Mixed Non-Bayesian and DeGroot Learning

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1 Introduction

[2] [1]

2 The Model

2.1 The Network

(Mixed Non-Bayesian and DeGroot nodes)

2.2 States and Signals

We will consider the state of the world to be a probability distribution $p^*(t) \in \mathbb{P}$. $\mathbb{P}$ is a pre-defined class of distributions where each $p^*(t) \in \mathbb{P}$ is distinguishable by a single parameter. This single parameter m^* where $\{m^* \in \mathbb{R} : m_{min} <= m^* <= m_{max}\}$ will be the state which the nodes in the network are attempting to learn. That is, m^* is the "true" state of the world. *A priori*, the nodes know the class of distribution but not the specific value of m^*; only that it lies uniformly in the range $[m_{min}, m_{max}]$.

Our model allows for $\mathbb{P}$ to be any class of probability distribution as long as it can be completely specified by a single parameter m^*. Examples include the Poisson distribution with parameter m^* and Gaussian distribution with mean m^* and constant variance. We also introduce a trapezoidal probability distribution in section 2.2.1 which serves as a valid choice for $\mathbb{P}$.

At each timestep t, every Non-Bayesian node i will receive a signal $s_{i,t}$ which is a value drawn from $p^*(x)$. These observations will be used by the nodes in the network to construct a belief about m^*.

2.2.1 The Trapezoidal Distribution

As an example of a class of distributions which can be used in our model we present the following trapezoidal distribution.

$$p^*(x) = \begin{cases} m^* \cdot x + \frac{1}{2}(2 - m^*) & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

The slope $\{m^* \in \mathbb{R} : m_{min} = -2 \leq m^* \leq 2 = m_{max}\}$ will be the defining characteristic of the distribution. We have chosen to bound m^* between -2 and 2 and add the term $\frac{1}{2}(2-m^*)$ so that the support of $p^*(x)$ is always over $[0, 1]$.

2.3 Representing Belief States

For a DeGroot node i, we represent the belief state of i at time t as a single value $m_{i,t} \in \mathbb{R}$. However, belief of Non-Bayesian nodes is a probability distribution over a finite set of possible states of the world. There is thus a tension between these two schemes and it is necessary to develop a conversion between the belief states of Non-Bayesian and DeGroot nodes.

2.3.1 Belief in Non-Bayesian Nodes

To represent a belief about the continuous slope value m^* for a Non-Bayesian node we discretize m in the following manner. Let $\Theta = \{\theta_0, \theta_1, ...\theta_n\}, n < \infty$ be the discrete set of possible states of the world over which each Non-Bayesian node holds a distribution of belief. Each θ_k corresponds to the belief that the slope of $p^*(x)$ is equal to m_k where

$$\hat{m_k} = \frac{(m_{max} - m_{min})k}{n - 1} + m_{min} \tag{2}$$

As an example, if we take $n = 21$ belief states, $m_{min} = -2$, and $m_{max} = 2$, then we have

$$\begin{cases} \theta_0 : m^* = -2.0 \\ \theta_1 : m^* = -1.9 \\ ... \\ \theta_{20} : m^* = 2.0 \end{cases}$$

For each Non-Bayesian node i, the belief at time t that θ_k is the true state of the world is denoted by $\mu_{i,t}(\theta_k)$. Thus $\{\mu_{i,t}(\theta_1), \mu_{i,t}(\theta_2), ...\mu_{i,t}(\theta_n)\}$ is a probability distribution over the set of world states for fixed i, t.

2.4 Learning in DeGroot Nodes

DeGroot nodes in our model do not directly receive signals; they act merely as message-passers in the network. For the most part our DeGroot nodes behave in a fashion similarly to nodes in a standard DeGroot-style network. That is, a DeGroot node's i's belief of the true slope value m^* is given at time $t+1$ as

$$m_{i,t+1} = \sum_{j \in N(i)} a_{ij} m_{j,t} \tag{3}$$

where $N(i)$ denotes the neighbors of i. a_{ij} represents the level of "trust" that a DeGroot node i has in neighbor j, such that

$$\sum_{j \in N(i)} a_{ij} = 1$$

for all DeGroot nodes i. Thus the DeGroot update in equation 3 is simply a weighted average of its neighbors' beliefs in the previous timestep.

When a DeGroot node with a Non-Bayesian neighbor performs an update it must obtain a value $m'_{j,t}$ for the belief of this neighbor. This is calculated as

$$m'_{j,t} = \sum_{k=1}^{n} \mu_{j,t}(\theta_k)\hat{m}_k \tag{4}$$

Here we are taking a weighted average of the $\hat{m}_k$ slope values represented by each state θ_k as defined in equation 2, where the weights are given by j's level of belief in each state.

