations. Utilization of this approach is well founded. However, it is important to indicate that, in practice, there are situations where the fuzzy quantities $F(X_k)$ and $F(X_l)$ have trapezoidal membership functions that are located in such a way that it is not possible to distinguish X_k from X_l. For instance, we can observe that for the situation shown in Figure 5.4 the alternatives are indistinguishable since

$$\eta\{\mu[F(X_1)],\mu[F(X_2)]\}=\eta\{\mu[F(X_2)],\mu[F(X_1)]\}=\alpha \tag{5.31}$$

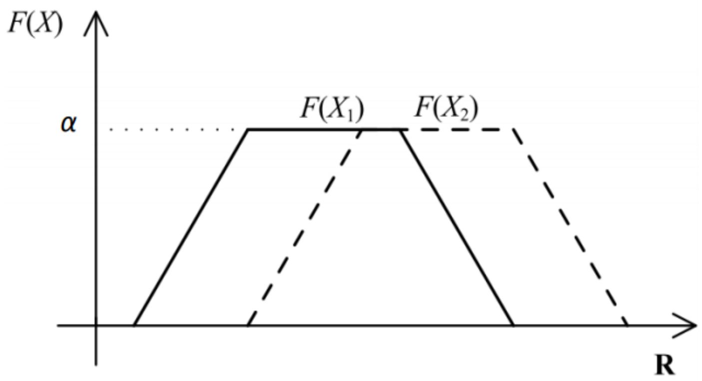


Figure 5.4 Alternatives with trapezoidal membership functions.

As will be further discussed in this chapter, in such situations, the algorithms applied to solve the problems with fuzzy coefficients do not allow one to obtain unique solutions because they “stop” when conditions like (5.31) arise. This should be considered to be a natural consequence of the existence of decision uncertainty regions, produced by a combination of the uncertainty and the relative stability of optimalsolutions (Ekeland Popov, 1985; Popov and Ekel,1987). In this connection, other indices may be used as additional means for the ranking of fuzzy numbers.

The reviews on techniques used for ranking of fuzzy numbers can be found in (Dubois and Prade, 1999; Wang and Kerre, 2001).

The authors of (Wang and Kerre, 2001) enumerate more than 35 fuzzy number ranking indices and conclude the following: unlike the case of real numbers, fuzzy quantities have no natural order. The basic idea behind the methods for the ordering of fuzzy quantities consists of converting each fuzzy quantity into a real number and realizing the comparison of fuzzy quantities on the basis of the resulting real numbers. However, each approach for realizing such a conversion focuses on an intricate aspect inherent to fuzzy quantities. As a consequence, each approach suffers from some weakness associated with the loss of information inherent to the conversion of a fuzzy quantity to a single real number. The authors of (Wang and Kerre, 2001) support this point of view by citing the author of (Freeling, 1980): "by reducing the whole of our analysis to a single number, we are losing much of the information we have purposely been keeping throughout calculations". The authors of works (Cheng, 1998; Lee-Kwang, 1999) share this opinion as well. The author of (Cheng, 1998) also indicates that many of indices produce different rankings for the same problem. The authors of (Cheng, 1998; Ekel, Pedrycz, and Schinzinger, 1998; Lee-Kwang, 1999; Chen and Lu, 2002) underline that fuzzy number ranking indices occasionally result in choices which appear inconsistent with intuition. The authors of (Chen and Lu, 2002) indicate that the majority of methods for the ranking of fuzzy numbers suppose that membership functions of fuzzy numbers are normalized. However, this limitation is not always adequate. The authors of Tseng and Klein (1989) indicate that the ranking methods may not reflect the preferences of interests of a DM. Further, many techniques of ordering help one only to observe an order among fuzzy quantitatives; however, they do not permit one to measure the degree of dominance among them, requiring a significant volume of valculations (Chen and Klein, 1997). Finally, it is necessary to mention that the majority of indices for the ranking of fuzzy quantities has been proposed with the aspiration for obligatory distinguishing the alternatives, which is always questionable because the uncertainty of information creates inherent decision uncertainty regions. Taking this into account, the possibility of identifying situations where the compared alternatives cannot be distinguished should be considered as a merit of the fuzzy number ranking index based on the conception of a membership function of a generalized preference relation.

Taking the above into account, the fuzzy number ranking index based on the conception of a membership function of a generalized preference relation is used in the present book for ordering of fuzzy quantities and, on its basis, for the construction of fuzzy preference relations.

5.4 Preference Formats

Fuzzy preference relations are not a unique form of preference representation. In real-world applications, every professional involved in any decision process has own perception of the problem, a different way of thinking and usually has access to different information sources. As a consequence, it is natural to meet circumstances where every DM selects a different format to express own preferences. Furthermore, several factors may lead a DM to select a different format for expressing own preferences on each criterion. Among these factors we can list the following (Pedrycz, Ekel, and Parreiras, 2011):

▪ Each criterion comes with its significance (a fundamental feature which provides significance to the difference between two degrees evaluated on this criterion). Depending on whether this significance has a quantitative or qualitative character, the use of certain preference formats can make the preference elicitation process easier and also more reliable.
▪ Each criterion is associated with information arising from different sources and with information having different levels of its uncertainty.
▪ A DM may find that his/her preference strengths can be better reflected or quantified by a specific preference format.
▪ The fact that a DM may possess previous knowledge or experience in expressing a specific preference format can motivate him/her to choose it again.

Considering this, it should be noted that, for instance, the authors of (Zhang, Chen, and Chong, 2004; Zhang, Wang, and Yang, 2007) distinguish eight preference formats which can be used to establish preferences among analyzed alternatives. With the availability of different formats, a DM can select the one that makes hem/her feel more comfortable for articulating his/her own preferences (Pedrycz, Ekel, and Parreiras, 2011). In this book, we deal with five preference formats, which, in our opinion, cover most part of real situations in preparing information for decision making. They include:

▪ ordering of the alternatives;
▪ utility values;
▪ fuzzy estimates;
▪ multiplicative preference relations;
▪ fuzzy preference relations.

The last preference format was considered above. Below, we briefly consider other formats.

5.4.1 Ordering of Alternatives

In certain cases, a DM is ready to provide a direct ranking of alternative (from the point of view of some indicators, criteria, etc.) in accordance with his/here preferences. Sometimes this fact should be regarded as positive. In other cases, this capacity of a DM is to be considered as a necessary measure. In particular, when a DM has difficulties in assessing the strength of preferences quantitatively, it is advantageous to use information of purely ordinal character. By asking a DM to provide a complete ranking of the alternatives, he/she is released from having to quantify the difference in his/her preference strengths between any two alternatives. In this way, the chances of deriving recommendations based on incorrect information are reduced (Pedrycz, Ekel, and Parreiras, 2011).

The ordering of alternatives from best to worst can be represented as an array $O=\{O(X_1)\ O(X_2)\ ...\ O(X_n)\}$, with $O(X_k)$ being a permutation function which returns the position of alternative X_k among the integer values $\{1,2,\ldots,n\}$ (Chiclana, Herrera, and Herrera-Viedma, 1998).

Example 5.10. A DM is to provide the ordering of six alternatives $X=\{X_1, X_2, X_3, X_4\}$ from the best (first position) to the worst (last position) on a given criterion F. Table 5.5 shows DM judgments which permitted the construction of the following ordered array: $O=\{3\ 1\ 2\ 4\}$.

Table 5.5 Ordering of alternatives

Alternatives	X_1	X_2	X_3	X_4
Positions	Third	First	Second	Fourth

5.4.2 Utility Values

The terms "utility function" and "value function" are related to two types of preference models. Utility theory deals with preference models for risky decisions that is, decisions involving alternatives whose consequences are uncertain and (as a consequence) involve risks (Pedrycz, Ekel, and Parreiras, 2011). The value theory is considered as a simplification of utility theory for dealing with decision under certainty. In this book, we focus on preference models based on utility functions and value functions, along with their preference eliciting procedures. However, we consider only the preference elicitation process of value functions, indicating (Keeney and Raiffa, 1976; Von Winterfeldt and Edwards, 1986) for preference elicitation techniques associated with the construction of utility functions. The elicitation techniques for these two types of models are enough different. However, there is no need for making a distinction among them from the point of view of their converting to fuzzy preference relations on the basis of applying transformation functions (Pedrycz, Ekel, and Parreiras, 2011). Taking this into account, for simplifying, we use the term “utility” to refer to both types of models to provide the text coherent with some relevant references on the corresponding transformation functions (Chiclana, Herrera, and Herrera-Viedma, 1998; Chiclana, Herrera, and Herrera-Viedma, 2001; Zhang, Wang, and Yang, 2007).

Let us consider the representation of DM preferences with the use of the preference function named utility function $U(X)$ (by convention, the higher value of the utility function is equal to 1 and its lower value is equal to 0). In the literature, it is possible distinguish two main types of utility functions: ordinal and cardinal (Pedrycz, Ekel, and Parreiras, 2011). The ordinal utility function is related to the ordering of the alternatives rather than reflecting the preference strength of one alternative over another. In real-world applications, the ordinal utility function is usually modeled as a monotonically increasing (or decreasing) function, such as a maximizing profit function (or a minimizing cost function), defined over the significance axis of the considered criterion (Dyer, 2005).

It is assumed that the ordinal utility function preserves the preference ordering of alternatives in such a way that

if $U(X_k) > U(X_l)$, then X_k is preferred to X_l (5.32)

if $U(X_k) = U(X_l)$, then X_k is indifferent to X_l (5.33)

Since in the use of ordinal utility functions, the ranking of the numbers is all that matters, any monotonic transformation of this function is considered equivalent to it. Thus, the main weakness of ordinal utility functions is associated with the fact that different ordinal utility functions can be utilized for reflecting the same ordering of the alternatives (Pedrycz, Ekel, and Parreiras, 2011). In the aggregation across the criteria in the multicriteria analysis, each one of these admissible functions may lead to different outcomes. However, this ambiguity can be reduced by using the measurable or cardinal utility function for capturing the strength of preferences.

The most commonly utilized cardinal utility function (Farquhar and Keller, 1989; Belton, 1999; Dyer, 2005) is based on differences in preference strengths, in such a way that, given a measurable utility function, if we have that

$$U(X_j) - U(X_k) > U(X_k) - U(X_l) \tag{5.34}$$

then we can conclude that the difference in the preference between X_j and X_k is greater than the difference in preference between X_k and X_l.

It should be noted that the ratio between two preference degrees expressed as cardinal utilities based on interval scales is not meaningful, only the ratio between their differences is significant. The ratio between preference strengths makes sense only when utilities are measured on a ratio scale. This is the most rigorous type of preference measure, being admissible only when they are measured on an appropriate scale with an absolute zero (Pedrycz, Ekel, and Parreiras, 2011). Considering this, one can state the preference for any alternative X_k over another alternative X_l as a ratio $U(X_k)/U(X_l)$ between their respective utilities, that is, it is possible to determine how many times alternative X_k is better (or worse) than the other alternative X_l by means of this ratio.

The interval scale cardinal utility function can be determined by several manners, for instance, (Winterfeldt and Edwards,1986; Belton, 1999). Here we consider one of them: the direct rating (Pedrycz, Ekel, and Parreiras, 2011). In its use, a DM is supposed to compare differences in preference strengths in order to determine the utility of each alternative.

The essence of the direct rating technique consists of obtaining utility values, taking as reference only two anchor points, in such a way that it is unnecessary to define a scale for characterizing other performances rather than of the alternatives being evaluated (V. Belton, 1999). It is associated with the following sequence of steps.

▪ **Step 1.** A DM identifies two anchors, which correspond to the worst evaluated alternative and the best evaluated alternative from the point of view of the considered criterion. The values 0 and 1 are respectively assigned to them.
▪ **Step 2.** A DM rates the remaining alternatives in between the extreme points of the scale, in such a way that the spacing between the alternatives reflects the strength of preferences of one alternative over another.
▪ **Step 3.** A DM should review the assessments and, whenever necessary, update them (the process stops only if a DM is in complete accordance with the elicited utility values). If the criterion under consideration can be captured by an attribute which can be measured on a numerical scale, it is possible to plot the assessed points and draw a smooth curve passing through them.

Example 5.11. A DM has modified the considerations related to the four apartments, which are the subject of Example 5.5. In particular, taking into account that X_1 is closer than X_4 to the municipal park, a DM, in Step 1 of applying the direct ratng technique, identifies the apartments X_1 and X_2 as the anchors (X_1 is the best alternative and X_2 is the worst one).

In Step 2, a DM thinks that the difference in preference of X_1 over X_4 is much lover than the difference in preference of X_4 over X_2, which is characterized by the lack of nearbly school (the family has two children 9 and 11 years old). Besides, a DM thinks that the difference in preference of X_1 over X_3 is more or less iqual to difference in preference of X_3 over X_2. Finally, a DM judges that X_4 is preferred to X_3. X_3 is characterized by the absence of nearby cinemas which is more significant than the large distance to the municipal park. These considerations are reflected by Figure 5.5. In Step 3, the given assessments are confirmed with the preferences of a DM.

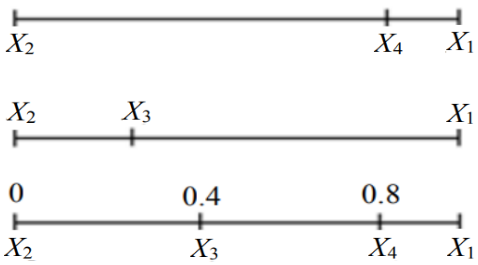


Figure 5.5 Utilities elicited with applying the direct rating technique.

5.4.3 Fuzzy Estimates

The elements of X can be evaluated with the use of fuzzy estimates $L = \{l(X_1),\ l(X_2),\ldots,l(X_n)\}$, where $l(X_k)$ is the fuzzy estimate associated with alternative X_k completed from the point of view of a given criterion F. The fuzzy estimate $l(X_k)$ refers to a fuzzy number that can be directly specified by a DM or indirectly specified by means of linguistic terms from a set S such as, for instance, $S(F) = \{\textit{low velocity}, \textit{average velocity}, \textit{high velocity}\}$. The linguistic terms are to be converted into fuzzy estimates, as it is required to perform the analysis of the problem (Pedrycz, Ekel, and Parreiras, 2011). Although it is possible to conclude that the use of linguistic terms makes the preference elicitation process more intuitive, it is important to indicate that the effectiveness of the elicitation can be diminished due to the existence of differences between numerical interpretation of the linguistic terms in the experts' mind and their numerical representation in the model being utilized (Pedrycz, Ekel, and Parreiras, 2011). In this context, the techniques discussed in Chapter 3 for constructing and equalizing fuzzy sets may be helpful for reducing this type of elicitation error.

Example 5.12. The DM utilized linguistic terms from a set $S(F) = \{$very poor, poor, average, good, very good$\}$ to evaluate the inftustructure convenience for each apartment from the previous example. The set of linguistic terms along with their respective representation through fuzzy sets are shown in Figure 5.6. The elicited preferences are given in Table 5.6.

Table 5.6 Evaluation of alternative by means of linguistic terms

Apartment	Evaluation
X_1	Very good
X_2	Poor
X_3	Average
X_4	Good

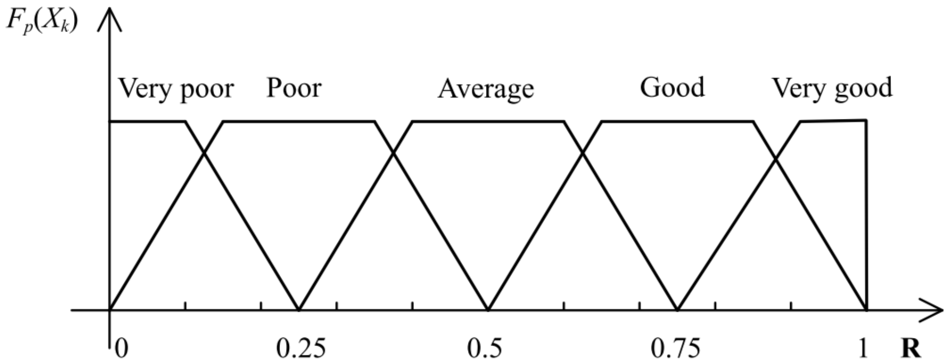


Figure 5.6 Set of linguistic terms.

