23 BOOTSTRAPPING: LEARNING FROM THE SAMPLE

In the regression models we have discussed throughout this book, we assumed that we have a random sample that is drawn from some population. The usual objective of the analysis is to develop a suitable model (based on some theory), estimate parameters of the model, estimate their standard errors, develop a suitable test statistic (such as *t*, *F*, and χ^2), and test hypotheses about the (population) values of the parameters. In conducting this exercise, we made several assumptions, which may not be tenable in practice.

For example, in the classical linear regression model (CLRM) discussed in Sec. 1.4, we made several assumptions (see Assumptions A-1 to A-7). Given these assumptions, we stated that OLS estimators of the (populations) regression coefficients are *Best Linear Unbiased Estimators* (BLUE). For hypothesis testing, we further assumed that the regression error terms (u_i) are independently and identically normally distributed with zero mean and constant variance, that is, $u_i \sim N(0,\sigma^2)$.

With the normality assumption, we stated that OLS estimators are not only BLUE but are also *Best Unbiased Estimators* (BUE) – they have minimum variance in the entire class of unbiased estimators, linear or not. Under the normality assumption, we also showed that OLS estimators of the regression parameters are also *maximum-likelihood* (ML) estimators.[1] As a result, we also showed that the estimators of the regression coefficients are themselves normally distributed. This assumption enabled us to test hypotheses about the true values of the parameters in the population from which we obtained the sample. Such an approach to statistical analysis is known as **parametric statistical analysis**.

In practice, one or more of the CLRM assumptions may not hold. In several chapters, we discussed the consequences of violating one or more of the classical assumptions and suggested remedial measures. In particular, the assumption that the error terms are normally distributed may be unrealistic in many situations. In such cases, we resorted to large sample, or asymptotic, test procedures. The asymptotic tests, however, do not work well in finite samples, for if the sample is too small, the *central limit theorem* (CLT) may not be of much help.

Even in large samples, the asymptotic sampling distribution of a statistic may be difficult to obtain. For example, in normally distributed populations, the variance of the *median* (a measure of central tendency) is 1.25, which is greater than the variance of the mean. That is, the mean is more efficient than the median. However, in samples from non-normal populations, it is analytically difficult to estimate the variance

1 However, the ML estimator of the error variance is a biased estimator of the true variance, although in large samples the bias tends to be small.

of the median. In cases of heteroscedasticity, the OLS estimators, although unbiased, are not efficient. If the sample is very large, we can use the White-heteroscedasticity corrected standard errors, but that is of little consolation in finite samples. Similarly, if there are outliers in the data, it is not easy to estimate the standard errors of the estimators. In non-linear parameter regression models it is difficult to devise test procedures and confidence intervals in finite samples. The same is true of variables in ratio forms.

For all these reasons, we need an alternative method of statistical inference that relies on less restrictive assumptions. One such alternative is **nonparametric statistical analysis**, also known as **distribution free statistical analysis**.

In nonparametric statistics, we deal with variables without making assumptions about the form of their (probability) distributions.[2] In the regression context, this would mean that we do not necessarily assume that the regression error, u_i, is normally distributed, as in CLRM.

In the remainder of this chapter, we discuss some approaches to nonparametric statistical inference, first applied to a simple example and then to regression models, which are the thrust of this book.

23.1 Nonparametric approach to statistical inference

There are several methods of nonparametric statistical inference, such as the **Randomization Exact Test** (also known as the **Permutation Test**), **Cross-Validation**, and the **Jackknife** (also known as the **Quenouille–Tuckey Jackknife**).[3] The method that has received wide currency, and which is often used in econometrics, is the method of **bootstrap sampling**.[4] In the rest of this chapter, we will discuss bootstrapping in some detail. At the outset, it may be stated that the topic of bootstrapping is vast and evolving. Keeping the practical nature of this book, the discussion is kept to a bare minimum, and is often heuristic, but references are provided for more advanced discussion of this subject.

The idea behind the bootstrap is quite simple. Treat a randomly drawn sample as the population itself and draw repeated samples from this "population", equal in size to the given sample, *but with the proviso that every time we draw a member from this sample we put it back before drawing another member from this sample*. This procedure, called **sampling from within a sample**, or "recycling", or "regurgitating", if you will, is called bootstrap sampling. The commonly used expression, "pull yourself up

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2 For a discussion of parametric and nonparametric statistics, see Sheskin, D. J., *Handbook of Parametric and Nonparametric Statistical Procedures*, CRC Press, Boca Raton, 1997.

3 For an informal discussion of these tests, see Mooney, C. Z. and Duval, R. D., *Bootstrapping: A Nonparametric Approach to Statistical Inference*, Sage University Paper, 1993. For an advanced discussion, see Cameron, A. C. and Trivedi, P. K., *Microeconometrics: Methods and Applications*, Chapter 11, Cambridge University Press, New York, 2005; Davidson, R. and MacKinnon, J. G., *Econometric Theory and Methods*, Chapter 5, Oxford University Press, New York, 2004. See also Vinod H. D., Bootstrap methods: applications in econometrics, in G. S. Maddala, C. R. Rao, and H. D. Vinod (eds.), *Handbook of Statistics*, Vol. 11, Elsevier Science, Amsterdam, 1993, pp. 629–61. For an hands-on discussion of bootstrapping using the *R* language, see Vinod, H. D., *Hands-on Intermediate Econometrics using R*: Templates for Extending Dozens of Practical Examples, World Scientific Publishing, Singapore, 2008. This is an excellent book on various econometric techniques used in various fields.