2.5 Learning in Non-Bayesian Nodes

The update of a Non-Bayesian node in our model is handled similarly to the method laid out in [2]. We denote the belief of a node i that it will receive the signal s_i at time t as $m_{i,t}(s_i)$, defined as follows:

$$m_{i,t}(s_i) = \int_{\Theta} \ell_i(s_i|\theta) d\mu_{i,t}(\theta) = \sum_{k=1}^{n} \ell_i(s_i|\theta_k)\mu_{i,t}(\theta_k) \tag{5}$$

The likelihood function $\ell_i(s_i|\theta_k)$ can be obtained from the probability distribution function represented by the belief state θ_k. As an example, for the trapezoidal distribution described in section 2.2.1, the likelihood function would be

$$\ell_i(s_i|\theta_k) \propto \hat{m_k} s_i + \frac{1}{2}(2 - \hat{m_k}) \tag{6}$$

where $\hat{m_k}$ is defined as in equation 2.

The update of a node's belief in each state θ_k for a given time period t is then given by

$$\mu_{i,t+1}(\theta_k) = a_{ii}\mu_{i,t}(\theta_k)\frac{\ell_i(\omega_{i,t+1}|\theta_k)}{m_{i,t}(\omega_{i,t+1})} + \sum_{j\in N(i)} a_{ij}\mu_{j,t}(\theta_k) \tag{7}$$

Here the first term is the Bayesian update of the belief $\mu_{i,t}(\theta_k)$ after observing the signal $\omega_{i,t+1}$, multiplied by the node's self reliance a_{ii}. The summation is the linear incorporation of the beliefs of i's neighbors.

This update differs from the previous work in two major ways. Firstly, we provide each Non-Bayesian node with only a single signal per timestep, where this signal is a single draw from the probability distribution $p^*(x)$.

Secondly, we must define $\mu_{j,t}(\theta_k)$ when j is a DeGroot node. We consider two methods for doing so, defined in sections 2.5.1 and 2.5.2 respectively.

2.5.1 Belief Distribution from Draw of Probability Distribution

We require a method for converting a DeGroot node j's belief $m_{j,t} \in \mathbb{R}$ to a series of priors which can be included in a Non-Bayesian node i's belief update. The first method we consider involves generating a probability distribution $p_{j,t}$ from j's belief at time t and drawing from this distribution. The generated distribution is the $p_{j,t} \in \mathbb{P}$ where j's belief $m_{j,t}$ is the single parameter defining the distribution. If $\mathbb{P}$ is the trapezoidal distribution defined in section 2.2.1, then for each DeGroot neighbor of i at time t we have

$$p_{j,t}(x) = \begin{cases} m_{j,t}x + \frac{1}{2}(2 - m_{j,t}) & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases} \tag{8}$$

This is simply the distribution which j believes to be the state of the world. We then draw a single signal $s'_{j,t}$ from this distribution. We create a belief distribution $\mu'_{j,t}$ from $s'_{j,t}$ by performing a Bayesian-style update with equal priors.

$$\mu'_{j,t}(\theta_k) = \frac{\ell_i(s'_{j,t}|\theta_k)}{\sum_{i=1}^{n} \ell_i(s'_{j,t}|\theta_i)} \tag{9}$$

This generated belief distribution $\mu'_{j,t}$ is then used for the values of $\mu_{j,t}(\theta_k)$ for the DeGroot node j in the Non-Bayesian update of equation 7.

2.5.2 Belief Distribution from Likelihood Distance Metric

The second method we consider for creating a probability distribution over the set of belief states Θ is to define an exponential distance metric such that

$$\mu'_{j,t}(\theta_k) \propto e^{-(m_{j,t}-\hat{m_k})^2} \tag{10}$$

where $\mu'_{j,t}$ is the belief distribution which is used for the DeGroot node j's values of $\mu_{j,t}(\theta_k)$ in the Non-Bayesian update of equation 7.