5.4.4 Multiplicative Preference Relations

The multiplicative preference relation can be represented as a $n \times n$ positive reciprocal matrix MR reflecting the preference intensity ratio between the alternatives in accordance with the Analityc Hierarrchy Process (AHP) approach (Saaty, 1980) presented in Chapter 3 as the Saaty's priority method. Each entry $MR(X_k, X_l)$ of this matrix represents a preference intensity ratio and can be interpreted as: " X_k is $MR(X_k, X_l)$ times more dominant than X_l" (Saaty, 1980) or, following (Chiclana, Herrera, and Herrera-Viesma, 2001), as " X_k is $MR(X_k, X_l)$ times as good as X_l".

The elicitation process within the AHP approach permits a DM to express preferences verbally, applying the corresponding linguistic terms, or numerically, utilizing different ratio scales (Saaty, 1980). If a DM uses linguistic terms, the judgments are to be converted into numbers, in order to realize the analysis of the decision making problem. In this way, regardless of how the elicitation process is carried out, it is necessary to define an adequate ratio scale (Pedrycz, Ekel, and Parreiras, 2011). The selection of a proper ratio scale is to be done by considering the entire set of objects about which ratio comparisons are to be performed (Harker and Vargas, 1987; Salo and Hämäläinen, 1997).

Under the situation of multiplicative reciprocity, once a DM provides $MR(X_k, X_l)$, the value of $MR(X_l, X_k)$ is automatically inferred as $MR(X_l, X_k) = 1/MR(X_k, X_l)$. Although multiplicative preference relations accommodate intransitivity, it is desirable to collect judgments as much consistent as possible, since the methods for the analysis of such relations usually require them to be transitive in order to guarantee results of high quality (Pedrycz, Ekel, and Parreiras, 2011). As it was already indicated in Chapter 3, the perfect consistency of a multiplicative preference relation is reflected as the satisfaction of the multiplicative transitivity property, that is,

$$MR(X_k, X_j) = MR(X_k, X_l) \cdot MR(X_l, X_j) \quad \forall j,k,l \in \{1,2,...,n\} \tag{5.35}$$

In the elicitation process, it is necessary to collect $n(n-1)/2$ pairwise comparisons or else, by enforcing multiplicative transitivity, it is possible to collect only $(n-1)$ pairwise comparisons and estimate the missing ones with the use of the condition (5.35). As it is described in Chapter 3, when a DM provides all pairwise comparisons, if they do not satisfy (5.35), it is possible to identify the inconsistent pairwise comparisons so that a DM can review them or else. It is also possible to apply an automated method to improve the consistency of the constructed multiplicative preference relations (Zeshui and Cuiping, 1999).

Example 5.13. Let us consider the preference elicitation process by means of multiplicative preference relation based on the 1-9 Saaty scale, considered in Chapter 3, taking into account the same inftustructure convenience for the four apartments. The pairwise judgments provided by a DM, for instance, are the following:

- X_1 is 2 times better than X_2;
- X_1 is 4 times better than X_3;
- X_1 is 8 times better than X_4;
- X_2 is 2 times better than X_3;
- X_2 is 4 times better than X_4;
- X_3 is 8 times better than X_4.

These judgments are reflected in the form of the following matrix:

$$MR = \begin{bmatrix} 1 & 2 & 4 & 8 \\ 1/2 & 1 & 2 & 4 \\ 1/4 & 1/2 & 1 & 2 \\ 1/8 & 1/4 & 1/2 & 1 \end{bmatrix} \tag{5.36}$$

To verify the consistency of the obtained multiplicative preference relation, we can utilize the index of inconsistency proposed by Saaty and considered in Chapter 3. Since the maximal eigenvalue of the matrix (5.36) is 4, the index inconsistency calculated applying (3.X) is

$$\nu = \frac{(4-4)}{3} = 0 \tag{5.37}$$

In this way, we can speak about the absolute consistency of (5.36).

Example 5.14. Suppose that a DM changed his/here opinions. The new judgments are the following:

- X_1 is 8 times better than X_2;
- X_1 is 4 times better than X_3;
- X_1 is 2 times better than X_4;
- X_2 is 2 times better than X_3;
- X_2 is 4 times better than X_4;
- X_3 is 8 times better than X_4.

These judgments are reflected as follows:

$$MR = \begin{bmatrix} 1 & 8 & 4 & 2 \\ 1/8 & 1 & 2 & 4 \\ 1/4 & 1/2 & 1 & 2 \\ 1/2 & 1/4 & 1/2 & 1 \end{bmatrix} \tag{5.38}$$

The maximal eigenvalue of the matrix (5.38) is 5.35. The corresponding index of inconsistency is

$$\nu = \frac{(5,35-4)}{3} = 0.45 \tag{5.39}$$

Therefore, as ν is much higher than 0.1, the inconsistency level of (5.38) is not acceptable.

5.5 Transformation Functions and Their Application to Unifying Different Preference Formats

In procedures of decision-making, when different preference formats are utilized, the information is to be uniformed with the use of adequate transformation functions to be aggregated. These transformation functions serve for converting heterogeneous preference information, which may be quantitative or qualitative, two valued or fuzzy, ordered or nonordered, ordinal or cardinal, and even based on different types of scales, into fuzzy preference relations (Pedrycz, Ekel, and Parreiras, 2011).

It was shown in the previous section, how fuzzy preference relations can be obtained from the evaluation of the alternatives on the basis of fuzzy estimates. Below we discuss questions of constructing fuzzy preference relations from the evaluation of the alternatives on the basis of other preference formats as well as quantitative information. First, we consider certain transformation functions for converting the preference information from the ordering of the alternatives, utility values, multiplicative preference relations, and nonreciprocal fuzzy preference relations into the additive reprocical format. We will not consider all the existing transformation functions but just some of them. Then, we derive from those selected transformation functions other transformation functions, which can be used for converting the preference information from those different formats (including the additive reprocical fuzzy preference relations) into the nonreprocical format (Pedrycz, Ekel, and Parreiras, 2011).

5.5.1 Transformation of the Ordered Array into the Additive Reciprocal Fuzzy Preference Relation

The following transformation function, which helps one to convert the ordered array into the additive reciprocal fuzzy preference relation, is proposed in (Chiclana, Herrera, and Herrera-Viedma, 1998):

$$RR(X_k, X_l) = \frac{1}{2}\left(1 + \frac{O(X_l) - O(X_k)}{n-1}\right) \tag{5.40}$$

Some properties of this transformation function are discussed in (Chiclana, Herrera, and Herrera-Viedma, 1998; Pedrycz, Ekel, and Parreiras, 2011). In particular, the use of (5.40) produces the additive reciprocal fuzzy preference relation that satisfies the additive transitivity.

Example 5.15. Let us consider the ordered array of Example 5.10, presented in Table 5.5. For instance, applying (5.40) to the pair X_2 and X_3, we can obtain the following results:

$$RR(X_2, X_3) = \frac{1}{2}\left(1 + \frac{2-1}{3}\right) = 0.67 \tag{5.41}$$

$$RR(X_3, X_2) = \frac{1}{2}\left(1 + \frac{1-2}{3}\right) = 0.33 \tag{5.42}$$

The application of (5.40) to other pairs of alternatives permits us to construct the following additive reprocical fuzzy preference relation:

$$RR = \begin{bmatrix} 0.50 & 0.17 & 0.33 & 0.67 \\ 0.83 & 0.50 & 0.67 & 1 \\ 0.67 & 0.33 & 0.50 & 0.83 \\ 0.33 & 0 & 0.17 & 0.50 \end{bmatrix} \tag{5.43}$$

The reader can see that the additive reprocical fuzzy preference relation (5.43) is coherent with the array provided by the DM.

5.5.2 Transformation of the Utility Values into the Additive Reciprocal Fuzzy Preference Relation

If the utility values are estimated on a ratio scale, the additive reciprocal fuzzy preference relation can be derived (Tanino, 1984; Chiclana, Herrera, and Herrera-Viedma, 1998; Pedrycz, Ekel, and Parreiras, 2011) from the corresponding array with applying the following correlations:

$$RR(X_k, X_l) = \begin{cases} \dfrac{U(X_k)}{U(X_k)+U(X_l)} & \text{if } U(X_k)+U(X_l) \neq 0 \\ 0.5 & \text{if } U(X_k) = U(X_l) = 0 \end{cases} \tag{5.44}$$

$$RR(X_k, X_l) = \begin{cases} \dfrac{[U(X_k)]^2}{[U(X_k)]^2+[U(X_l)]^2} & \text{if } U(X_k)+U(X_l) \neq 0 \\ 0.5 & \text{if } U(X_k) = U(X_l) = 0 \end{cases} \tag{5.45}$$

It is necessary to indicate that the transformation functions (5.44) and (5.45) generate the additive reciprocal fuzzy preference relations which do not satisfy the additive transitivity, but satisfy the multiplicative transitivity (Tanino,1984; Pedrycz, Ekel, and Parreiras, 2011).

As an example from (Pedrycz, Ekel, and Parreiras, 2011) presented below shows, the main difference between (5.44) and (5.45) is associated with the fact that the strength of the additive reciprocal fuzzy preference relation calculated on the basis of (5.45) tends to be farther from the indifference judgment than the corresponding strength of the additive reciprocal fuzzy preference relation calculated with the use of (5.44).

Example 5.16. The application of (5.44) and (5.45) to the utility array $U = \{0.2\ 0.4\ 0.1\ 0.3\}$ provides the following results:

$$RR = \begin{bmatrix} 0.50 & 0.33 & 0.67 & 0.4 \\ 0.67 & 0.50 & 0.8 & 0.57 \\ 0.33 & 0.2 & 0.50 & 0.25 \\ 0.6 & 0.43 & 0.75 & 0.50 \end{bmatrix} \tag{5.46}$$

$$RR = \begin{bmatrix} 0.50 & 0.20 & 0.80 & 0.31 \\ 0.80 & 0.50 & 0.94 & 0.64 \\ 0.50 & 0.06 & 0.50 & 0.10 \\ 0.69 & 0.36 & 0.90 & 0.50 \end{bmatrix} \quad (5.47)$$

As can be seen, the additive reciprocal fuzzy preference relations (5.46) and (5.47) are coherent with the utility array. However, in the use of (5.47), more than in applying (6.24), the strength of nonstrict preference of each alternative over another tends to the more distant from the judgment of indifference,being more higher or more lower than 0.50.

If we process a vector of cardinal utility values, defined on an interval scale normalized in [0, 1], the additive reciprocal fuzzy preference relation can be obtained by applying (Tanino, 1984; Chiclana, Herrera, and Herrera-Viedma, 1998; Pedrycz, Ekel, and Parreiras, 2011) as follows:

$$RR(X_k, X_l) = \frac{1}{2}(1 + U(X_k) - U(X_l)) \quad (5.48)$$

Example 5.17. Let us recall the utility values from the Example 5.11, that is $U = \{1 \; 0 \; 0.5 \; 0.8\}$. Applying (5.48), we obtain the following additive reciprocal fuzzy preference relation:

$$RR = \begin{bmatrix} 0.50 & 1 & 0.75 & 0.60 \\ 0 & 0.50 & 0.25 & 0.10 \\ 0.25 & 0.75 & 0.50 & 0.35 \\ 0.40 & 0.90 & 0.65 & 0.50 \end{bmatrix} \quad (5.49)$$

The reader can note that (5.49) is coherent with the vctor assessed by the DM.

5.5.3 Transformation of the Multiplicative Preference Relation into the Additive Reciprocal Fuzzy Preference Relation

The authors of (Chiclana, Herrera, and Herrera-Viedma, 2001; Herrera-Viedma et al., 2004) have proposed the following transformation function to convert the multiplicative preference relation into the additive reciprocal fuzzy preference relation:

$$RR(X_k, X_l) = \frac{1}{2}\left(1 + \log_m MR(X_k, X_l)\right) \quad (5.50)$$

where m is an upper limit of the ratio scale.

Accepting the scale used in the AHP approach (Saaty, 1980) with m =9 it is possible to transform (5.50) as follows:

$$RR(X_k, X_l) = \frac{1}{2}\left(1 + \log_9 MR(X_k, X_l)\right) \quad (5.51)$$

It is necessary to indicate that if the multiplicative preference relation satisfies multiplicative transitivity, then (5.50) generates the additive reprocical fuzzy preference relation that verify the property of additive transitivity (Herrera-Viedma et al., 2004).

Example 5.18. Let us transform the multiplicative preference relation (5.36) in Example 5.13 into the additive reprocical fuzzy preference relation on the basis of applying of (5.51):

$$RR = \begin{bmatrix} 0.5 & 0.66 & 0.82 & 0.97 \\ 0.34 & 0.5 & 0.66 & 0.82 \\ 0.18 & 0.34 & 0.5 & 0.66 \\ 0.03 & 0.18 & 0.34 & 0.5 \end{bmatrix} \tag{5.52}$$

5.5.4 Transformation of the Nonreciprocal Fuzzy Preference Relation into the Additive Reciprocal Fuzzy Preference Relation

The following three transformation functions, which permits one to convert the nonreciprocal fuzzy preference relation into the additive reciprocical fuzzy preference reation, are considered and analysed in (Pedrycz, Ekel, and Parreiras, 2011):

$$RR(X_k, X_l) = \frac{1}{2}(1 + NR(X_k, X_l) - NR(X_l, X_k)) \tag{5.53}$$

$$RR(X_k, X_l) = \frac{NR(X_k, X_l)}{NR(X_k, X_l) + NR(X_l, X_k)} = \frac{1}{1 + \dfrac{NR(X_l, X_k)}{NR(X_k, X_l)}} \tag{5.54}$$

$$RR(X_k, X_l) = \frac{[NR(X_k, X_l)]^2}{[NR(X_k, X_l)]^2 + [NR(X_l, X_k)]^2} = \frac{1}{1 + \left(\dfrac{NR(X_l, X_k)}{NR(X_k, X_l)}\right)^2} \tag{5.55}$$

The transformation functions (5.54) and (5.55) are analysed in (Herrera-Viedma et al., 2007) and the transformation functions (5.53) has been proposed in (Queiroz, 2009).

The selection of an adequate transformation function among (5.53), (5.54), and (5.55) depends upon whether the ratio $NR(X_k, X_l)/NR(X_l, X_k)$ or the difference $NR(X_k, X_l) - NR(X_l, X_k)$ is meaningful (Pedrycz, Ekel, and Parreiras, 2011). In particular, the transformation function (5.53) may be utilized when the difference $NR(X_k, X_l) - NR(X_l, X_k)$ is meaningful. At the same time, the transformation functions (5.54) and (5.55) are applicable when the ratio $NR(X_k, X_l)/NR(X_l, X_k)$ is meaningful. The authors of (Pedrycz, Ekel, and Parreiras, 2011) note that the main difference between (5.54) and (5.55) lies in the fact that the strength of $NR(X_k, X_l)$ obtained on the basis of (5.55) tends to the farther from the indifference judgment than the corresponding strength of $NR(X_k, X_l)$ calculated on the basis of (5.54).

Furthermore, it is necessary to indicate that the additive reciprocal fuzzy preference relations constructed with the use of (5.54) or (5.55) satisfy multiplicative transitivity if the corresponding nonreciprocal fuzzy preference relations also satisfy multiplicative transitivity.

At the same time, the additive reciprocal fuzzy preference relations obtained on the basis of applying (5.53) satisfy additive transitivity only if the corresponding nonreciprocal fuzzy preference relations satisfy the following condition (Pedrycz, Ekel,and Parreiras, 2011):

$$(NR(X_k, X_j) - NR(X_j, X_k)) + (NR(X_j, X_l) - NR(X_l, X_j))$$
$$+(NR(X_k, X_l) - NR(X_l, X_k)) = 0 \tag{5.56}$$

Example 5.19. Let us consider the nonreciprocal fuzzy preference relation constructed in Example 5.6. Taking into account that differences $NR(X_k, X_l) - NR(X_l, X_k)$ can be considered significant, it is possible to apply (5.53) to transform (5.23) into the additive reciprocal fuzzy preference relation. In particular, the result of the use of (5.52) is the following:

$$RR = \begin{bmatrix} 0.50 & 1 & 0.85 & 0.50 \\ 0 & 0.50 & 0.40 & 0 \\ 0.15 & 0.60 & 0.50 & 0.15 \\ 0.50 & 1 & 0.85 & 0.50 \end{bmatrix} \tag{5.57}$$

The reader can see that (5.57) is well compatible with (5.23).