4 The term was originally coined by Effron: see Effron, B. (1979) Bootstrap methods: another look at the jackknife, *Annals of Statistics*, 7, 1–26. See also Efron, B. and Tibshirani, R. J., *An Introduction to Bootstrap*, Chapman and Hall, New York, 1993.

by your own bootstraps" – that is, rely on your own resources (i.e. a given sample) – is the essence of bootstrap sampling methods.[5]

Bootstrapping can be used to learn about the *sampling distribution of a statistic* (e.g. the sample mean) without imposing external assumptions, as in the case of CLRM.[6] If it is not possible to develop sampling distributions of test statistics, it is difficult to draw statistical inferences (i.e. test hypotheses). In the sections that follow, we will explain how we do this. Bootstrapping is of general application in that we can even obtain the sampling distribution of the standard error itself.

Another reason for bootstrapping is well stated by Draper and Smith:

> Via resampling [i.e. bootstrapping], we can reexamine a regression analysis already made, by comparing it with a population of results that might have been obtained under certain assumed circumstances.[7]

If there are substantial differences in the results based on, say, the classical linear model and those based on the bootstrap methodology, this would suggest that perhaps one or more assumptions of the CLMR are violated. In short, bootstrap can be used as a form of *sensitivity analysis*, or what Gifi calls "stability under data selection".[8]

There are two types of bootstrapping – parametric and nonparametric.

In **parametric bootstrapping**, the probability distribution of the underlying population is assumed known. In this case, we can establish confidence intervals in the traditional manner, except that we use the bootstrapped standard errors (to be explained shortly) to compute this interval.

In the case of **nonparametric bootstrapping**, we do not know the distribution from which the (original) sample was obtained. It is from the given sample, with repeated bootstrap samples, that we try to estimate the shape of the true distribution by considering some key features, such as the mean and standard deviation values of that distribution. In practice, this is more interesting than parametric bootstrapping, for very often we do not know the true population, also known as the **data generating process** (**DGP**), which generated the sample at hand. We will show this with a concrete example after we discuss parametric bootstrapping first.

23.2 A simple example of parametric bootstrapping: the distribution of the sample mean from a normal population

Suppose we draw a random sample of 25 observations from the standard normal distribution $N(0,1)$ and compute the mean value from this sample. From statistical theory, we know that the sample mean is also distributed normally with mean zero and variance equal to $1/n$, where n is the sample size. Symbolically, if

$$X_i \sim N(0,1) \tag{23.1}$$

then

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5 It seems the phrase is based on the famous story *Adventures of Baron Munchausen* by Rudolph Eric Raspe. According to the story, the Baron had fallen into a swamp and saved himself by pulling himself up by his hair. See Chong, F. *et al.* (2011) *Proceedings of Singapore Healthcare*, **20**(3).

6 Sampling distribution describes the variation in the values of a statistic over all possible samples.

7 Draper, N. and Smith, H., *Applied Regression Analysis*, 3rd edn, Wiley, New York, 1998, p. 585.

8 See Gifi, A., *Nonlinear Multivariate Analysis*, Wiley, New York, 1990, Section 1.5.3.

$$\bar{X} \sim N(0,1/n) \tag{23.2}$$[9]

How do we establish this empirically? One way is to draw several samples, each of 25 observations from N (0, 1), and compute the sample mean for each sample thus drawn. How many samples do we draw? In practice, we can draw say 500 samples and thus compute 500 sample means and compute the variance of these 500 means. We can also draw an (empirical) histogram of these sample means and see if it resembles the picture of a normal distribution. The variance of these sample means is computed as follows:

$$\text{var}(\bar{X}) = S_{\bar{X}}^2 = \sum_{b=1}^{b=m} [\bar{X}^{(b)} - \bar{\bar{X}}]^2 / (m-1) \tag{23.3}$$

where m is the number of samples, $\bar{X}^{(b)}$ is the sample mean from the bth sample, $\bar{\bar{X}}$ is the *grand mean*, that is, the mean value of all the sample means and $S_{\bar{X}}^2$ is the (sample) variance of the sample means. *The square root of Eq. (23.3) gives the standard error*, which is simply the (sample) standard deviation. This method of drawing samples is known as **Monte Carlo simulation**.

Instead of drawing 500 independent samples from the standard normal distribution, suppose we draw just one sample of 25 observations and from this single sample we draw, say, 500 samples, each of 25 observations, *with the proviso* that when we draw a member of this sample, we put that member back into the sample before we draw another sample. This is called **sampling with replacement**. This is the essence of bootstrap sampling. In the literature, the terms "bootstrapping" and "sampling with replacement" are sometimes used interchangeably. For brevity of discussion, we will call sampling with replacement simply **resampling**, it being understood that we are drawing repeated samples from the original sample.

Since bootstrap involves sampling with replacement, the bootstrapped sample will contain the same observation more than once and some observations may not be repeated at all. Also, note that bootstrapping is a computer-intensive procedure, since we will have to draw several resamples from a given sample.

There is no definitive rule about the number of bootstrap samples to be drawn. Efron and Tibshirani state that for standard error estimation 50 bootstrap samples are enough to give a good estimate, but seldom more than 200 replications.[10] In practice, it is common to find bootstrap samples as high as 2,000 or more.

To illustrate this, we drew a random sample of 25 observations from the standard normal distribution, as shown in the first column of Table 23.1 (given in the appendix to this chapter). Columns 2 to 26 of this table show 25 bootstrap samples, each of 25 observations, drawn from the first column of (original) observations, noting that this is *sampling with replacement*. In practice, we draw several thousand (bootstrap) samples. However, to keep the discussion simple, for now we only consider 25 bootstrap samples. The bootstrap samples are numbered B1, B2, and B25.

At the bottom of this table, for each sample (including the original), we give its (sample) mean value, $\bar{X}$, the (sample) standard deviation S_x and the (sample) standard error of the mean, which is $S_x / \sqrt{n}$, where n = 25 (the sample size).

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9 In general, if $X_i \sim N(\mu,\sigma^2)$, then $\bar{X} \sim N(\mu,\sigma^2/n)$, where μ, σ^2, and n are the mean, variance, and the sample size respectively.

