Mixed Non-Bayesian and DeGroot Learning

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1 Introduction

[2] [1]

2 The Model

2.1 The Network

(Mixed Non-Bayesian and DeGroot nodes)

2.2 States and Signals

We will represent the state of the world as a probability distribution:

$$p^*(x) = \begin{cases} m^* \cdot x + \frac{1}{2}(2 - m^*) & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

The slope $\{m^* \in \mathbf{R} : -2 \leq m^* \leq 2\}$ will be the defining characteristic of the distribution. We have chosen to bound m^* between -2 and 2 and add the term $\frac{1}{2}(2 - m^*)$ so that the support of $p^*(x)$ is always over $[0, 1]$. The value of the parameter m^* is the "true" state that each node is trying to learn.

At each timestep t, every Non-Bayesian node i will receive a signal $s_{i,t}$ which is a value drawn from $p^*(x)$. These observations will be used by the nodes in the network to construct a belief about m^*.

2.3 Representing Belief States

For a DeGroot node i, we represent the belief state of i at time t as a single value $m_{i,t} \in \mathbf{R}$. However, belief of Non-Bayesian nodes is a probability distribution over a finite set of possible states of the world. There is thus a tension between these two schemes and it is necessary to develop a conversion between the belief states of Non-Bayesian and DeGroot nodes.

2.3.1 Belief in Non-Bayesian Nodes

To represent a belief about the continuous slope value m^* for a Non-Bayesian node we discretize m in the following manner. Let $\Theta = \{\theta_0, \theta_1, ...\theta_n\}, n < \infty$ be the discrete set of possible states of the world over which each Non-Bayesian node holds a distribution of belief. Each θ_k corresponds to the belief that the slope of $p^*(x)$ is equal to m_k where

$$\hat{m_k} = \frac{4k}{n-k} - 2 \tag{2}$$

As an example, if we take n = 21 belief states, then we have

$$\begin{cases} \theta_0 : m^* = -2.0 \\ \theta_1 : m^* = -1.9 \\ ... \\ \theta_{20} : m^* = 2.0 \end{cases}$$

For each Non-Bayesian node i, the belief at time t that θ_k is the true state of the world is denoted by $\mu_{i,t}(\theta_k)$. Thus $\{\mu_{i,t}(\theta_1), \mu_{i,t}(\theta_2), ...\mu_{i,t}(\theta_n)\}$ is a probability distribution over the set of world states for fixed i, t.

2.4 Learning in DeGroot Nodes

DeGroot nodes in our model do not directly receive signals; they act merely as message-passers in the network. For the most part our DeGroot nodes behave in a fashion similarly to nodes in a standard DeGroot-style network. That is, a DeGroot node's i's belief of the true slope value m^* is given at time $t+1$ as

$$m_{i,t+1} = \sum_{j \in N(i)} a_{ij} m_{j,t} \tag{3}$$

where $N(i)$ denotes the neighbors of i. a_{ij} represents the level of "trust" that a DeGroot node i has in neighbor j, such that

$$\sum_{j \in N(i)} a_{ij} = 1$$

for all DeGroot nodes i. Thus the DeGroot update in equation 3 is simply a weighted average of its neighbors' beliefs in the previous timestep.

When a DeGroot node with a Non-Bayesian neighbor performs an update it must obtain a value $m'_{j,t}$ for the belief of this neighbor. This is calculated as

$$m'_{j,t} = \sum_{k=1}^{n} \mu_{j,t}(\theta_k)\hat{m}_k \tag{4}$$

Here we are taking a weighted average of the $\hat{m}_k$ slope values represented by each state θ_k as defined in equation 2, where the weights are given by j's level of belief in each state.