Example 5.20. Let us transform the following nonreciprocal fuzzy preference relation:

$$NR = \begin{bmatrix} 1 & 0.4 & 0.2 \\ 1 & 1 & 1 \\ 1 & 0.5 & 1 \end{bmatrix} \tag{5.58}$$

Taking into account that, for instance, the ratio $NR(X_2, X_1)/NR(X_1, X_2)$ is high, the transformation is to be made by (5.54) or (5.55). In particular, the use of (5.54) leads to

$$RR = \begin{bmatrix} 0.50 & 0.29 & 0.17 \\ 0.71 & 0.50 & 0.67 \\ 0.83 & 0.33 & 0.50 \end{bmatrix} \tag{5.59}$$

At the same time, the application of (5.55) provides

$$RR = \begin{bmatrix} 0.50 & 0.14 & 0.04 \\ 0.86 & 0.50 & 0.80 \\ 0.96 & 0.20 & 0.50 \end{bmatrix} \tag{5.60}$$

5.5.5 Transformation of the Additive Reciprocal Fuzzy Preference Relation into the Nonreciprocal Fuzzy Preference Relation

In (Pedrycz, Ekel, and Parreiras, 2011), the following transformation functions are considered which permit one to convert the additive reciprocal fuzzy preference relation into the nonreciprocal fuzzy preference relation:

$$NR(X_k, X_l) = \begin{cases} 1 + RR(X_k, X_l) - RR(X_l, X_k) & \text{if } RR(X_k, X_l) < 0.5 \\ 1 & RR(X_k, X_l) \geq 0.5 \end{cases} \tag{5.61}$$

$$NR(X_k, X_l) = \begin{cases} \dfrac{RR(X_k, X_l)}{RR(X_l, X_k)} & \text{if } RR(X_k, X_l) < 0.5 \\ 1 & \text{if } RR(X_k, X_l) \geq 0.5 \end{cases} \tag{5.62}$$

$$NR(X_k, X_l) = \begin{cases} \left(\dfrac{RR(X_k, X_l)}{RR(X_l, X_k)}\right)^{0.5} & \text{if } RR(X_k, X_l) < 0.5 \\ 1 & \text{if } RR(X_k, X_l) \geq 0.5 \end{cases} \tag{5.63}$$

These expressions (the construction of (5.62) was proposed in (Queiroz, 2009)) represent the transformation functions that permit one to reverse conversions of (5.53)-(5.55). The recommendations on the use of (5.61)-(5.63) are the following (Pedrycz, Ekel, and Parreiras, 2011).

The of (5.61) may be utilized when the additive reciprocal fuzzy preference relation is defined in such a way that the difference $RR(X_k, X_l) - RR(X_l, X_k)$ has a sense. The transformation functions (5.62) and (5.63) may utilized when the additive reciprocal fuzzy preference relation is defined in such a way that the ratio $RR(X_k, X_l) / RR(X_l, X_k)$ has sense, indicating how many times X_k is preferred to X_l. The main difference between (5.62) and (5.63) is associated with with the fact that each pairwise judgment $R(X_k, X_l)$ produced by (5.62) tend to be closer to the indifference judgment than each corresponding pairwise judgment produced with the application of (5.63), as it can be confirmed in the following example.

Example 5.21. The nonreciprocal fuzzy preference relation (5.23) constructed in Example 5.6 has been transformed in Example 5.19 in the additive reciprocal fuzzy preference relation (5.57). Let us transform (5.57) in the nonreciprocal fuzzy preference relations, applying (5.61)-(5.63).

It is not diificult to verify that the use of (5.61) generates the nonreciprocal fuzzy preference relations (5.23) that is natural.

The use of (5.62) to convert (5.57) generates the following nonreciprocal fuzzy preference relation:

$$NR = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0.67 & 0 \\ 0.18 & 1 & 1 & 0.18 \\ 1 & 1 & 1 & 1 \end{bmatrix} \tag{5.64}$$

At the same time, the use of (5.63) permits one to obtain

$$NR = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0.45 & 0 \\ 0.03 & 1 & 1 & 0.03 \\ 1 & 1 & 1 & 1 \end{bmatrix} \tag{5.65}$$

The presented below transformation functions permit one to convert the preference information expressed in terms of the different formats directly into the nonreciprocal fuzzy preference relation (Pedrycz, Ekel, and Parreiras, 2011) by substituting into (5.61) or (5.62) or (5.63), the expressions presented above for conversion from the different preference formats into the additive reciprocal fuzzy preference relation.

5.5.6 Transformation of the Ordered Array into the Nonreciprocal Fuzzy Preference Relation

In order to convert preferences expressed in terms of the ordered array into the nonreciprocal fuzzy preference relation, it is possible to use the following transformation function (Pedrycz, Ekel, and Parreiras, 2011):

$$NR(X_k, X_l) = \begin{cases} \dfrac{1}{2} + \dfrac{O(X_l) - O(X_k)}{2(n-1)} & \text{if } O(X_k) > O(X_l) \\ 1 & \text{if } O(X_k) \le O(X_l) \end{cases} \tag{5.66}$$

which is obtain by substituting (5.40) into (5.61). The advantage of applying (5.40) combined with (5.61) and not with (5.62) or (5.63) is associated with the need to preserve the meaning of the differences $O(X_l) \le O(X_k)$ among the positions of two alternatives in the conversion from the ordered array into the nonreciprocal fuzzy preference relation (Pedrycz, Ekel, and Parreiras, 2011).

Example 5.22. The application of (5.66) to convert the ordered array considered in Example 5.10 permits one to form the following nonreciprocal fuzzy preference relation:

$$NR = \begin{bmatrix} 1 & 0.17 & 0.33 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0.33 & 1 & 1 \\ 0.33 & 0 & 0.17 & 1 \end{bmatrix} \tag{5.67}$$

At the same time, Example 5.15 includes the result (5.43) of converting the same ordered array of Example 5.10 to the additive reciprocal fuzzy preference relation on the basis of (5.40). Considering this, it is possible to observe that the application of the transformation function (5.53) to the nonreciprocal fuzzy preference relation (5.67) generates the additive reciprocal fuzzy preference relation coinciding with (5.43).

5.5.7 Transformation of the Utility Values into the Nonreciprocal Fuzzy Preference Relation

The preferences presented as a vector of utilities defined on a ratio scale and normalized in the interval [0, 1], can be converted into a nonreciprocical fuzzy preference relation on the basis of one of the following expressions (Pedrycz, Ekel, and Parreiras, 2011):

$$NR(X_k, X_l) = \begin{cases} \dfrac{U(X_k)}{U(X_l)} & \text{if } U(X_k) < U(X_l) \\ 1 & \text{if } U(X_k) \geq U(X_l) \end{cases} \tag{5.68}$$

$$NR(X_k, X_l) = \begin{cases} \left(\dfrac{U(X_k)}{U(X_l)}\right)^{0.5} & \text{if } U(X_k) < U(X_l) \\ 1 & \text{if } U(X_k) \geq U(X_l) \end{cases} \tag{5.69}$$

$$NR(X_k, X_l) = \begin{cases} \left(\dfrac{U(X_k)}{U(X_l)}\right)^{2} & \text{if } U(X_k) < U(X_l) \\ 1 & \text{if } U(X_k) \geq U(X_l) \end{cases} \tag{5.70}$$

The expressions (5.68) and (5.69) have been obtained by substituting (5.44) into (5.62) and (5.63), respectively. At the same time, the expression (5.70) has been obtained by the substituting (5.45) into (5.62). It is necessary to indicate that the substitution (5.45) into (5.63) also corresponds to (5.62). In order to preserve the meaning of the ratio $U(X_k)/U(X_l)$ in the conversion from the utility values into a nonreciprocal fuzzy preference relation, substituition of (5.44) or (5.45) into (5.62) or (5.63) is preferable to substituition of (5.44) or (5.45) into (5.61) (Pedrycz, Ekel, and Parreiras, 2011).

Caracterizing the transformation functions (5.68)-(5.70), it is important to indicate that (5.69) produces pairwise judgments tending somewhat more to an indifference judgment than the other transformation functions do (Pedrycz, Ekel, and Parreiras, 2011). At the same time, (5.70) produces pairwise judgments tending somewhat more to a strict preference than the other transformation functions do. Finally, (5.68) can be considered an intermediate case between the extreme cases, (6.46) and (6.47), respectively. This is a consequence of the fact that the value of $R(X_k, X_l)$, when $U_p(X_k) < U_p(X_l)$, can be given by one of the ratios satisfying the relationships

$$\left(\frac{U(X_k)}{U(X_l)}\right)^2 \leq \frac{U(X_k)}{U(X_l)} \leq \left(\frac{U(X_k)}{U(X_l)}\right)^{0.5} \tag{5.71}$$

The example from (Pedrycz, Ekel, and Parreiras, 2011) given below confirms the validity of (5.71).

Example 5.23. By applying (5.68)-(5.70) to convert the utility array $U = \{0.2\ 0.4\ 0.1\ 0.3\}$ from Example 5.16 into nonreprocical fuzzy preference relation, we obtain

$$NR = \begin{bmatrix} 1 & 0.5 & 1 & 0.67 \\ 1 & 1 & 1 & 1 \\ 0.5 & 0.25 & 1 & 0.33 \\ 1 & 0.75 & 1 & 1 \end{bmatrix} \tag{5.72}$$

$$NR = \begin{bmatrix} 1 & 0.71 & 1 & 0.82 \\ 1 & 1 & 1 & 1 \\ 0.71 & 0.5 & 1 & 0.58 \\ 1 & 0.87 & 1 & 1 \end{bmatrix} \tag{5.73}$$

$$NR = \begin{bmatrix} 1 & 0.25 & 1 & 0.44 \\ 1 & 1 & 1 & 1 \\ 0.25 & 0.06 & 1 & 0.11 \\ 1 & 0.56 & 1 & 1 \end{bmatrix} \tag{5.74}$$

respectively. It is possible to observe that the obtained nonreprocical fuzzy preference relations are coherent with the analysis of the transformation functions (5.68)-(5.70), presented above.

Furthermore, the authors of (Pedrycz, Ekel, and Parreiras, 2011) analyze the possibility of converting the preferences given in terms of a vector of utilities defined on an interval scale into a nonreprocical fuzzy preference relation as follows:

$$NR(X_k, X_l) = \begin{cases} 1 + U(X_k) - U(X_l) & \text{if } U(X_k) < U(X_l) \\ 1 & \text{if } U(X_k) \ge U(X_l) \end{cases} \tag{5.75}$$

5.5.8 Transformation of the Multiplicative Preference Relation into the Nonreciprocal Fuzzy Preference Relation

One of the possible ways of transforming the multiplicative preference relation into a nonreprocical preference relation (Pedrycz, Ekel, and Parreiras, 2011) is associated with the use of

$$NR(X_k, X_l) = \begin{cases} 1 + \dfrac{1}{2}\log_m \dfrac{M(X_k, X_l)}{M(X_l, X_k)} & \text{if } \log_m M(X_k, X_l) < 0 \\ 1 & \text{if } \log_m M(X_k, X_l) \ge 0 \end{cases} \tag{5.76}$$

The expression has been obtained by substituting (5.50) into (5.61). The advantage of considering (5.61), rather than (5.62) or (5.63) lies in the fact that the meaning of the ratio $M(X_k, X_l)/M(X_l, X_k)$ is to some extent preserved, as can be seen in (5.76) (Pedrycz, Ekel, and Parreiras, 2011).

Example 5.19. Let us convert the multiplicative preference relation (5.36), constructed in Example 6.13, into a nonreprocical fuzzy preference relation. The use of (5.76) , considering that $m = 9$, generates

$$MR = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0.685 & 1 & 1 & 1 \\ 0.369 & 0.685 & 1 & 1 \\ 0.053 & 0.369 & 0.685 & 1 \end{bmatrix} \tag{5.77}$$

which is coherent with (5.36).

5.5.9 Transformation of the Quantitative Information into the Fuzzy Preference Relation

Assume we have an objective function $F(X)$, which is to be maximized, and we are to contruct an additive reprocical fuzzy preference relation for two alternatives X_k and X_l . Then, it is possible (Zhukovin, 1988; Ekel, 2002) to construct the following expression:

$$RR(X_k, X_l) = \alpha[F(X_k) - F(X_l)] + \beta \tag{5.78}$$

The compliance with the condition (5.20) for the additive reprocical preference relations leads to $\beta = 0.5$. It permits one to obtain

$$\alpha[\max_{X \in L} F(X_k) - \min_{X \in L} F(X_l)] + 0.5 = 1 \tag{5.79}$$

and

$$\alpha = \frac{1}{2[\max_{X \in L} F(X_k) - \min_{X \in L} F(X_l)]} \tag{5.80}$$

leading to

$$RR(X_k, X_l) = \frac{F(X_k) - F(X_l)}{2[\max_{X \in L} F(X) - \min_{X \in L} F(X)]} + 0.5 \tag{5.81}$$

It is not difficult to understand that for $F_p(X)$, which is to be minimized, it is possible to write the following expression:

$$RR(X_k, X_l) = \frac{F(X_l) - F(X_k)}{2[\max_{X \in L} F(X) - \min_{X \in L} F(X)]} + 0.5 \tag{5.82}$$

The application of (5.61) to (5.81) or (5.82) permits one to construct the corresponding nonreciprocal fuzzy preference relations.

5.6 Optimization Problems with Fuzzy Coefficients and Their Analysis

In Chapter 1, the general issues related to the necessity of setting up and solving multicriteria problems have been discussed. In particular, one of the classes of situations, which need the application of a multicriteria approach, is associated with problems that may be solved on the basis of a single criterion or several criteria. However, if information uncertainty does not permit one to derive unique solutions, it is possible to transform these problems, applying additional criteria, including criteria of qualitative character, to reduce the decision uncertainty regions. Taking this into account, let us consider a model, which includes fuzzy coefficients in an objective function and constraints. There exist diverse problems related to system design, planning, and control, which can be formalized within the framework of this type of models. Besides, although there are diverse formulations of optimization problems with fuzziness (Dubois and Prade, 1980; Orlovsky, 1981; Delgado et al., 1994; Zimmermann, 1996; Zimmermann, 2008, for instance), in opinion of the authors of (Orlovsky, 1981; Pedrycz and Gomide, 1998), the problems with fuzzy coefficients in objective functions and constraints are to be considered as a general problem of fuzzy mathematical programming. The problem can be formulated as follows:

$$\text{maximize } F(x_1, x_2, ..., x_n) \tag{5.83}$$

subject to constraints

$$G_j(x_1, x_2, ..., x_n) \subseteq B_j, \quad j = 1, 2, ..., m \tag{5.84}$$

where the objective function (5.83) and constraints (5.84) include fuzzy coefficients.

Let us consider the question of considering constraints of different nature and, primarily, the functional constraints. For simplicity, we can begin from only a single constraint of the following form (Pedrycz, Ekel, and Parreiras, 2011):

$$\sum_{i=1}^{n} G_i x_i \subseteq B \tag{5.85}$$

where $G_i, i = 1, 2, ..., n$, and B are fuzzy numbers with membership functions $\mu(G_i)$, $i = 1, .2, .., n$, and $\mu(B)$, respectively.

An approach to handling constraints of the form (5.85) has been proposed in (Negoita and Ralescu, 1975). In particular, if certain conditions are satisfied (specifically, with regard to the convexity of the fuzzy coefficients $G_i, i = 1, 2, ..., n$, and B), and we assume the possibility of ordering

$$0 \le \alpha_1 < \ldots < \alpha_k < \ldots < \alpha_K \le \min\{\min_{1\le i\le n} \ \sup \ \mu(G_i), \ \mu(B)\} \tag{5.86}$$

then the constraint (5.85) can be modified to obtain the following system of deterministic inclusions:

$$\sum_{i=1}^{n} G_{i,\alpha_k} x_i \subseteq B_{\alpha_k}, \quad k = 1, 2, \ldots, K \tag{5.87}$$

where G_{i,α_k} and B_{α_k}, $k = 1, 2, \ldots, K$, are sets of the α_k-level of G_i, $i = 1, 2, \ldots, n$, and B, respectively.