10 See Efron and Tibshirani, *op cit.*, p. 52. See also Poi, B. P. (2004) Some bootstrapping techniques. *Stata Journal*, **4**, 312–28.

If you examine Table 23.1, there are quite a few repeated numbers, which should not be surprising because the sampling is done with replacement. That is why the sample means and sample standard deviations of the individual resamples are different.

To get some idea about the distribution of the bootstrap sample means, in Figure 23.1 we give the histogram of the 25 bootstrap sample means.

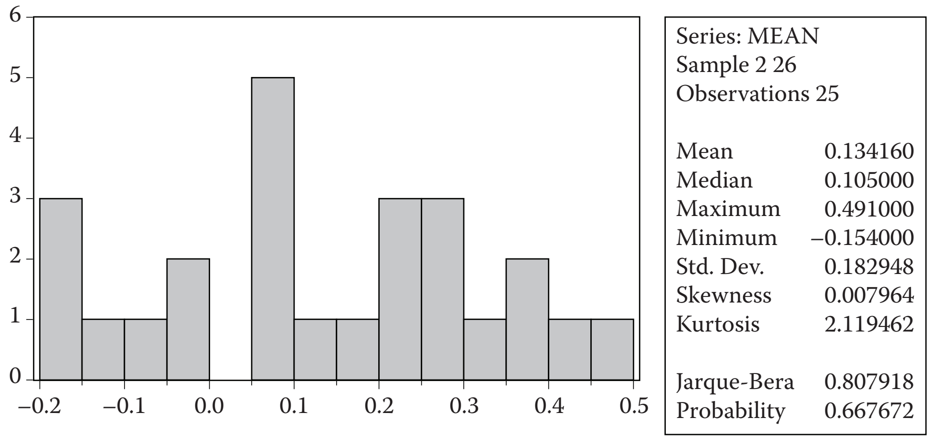


Figure 23.1 Histogram of 25 bootstrap means.

This histogram does not look quite the picture of a normally distributed variable, even though the Jarque–Bera (JB) statistic does not reject the normality assumption. Of course, as noted in Chapter 7, the JB statistic is a large sample statistic, which is not the case here.[11]

However, if we extend the analysis to 2,000 bootstrap sample means, we obtain the picture shown in Figure 23.2. If you sketch a curve through this histogram, it will resemble the normal curve, the approximation being better the larger the sample size and higher the number of bootstrap samples we draw. We can call the histogram in this figure the **empirical frequency distribution** (EFD) of the sample means. From this EFD we can obtain the **cumulative empirical distribution function** (**CEDF**), the counterpart of the theoretical **cumulative distribution function** (**CDF**) (see the statistical appendix in the text).

The mean of the 2000 sample means is not centered on zero (as in theory), although it is close to it. *But it should be noted that the bootstrap estimate of the population parameter need not be unbiased*. We will visit this topic later. The theoretical variance of the sample means as given by Eq. (23.2) is 1/5 and therefore the standard deviation is $1/\sqrt{n} = 1/\sqrt{25} = 0.2$.

23.3 Nonparametric bootstrapping

The bootstrap procedure that we have discussed is called **parametric bootstrapping**, for we know the theoretical distribution of the underlying population (normal in our case). However, in many applications we may not know the particular distribution

11 For a finite sample distribution of the JB statistic, see Godfrey, L., *Bootstrap Tests for Regression Models*, Palgrave-Macmillan, 2009, pp. 83–8.

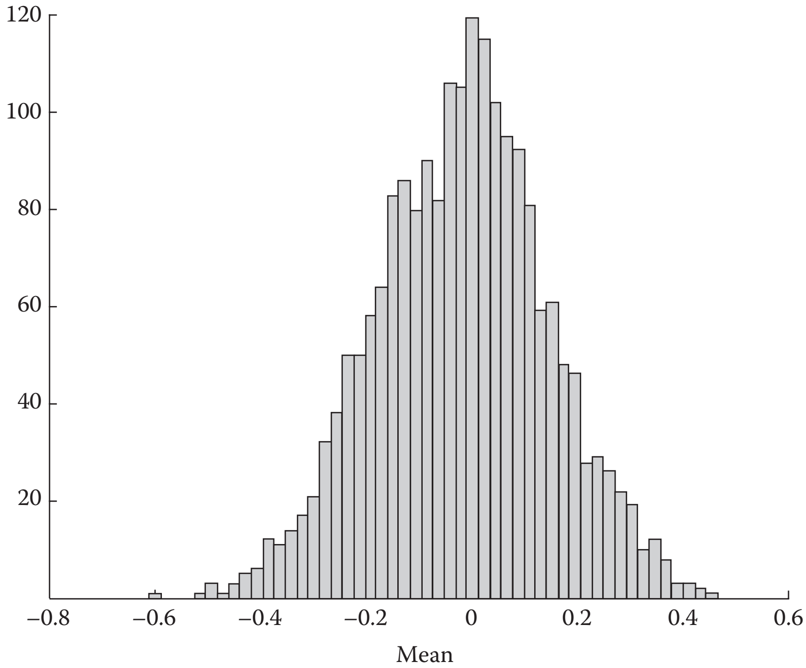


Figure 23.2 Histogram of 2000 means drawn from *N*(0,1).

from which we obtained the sample. In this situation, which is more likely in practice, bootstrapping may help us to obtain the **empirical frequency distribution** (**EDF**) of the statistic of interest, say, the sample mean. From the empirical distribution, we can estimate the standard error (i.e. standard deviation) of the statistic of interest.[12]

This approach is called **non-parametric bootstrapping**, because we do not make any assumptions regarding the population underlying the sample, such as the normal, *t*, *F*, and chi-square distributions. On the contrary, we let the sample lead us to the underlying population, which we obtain by intensive bootstrap sampling.