2.5 Learning in Non-Bayesian Nodes

The update of a Non-Bayesian node in our model is handled similarly to the method laid out in [2]. We denote the belief of a node i that it will receive the signal s_i at time t as $m_{i,t}(s_i)$, defined as follows:

$$m_{i,t}(s_i) = \int_{\Theta} \ell_i(s_i|\theta)d\mu_{i,t}(\theta) = \sum_{k=1}^{n} \ell_i(s_i|\theta_k)\mu_{i,t}(\theta_k) \tag{5}$$

In this case the likelihood function $\ell_i(s_i|\theta_k)$ can be obtained from the probability distribution function represented by the belief state θ_k.

$$\ell_i(s_i|\theta_k) = \hat{m}_k s_i + \frac{1}{2}(2 - \hat{m}_k) \tag{6}$$

where $\hat{m}_k$ is defined as in equation 2.

The update of a node's belief in each state θ_k for a given time period t is then given by

$$\mu_{i,t+1}(\theta_k) = a_{ii}\mu_{i,t}(\theta_k)\frac{\ell_i(\omega_{i,t+1}|\theta_k)}{m_{i,t}(\omega_{i,t+1})} + \sum_{j \in N(i)} a_{ij}\mu_{j,t}(\theta_k) \tag{7}$$

Here the first term is the Bayesian update of the belief $\mu_{i,t}(\theta_k)$ after observing the signal $\omega_{i,t+1}$, multiplied by the node's self reliance a_{ii}. The summation is the linear incorporation of the beliefs of i's neighbors.

This update differs from the previous work in two major ways. Firstly, we provide each Non-Bayesian node with only a single signal per timestep, where this signal is a single draw from the probability distribution $p^*(x)$.

Secondly, we must define $\mu_{j,t}(\theta_k)$ when j is a DeGroot node. We consider two methods for doing so, defined in sections 2.5.1 and 2.5.2 respectively.

2.5.1 Belief Distribution from Draw of Probability Distribution

We require a method for converting a DeGroot node j's belief $m_{j,t} \in \mathbf{R}$ to a series of priors which can be included in a Non-Bayesian node i's belief update. The first method we consider involves generating a probability distribution from the j's current belief and drawing from this distribution. For each DeGroot neighbor of i at time t we define the probability distribution

$$p_{j,t}(x) = \begin{cases} m_{j,t}x + \frac{1}{2}(2 - m_{j,t}) & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases} \tag{8}$$

This is simply the distribution which j believes to be the state of the world. We then draw a single signal $s'_{j,t}$ from this distribution. We create a belief distribution $\mu'_{j,t}$ from $s'_{j,t}$ by performing a Bayesian-style update with equal priors.

$$\mu'_{j,t}(\theta_k) = \frac{\ell_i(s'_{j,t}|\theta_k)}{\ell_i(s'_{j,t}|\theta_0) + \ell_i(s'_{j,t}|\theta_1) + ... + \ell_i(s'_{j,t}|\theta_n)} \tag{9}$$

This generated belief distribution $\mu'_{j,t}$ is then used for the values of $\mu_{j,t}(\theta_k)$ for the DeGroot node j in the Non-Bayesian update of equation 7.

2.5.2 Belief Distribution from Likelihood Distance Metric

The second method we consider for creating a probability distribution over the set of belief states Θ is to define an exponential distance metric such that

$$\mu'_{j,t}(\theta_k) \propto e^{-(m_{j,t} - \hat{m_k})^2} \tag{10}$$

where $\mu'_{j,t}$ is the belief distribution which is used for the DeGroot node j's values of $\mu_{j,t}(\theta_k)$ in the Non-Bayesian update of equation 7.

Since $\mu'_{j,t}$ must be a probability distribution we must normalize the values for each state with the value

$$q_{j,t} = \sum_{k=1}^{n} e^{-(m_{j,t} - \hat{m_k})^2} \tag{11}$$

Thus the calculation to create the DeGroot node's level of belief in each state θ_k is given by

$$\mu'_{j,t}(\theta_k) = e^{-(m_{j,t}-\hat{m}_i)^2}/q_{j,t} \tag{12}$$

References

[1] Morris H. DeGroot. Reaching a consensus. *Journal of American Statistical Association*, pages 118–121, 1974.

[2] Ali Jadbabaie, Pooya Molavi, Alvaro Sandroni, and Alireza Tahbaz-Salehi. Non-bayesian social learning. *Games and Economic Behavior*, pages 210–225, 2012.