Considering the definition of sets of a α_k-level, see Chapter 2, it is possible to obtain from (5.87):

$$\sum_{i=1}^{n} [g_{i_1,\alpha_k}, g_{i_2,\alpha_k}] x_i \subseteq [b_{1,\alpha_k}, b_{2,\alpha_k}], \quad k = 1, 2, \ldots, K \tag{5.88}$$

which means that

$$\sum_{i=1}^{n} g_{i_2,\alpha_k} x_i \le b_{2,\alpha_k}, \quad k = 1, 2, \ldots, K \tag{5.89}$$

and

$$\sum_{i=1}^{n} g_{i_1,\alpha_k} x_i \ge b_{1,\alpha_k}, \quad k = 1, 2, \ldots, K \tag{5.90}$$

Using the principle of explicit domination, realized on the basis of (5.94) and (5.95) given below, we can reduce the dimensionality of the sets of inequalities (5.89) and (5.90). As a result of normalization (Ekel, Pedrycz, and Schinzinger, 1998), carried out in accordance with the expression

$$h_{i,\alpha_k} = g_{i,\alpha_k} \frac{b}{b_{\alpha_k}}, \quad k = 1, 2, \ldots, K, \quad i = 1, 2, \ldots, n \tag{5.91}$$

we can consider, instead of (5.89) and (5.90), the sets of constraints

$$\sum_{i=1}^{n} h_{i_2,\alpha_k} x_i \le b, \quad k = 1, 2, \ldots, K \tag{5.92}$$

and

$$\sum_{i=1}^{n} h_{i_1,\alpha_k} x_i \ge b, \quad k = 1, 2, \ldots, K \tag{5.93}$$

respectively. In the expressions (5.91)-(5.93), $b > 0$ is a normalizing factor.

If, as a result of analyzing the set of constraints (5.92) with $h_{i_2,\alpha_k} \geq 0$, it turns out that

$$h_{i_2,\alpha_q} \leq h_{i_2,\alpha_p}, \quad q \neq p, \quad i = 1, 2, ..., n \tag{5.94}$$

then the pth constraint, for a purposeful increase in the variables x_i, $i = 1, 2, ..., n$, is disturbed earlier than the qth constraint. For this reason, the qth constraint can be eliminated from further consideration.

In a similar way, the condition of eliminating the qth constraint from consideration in the case of analyzing the set of constraints (5.93) is the following:

$$h_{i_1,\alpha_q} \geq h_{i_1,\alpha_p}, \quad q \neq p, \quad i = 1, 2, ..., n \tag{5.95}$$

According to the essence of the optimization problem, one may replace constraints (5.84) with constraints

$$g_j(x_1,...,x_n) \leq b_j, \quad j = 1, 2,..., m' \geq m \tag{5.96}$$

or constraints

$$g_j(x_1,...,x_n) \geq b_j, \quad j = 1, 2,..., m'' \geq m \tag{5.97}$$

In this way, the problem with constraints containing fuzzy coefficients, one can construct an equivalent nonfuzzy analog of the problem whose dimension is reduced by using the principle of explicit domination (5.94) or (5.95).

The solution of problems containing fuzzy coefficients in objective functions alone is possible by a modification of traditional mathematical programming methods (Ekel, Pedrycz, and Schinzinger, 1998; Ekel, 2002).

When applying optimization methods for fuzzy problems, one needs to compare solutions on the levels of the objective function (in essence, to compare or rank corresponding fuzzy numbers to choose the largest or smallest one). If we talk about a problem of linear programming and the corresponding modification of the simplex method for its solution, it is necessary to compare coefficients of nonbasic variables with zero at any cycle of the optimization process.

Taking into consideration the discussion above, we can apply (5.27) and (5.28) (or (5.29) and (5.30)) for this comparison. However, we have to keep in mind that if the membership functions of the solutions (fuzzy numbers) $F(X_k)$ and/or $F(X_l)$ compared are trapezoidal or flat fuzzy numbers, these solutions can be indistinguishable considering the condition (5.31). In such situations, algorithms based on the modification of traditional optimization methods do not allow one to obtain unique solutions because they "stop" when conditions like (5.31) arise (Ekel, Pedrycz, and Schinzinger, 1998; Galperin and Ekel, 2005; Pedrycz, Ekel, and Parreiras, 2011)). This is natural because a combination of the uncertainty and the relative stability of optimal solutions can produce decision uncertainty regions. This is illustrated by the following simple example (Galperin and Ekel, 2005; Pedrycz, Ekel, and Parreiras, 2011)) where appropriate modification of the simplex method of linear programming is applied.

Example 5.20. Consider the following problem:

$$\text{maximize } F(x_1, x_2) = C_1 x_1 + C_2 x_2 \tag{5.98}$$

subject to

$$G_{11} x_1 + G_{12} x_2 \subseteq B_1 \tag{5.99}$$

$$G_{21} x_1 + G_{22} x_2 \subseteq B_2 \tag{5.100}$$

$$x_1 \geq 0, \quad x_2 \geq 0, \quad \text{(numeric)} \tag{5.101}$$

where all coefficients in (5.98)-(5.100) are trapezoidal fuzzy numbers defined as $\mu(C_1) = \{1.2,\ 1.3,\ 1.6,\ 1.7\}$, $\mu(C_2) = \{2.1, 2.2, 2.7,\ 2.8\}$, $\mu(G_{11}) = \{9, 10, 11, 12\}$, $\mu(G_{12}) = \{5, 6, 8, 9\}$, $\mu(B_1) = \{24,\ 29, 49, 53\}$, $\mu(G_{21}) = \{6, 7, 9, 10\}$, $\mu(G_{22}) = \{6, 7, 9, 10\}$, and $B_2 = \{25,\ 29,\ 48,\ 52\}$.

Considering that G_{11}, G_{12}, B_1, G_{21}, G_{22}, and B_2 are trapezoidal fuzzy numbers, it is sufficient to consider constraints (5.99) and (5.100) for $\alpha_1 = 0$ and $\alpha_2 = \alpha_K = 1$. For this reason, using (5.92) and (5.93), it is possible to rewrite (5.99) as follows:

$$12x_1 + 9x_2 \leq 53 \tag{5.102}$$

$$11x_1 + 8x_2 \leq 49 \tag{5.103}$$

and

$$9x_1 + 5x_2 \geq 24 \tag{5.104}$$

$$10x_1 + 6x_2 \geq 29 \tag{5.105}$$

At the same time, the constraint (5.100) can be presented by the following inequalities:

$$10x_1 + 10x_2 \leq 52 \tag{5.106}$$

$$9x_1 + 9x_2 \leq 48 \tag{5.107}$$

and

$$6x_1 + 6x_2 \geq 25 \tag{5.108}$$

$$7x_1 + 7x_2 \geq 29 \tag{5.109}$$

Taking into consideration that we have to maximize the objective function with positive coefficients, it is possible to ignore (5.104) and (5.105) as well as (5.108) and (5.109). The principle of explicit domination (5.94) applied to (5.102) and (5.103) allows us to eliminate (5.103) from further consideration. At the same time, the application of the principle of explicit domination (5.94) to (5.106) and (5.107) permits us to eliminate (5.107).

Finally, introducing the slack variables $x_3 \geq 0$ and $x_4 \geq 0$, we transform (5.102) and (5.106) to

$$12x_1 + 9x_2 + x_3 \leq 53 \tag{5.110}$$

$$10x_1 + 10x_2 + x_4 \leq 52 \tag{5.111}$$

respectively.

To apply the modification of the version of the simplex method given in (Rao, 1996), we have to consider the minimization problem instead of (5.98). Thus, our problem is

$$\text{minimize } [-F(x_1, x_2)] = -C_1 x_1 - C_2 x_2 \tag{5.112}$$

subject to (5.110), (5.111), and

$$x_i \geq 0, \quad i = 1,...,4 \tag{5.113}$$

Using the above-mentioned modification of the simplex method with executing necessary operations on the fuzzy coefficients, discussed in Chapter 3, of the objective function, for the first cycle we obtain: $x_1 = 4.42$, $x_4 = 7.85$ (basic variables) and $x_2 = 0$, $x_3 = 0$ (nonbasic variables) with $C_2 = \{-1.90, \ -1.73, \ -1.00, -0.83\}$ and $C_3 = \{0.10, \ 0.11, \ 0.13, \ 0.14\}$. Since $C_2 < 0$, we are to continue to perform the second cycle: $x_1 = 2.07$, $x_2 = 3.13$ (basic variables) and $x_3 = 0$, $x_4 = 0$ (nonbasic variables) with $C_3 = \{-0.53, -0.47, -0.20, -0.13\}$, $C_4 = \{0.33, \ 0.40, \ 0.69, \ 0.76\}$. Since $C_3 < 0$, we are to continue, obtaining in the third cycle: $x_2 = 5.20$, $x_3 = 6.22$ (basic variables) and $x_1 = 0$, $x_4 = 0$ (nonbasic variables) with $C_1 = \{0.40, \ 0.60, \ 1.40, \ 1.60\}$, $C_4 = \{-0.15, \ -0.02, \ 0.51, \ 0.64\}$. Taking into account that in comparison of C_4 with zero, the situation (6.31) takes place, the simplex method "stops"; namely it "cannot identify" if the optimal solution has been obtained or not.

The possible way to overcome this type of situations or, at least, to contract the decision uncertainty regions to the highest extent, is associated with the approach based on formulating and solving one and the same problem within the framework of mutually related models (Ekel, Pedrycz, and Schinzinger, 1998; Ekel, 2002; Pedrycz, Ekel, and Parreiras, 2011). In particular, the problem (5.83) with the constraints (5.84) approximated by (5.96) and the problem

$$\text{minimize } F(x_1,...,x_n) \tag{5.114}$$

subject to the same constraints (5.84) approximated by (5.97) can serve as a mutually related model.

This approach is applicable for solving problems with continuous as well as discrete variables. To understand its essence, let us proceed with an analysis of a certain discrete optimization problem.

The desirability of allowing for constraints on the discreteness of variables in the form of discrete sequences

$$x_{s_i}, \alpha_{s_i}, \beta_{s_i}, \ldots, \quad s_i = 1, 2, \ldots, r_i \tag{5.115}$$

has been validated in (Zorin and Ekel, 1980); here $\alpha_{s_i}, \beta_{s_i}, \ldots$, are technical and economic characteristics required for formation of objective functions, constraints and their increments that correspond to the *s*th standard value of the variable x_i.

It is expedient to utilize discrete sequences of the type (5.115) because the characteristics $\alpha_{s_i}, \beta_{s_i}, \ldots$, cannot always be fitted closely to the analytical relationships in terms of x_{s_i}, but in discrete sequences of the type (5.115) these characteristics may be taken to be exact. Besides, a flexible formalization of combinatorial type problems is possible on the basis of the discrete sequences because they can be different for different variables. Examples of this flexible application of the discrete sequences are given in (Zorin and Ekel, 1980; Ekel and Schuffner Neto, 2006).

Taking the above into account with respect to the expediency of using discrete sequences and by analogy with the problem (5.83), (5.84), the maximization problem can be formulated as follows.

Assume we are given discrete sequences of the type (5.115) (depending on the problem statement the sequences could be increasing or decreasing). From these sequences it is necessary to choose elements such that the objective

$$\text{maximize } F(x_{s_1}, \alpha_{s_1}, \beta_{s_1}, \ldots, \ldots, x_{s_n}, \alpha_{s_n}, \beta_{s_n}, \ldots) \tag{5.116}$$

is met while satisfying the constraints

$$G_j(x_{s_1}, \alpha_{s_1}, \beta_{s_1}, \ldots, \ldots, x_{s_n}, \alpha_{s_n}, \beta_{s_n}, \ldots) \subseteq B_j, \quad j = 1, 2, \ldots, m \tag{5.117}$$

Given a maximization problem of the type (5.115)-(5.117) considered above and by analogy with (5.114), we can formulate a mutually related problem with the objective

$$\text{minimize } F(x_{s_1}, \alpha_{s_1}, \beta_{s_1}, \ldots, \ldots, x_{s_n}, \alpha_{s_n}, \beta_{s_n}, \ldots) \tag{5.118}$$

while satisfying the constraints (5.117).

Taking the above into account, the constraints (5.117) may be reduced to the set of nonfuzzy (numeric) constraints

$$g_j(x_{s_1}, \alpha_{s_1}, \beta_{s_1}, ..., ..., x_{s_n}, \alpha_{s_n}, \beta_{s_n}, ...) \le b_j, \quad j = 1, 2, ..., m' \ge m \tag{5.119}$$

and

$$g_j(x_{s_1}, \alpha_{s_1}, \beta_{s_1}, ..., ..., x_{s_n}, \alpha_{s_n}, \beta_{s_n}, ...) \ge b_j, \quad j = 1, 2, ..., m'' \ge m \tag{5.120}$$

Below, we consider an example from (Ekel, Pedrycz, and Schinzinger, 1998; Pedrycz, Ekel, and Parreiras, 2011) to demonstrate the analysis of the mutually related models (5.116), (5.119) (with the increasing (decreasing) discrete sequences (5.115)) and (5.118), (5.120) (with the decreasing (increasing) discrete sequences (5.115)) and the results based on its application.

Example 5.21. Assume we are given the discrete sequence

$$\begin{array}{llll} & x_{s_i} & \alpha_{s_i} & \beta_{s_i} \\ s_i=1: & 0, & 0, & 26 \\ s_i=2: & 1, & 3, & 25 \\ s_i=3: & 2, & 6, & 23 \\ s_i=4: & 3, & 9, & 19 \\ s_i=5: & 4, & 12, & 14 \\ s_i=6: & 5, & 15, & 9 \end{array} \tag{5.121}$$

From this sequence it is necessary to choose elements, which maximize the objective function

$$F(x_1, x_2) = [(C_1\alpha_{s_1} + \beta_{s_1}) + (C_2\alpha_{s_2} + \beta_{s_2})] \tag{5.122}$$

subject to the following constraints:

$$G_{11}x_{s_1} + G_{12}x_{s_2} \subseteq B_1 \tag{5.123}$$

$$G_{21}x_{s_1} + G_{22}x_{s_2} \subseteq B_2 \tag{5.124}$$

where coefficients in (5.122)-(5.124) are trapezoidal fuzzy numbers defined as $C_1 = \{1.2,\ 1.4, 1.5, 1.7\}$, $C_2 = \{2.1,\ 2.4,\ 2.5,\ 2.8\}$, $G_{11} = \{5,\ 6,\ 8,\ 9\}$, $G_{12} = \{9,\ 10,\ 11,\ 12\}$, $B_1 = \{24,\ 29,\ 49,\ 53\}$, $G_{21} = \{6,\ 7,\ 9,\ 10\}$, $G_{22} = \{6,\ 7,\ 9,\ 10\}$, and $B_2 = \{25,\ 29,\ 48,\ 52\}$.

As in Example 5.20, it is sufficient to consider the constraints (5.123) and (5.124) for $\alpha_1 = 0$ and $\alpha_K = \alpha_2 = 1$. Considering this, we can write for (5.123)

$$12x_{s_1} + 9x_{s_2} \le 53 \tag{5.125}$$

$$11x_{s_1} + 8x_{s_2} \le 49 \tag{5.126}$$

and

$$9x_{s_1} + 5x_{s_2} \geq 24 \tag{5.127}$$

$$10x_{s_1} + 6x_{s_2} \geq 29 \tag{5.128}$$

Similarly, it is possible to go from the constraints (5.124) to the following inequalities:

$$10x_{s_1} + 10x_{s_2} \leq 52 \tag{5.129}$$

$$9x_{s_1} + 9x_{s_2} \leq 48 \tag{5.130}$$

and

$$6x_{s_1} + 6x_{s_2} \geq 25 \tag{5.131}$$

$$7x_{s_1} + 7x_{s_2} \geq 29 \tag{5.132}$$

The principle of explicit domination (5.94) applied to (5.125) and (5.126) permits one to eliminate (5.126) from further consideration. The application of the principle of explicit domination (5.94) to (5.129) and (5.130) allows us to eliminate (5.130).