How do we engage in statistical inference – that is, testing hypotheses – based on the bootstrap EDF? There are *four approaches*, which we discuss sequentially.[13]

1. Confidence intervals based on the normal approximation

If it is assumed that a statistic of interest is normally distributed, but it is difficult to find an analytical method to estimate its standard error, we can use the standard error obtained from the EDF based on the bootstrap samples, as shown in Eq. (23.3). Thus, the (1 – α) confidence interval for the population mean will be the traditional confidence interval formula *but with the standard error computed from the bootstrapped EDF*.

$$\Pr(\bar{X} - z_{\alpha/2} S_{\bar{X}} < \mu_x < \bar{X} + z_{\alpha/2} S_{\bar{X}}) = (1-\alpha) \tag{23.4}$$

12 A theorem, known as Glivenko-Cantelli theorem, asserts that the empirical distribution converges uniformly with probability one to its theoretical distribution if the observations are independent and identically distributed. See Chung, K. L., *A Course in Probability Theory*, 2nd edn, Academic Press, New York, 1974, p. 133.

13 For details, see Mooney and Duval, *op cit.*, pp. 34–40 and Cameron and Trivedi, *op cit.*, pp. 415–43.

where $S_{\bar{X}}$ is obtained from Eq. (23.3), α is the level of significance or a Type I error, and Z is the standard normal variate.

For the data in Table 23.1, the grand mean of 25 bootstrap means is 0.1342, $S_{\bar{X}}$ = 0.1829 and the Z value at 5% level is 1.96 (Check this out). Therefore, the 95 normal confidence for μ_x is [0.1342 ± 1.96(0.1829)] = (–0.2243, 0.4927).

Note that we have used the standard error of the mean based on bootstrap samples.

The mean value of the original sample in Table 23.1 is 0.152. Since we treat the original as population, we can treat this value as μ_x. This value lies in the confidence interval just established.

For the sample mean values shown in Figure 23.2, the 95% confidence interval is 0 ± 1.96(0.2), that is, (–0.392, 0.392) or about (–0.4, 0.4). From this figure, it is clear that the bulk of the 2,000 sample means lies in this range.

The main problem with this normal approximation method is that it relies heavily on a strong parametric assumption. Besides, this method does not take advantage of the entire EDF; it only considers the second moment of the distributions (i.e. the variance). However, in some situations, especially if the sample is quite large, Eq. (23.4) may be justified.

2. Confidence intervals based on bootstrap percentiles[14]

We continue with our example of the sample mean discussed earlier. Suppose we have obtained, say, 1,000 bootstrap sample means from the original sample, as in Column 1 of Table 23.1. We order the bootstrap sample means from the smallest to the largest as follows:

$$\bar{X}^{b1} \leq \bar{X}^{b2} \leq \ldots \leq \bar{X}^{B} \tag{23.5}$$

where B is the total number of bootstraps.

For our numerical example, the means ordered in ascending order are given in Table 23.2.

Table 23.2 Twenty-five bootstrap mean values from the smallest to the largest.

–0.155	–0.154	–0.153	–0.102	–0.056
–0.02	–0.013	0.052	0.054	0.093
0.098	0.099	0.105	0.166	0.202
0.213	0.224	0.254	0.260	0.269
0.309	0.351	0.356	0.41	0.491

An easy way to compute confidence interval for the population mean μ_x is to compute the percentiles of the sample means.

To illustrate this method, we examine the 25 bootstrapped sample means given in Table 23.2. Although, this sample is much too small, the percentile confidence intervals based on this sample will illustrate the mechanics of this method.

The selected percentiles for our example are given in Table 23.3.

14 For details, see Fox, J., *App[lied Regression Analysis and Generalized Linear Models*, Chapter 11, Sage Publications, 2008.

Table 23.3 Percentiles of 25 bootstrapped sample means.

Obs	Percentile	Centile	[95% Conf. Interval]	
	5	−.1547	−.155	−.1205467*
	10	−.1534	−.155	−.0433598*
	15	−.1323	−.1548972	.0520535
	20	−.0928	−.1541012	.0828426
	25	−.038	−.1532834	.0974932
	30	.0376	−.1378061	.1033127
	35	.0579	−.1108248	.167027
	40	.0946	−.0660879	.2068678
	45	.0984	−.0215819	.2208397
	50	.105	.0522083	.2508753
	55	.1768	.0652049	.2603955
	60	.2086	.0952299	.277772
	65	.2229	.0989429	.3222372
	70	.2552	.1221546	.352697
	75	.2645	.1931217	.3715859
	80	.301	.2158649	.4190967
	85	.3515	.2531981	.4827734
	90	.378	.26584	.491*
	95	.467	.33682	.491*
	100	.491	.491	.491*

* Lower (upper) confidence limit held at minimum (maximum) of sample

From this table, we can compute the (percentile) confidence interval at any level of significance that we choose. For instance, the 95th quantile (or percentile) value is 0.467 and the fifth quantile (or percentile) value is –0.1547. Therefore, the interval (–0.1547, 0.467) will give you the 90% confidence interval, and so on. As you can see from the 25 bootstrap mean values, this interval includes 90% of these values.

If, for example, the null hypothesis is that the true population mean value is 0.652, the preceding 90% confidence interval does not include this value. Hence, we can reject this null hypothesis.