Since $\mu'_{j,t}$ must be a probability distribution we must normalize the values for each state with the value

$$q_{j,t} = \sum_{k=1}^{n} e^{-(m_{j,t}-\hat{m_k})^2} \tag{11}$$

Thus the calculation to create the DeGroot node's level of belief in each state θ_k is given by

$$\mu'_{j,t}(\theta_k) = e^{-(m_{j,t}-\hat{m_i})^2}/q_{j,t} \tag{12}$$

2.6 Convergence

In this section we discuss theoretical results concerning convergence. Here we define convergence as consisting of two parts: convergence for Non-Bayesian nodes and convergence for DeGroot nodes. For a Non-Bayesian node b, we consider b to have converged to a belief state θ_k if $\mu_{b,t}(\theta_k) \to 1$ as $t \to \infty$. We consider a DeGroot node d to have converged to the belief value m if $m_{d,t} \to m$ as $t \to \infty$. If these conditions hold for every Non-Bayesian and DeGroot node in the network, we consider the network to have converged.

We will demonstrate that convergence occurs for a very simple instance of our model, and then discuss difficulties encountered when attempting to extend these results.

2.6.1 Convergence in a Simple Case

Consider a network consisting of a Non-Bayesian node b and a DeGroot node d where b and d are connected and d has a nonzero prior on the true state. Suppose that a_{dd} and a_{bd} are strictly positive but that $a_{db} = 0$. That is, the DeGroot node listens to the Non-Bayesian node but the Non-Bayesian does not consider the belief of the DeGroot node.

The update calculation for the Non-Bayesian node will then be

$$\mu_{b,t+1}(\theta_k) = a_{bb}\mu_{b,t}(\theta_k)\frac{\ell_i(\omega_{b,t+1}|\theta_k)}{m_{b,t}(\omega_{b,t+1})} \tag{13}$$

This is simply a Bayesian update based on b's signal received at every timestep, so follows that b's belief will converge to the correct value. That is, $\mu_{b,t}(\theta^*) \to 1$ as $t \to \infty$ where θ^* is the correct belief state.

The DeGroot node d's update calculation will be

$$m_{d,t+1} = a_{db} \sum_{i=1}^{n} \mu_{d,t}(\theta_i) \hat{m}_k \tag{14}$$

Once b has converged to the correct state this will reduce to

$$m_{d,t+1} = a_{db} \hat{m_{\theta^*}} = m^* \tag{15}$$

Thus we observe that the network consisting of a DeGroot node listening to a Non-Bayesian node will converge to the correct belief.

2.6.2 Difficulties in Extending Convergence Results

Consider the simple network described in the previous section with the modification that a_{bd} is strictly positive, so that the Non-Bayesian node takes into account the belief of the DeGroot node. We then have

$$\mu_{b,t+1} = a_{bb} \mu_{b,t}(\theta_w) \frac{\ell_i(\omega_{b,t+1}|\theta_k)}{m_{b,t}(\omega_{b,t+1})} + a_{bd} \mu'_{d,t}(\theta_w) \tag{16}$$

where the term $\mu'_{d,t}(\theta_w)$ depends on the method used to convert DeGroot to Non-Bayesian belief. The methods we consider in sections 2.5.1 and 2.5.2 both inject a level of noise into the network during each belief update through this term.

Noise from section 2.5.1's method results from the fact that we are performing a random draw from a probability distribution $p_{j,t}$ created from the parameter $m_{j,t}$ which may be different from the true value m^*. Even if many draws are performs and averaged so that the "signal-to-noise" ratio is increased, the inclusion of data from a potentially incorrect probability distribution will still inject error into the network.

The likelihood metric method discussed in section 2.5.2 also introduces error. Even if the DeGroot node d's belief $m_{d,t}$ has converged to the correct value m^*, we still have

$$\mu'_{d,t}(\theta_w) \propto e^{-(m_{d,t} - \hat{m_w})^2} > 0 \tag{17}$$

for each incorrect state $\{\theta_w \in \Theta : \theta_w \neq \theta^*\}$. This will result in a small amount of noise added to each update of d's neighbors' beliefs.

We have observed in simulation that the noise from the DeGroot to Non-Bayesian conversion can overcome the rate of learning from the signal observations and thus prevent convergence of the network. However, it is difficult to determine exactly what conditions must hold for this to be the case. The level of impact from the conversion noise will depend on many factors including network structure, trust values between nodes, and the number of states $\theta_i \in \Theta$ used to quantize possible values of m^*. We have therefore found it difficult to prove convergence results with any significant level of generality.

References

[1] Morris H. DeGroot. Reaching a consensus. *Journal of American Statistical Association*, pages 118–121, 1974.

[2] Ali Jadbabaie, Pooya Molavi, Alvaro Sandroni, and Alireza Tahbaz-Salehi. Non-bayesian social learning. *Games and Economic Behavior*, pages 210–225, 2012.