In the same way, applying the principle of explicit domination (5.95) to (5.127) and (5.128), we can eliminate (5.127) from the further consideration. The application of the principle of explicit domination (5.95) to (5.131) and (5.132) permits one to eliminate (5.132).

Thus, the problem is consists of the maximization of (5.122) subjected to the constraints (5.128) and (5.131). At the same time, the mutually related problem consists of minimizing

$$F(x_1, x_2) = [-(C_1\alpha_{s_1} + \beta_{s_1}) - (C_2\alpha_{s_2} + \beta_{s_2})] \tag{5.133}$$

subject to the constraints (5.128) and (5.131) with applying the discrete sequence that is decreasing on s_i:

	x_{s_i}	α_{s_i}	β_{s_i}
s_i =1:	5,	15,	26
s_i =2:	4,	12,	14
s_i =3:	3,	9,	19
s_i =4:	2,	6,	23
s_i =5:	1,	3,	25
s_i =6:	0,	0,	26

(5.134)

All steps of the process of solving the problem (5.121), (5.122), (5.125), and (5.129) on the basis of modifying the generalized algorithms of discrete optimization (Ekel, Pedrycz, and Schinzinger, 1998; Ekel and Schuffner Neto, 2006) are given in (Ekel, Pedrycz, and Schinzinger, 1998). In particular, this process "stops" when we meet a situation of the impossibility to distinguish two solutions $X_1 = \{x_1 = 2,\ \ x_2 = 3\}$ and $X_2 = \{x_1 = 1,\ \ x_2 = 4\}$. At the same time, the solution of the mutually related problem (5.134), (5.133), (5.128), and (5.131) leads to the solution $X_3 = \{x_1 = 0, x_2 = 5\}$. As it is shown in (Ekel, Pedrycz, and Schinzinger, 1998), there are no more solutions which are competitive. Thus, the decision uncertainty region $X = \{X_1, X_2, X_3\}$ is a formal solution of the problem (5.121)-(5.124).

Figure 5.7 demonstrats the essence of the approach discussed above: the solutions dominated by the initial objective function are cut off from below as well as from above to the greatest degree.

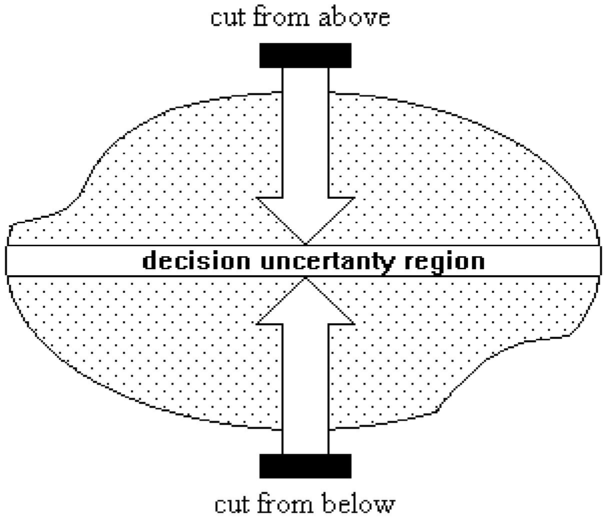


Figure 5.7 Cutting dominated alternatives.

Thus, it was demonstrated that the uncertainty of information, particularly reflected by fuzzy coefficients in objective function and constraints of monocriteria problems, generates the decision uncertainty regions. Their contraction, as it was indicated above, is possible on the basis of reducing the problem to multicriteria decision making with applying additional criteria, including criteria of qualitative character. It is natural that the problem of the evaluation, comparison, choice, and/or ordering of alternatives can be initially stated as a multicriteria problem.

5.7 <*X*, *R*> Models of Multicriteria Decision Making

The problem of multicriteria (multiattribute) analysis of alternatives in a fuzzy environment can be presented as follows.

We are given a set X of alternatives from the decision uncertainty region or predetermined alternatives, which are to be examined by q criteria of quantitative and/or qualitative character. That is, indices $F_p(X_k),\ p = 1, 2,...,q,\ X_k \in X$, are to be analyzed to make a selection among alternatives. The problem of decision making, as it was indicated in Chapter 1, may be presented as a pair $< X, R >$ where $R = \{R_1, R_2, ..., R_p, ..., R_q\}$ is a vector fuzzy preference relation (Orlovsky, 1981; Fodor and Roubens, 1994; Pedrycz, Ekel, and Parreiras, 2011). In this case we have

$$R_p = [X \times X,\ R_p(X_k, X_l)], \quad p = 1, 2, ..., q, \quad X_k, X_l \in X \tag{5.135}$$

where $R_p(X_k, X_l)$ is a membership function of the pth fuzzy preference relation.

Above, we discussed the utilization of different preference formats for the presentation of initial information for decision making and the rationality of using fuzzy preference relations for a uniform preference representation as well as ways of converting diverse preference formats to fuzzy preference relations. Considering this, below, we consider techniques for carrying out the evaluation, comparison, choice, prioritization, and/or ordering of alternatives on the basis of information presented by (5.135).

Below, we discuss five techniques of multiattribute analysis of alternatives in a fuzzy environment (techniques of analysing $< X, R >$ models). The first three techniques are directly based on the notion of the Orlovsky choice function (Orlovsky, 1978; Orlovsky, 1981). The fourth technique is also based on applying the notion the Orlovsky choice function. However, this technique allows a DM to present information related to the importance of fuzzy preference relations in a fuzzy form, particularly, in the form of nonreciprocal fuzzy preference relations. The fifth techniques can be considered as the generalization of the Orlovsky choice function on the basis of applying the ordered weighted average (OWA) operator, discussed in Chapter 4.

Below, we utilize R to denote fuzzy nonstrict preference relations. As we only make use of nonreciprocal fuzzy preference relations below, we do not use any particular notation to indicate whether the fuzzy preference relation unde consideration is an additive reciprocal fuzzy preference relation or a nonreciprocal fuzzy preference relation.

5.8 Techniques for Analyzing <*X*, *R*> Models

In this section, we consider techniques of the analysis of $< X, R >$ models, which are based on the application of the notion of the Orlovsky choice function. This notion was introduced by Orlovsky (Orlovsky, 1978; Orlovsky, 1981) and afterward studied by many researches. In particular, its axiomatic characterization is given, for instance, in (Banerjee, 1993; Bouyssou, 1997; Sengupta, 1998). The authors of (Barret, Bouyssou, Salles, 1990) demonstrated many interesting and desirable properties of the Orlovsky choice function.

Let us consider the situation of setting up a single fuzzy nonstrict preference relation R given in one of the following forms (Orlovsky, 1981):

- $(X_k, X_l) \in R$ or $X_k \succcurlyeq X_l$ that means "X_k is not worse than X_l";
- $(X_l, X_k) \in R$ or $X_l \succcurlyeq X_k$ that means "X_l is not worse than X_k";
- $(X_k, X_l) \notin R$ or $(X_l, X_k) \notin R$ that means "X_k and X_l are not comparable".

A fuzzy nonstrict preference relation R can be processed to obtain a fuzzy strict preference relation P. In particular, $(X_k, X_l) \in P$ means that X_k is strictly better than X_l (or X_k dominates X_l, i.e., $X_k \succ X_l$). It is natural that the alternative $X_k \in X$ is nondominated in $< X, R >$ if $(X_k, X_l) \in P$ for any $X_l \in X$.

As it was discussed above, it is possible to construct the fuzzy strict preference relation P only in terms of the fuzzy nonstrict preference relation R. In particular, it is possible to do it on the basis of applying (5.13), which can be formally obtained (Orlovsky, 1978; Orlovsky, 1981) as follows:

$$P = R \setminus R^{-1} \tag{5.136}$$

The expression (5.13) plays an important role in this section: it can serve as the basis for the evaluation, comparison, choice, prioritization, and/or ordering of alternatives. In particular, it is possible to observe that $P(X_l, X_k),\ \forall X_k \in X$ is the membership function of the fuzzy set of all X_k which are strictly dominated by X_l. It is natural that that the complemantary relation $P^c(X_l, X_k) = 1 - P(X_l, X_k),\ \forall X_k \in X$, generates the fuzzy set of alternatives which are not dominated by X_l. Considering this, in order to obtain the set of alternatives from X that are not dominated by other alternatives, it is possible to find the fuzzy preference relation that corresponds to the intersection of all $P^c(X_l, X_k),\ \forall X_k \in X$, on all $X_l \in X$ (Orlovsky, 1978; Orlovsky, 1981). This intersection, which corresponds to the fuzzy set of nondominated alternatives, can be presented in the following form:

$$ND(X_k) = \inf_{X_l \in X} [1 - P(X_l, X_k)] = 1 - \sup_{X_l \in X} P(X_l, X_k) \tag{5.137}$$

The use of (5.137) permits one to reflect the level of nondominance of each alternative X_k. Considering this, it is natural to choose alternatives providing the highest level of nondiminance X^{ND} as follows:

$$X^{ND} = \{X_k^{ND} \mid X_k^{ND} \in X,\ ND(X_k^{ND}) = \sup_{X_k \in X} ND(X_k)\} \tag{5.138}$$

Example 5.22. Consider the following nonstrict fuzzy preference relation:

$$R = \begin{bmatrix} 1 & 0.2 & 0.3 & 0.1 \\ 0.5 & 1 & 0.2 & 0.6 \\ 0.1 & 0.6 & 1 & 0.3 \\ 0.6 & 0.1 & 0.5 & 1 \end{bmatrix} \tag{5.139}$$

defined on a set of alternatives $X = \{X_1, X_2, X_3, X_4\}$ to order them and to select the best one.

Applying (5.13) to (5.139), it is possible to obtain the membership function of the fuzzy strict preference relation

$$P = \begin{bmatrix} 0 & 0 & 0.2 & 0 \\ 0.3 & 0 & 0 & 0.5 \\ 0 & 0.4 & 0 & 0 \\ 0.5 & 0 & 0.2 & 0 \end{bmatrix} \tag{5.140}$$

The use of (5.137) generates the following membership function of the fuzzy set of nondominated alternatives:

$$ND = \begin{bmatrix} 0.5 & 0.6 & 0.8 & 0.5 \end{bmatrix} \tag{5.141}$$

It permits us to determine $X_3 \succ X_2 \succ X_1 \sim X_4$ and concluir that $X^{ND} = \{X_3\}$.

In (Orlovsky, 1981), it has been introduced the notion of a set of nonfuzzy nondominated alternatves. In particular, if $\sup\limits_{X_k \in X} ND(X_k) = 1$, then the alternatives

$$X^{NFND} = \{X_k^{NFND} \mid X_k^{NFND} \in X,\ ND(X_k^{NFND}) = 1\} \tag{5.142}$$

are nonfuzzy nondominated. These alternatives can be considered as a nonfuzzy solution to the fuzzy problem.

Example 5.23. Assume there is a set of alternatives $X = \{X_1, X_2, X_3, X_4\}$ and assigned to them the following nonstrict fuzzy preference relation:

$$R = \begin{bmatrix} 1 & 0.8 & 0.5 & 0.5 \\ 0 & 1 & 0 & 0.2 \\ 0.6 & 0.9 & 1 & 0.6 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{5.143}$$

Applying (5.13) to (5.143), we obtain the membership function of the fuzzy strict preference relation

$$P = \begin{bmatrix} 0 & 0.8 & 0 & 0.5 \\ 0 & 0 & 0 & 0.2 \\ 0.1 & 0.9 & 0 & 0.6 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{5.144}$$

The utilization of (5.137) permits us to generate the following membership function of the fuzzy set of nondominated alternatives:

$$ND = \begin{bmatrix} 0.9 & 0.1 & 1 & 0.4 \end{bmatrix} \tag{5.145}$$

It allows us to obtain $X^{NFND} = \{X_3\}$.

If the fuzzy preference relation R satisfies weak transitivity, then $X^{NFND} \neq \varnothing$. Taking this into account, it should be noted that when the preferences of a DM are exspressed as the ordering of alternatives, utility values or fuzzy estimates and subsequently converted into fuzzy preference relations with the use of the corresponding transformation functions, then $X^{NFND} \neq \varnothing$, since these transformation functions guarantee weak transitivity of the constructed fuzzy preference relation (Pedrycz, Ekel, and Parreiras, 2011). However, it is possible to obtain $X^{NFND} = \varnothing$ if a DM provides the preferences as a multiplicative preference relation or a fuzzy preference relation that does not satisfy weak transitivity, because the corresponding transformation functions transmit existent inconsistencies to the fuzzy preference relations. Thus, the fact of $X^{NFND} = \varnothing$ permits one to detect contraditions in the expert's estimates (Ekel, 2002; Pedrycz, Ekel, and Parreiras, 2011).

The expressions (5.13), (5.137), and (5.138) may be applied to solve choice problems as well as other problems, related to the evaluation, comparison, prioritization, and/or ordering of alternatives with a single fuzzy preference relation. These expressions may also be utilized when R is a vector of fuzzy preference relations.

Let us consider the ***first technique*** for dealing with a vector fuzzy preference relations R (Orlovsky, 1981). The expressions (5.13), (5.137), and (5.138) are applicable if we take $R = \bigcap_{p=1}^{q} R_p$ with the membership function

$$R(X_k, X_l) = \min_{1 \le p \le q} R_p(X_k, X_l), \quad X_k, X_l \in X \tag{5.146}$$

The use of an intersection operator in (5.146) is associated with the need to satisfy all criteria simultaneously. When using (5.146), the set X^{ND} fulfils the role of a Pareto set (Orlovsky, 1981). It is possible to contract it on the basis of differentiating the importance of R_p, $p = 1,..., q$ with the use of the following convolution:

$$T(X_k, X_l) = \sum_{p=1}^{q} \lambda_p R_p(X_k, X_l), \quad X_k, X_l \in X \tag{5.147}$$

where $\lambda_p \ge 0$, $p = 1, 2,..., q$, are importance factors for the corresponding criteria, defined as (4.15) and (4.16).

The construction of $T(X_k, X_l)$, $X_k, X_l \in X$, allows one to obtain the membership function $ND'(X_k)$ of the fuzzy set of nondominated alternatives according to an expression similar to (5.137). The intersection

$$Q(X_k) = \min\{ND(X_k),\ ND'(X_k)\}, \quad X_k \in X \tag{5.148}$$

provides us with

$$X^{ND} = \{X_k^{ND} \mid X_k^{ND} \in X, Q(X_k^{ND}) = \sup_{X_k \in X} Q(X_k)\} \tag{5.149}$$

The expressions (5.137) and (5.138) can serve as the basis for building the ***second technique***. This technique is of a lexicographic character and is directed at step-by-step application of criteria for comparing alternatives. The technique permits one (Ekel, Pedrycz, and Schinzinger, 1998; Ekel, 2001; Ekel, 2002) to construct a sequence $X^1, X^2,..., X^q$ so that $X \supseteq X^1 \supseteq X^2 \supseteq ... \supseteq X^q$ with using the following expressions:

$$ND^p(X_k) = \inf_{X_l \in X^{p-1}} [1 - P_p(X_l, X_k)] = 1 - \sup_{X_l \in X^{p-1}} P_p(X_l, X_k), \quad p = 1, 2, ..., q \tag{5.150}$$

$$X^p = \{X_k^{ND,p} \mid X_k^{ND,p} \in X^{p-1}, ND^p(X_k^{ND,p}) = \sup_{X_l \in X^{p-1}} ND^p(X_k)\} \tag{5.151}$$

Characterizing the *second technique*, it should be noted that if R_p satisfies weak transitivity, it is possible to bypass the pairwise comparison of alternatives at the *p*th step. In this situation, the comparison can be conducted on a serial basis (directly on the basis of (5.29) and (5.30)) by memorizing the best alternatives (Ekel, 2002).