In general, if $B = 1000$, and the Type I error is 5% ($\alpha = 0.5$), the lower and upper confidence band for the true mean is given by the quantiles $B\alpha/2$ and $B(1 - \alpha/2)$. Thus, for $B = 1000$, and $\alpha = 0.05$ (or 5%), the lower and upper bounds of 95% confidence interval will be the 25th and the 975th percentiles of the bootstrap (sampling) distribution of the sample mean. Symbolically, we can write this as:

$$[\bar{X}^{lr}_{B\alpha/2}, \bar{X}^{ur}_{B(1-\alpha/2)}] \tag{23.6}$$

where $\bar{X}^{lr}$ will be the 25th (or lower) quantile value and $\bar{X}^{ur}$ will be the 975th (upper) quantile value of the bootstrap distribution of the sample mean. So once the sample means are ordered, it is easy to find out the 25th and 975th quantiles of their distribution, as we showed in Table 23.3. *Stata* and other similar programs can easily compute the quantile values (see Chapter 20 on quantile regression).

The percentile method has these advantages: (1) it avoids making parametric assumptions of the traditional and normal approximation methods; (2) it is simple to implement in that we only calculate the statistic of interest, say, the mean value, and arrange it as in Eq. (23.5) to calculate the appropriate percentile(s); and (3) it is also intuitive, because from EDF based on the bootstrap method we can easily calculate the percentiles (see also footnote 12).

The percentile method is not free from drawbacks. (1) The interval, such as Eq. (23.6), may be biased, especially if the sample is small and or the EDF is skewed, and therefore, a 95% confidence interval, for example, may not in fact be a 95% confidence interval; (2) this method assumes that the EDF is an unbiased estimate of the true PDF (but see footnote 12); and (3) compared with the normal approximation method, the percentile method requires a large number of bootstrap samples, which may not be a problem with high-speed computers.

3. Bias-corrected (BC) and accelerated bias-corrected (ABC) confidence intervals

To correct for the bias that may result if the EDF is not an unbiased estimate of the true PDF, Efron suggested the BC method. The ABC method takes into account bias as well as the skewness of the EDF. We will not discuss the details of this method here, for they are complicated.[15] We will present an example of the BC correction method, using *Stata* (see Table 23.7).

4. The percentile bootstrap-*t* method

To avoid some of the drawbacks of the previous methods, Efron, among others, has developed an equivalent of the classical t test, denoted as t_b^*, which is defined as follows:

$$t_b^* = \frac{\bar{X}_b - \mu_x}{S_{\bar{X}}} \quad \text{for } b = 1, \ldots, B \tag{23.7}$$

In this expression, $\bar{X}_b$ is the sample mean from the bth bootstrap sample, μ_x is the population mean, and $S_{\bar{X}}$ is the standard error of the mean, as computed from Eq. (23.3). Incidentally, note that t and also t_b^* are known as **pivotal statistics**. A pivotal statistic does not depend on any parameters of the underlying PDF; all quantities are sample-based.

Note that we use the t rather than the normal z test because we are estimating the standard error from the sample. This method is simple and has a higher order of accuracy than Efron's percentile method. That is why it is popular in practice. *Since we treat the original sample as the "population", we can treat the mean value of the original sample as* μ_x. For the data in Table 23.1, this value is 0.152. We will use this value to compute t_b^* as shown in Eq. (23.7).

How does this type of bootstrapping help us develop test statistic(s) and establish confidence intervals? We know that the grand mean, the mean of sample means, has variance as shown in Eq. (23.3). The square root of the variance gives us an estimate

15 For a somewhat accessible discussion of this, see Mooney and Duval, *op cit*., pp. 37–40.

of the **bootstrap standard error**. Therefore we can use the equivalent of the classical t test: call it t_b^*; the subscript b reminds us that it is a bootstrapped t value.

Equation (23.7) looks very much like the usual t statistic that we use in parametric statistical inference, *but we cannot use it in the traditional manner* to establish confidence intervals because the usual t test is based on the assumption that the underlying population is Normal. In other words, we need to find the sampling distribution of t_b^* before we can engage in hypothesis testing. For this purpose, we can use the **percentile bootstrap-*t* method**.

In this method, we obtain an *approximate* sampling distribution of t_b^* by arranging the computed t_b^* in an ascending order as shown below

$$t_{b1}^* \le t_{b2}^* \le t_{b3}^* \le \ldots \le t_B^* \tag{23.8}$$

We establish a two-sided equal-tailed confidence interval by finding the lower and upper confidence bounds $B\alpha/2$ and $B(1 - \alpha/2)$ of the ordered elements in Eq. (23.8), respectively, where B is the number of bootstrap samples and α is the level of significance (a Type I error). For example, for $B = 1000$, and $\alpha = 0.05$ (or 5%), these are the 25th and the 975th ordered elements in Eq. (23.6). Therefore, the estimated $(1 - \alpha)$ (here, 95%) confidence interval for the true t is:

$$(t_{B\alpha/2}^*, t_{B(1-\alpha/2)}^*) \tag{23.9}$$

It should be noted that this confidence interval is not necessarily symmetric.

If the hypothesized t value lies outside this confidence interval, we can reject the null hypothesis.

From this we can estimate the 0.025 and 0.975 quantiles, which will give us the middle 95% of the t_b^* values. Letting $l = 0.025B$ and $u = B - l$, estimates of the 0.025 and 0.975 quantiles of t_b^* are t_{l+1}^* and t_u^*. Then the 95% confidence interval for μ_x is

$$\left[\bar{X} - t_u^* \frac{s}{\sqrt{n}}, \bar{X} - t_{l+1}^* \frac{s}{\sqrt{n}}\right] \qquad (23.10)^{16}$$

Note: The lower end of the confidence interval is based on t_u^*, and t_{l+1}^* is used to compute the upper end.

To test the hypothesis that $\mu_x = \mu_0$, we compute the t_b^* value as shown in Eq. (23.7) and reject the null hypothesis if

$$t_b^* \le t_{l+1}^*$$

or if

$$t_b^* \ge t_u^*$$

This method of nonparametric bootstrap inference is known as the **Bootstrap-*t* method**.

Returning to our example, the percentile values calculated from Eq. (23.7) are shown in Table 23.4.

For this example, the 5% quantile value is –1.5755 and the 95% quantile value is 1.8197. Therefore, the interval (–1.5755, 1.8197) provides a 90% confidence interval for t. We will not reject the null hypothesis if this interval includes the hypothesized t value.

16 On this see, Wilcox, R., *Modern Statistics for the Social and Behavioral Sciences: A Practical Introduction*, CRC Press, New York, 2012, pp. 268–9.

Table 23.4 Percentiles of 50 bootstrap *t* values.

				Binom. Interp. --	
Variable	**Obs**	**Percentile**	**Centile**	**[95% Conf. Interval]**	
var1	25	5	–1.575506	–1.575506	–1.206285*
		10	–1.572225	–1.575506	–.8294271*
		15	–1.319081	–1.575506	–.4489147
		20	–1.000328	–1.575506	–.280576
		25	–.823729	–1.571587	–.197704
		30	–.5202842	–1.385834	–.1701943
		35	–.4169489	–1.11896	.1796989
		40	–.2141062	–.8860298	.3975275
		45	–.197704	–.8062748	.4739179
		50	–.159431	–.448068	.6381366
		55	.2331327	–.3770098	.6901883
		60	.4069982	–.2098014	.785194
		65	.4851828	–.197704	1.028306
		70	.661782	–.0656387	1.194844
		75	.7126295	.3223709	1.298118
		80	.9121922	.4467187	1.557882
		85	1.1883	.6508366	1.906032
		90	1.333187	.7199556	1.951011*
		95	1.819792	1.108037	1.951011*
		100	1.951011	1.951011	1.951011*

* Lower (upper) confidence limit held at minimum (maximum) of sample

We also present the histogram of the estimated t_b^* values along with the J-B test of normality: Figure 23.3.

The distribution of t_b^* values seems to be approximately normal. This is also confirmed by the Jarque-Bera statistic. However, keep in mind that our bootstrap sample is not very large.

To summarize, we have four methods of establishing confidence intervals and test hypotheses based on the bootstrap sampling method. Each method has its advantages and disadvantages. However, the following guidelines may be helpful:[17]

1. If the bootstrap distribution (of a statistic) is approximately normal, all these methods should give similar confidence interval as the number of replications becomes large. The **pnorm** command in *Stata* can be used to find out if the bootstrap distribution closely follows a normal distribution.
2. If the statistic is an unbiased estimator of its population value, the percentile and biased-corrected (BC) methods should give similar results.

17 For details, see *Stata Reference Manual for Bootstrapping*, pp. 222–3.

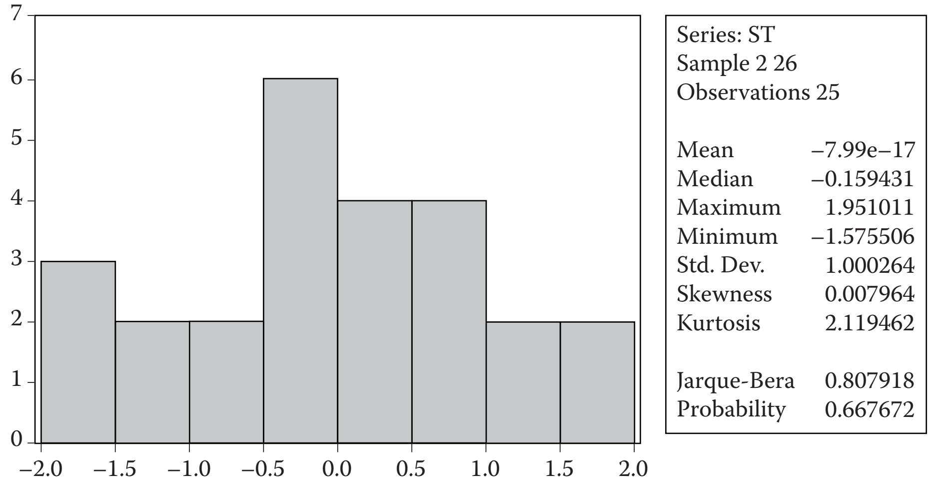


Figure 23.3 Distribution of 25 bootstrapped *t* values.

3 If the observed value of a statistic is equal to the median of the bootstrap distribution, the BC and percentile confidence intervals will be the same.

4 For biased statistics, the BC method should yield confidence intervals with better coverage probability closer to the nominal value of 95% or other specified confidence coefficient.

Some authors question the utility of estimating bootstrapped standard errors. For instance, Jeong and Maddala argue that standard errors are of interest only if the distribution is normal. Otherwise, if one wants to make confidence interval statements and test hypotheses, one should use bootstrap methods directly and skip the standard errors, which are useless.[18] As Richard Berk notes, "it [bootstrapping] cannot be used to undertake statistical inference when the data are a convenience sample or a population".[19]

23.4 A critique of bootstrapping[20]

There is no question that resampling technique has some advantages. (1) It does not rely on theoretical distributions; (2) conceptually it is simple; (3) unlike classical procedures, resampling is valid for random and non-random data, although there is some controversy about this;[21] and (4) bootstrapping treats small samples as the virtual population to generate more observations, although there is no clear-cut guideline about the size of the sample.

---

18 Jeong, J. and Maddala, G. S., A perspective on application of bootstrap methods in econometrics, in G. S. Maddala, C. R. Rao, and H. D. Vinod (eds.), *Handbook of Statistics*, Elsevier, 1993, pp. 573–610. See also Horowitz, J., Bootstrap methods in econometrics: theory and numerical performance, *Working Paper Series* #95-10, Department of Economics, University of Iowa, 1995.

19 Berk, R. A., *Regression Analysis: A Constructive Critique*, Sage Publications, California, 2004, p. 78.

20 For details, see Yu, C. H. (2002) Resampling methods: concepts, applications, and justification. *Practical Assessment, Resarch and Evaulation*, pp. 1–21. http://pareonline.net/getvn.asp?v=8&n=19. The following discussion draws on this paper.