The ***third technique*** is based on the use of (5.137) which can be represented in the following form:

$$ND(X_k) = 1 - \sup_{X_l \in X} P_p(X_l, X_k), \quad p = 1,2,...,q \tag{5.152}$$

In this way, we can construct the membership functions of the fuzzy sets of nondominated alternatives for each fuzzy preference relation.

The membership functions $ND_p(X_k)$, $p = 1, 2,...,q$ can be considered as the membership functions replacing objective functions $F_p(x)$, $p = 1, 2,...,q$ in analyzing $< X, F >$models (Ekel, Schuffner, 2006). Therefore, it is possible to construct

$$ND(X_k) = \min_{1 \le p \le q} ND_p(X_k) \tag{5.153}$$

to obtain X^{ND}. If necessary to differentiate the importance of different preference relations, it is possible to transform (5.153) as

$$ND(X_k) = \min_{1 \le p \le q} \left[ND_p(X_k) \right]^{\lambda_p} \tag{5.154}$$

The use of (5.154) does not require the normalization of $\lambda_p \ge 0$, $p = 1, 2, ..., q$ in the way similar to (4.16).

Example 5.24. Assume we are given a set of alternatives $X = \{X_1, X_2, X_3, X_4\}$ and assigned to them the following nonstrict fuzzy preference relations:

$$R_1 = \begin{bmatrix} 1 & 0.8 & 0.5 & 0.1 \\ 1 & 1 & 0.8 & 0.6 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \tag{5.155}$$

$$R_2 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0.4 & 1 & 0.6 & 0.8 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \tag{5.156}$$

$$R_3 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0.7 \\ 1 & 1 & 1 & 0.8 \\ 1 & 1 & 1 & 1 \end{bmatrix} \tag{5.157}$$

Let us consider the solution of the problem on the basis of the *first technique*. The intersection of the fuzzy nonstrict preference relations (5.155)-(5.157), applying (5.146), generate

$$R = \begin{bmatrix} 1 & 0.8 & 0.5 & 0.1 \\ 0.4 & 1 & 0.6 & 0.6 \\ 1 & 1 & 1 & 0.8 \\ 1 & 1 & 1 & 1 \end{bmatrix} \tag{5.158}$$

The utilization of (5.13) permits us to construct the following fuzzy strict preference relation:

$$P = \begin{bmatrix} 0 & 0.4 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0.5 & 0.4 & 0 & 0 \\ 0.9 & 0.4 & 0.2 & 0 \end{bmatrix} \tag{5.159}$$

The application of (5.137) generates

$$ND = [0.1 \quad 0.6 \quad 0.8 \quad 1] \tag{5.160}$$

and $X^{ND} = \{X_4\}$.

Let us consider the application of the *second technique*, assuming that the criteria are arranged, for example, in the following order of importance: $p = 1$, $p = 2$, and $p = 3$.

Consistently applying (5.13), (5.137), and (5.138), we obtain

$$P_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0.2 & 0 & 0 & 0 \\ 0.5 & 0.2 & 0 & 0 \\ 0.9 & 0.4 & 0 & 0 \end{bmatrix} \qquad (5.161)$$

$$ND^1 = [0.1 \quad 0.6 \quad 1 \quad 1] \qquad (5.162)$$

and $X^1 = \{X_3,\ X_4\}$.

For the second step, we can construct the fuzzy nonstrict preference rtelation, the fuzzy strict preference relation, and the fuzzy set of nondominated alternatives, considering only alternatives X_3 and X_4 as follows:

$$R^2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \qquad (5.163)$$

$$P_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad (5.164)$$

$$ND^2 = [1 \quad 1] \qquad (5.165)$$

and $X^2 = \{X_3,\ X_4\}$. Thus, the second step does not allow us to narrow down the decision uncertainty region.

Let us construct the fuzzy nonstrict preference rtelation, the fuzzy strict preference relation, and the fuzzy set of nondominated alternatives for the third step as follows:

$$R^3 = \begin{bmatrix} 1 & 0.8 \\ 1 & 1 \end{bmatrix} \qquad (5.166)$$

$$P_3 = \begin{bmatrix} 0 & 0 \\ 0.2 & 0 \end{bmatrix} \qquad (5.167)$$

$$ND^3 = [0.8 \quad 1] \qquad (5.168)$$

and $X^2 = \{X_4\}$.

Let us consider the application of the *third technique*. The membership function ND_1 of the set of nondiminated alternatives for the first fuzzy nonstrict R_1 is (5.162). The analysis of the fuzzy nonstrict preference relation (5.156) generates the following membership function of nondominated alternatives:

$$ND_2 = [1 \quad 0.4 \quad 1 \quad 1] \qquad (5.169)$$

The analysis of the fuzzy nonstrict preference relation (5.157) leads to

$$ND_2 = [1 \quad 0.7 \quad 0.8 \quad 1] \tag{5.170}$$

Thereby, all three methods allowed us to obtain the same result. Taking this into account, it should be noted that the application of the *second technique* may lead to solutions different from the results obtained on the basis of the *first technique*. However, solutions based on the *first technique* and the *third technique*, which share a single generic basis, may in some cases also differ from each other (Ekel and Schuffner Neto, 2006). At the same time, the *third technique* is more preferential from the substantial point of view. In particular, the use of the *first technique* can lead to choosing alternatives with the degree of nondominance equal to one, though these alternatives are not the best ones from the point of view of all preference relations. The *third technique* can generate this result only for alternatives that are the best solutions from the point of view of all fuzzy preference relations. It should be stressed that the fact of the possibility to obtain different solutions on the basis of different approaches (as it is demonstrated by an example in Section 5.9) is to be considered natural, and the choice of the approach is a prerogative of DM.

The described techniques have been implemented within the framework of interactive decision making system for multicriteria analysis of alternatives in a fuzzy environment, named MDMS (Pedrycz, Ekel, and Parreiras, 2011). These techniques are of a universal nature and are already being used to solve problems in power engineering (Canha et al., 2007; Ekel et al., 2016), naval engineering (Botter and Ekel, 2005), and management (Berredo et al., 2005).

All three techniques aimed at analyzing <*X*, *R*> models described above require the explicit direct or indirect ordering of the criteria. Considering this, it is necessary to distinguish the results of (Orlovsky, 1981), which allow one to present information related to the importance of the criteria as a nonreprocical fuzzy preference relation:

$$\Lambda = [\lambda \times \lambda,\ \Lambda(\lambda_p,\ \lambda_t)], \quad p,t = 1,\ 2,...,q \tag{5.171}$$

Using the membership functions of the fuzzy sets of nondominated alternatives for all preference relations (5.137), it is possible to construct the following fuzzy preference relation induced by the preference relations (5.137) and (5.171):

$$R_\Lambda(X_k, X_l) = \sup_{\lambda_p,\lambda_t \in \Lambda} \min_{X_k, X_l \in X} \{ND_p(X_k),\ ND_t(X_l),\ \Lambda(\lambda_p,\ \lambda_t)\}, \quad p,t = 1,\ 2,...,q \tag{5.172}$$

The fuzzy preference relation (7.93) can be considered (Ekel, et al., 2006) as a result of aggregating the family of $R_p, p = 1, 2,..., q$, with the use of information reflecting the relative importance of criteria given in the form (5.171).

Applying (5.13) and (5.137) to (5.172), it is possible to construct the fuzzy set of nondominated alternatives $ND'_\Lambda(X_k)$. As it is shown in (Orlovsky, 1981), the set $ND'_\Lambda(X_k)$ is to be modified in accordance with the following relationship:

$$ND_\Lambda(X_k) = \min\{ND'_\Lambda(X_k), R_\Lambda(X_k, X_k)\} \tag{5.173}$$

To better understand the fourth technique, consider an example from (Pedrycz, Ekel, and Parreiras, 2011).

Example 5.25. We are given a set of alternatives $X = \{X_1, X_2, X_3\}$ which are to be compared applying three criteria. The corresponding nonstrict fuzzy preference relations are the following:

$$R_1 = \begin{bmatrix} 1 & 1 & 1 \\ 0.8 & 1 & 0.6 \\ 0.4 & 0.6 & 1 \end{bmatrix} \tag{5.174}$$

$$R_2 = \begin{bmatrix} 1 & 0.4 & 0.3 \\ 1 & 1 & 0.7 \\ 0.9 & 0.6 & 1 \end{bmatrix} \tag{5.175}$$

$$R_3 = \begin{bmatrix} 1 & 0.8 & 0.7 \\ 0.4 & 1 & 0.8 \\ 0.3 & 0.6 & 1 \end{bmatrix} \tag{5.176}$$

The information related to the importance of criteria is presented as follows:

$$\Lambda = \begin{bmatrix} 1 & 0.8 & 0.7 \\ 1 & 1 & 0.5 \\ 0.9 & 0.7 & 1 \end{bmatrix} \tag{5.177}$$

The application of (5.13) to (5.175)-(5.177), permits one to construct the following fuzzy strict preference relations:

$$P_1 = \begin{bmatrix} 0 & 0.2 & 0.6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{5.178}$$

$$P_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0.6 & 0 & 0.1 \\ 0.6 & 0 & 0 \end{bmatrix} \tag{5.179}$$

and

$$P_3 = \begin{bmatrix} 0 & 0.4 & 0.4 \\ 0 & 0 & 0.2 \\ 0 & 0 & 0 \end{bmatrix} \tag{5.180}$$

for the first, second, and third criteria, respectively. Applying (5.137) to (5.178)-(5.180), we obtain the following membership functions of the fuzzy sets of nondominated alternatives:

$$ND_1 = \begin{bmatrix} 1 & 0.8 & 0.4 \end{bmatrix} \tag{5.181}$$

$$ND_2 = \begin{bmatrix} 0.4 & 1 & 0.9 \end{bmatrix} \tag{5.182}$$

and

$$ND_3 = \begin{bmatrix} 1 & 0.6 & 0.6 \end{bmatrix} \tag{5.183}$$

for the first, second, and third criteria, respectively.

The utilization of (5.172) for the processing of (5.181)-(5.183) together with (5.177), permits us to obtain

$$R_\Lambda = \begin{bmatrix} 1 & 0.8 & 0.8 \\ 1 & 1 & 0.9 \\ 0.9 & 0.9 & 0.4 \end{bmatrix} \tag{5.184}$$

The use of (5.13) permits us to construct the corresponding fuzzy strict preference relation

$$P_\Lambda = \begin{bmatrix} 0 & 0 & 0 \\ 0.2 & 0 & 0 \\ 0.1 & 0 & 0 \end{bmatrix} \tag{5.185}$$

which, in accordance with (5.137), leads to

$$ND'_\Lambda = \begin{bmatrix} 0.8 & 1 & 1 \end{bmatrix} \tag{5.186}$$

Finally, the application of (5.173), taking into account that

$$R_\Lambda(X_k, X_k) = \begin{bmatrix} 1 & 1 & 0.4 \end{bmatrix} \tag{5.187}$$

generates

$$ND_\Lambda = \begin{bmatrix} 0.8 & 1 & 0.4 \end{bmatrix} \tag{5.188}$$

Finally, applying the results discussed in Section 4.6 as well as (Grabisch, Orlovski, and Yager, 1998; Pedrycz, Ekel, and Parreiras, 2011; Ekel *et al*., 2016), we can speak about the analysis of $< X,\ R >$ models with applying the OWA operator. In particular, in this case, (4.39) is transformed in the following expression:

$$\mathrm{OWA}(R_1(X_k, X_l), R_2(X_k, X_l), ..., R_q(X_k, X_l)) = \sum_{i=1}^{q} w_i b_i \tag{5.189}$$

Exampes of applying the OWA operator for multicriteria analysis of alternatives in a fuzzy environment can be found in (Pedrycz, Ekel, and Parreiras, 2011).

5.9 Practical Examples of Analyzing <*X, R*> Models

The example (Pedrycz, Ekel, and Parreiras, 2011) given below demonstrates three techniques for analyzing <*X*, *R*> models based on the concept of the strict fuzzy preference relation.

Example 5.26. The problem of substation planning in power systems with considering the uncertainty of information is analyzed in (Fontoura Filho, Ales, and Tortelly, 1994). Its practical application is associated with a group of substations 138/13.8 kV of a power utility. In particular, careful analysis has been executed to select a solution from three alternatives on the basis of their total costs where the uncertainty of interest rates is modeled as trapezoidal membership functions. The details about the membership functions of alternative costs are given in Table 5.7 and are also illustrated in Figure 5.8.

Table 5.7 Total costs (US$ thousand) of alternatives

Alternative	1	2	3	4
1	20,291	22,007	22,769	27,054
2	21,058	21,831	22,378	25,865
3	21,977	22,749	23,098	24,276

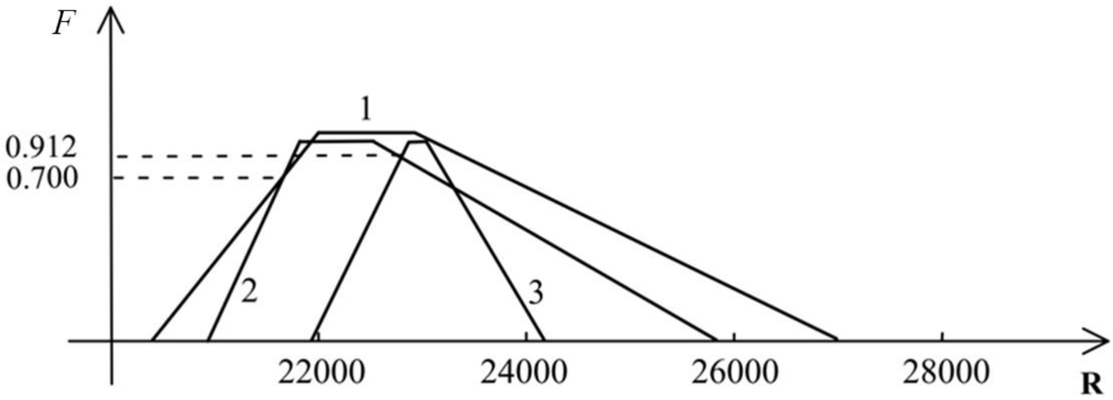


Figure 5.8 Total alternative costs.

It is evident that the selection of the most preferable alternative is hampered: the difference between the alternatives 1 and 2 is equal to 0.38% for the left bounds of the corresponding membership functions for certainty of 70%, that has been accepted in (Fontoura Filho, Ales, and Tortelly, 1994), but does not give ground to proceed with a convincing decision. This may also be illustrated by analyzing a nonstrict fuzzy preference relation

$$R_1 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0.912 & 1 \end{bmatrix} \tag{5.190}$$

constructed on the basis of Figure 5.8. Applying (5.13) to (5.190), we obtain the following strict fuzzy preference relation:

$$P_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0.088 \\ 0 & 0 & 0 \end{bmatrix} \tag{5.191}$$

Using (5.137), we can obtain the membership function of the fuzzy set of nondominated alternatives

$$ND = \begin{bmatrix} 1 & 1 & 0.912 \end{bmatrix} \tag{5.192}$$

which indicates that the alternatives 1 and 2 are indistinguishable.

Taking this into account, it is possible to consider indices "*Flexibility of Development*" and "*Damage to Agriculture*" as additional criteria denoted by $F_2(X_k)$ and $F_3(X_k)$. The membership functions corresponding to the normalized fuzzy values $S(F)$= (*very small*, *small*, *middle*, *large*, and *very large*) of the linguistic variables *Flexibility of Development* and *Damage to Agriculture*, which can be used to estimate $F_2(X_k)$ and $F_3(X_k)$, are given in Figure 5.9.

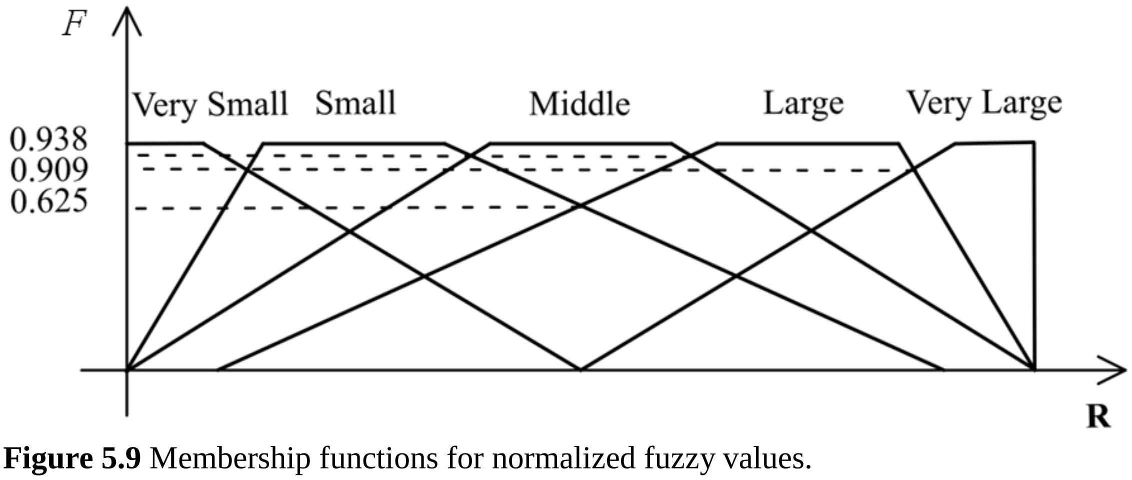


Figure 5.9 Membership functions for normalized fuzzy values.

Assume that the alternatives have received the following estimates: $F_2(X_1) =$*large*, $F_2(X_2) =$*large*, and $F_2(X_3) =$*very large* for the second criterion and $F_3(X_1) =$*small*, $F_3(X_2) =$*middle*, and $F_3(X_3) =$*Large* for the third criterion. Considerin this as well as the necessity to maximize $F_2(X_k)$ and to minimize $F_3(X_k)$, it is possible to construct the matrices of the nonstrict fuzzy preference relations

$$R_2 = \begin{bmatrix} 1 & 1 & 0.909 \\ 1 & 1 & 0.909 \\ 1 & 1 & 1 \end{bmatrix} \tag{5.193}$$

and

$$R_3 = \begin{bmatrix} 1 & 1 & 1 \\ 0.938 & 1 & 1 \\ 0.625 & 0.938 & 1 \end{bmatrix} \tag{5.194}$$

for the second and third criterion, respectively.

Applying the *first technique* presented in Section 5.8, it is possible to obtain the intersection of (5.190), (5.193), and (5.194) as follows:

$$R = \begin{bmatrix} 1 & 1 & 0.909 \\ 0.938 & 1 & 0.909 \\ 0.625 & 0.912 & 1 \end{bmatrix} \tag{5.195}$$

Applying (5.13) to (5.195), one can construct the following strict fuzzy preference relation:

$$P = \begin{bmatrix} 0 & 0.062 & 0.274 \\ 0 & 0 & 0 \\ 0 & 0.03 & 0 \end{bmatrix} \tag{5.196}$$

that permits us, using (5.137), to obtain the membership function of the fuzzy set of nondominated alternatives

$$ND = \begin{bmatrix} 1 & 0.938 & 0.716 \end{bmatrix} \tag{5.197}$$

The alternative 1 has the maximum degree of nondominance and it is natural to consider it as the solution, i.e., $X^{ND} = \{X_1\}$. Thus, we have obtained the solution without applying the convolution (5.147).

Let us consider the application of the *second technique* if the criteria are arranged, for example, in the following order of their importance: p=1, p=2, and p=3.

Applying (5.13), (5.150), and (5.151), we obtain, from (5.190), the result coinciding with (5.192). This is obvious, and $X^1 = \{X_1, X_2\}$. Thus, the alternatives X_1 and X_2 are to be considered for a sequent analysis, and from (5.193) we can proceed with the second step:

$$R^2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \tag{5.198}$$

that leads to

$$ND^2 = \begin{bmatrix} 1 & 1 \end{bmatrix} \tag{5.199}$$

and $X^2 = \{X_1, X_2\}$. The second step does not allow us to narrow down the decision uncertainty region.

From (5.194), we can proceed with the third step

$$R^3 = \begin{bmatrix} 1 & 1 \\ 0.938 & 1 \end{bmatrix} \tag{5.200}$$

Then

$$ND^3 = \begin{bmatrix} 1 & 0.938 \end{bmatrix} \tag{5.201}$$

leading to $X^3 = \{X_1\}$.

It is natural that if the criteria are arranged in another order, it is possible to obtain another solution. For instance, it is not difficult to verify if *p*=2, *p*=3, and *p*=1, then $X^1 = \{X_3\}$.

Finally, let us apply the *third technique*. The membership function of the set of nondominated alternatives for the first fuzzy preference relation $ND_1(X_k)$ is (5.192). Processing (5.193), it is possible to build

$$ND_2 = \begin{bmatrix} 0.909 & 0.909 & 1 \end{bmatrix} \tag{5.202}$$

In an analogous way, the processing of (5.194) leads us to

$$ND_3 = \begin{bmatrix} 1 & 0.938 & 0.625 \end{bmatrix} \tag{5.203}$$

The intersection of (5.192), (5.202), and (5.203) produces

$$ND = \begin{bmatrix} 0.909 & 0.909 & 0.625 \end{bmatrix} \tag{5.204}$$

and generates $X^{ND} = \{X_1, X_2\}$.

In this way, the *third technique* does not permit one to choose a unique alternative. It allowed us only to exclude alternative X_3 from further considerations: information given by (5.190), (5.193), and (5.194) is not sufficient to choose a unique alternative.

Therefore, the *first technique* allows one to choose the alternative X_1. The *second technique* indicates the alternative X_1 (for the order of importance: *p*=1, *p*=2, and *p*=3) as well. The *third technique* only permits one to eliminate the alternative X_3.

The example (Borisov, Krumberg, and Fedorov, 1990) given below demonstrates the applicability of the results presented in Section 5.8 to analyze problems associated with nonfuzzy preference relations as well.

Example 5.27. The direct development of an integrated project by an enterprise presents considerable difficulties. There exists several ways around these difficulties:

1. Training proper professionals;
2. Inviting new professionals capable of developing a project;
3. Contracting another enterprise with the necessary profile.

The decision of a manager is to be based on applying the following criteria:

1. Project development duration;
2. Financial expenditures;
2. Project development quality.

Thus, we have three alternatives $X_1, X_2,$ and X_3, which are to be analyzed from the point of view of the indicated above criteria. Let us apply the *first technique* given in Section 5.8 by taking into account, if necessary, the following importance factors: $\lambda_1 = 0.6$, $\lambda_2 = 0.2$, and $\lambda_3 = 0.2$.

From the point of view of the first criterion, a manager has an opinion that X_1 is as good as X_2 and X_3 is better than X_2. This information permits one to construct the following nonstrict preference relation:

$$R_1 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \tag{5.205}$$

The preferences expressed from the point of view of the second criterion are the following: X_1 is better than X_2 and X_3, and X_2 is better than X_3. These preferences permit one to construct the second nonstrict preference relation

$$R_2 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \tag{5.206}$$

Finally, the preferences from the point of view of the third criterion are presented as follows: X_1 is as good as X_2 and X_3 is better than X_1. It allows one to construct the third nonstrict preference relation

$$R_3 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \tag{5.207}$$

The intersection of (5.205)-(5.207) leads to the formation of the following nonstrict preference relation:

$$R = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{5.208}$$

The use of (5.13) to process (5.208) permits one to construct the following strict preference relation:

$$P = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{5.209}$$

providing

$$ND = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} \tag{5.210}$$

on the basis of (5.137).

In such a manner, we have to introduce the convolution (5.147) into consideration by applying the importance factors given above as follows:

$$T = 0.6\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} + 0.2\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} + 0.2\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0.2 \\ 0.8 & 1 & 0.2 \\ 0.2 & 0.6 & 1 \end{bmatrix} \tag{5.211}$$

It is not difficult to understand that (5.211) generates the following strict preference relation:

$$P = \begin{bmatrix} 0 & 0.2 & 0 \\ 0 & 0 & 0 \\ 0 & 0.4 & 0 \end{bmatrix} \tag{5.212}$$

The membership function of the set of nondominated alternatives, which corresponds to (5.212), is the following:

$$ND' = \begin{bmatrix} 1 & 0.6 & 1 \end{bmatrix} \tag{5.213}$$

Finally, the intersection of (5.210) and (5.213) realized in accordance with (5.148) leads to

$$Q = \begin{bmatrix} 0 & 0.6 & 1 \end{bmatrix} \tag{5.214}$$

Given this, we define $X^{ND} = \{X_3\}$. It means that the suitable alternative is to recommend to contract another enterprise with the necessary profile to develop the project.

The example (Orudjev, 1983) given below demonstrates the analysis of the <*X*, *R*> models with fuzzy ordering of criteria.

Example 5.28. The problem of choosing a local reactive power source at a power system bus with reactive power shortage is considered in (Orudjev, 1983; Pedrycz, Ekel, and Parreiras, 2011). The following alternatives are considered:

1. Controlled thyristor reactor with constantly connected capacitor banks;
2. Controlled thyristor reactor with capacitor banks connected through the reactor;
3. Synchronous compensator;
4. Capacitor banks with smooth thyristor control.

The decision is to be made on the basis of utilizing the following criteria:

1. Reliability;
2. Investment;
3. Control rapidity.

The nonstrict fuzzy preference relations corresponding to the first, second, and third criteria are the following:

$$R_1 = \begin{bmatrix} 1 & 0.7 & 0.4 & 0.8 \\ 0 & 1 & 0.2 & 1 \\ 0.5 & 0.3 & 1 & 0.1 \\ 0.8 & 0.4 & 0.2 & 1 \end{bmatrix} \tag{5.215}$$

$$R_2 = \begin{bmatrix} 1 & 0.1 & 0.5 & 0.8 \\ 0 & 1 & 0.8 & 0.6 \\ 0.7 & 0.4 & 1 & 0.7 \\ 0.4 & 0.8 & 0.2 & 1 \end{bmatrix} \tag{5.216}$$

$$R_3 = \begin{bmatrix} 1 & 0.9 & 0.12 & 0.3 \\ 0.3 & 1 & 0.8 & 0.5 \\ 0.3 & 0.15 & 1 & 0.7 \\ 0.9 & 0.6 & 0.2 & 1 \end{bmatrix} \tag{5.217}$$

respectively.

The information related to the importance of considered criteria is presented in the following form:

$$\Lambda = \begin{bmatrix} 1 & 0.8 & 0.6 \\ 0.5 & 1 & 0.7 \\ 0.2 & 0.1 & 1 \end{bmatrix} \tag{5.218}$$

The application of (5.13) to (5.215)-(5.217) to construct the corresponding strict fuzzy preference relations and, then, the utilization of (5.137) yields the following membership functions of the fuzzy sets of nondominated alternatives:

$$ND_1 = \begin{bmatrix} 0.9 & 0.3 & 0.9 & 0.4 \end{bmatrix} \tag{5.219}$$

$$ND_2 = \begin{bmatrix} 0.8 & 0.8 & 0.6 & 0.5 \end{bmatrix} \tag{5.220}$$

and

$$ND_3 = \begin{bmatrix} 0.4 & 0.4 & 0.35 & 0.5 \end{bmatrix} \tag{5.221}$$

for the first, second, and third criterion, respectively.

The application of (5.172) to process (5.219)-(5.221) together with (5.218) leads to the following relation:

$$R_\Lambda = \begin{bmatrix} 1 & 0.8 & 0.9 & 0.5 \\ 0.8 & 0.9 & 0.5 & 0.5 \\ 0.9 & 0.8 & 0.9 & 0.8 \\ 0.5 & 0.5 & 0.5 & 0.5 \end{bmatrix} \tag{5.222}$$

The application of (5.13) to (5.222) to construct the corresponding strict fuzzy preference relations and, then, the use of (5.173) forms the following membership function of the fuzzy sets of nondominated alternatives:

$$ND' = \begin{bmatrix} 1 & 0.7 & 1 & 0.7 \end{bmatrix} \tag{5.223}$$

Finally, the application of (5.173), considering that

$$R_\Lambda = \begin{bmatrix} 1 & 0.9 & 0.9 & 0.5 \end{bmatrix} \tag{5.224}$$

leads to

$$ND_\Lambda = \begin{bmatrix} 1 & 0.7 & 0.9 & 0.5 \end{bmatrix} \tag{5.225}$$

Thus, a controlled thyristor reactor with constantly connected capacitor banks should be selected.

The last example (Ekel *et al*., 2019) demonstrates the analysis of the $<X, R>$ models formed on the basis of applying diverse preference formats and transformation functions to convert these formats to the nonstrict fuzzy preference relations providing homogeneous information for the use of decision making procedures.

Example 5.29. The expansion of electric transmission systems is fundamental to meet the observable growing demands for electricity. Such expansion is generally characterized by technical complexity, the necessity of considering socio-environmental factors, and the need for significant investments. Moreover, the process of their analysis has to be capable to prioritize the projects in accordance with a wide range of other considerations, including strategic goals of concessionaires and investors. Thus, there is a problem of managing the process of analyzing several investment alternatives in electric transmission, carried out by concessionaires or investors, which plan to participate in auctions of new transmission projects. The necessity of applying the express analysis is associated with the following considerations.

The granting authorities (usually, regulatory agencies) make available, together with auction announcements on bidding for electric transmission projects, reports containing information on the items that are part of each lot. In addition to the importance for future implantation, this information is related to decisions of competitors on participation or not participation in the auction, and on the composition of their proposals on lots of interest. However, the indicated above reports are usually published in a date close to the auction (for instance, one month of antecedence in Brazil). This does not allow one to effectively analyze the lots of interest. Thus, it is advantageous to carry out preliminarily studies in order to prepare the best proposals, with greater chances to get the corresponding lots.

In order to carry out efficient studies, without "waste" of resources invested in the analysis of ventures that prove to be very risky or generally unattractive, it is advisable to prioritize the projects to be studied. Given that, at this point, there is still insufficient information for the final definition of the corridor and the preferential routes of the lines, it is necessary to create a methodology for the convincing use of the information already available. Considering this, the results of (Ekel *et al*., 2019) are related to decision making procedures for supporting the evaluation, comparison, choice, prioritization, and/or ordering of electric transmission projects, in the prospecting stage, aimed at greater efficiency and effectiveness in the advanced preparation for auctions.

The process of elaborating decisions on transmission projects proposed in (Ekel *et al*., 2019) includes two stages. The first one is associated with studies related to the criteria relevant to the decision, which allow one to form the most rational estimates of the considered lots. These criteria often have a spatial nature. Considering this, the proposed models assume the use of Geographic Information Systems (GISs) which play an important role in supporting the analysis of such class of problems (Malczewski, 2006; Shafiullah *et al*., 2016). Another important aspect is the need to apply the techniques of multiobjective analysis under conditions of uncertainty. In particular, it is proposed in (Ekel *et al*., 2019) to use the generalization of the well-known Dijkstra's algorithm (Dijkstra, 1959) (related to graph optimization, allowing the construction of transmission line routes) to analyze multiobjetive problems. This generalization is complemented by the consideration of the uncertainty factor, applying a possibilistic approach. In particular, it is used a general scheme of decision making under conditions of uncertainty, discussed Chapter 7, which permits one to construct solutions, including robust solutions, to multiobjective problems.

The second stage is directed at defining a portfolio with the most appropriate and favorable areas for implanting the transmission lines from the set of lots analyzed at the first stage. This stage is based on applying techniques of preference modeling in a fuzzy environment discusses in the present chapter. Their application allows one to adequately consider quantitative as well as qualitative criteria, whose estimates are based on knowledge, experience, and intuition of a DM (individual or group).

In (Ekel *et al*., 2019), the first stage is based on applying the thematic maps which reflect technical-economic cost c^E and socio-environmental cost c^S.

Let us consider four auction lots in order to demonstrate the most important aspects of the two-stage multicriteria analysis of new transmission lines. Without discussing the execution of the first stage (it is considered in Chapter 7), its results are given in Table 5.8.

Table 5.8 Rational estimates of the analysed lots.

Lot	$F_1(X_k)=c^E$	$F_2(X_k)=c^S$
X_1	141,07	12,10
X_2	133,03	13,56
X_3	138,05	13,66
X_4	123,98	12,73

Let us consider the procedures of the second stage for the following situations:

1. Elaboration of the recommendation on the basis of applying the additional criterion "Expected Profit";
2. Elaboration of the recommendation on the basis of applying the criteria "Expected Profit" and "Time of Load Time Achievement".