21 See Edgington, E. S., *Randomization Tests*, Marcel Dekker, New York, 1995, and Lunneborg, C. E., *Data Analysis by Resampling: Concepts and Applications*, Duxbury Press, Pacific Grove, California, 2000.

Against this, we should note some skepticism:

1 According to Stephen Fienberg, "you are trying to get something for nothing. You use the same numbers repeatedly until you get an answer that you cannot get any other way. In order to do that, you have to assume something, and you may live to regret that hidden assumption later on".[22] But one could argue that classical statistics probably requires more assumptions than resampling does.
2 Since resampling is based on one sample, generalizations based on this sample may not go beyond the particular sample. But Fan and Wang argue that assessing the stability of test results is descriptive and inferential in nature.[23]
3 Bosch asserts that confidence intervals based on simple bootstrapping are always biased though the bias decreases with sample size.[24]
4 Some critics have argued that when the collected data are biased, resampling would repeat, and possibly magnify, the same mistakes.[25] However, if the data are biased, classical procedures also face the same problem as resampling.

When judging the pros and cons of resampling techniques, the researcher should be aware of the limitations and advantages of bootstrapping.[26]

With this background, we now turn our attention to the use of bootstrapping in regression modeling, which is the main thrust of this book.

23.5 Regression modeling with bootstrapping

We used the data in Table 1.1 to estimate a wage regression. Using the assumptions of the classical normal linear regression model (CNLRM), we obtained the wage regression shown in Table 1.2. We spent several chapters in Part II of this book finding out if the estimated model satisfied one or more assumptions of the CNLRM.

Now suppose the sample data for 1,289 wage earners given in Table 1.1 are drawn randomly from a population whose probability distribution is not known to us *a priori*. To put it differently, we assume that the sample of 1289 wage earners is drawn randomly from some population without knowing the true **data-generating process** (**DGP**) that produced that sample. Can we draw several samples of 1,289 observations from the original sample using the sampling with replacement procedure to draw inferences about the DGP? In other words, how can we use the bootstrapped samples for drawing inferences about the (probability) distribution of the population of the wage earners? How do we do that?

Toward that end, and to introduce the basics, we first consider the following simple wage regression:

$$\ln Y_i = B_1 + B_2 X_i + u_i \tag{23.11}$$

22 Cited by Peterson, I. (1991) Pick a sample, *Science News*, **140**, p. 57.

23 Fan, X. and Wang, L. (1996) Comparability of jackknife and bootstrap results: an investigation for a case of canonical correlation analysis, *Journal of Experimental Education*, **64**, 173–89.

24 Bosch, P. V. (2002) Personal communication to Chong Ho Yu.

25 See, Rodgers J. L. (1999) The bootstrap, the jackknife, and the randomization test: a sample taxonomy, *Multivariate Behavioral Research*, **34**, pp. 441–56. For a theoretical discussion of the power of the bootstrap tests and other finer points, see Davidson, R. and MacKinnon, J. G., *Econometric Theory and Methods*, Oxford University Press, New York, 2004, pp. 159–63.

26 For additional details, see Mooney and Duval, *op cit*., pp. 60–2.

where ln Y = log of hourly wage in dollars, X is education (in years), and u is the regression error term. We are using ln Y instead of Y because it is well known that the wage distribution is skewed.[27]

Since the sample is large enough for practical purposes, we may not be far off the mark if we assume that the error term follows the normal distribution with zero mean but with variance that may not be necessarily homoscedastic. In cross-section data, the error term is likely to be heteroscedastic. Therefore, we can estimate Eq. (23.11) by OLS, using the robust standard option in *Stata*. The results are given in Table 23.5.

Table 23.5 OLS semi-logarithmic wage regression.

regress lwage education,robust

Linear regression			Number of obs	=	1289	
			F(1, 1287)	=	279.79	
			Prob > F	=	0.0000	
			R-squared	=	0.2005	
			Root MSE	=	.52448	

lwage	**Coef.**	**Robust Std. Err.**	**t**	**P>t**	**[95% Conf.**	**Interval]**
education	.0933176	.0055789	16.73	0.000	.0823728	.1042623
_cons	1.115749	.0740643	15.06	0.000	.9704493	1.261049

The results of this regression are straightforward. The coefficient of education of 0.09 is highly statistically significant and suggests that if education increases by a year, the average wage goes up by about 9.3%. The R^2 value may seem low, but this is highly significant as judged by the F statistic. In a sample with several heterogeneous observations, R^2 values are typically low.

A caution is necessary here in that the education variable may be endogenous and thus correlated with the error term. For now, we proceed as if it is exogenous; in Chapter 19, on instrumental variables, we have already discussed this problem and shown how we can find an instrumental variable which is highly correlated with education but is uncorrelated with the error term.

Instead of assuming that the error is normally distributed, we can say that the random sample of 1,289 wage earners has come from a population of wage earners whose probability distribution we do not know. In that case, we will have to estimate the error variance and the distributions of the parameters B_1 and B_2 empirically. In this case, bootstrapping may come in handy.

Before we show how bootstrapping is implemented, we must distinguish two cases: (1) fixed regressors and (2) stochastic regressors. In the wage regression we are discussing, if we assume that the values of the education variable are fixed in repeated sampling, this is the case of fixed regressors. However, if we assume that the education variable is as random as the wage variable, we have stochastic regressors.

27 In Table 1.2 we used wage instead of log of wage because we had not discussed the log models in that chapter.

We discuss each case separately, for the strategies for implementing bootstrapping are different in the two cases.

Bootstrapping with fixed regressors

In this case, we proceed as follows:

1 Estimate the wage regression (23.11), obtaining the estimates b_1 and b_2 , the estimated value of ln Y_i [$\ln \hat{Y}_i = b_1 + b_2 X_i$] and the residuals $e_1, e_2, \ldots, e_{1289}$.