The alternative estimates from the point of view of the criterion "Expected Profit" can be constructed in the quantitative form. At the same time, the alternative estimates from the point of view of the criterion "Time of Planned Load Achievement" can be obtained in the qualitative form.

Taking into account that $F_1(X_k)$ and $F_2(X_k)$ are to be minimized, it is possible to apply (5.82) to construct the following additive reciprocal fuzzy preference relations:

$$RR_1 = \begin{bmatrix} 0.5000 & 0.2648 & 0.4175 & 0.0000 \\ 0.7352 & 0.5000 & 0.6527 & 0.2352 \\ 0.5825 & 0.3473 & 0.5000 & 0.0825 \\ 1.0000 & 0.7648 & 0.9173 & 0.5000 \end{bmatrix} \tag{5.266}$$

for $F_1(X_k)$ and

$$RR_2 = \begin{bmatrix} 0.5000 & 0.9679 & 1.0000 & 0.7019 \\ 0.0321 & 0.5000 & 0.5321 & 0.2340 \\ 0.0000 & 0.4679 & 0.5000 & 0.2019 \\ 0.2981 & 0.7660 & 0.7981 & 0.5000 \end{bmatrix} \tag{5.267}$$

for $F_2(X_k)$, respectively.

Applying the transformation function (5.83), it is possible to convert these additive reciprocal fuzzy preference relations can to the following nonreciprocal fuzzy preference relations:

$$NR_1 = \begin{bmatrix} 1.0000 & 0.5296 & 0.8323 & 0.0000 \\ 1.0000 & 1.0000 & 1.0000 & 0.4704 \\ 1.0000 & 0.6946 & 1.0000 & 0.1652 \\ 1.0000 & 1.0000 & 1.0000 & 1.0000 \end{bmatrix} \tag{5.268}$$

and

$$NR_2 = \begin{bmatrix} 1.0000 & 1.0000 & 1.0000 & 1.0000 \\ 0.0642 & 1.0000 & 1.0000 & 0.4680 \\ 0.0000 & 0.9358 & 1.0000 & 0.4038 \\ 0.5962 & 1.0000 & 1.0000 & 1.0000 \end{bmatrix} \tag{5.269}$$

respectively.

Figure 5.9 presents the estimates (membership functions of fuzzy values) for the alternatives from the point of view of the criterion "Expected Profit".

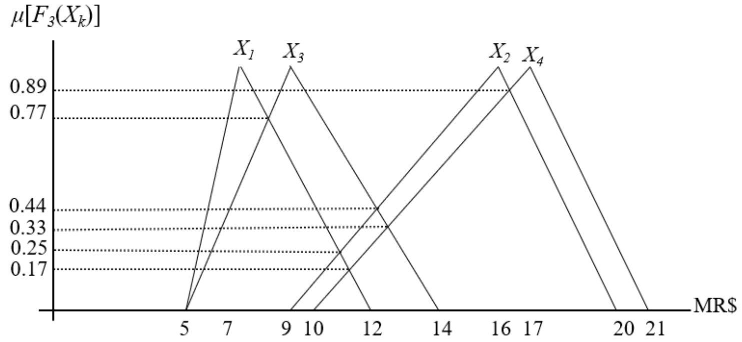


Figure 5.9 Membership functions of fuzzy values of estimates for the criterion "Expected Profit".

Applying (5.29) and (5.30), defined for $F_p(X_k) \geq F_p(X_l)$ and $F_p(X_l) \geq F_p(X_k)$, respectively, it is possible to obtain the following nonreciprocal fuzzy preference relation:

$$NR_3 = \begin{bmatrix} 1.00 & 0.25 & 0.77 & 0.17 \\ 1.00 & 1.00 & 1.00 & 0.89 \\ 1.00 & 0.44 & 1.00 & 0.33 \\ 1.00 & 1.00 & 1.00 & 1.00 \end{bmatrix} \tag{5.270}$$

Using the *first technique* for processing of NR_1, NR_2, and NR_3, it is possible to obtain

$$R = \begin{bmatrix} 1.0000 & 0.2500 & 0.7700 & 0.0000 \\ 0.0642 & 1.0000 & 1.0000 & 0.4680 \\ 0.0000 & 0.4400 & 1.0000 & 0.1652 \\ 0.5962 & 1.0000 & 1.0000 & 1.0000 \end{bmatrix} \tag{5.271}$$

on the basis of (5.146). Applying (5.13), one can build the fuzzy strict preference relation

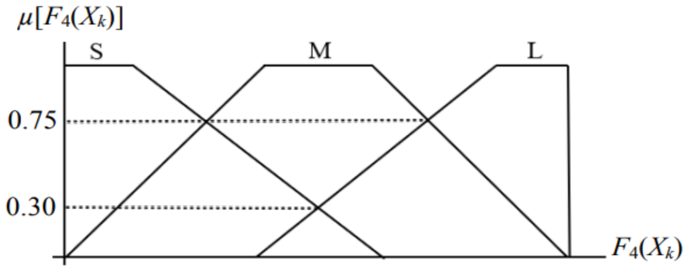
$$P = \begin{bmatrix} 0.0000 & 0.1858 & 0.7700 & 0.0000 \\ 0.0000 & 0.0000 & 0.5600 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.5962 & 0.5320 & 0.8348 & 0.0000 \end{bmatrix} \tag{5.272}$$

generating the set of non-dominated alternatives with the following membership function:

$$ND = \begin{bmatrix} 0.4038 & 0.4680 & 0.1652 & 1.0000 \end{bmatrix} \tag{5.273}$$

Therefore, the analysis on the basis of applying the three criteria permits us to elaborate the following recommendation: $X_4 \succ X_2 \succ X_1 \succ X_3$.

Now, the criterion "Time of Planned Load Achievement", which is of a qualitative character, is added. The membership functions of fuzzy values which can be used for estimating the alternatives from the point of view of the criterion "Time of Planned Load Achievement" are presented in Figure 5.10 (S – Small, M – Medium, and L – Large).



Figuew 5.10. Membership functions of fuzzy values of estimates for the criterion "Time of Planned Load Achievement".

The alternatives have received the following estimates: $X_1 - S,\ X_2 - L,\ X_3 - M,$ and $X_4 - L$. Applying (5.29) and (5.30) for these estimates, it is possible to construct the following nonreciprocal fuzzy preference relation:

$$NR_4 = \begin{bmatrix} 1.00 & 1.00 & 1.00 & 1.00 \\ 0.30 & 1.00 & 0.75 & 1.00 \\ 0.75 & 1.00 & 1.00 & 1.00 \\ 0.30 & 1.00 & 0.75 & 1.00 \end{bmatrix} \tag{5.274}$$

Using the *first technique* for NR_1, NR_2, NR_3 and NR_4, it is possible to obtain

$$R = \begin{bmatrix} 1.0000 & 0.2500 & 0.7700 & 0.0000 \\ 0.0642 & 1.0000 & 0.7500 & 0.4680 \\ 0.0000 & 0.4400 & 1.0000 & 0.1652 \\ 0.3000 & 1.0000 & 0.7500 & 1.0000 \end{bmatrix} \tag{5.275}$$

The application of (5.13) provides

$$P = \begin{bmatrix} 0.0000 & 0.1858 & 0.7700 & 0.0000 \\ 0.0000 & 0.0000 & 0.3100 & 0.0000 \\ 0.0000 & 0.0000 & 0.0000 & 0.0000 \\ 0.3000 & 0.5320 & 0.5848 & 0.0000 \end{bmatrix} \tag{5.276}$$

generating the set of non-dominated alternatives with the following membership function:

$$ND = \begin{bmatrix} 0.7000 & 0.4680 & 0.2300 & 1.0000 \end{bmatrix}$$

Thus, the introduction of the fourth criterion leads to the following recommendation: $X_4 \succ X_1 \succ X_2 \succ X_3,$ which is different from the recommendation obtained on the basis of applying three criteria. However, still the best alternative is X_4.

6.10 Conclusions

In this chapter, we have considered some selected issues related to constructing and analyzing <*X*, *R*> models serving for preference modeling through the binary fuzzy preference relations. The properties of the binary fuzzy preference relations as well as conditions of their transitivity have been discussed.

In dealing with <*X*, *R*> models, a fundamental question arises on how one can build fuzzy preference relations. The corresponding techniques are considered. Much attention has been paid to a natural and convincing approach to constructing fuzzy preference relations based on the ordering of fuzzy quantities and, in particular, on the use of the fuzzy number ranking index, generated by the concept of the membership function of the generalized preference relation.

The input of the preference information plays a fundamental role in decision making processes, as the recommendations are derived from the models, which are constructed in accordance with the provided information. Considering that this information is often subjective, vague, and uncertain, it is of high importance to provide involved experts with the means to articulate their preferences as a maximum truthful and accurate manner. Otherwise, if any expert is forced to express his/her preferences applying a preference format with which he/she does not feel comfortable, the input of the preference information can become a critical step in multicriteria analysis. Taking this into account, we have considered five main types of preference formats. The presence of different preference formats requires their conversion to a unique format to provide homogenious information for decision making procedures. Considering this, so-called transformation finctions are introduced. They permits one to reduce different preference formats to fuzzy preference relations as well as quantitative information to fuzzy preference relations.

We have discussed the questions of the emergence and importance of problems of multicriteria evaluation, comparison, choice, prioritization, and/or ordering of alternatives. There exist two types of situations which give rise to these problems. The first one is related to the direct statement of multiattribute decision-making problems when the consequences associated with solutions to problems cannot be estimated with a single criterion. The second class is related to problems that may be solved on the basis of a single criterion or on the basis of several criteria; however, if the uncertainty of information does not permit to obtain a unique solution (as it is demonstrated in the chapter by solving continuos and discrete problems of mathematical programming with fuzzy coefficients), it is possible to include additional criteria and thereby convert these problems into multiattribute tasks. Diverse techniques of multiattribute analysis in a fuzzy environment are discussed. These techniques can lead to different solutions. However, this is to be considered natural and intuitively appealing. The choice of a specific technique is a prerogative of the DM. This selection has to be based on the essence of the problem and the possible sources of information and its uncertainty.

Although these techniques are directly related to individual decision making, they can be applied to procedures of group decision making. The use of the presented results is illustrated by solving practical problems coming from different areas. One of this problems demands an integration of <*X*, *M*> and <*X*, *R*> models.

Exercises

Problem 5.1. Verify whether the fuzzy relation

$$R = \begin{bmatrix} 1 & 0.5 & 0.7 & 0.7 \\ 0.5 & 1 & 0.5 & 0.5 \\ 0.6 & 0.6 & 1 & 0.6 \\ 0.5 & 0.5 & 0.5 & 1 \end{bmatrix}$$

satisfies the conditions of a) min-transitivity, b) product transitivity, and c) Lukasiewicz-transitivity.

Problem 5.2. Verify whether the multiplicative preference relation

$$MR = \begin{bmatrix} 1 & 5 & 6 & 7 \\ 0.2 & 1 & 4 & 6 \\ 0.17 & 0.25 & 1 & 4 \\ 0.14 & 0.17 & 0.25 & 1 \end{bmatrix}$$

satisfies the condition of multiplicative transitivity.

Problem 5.3. Applying (5.13) and (5.14), construct the fuzzy strict preference relation and fuzzy indifference relation from the fuzzy nonstrict preference relation of Problem 5.1.

Problem 5.4. Applying (5.29) and (5.30), construct the nonreciprocal fuzzy preference relations from the comparison of the alternatives X_1, X_2, and X_3 which are to be evaluated from the point of view of a minimized criterion F. The corresponding levels of membership functions are given in Table 5.9.

Table 5.9 Membership functions of $F(X_1)$, $F(X_2)$, and $F(X_3)$

R	2	4	6	8	10	12	14	16
$\mu[F(X_1)]$	0	0.5	1	0.5	0.2	0	0	0
$\mu[F(X_2)]$	0	0.20	0.6	0.8	1	1	0.6	0
$\mu[F(X_3)]$	0	0	0.4	1	0.8	0.6	0.4	0

Problem 5.5. Convert the ordered array $O = \{2\ 1\ 3\ 4\ 5\}$ into an additive reprocical fuzzy preference relation.

Problem 5.6. Convert the array of utility values $U = \{1\ 0\ 0.6\ 0.4\}$ into nonreciprocal fuzzy preference relations, applying a) consistently (5.48) and (5.61) and b) (5.75) and compare obtained results.

Problem 5.7. Convert the multiplicative preference relation of Problem (5.2) into nonreciprocal fuzzy preference relations, applying a) consistently (5.51) and (5.61) and b) (5.76) (with $m = 9$) and compare obtained results.

Problem 5.8. Applying results given in Subsection 5.5.9, contruct nonreciprocal fuzzy preference relations related to indicators "Investment Level" and "Expected Profit Level" for the four industrial projects.

Table 5.10 Indicators of industrial projects (US$ thousand)

Project	Investment Level	Expected Profit Level
X_1	11,340.00	2600.00
X_2	8450.00	1950.00
X_3	9980.00	2080.00
X_4	10,760.00	2230.00

Problem 5.9. Construct the membership function of the fuzzy set of nondominated alternatives for the following fuzzy nonstrict preference relation:

$$R = \begin{bmatrix} 1 & 0.5 & 1 & 0.4 \\ 1 & 1 & 0.8 & 1 \\ 0.6 & 1 & 1 & 1 \\ 1 & 0.8 & 0.5 & 1 \end{bmatrix}$$

Problem 5.10. Verify the existence of the nonfuzzy solution in the problem described by the following fuzzy nonstrict preference relation:

$$R = \begin{bmatrix} 1 & 1 & 1 & 0.6 \\ 0.6 & 1 & 0.8 & 0 \\ 0.8 & 1 & 1 & 0.2 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Problem 5.11. Apply the *first technique* for analyzing <*X*, *R*> models to solve the problem which includes the following fuzzy nonstrict preference relations:

$$R_1 = \begin{bmatrix} 1 & 1 & 1 \\ 0.8 & 1 & 1 \\ 0.8 & 1 & 1 \end{bmatrix}$$

$$R_2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0.8 & 0.8 & 1 \end{bmatrix}$$

$$R_3 = \begin{bmatrix} 1 & 0.8 & 0.6 \\ 1 & 1 & 0.8 \\ 1 & 1 & 1 \end{bmatrix}$$

Take into account, if necessary, the following: $\lambda_1 = 0.4$, $\lambda_2 = 0.4$, and $\lambda_3 = 0.2$.

Problem 5.12. Utilize the *third technique* for analyzing <*X*, *R*> models to solve the problem which includes the fuzzy nonstrict preference relations given in Problem 5.11. If necessary, apply $\lambda_1 = 1$, $\lambda_2 = 2$, and $\lambda_3 = 1$.

Problem 5.13. Apply the *second technique* for analyzing <*X*, *R*> models to solve the problem which includes the fuzzy nonstrict preference relations given in Problem 5.11 with the following order of their importance:

Problem 5.14. Solve the Utilize the third technique for analyzing <*X*, *R*> models to solve the problem which includes the fuzzy nonstrict preference relations given in Problem 5.11. If necessary, apply $\lambda_1 = 1$, $\lambda_2 = 2$, and $\lambda_3 = 1$.

Problem 5.15. Solve the problem formulated in Example 5.26 if the alternatives have the following estimates:

$F_1(X_1) = middle$, $F_1(X_2) = middle$, and $F_1(X_3) = large$
$F_2(X_1) = small$, $F_2(X_2) = middle$, and $F_2(X_3) = large$
$F_3(X_1) = middle$, $F_3(X_2) = middle$, and $F_3(X_3) = large$

Problem 5.16. Verify the possibility of chainging the solution (alternative X_1) of the problem defined by Example 5.26, if the information related to the importance of criteria is presented in the following form:

$$\Lambda = \begin{bmatrix} 1 & 0.5 & 0.2 \\ 0.8 & 1 & 0.1 \\ 0.7 & 0.8 & 1 \end{bmatrix}$$

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