2 Bootstrap 1,289 residuals from the initial residuals, again with replacement. Call them $e_1^*, e_2^*, \ldots, e_{1289}^*$.

3 Obtain new values of Y_i, Y_i^*, as follows:

$$\ln Y_i^* = b_1 + b_2 X_i + e_i^*, \quad i = 1,2,\ldots,1289 \tag{23.12}$$

4 Estimate the preceding equation, obtaining the first set of bootstrap values of the regression coefficients.

5 Repeat this procedure, say, 500 times, thus obtaining 500 estimates of the bootstrapped regression coefficients.

6 Draw histograms of the 500 b_1 and b_2 values to learn something about the shape of the underlying probability distribution of each of these coefficients.

7 From the frequency distribution of the 500 estimates of the two regression parameters, we can estimate the mean values and the standard errors of the two estimators.

8 We can then proceed to establish confidence intervals in the manner suggested earlier.

Of course, this iterative procedure requires intensive computations, which is no longer a limitation with high-powered computers.

The bootstrapping just described is known appropriately as **residuals bootstrapping**.

Bootstrapping with stochastic regressors

In this case, we use a method called **paired sampling**. We proceed as follows:

1 As in the fixed regressor case, we first estimate the wage regression (23.11) by OLS, obtaining b_1, b_2 and the residuals, $e_1, e_2, \ldots, e_{1289}$.

2 From the original sample we draw n random pairs of ln Y_i and X_i, and call them $\ln Y_i^*$ and X_i^*, $i = 1, 2, \ldots, n$, with replacement from the rows of ln Y and X.

3 Use OLS to obtain b_1 and b_2, and call them b_1^* and b_2^*.

4 Repeat steps (2) and (3) m times, say, 500, which will give 500 values of b_1^* and b_2^*.

5 From the 500 bootstrap estimates of the intercept and slope coefficients, we can find out the shape of their distributions and compute the mean and variance values of each estimator.

6 We can proceed to establish confidence intervals for the intercept and the slope coefficient in the manner noted before.

Fortunately, we do not have to do this exercise manually, for software packages like *Stata* have built-in routines to do just that. We use *Stata* 12 in the following calculations. *Stata* used the paired bootstrap samples to carry out the computations, thus assuming that the regressor(s) are stochastic. We have already given the OLS results in Table 23.5 with the robust option of *Stata*.

The results obtained from 500 bootstrap samples from the original sample data given in Table 1.1 are given in Table 23.6; the *Stata* command is also given in this table.

Table 23.6 Bootstrap log-linear wage regression, 500 bootstrap samples.

```
bootstrap,reps(500): regress lwage education
(running regress on estimation sample)
Bootstrap replications (500)
1 ---+--- 2 ---+--- 3 ---+--- 4 ---+--- 5
.................................................. 50
.................................................. 100
.................................................. 150
.................................................. 200
.................................................. 250
.................................................. 300
.................................................. 350
.................................................. 400
.................................................. 450
.................................................. 500

Linear regression                 Number of obs   =     1289
                                  Replications    =      500
                                  Wald chi2(1)    =   275.40
                                  Prob > chi2     =   0.0000
                                  R-squared       =   0.2005
                                  Adj R-squared   =   0.1999
                                  Root MSE        =   0.5245
```

Observed	Bootstrap			Normal-based		
lwage	Coef.	Std. Err.	z	P>z	[95% Conf. Interval]	
education	.0933176	.0056231	16.60	0.000	.0822964	.1043387
_cons	1.115749	.0746742	14.94	0.000	.9693905	1.262108

The *Stata* bootstrap command is followed by the number of bootstrap samples used in the analysis (500 in the present case), followed by the usual regress command – the dependent variable followed by the regressor(s).

The regression output is similar to the OLS output given in Table 23.5. But the major difference is that the 95% confidence interval is based on the estimated bootstrap standard errors and not the traditional OLS confidence interval which is based on just one sample. However, in the present case, there is not much difference in the two sets of confidence intervals, which may not be surprising because our sample is large enough. This is probably a good sign that the OLS standard errors of the regression coefficients may not be distorted.

If we use the (bootstrap) percentile method, as discussed earlier, we obtain the results in Table 23.7.

Table 23.7 Bootstrap percentiles method.

. estat bootstrap,percentile

Linear regression			Number of obs	=	1289	
			Replications	=	500	
	Observed	**Bootstrap**				
lwage	**Coef.**	**Bias**	**Std. Err.**	**[95% Conf.**	**Interval]**	
education	.09331756	–.0000266	.00562315	.0823784	.1049625	(P)
_cons	1.1157493	–.0007787	.07467423	.962768	1.257678	(P)

(P) bias-corrected percentile confidence interval

As noted earlier, the estimated value of the education coefficient may be a biased estimate of the true value of the coefficient. The bias, if any, is shown in the bias column of Table 23.7. The bias in the present case seems minor. The biased-corrected percentile confidence intervals, denoted *P*, are shown in the above table. We do not see substantial differences in the two sets of confidence given in Tables 23.6 and 23.7. Again, this may due to the comparatively large sample we have.

By issuing the following command in *Stata* (Table 23.8), we can obtain the frequency distribution of say b_2, the slope coefficient of education in the wage regression.

In Figure 23.4, we give the empirical distribution of the education coefficient obtained from 500 bootstrap samples.

As this figure shows, it seems the estimated education coefficients are approximately normally distributed. That is why the OLS results given in Table 23.3 and those given in Tables 23.6 and 23.7 do not differ vastly.

Extended wage regression

We now present the results of wage regression that includes several regressors, as in Table 1.2. We first present the OL results with the robust option (Table 23.9).

All the regression coefficients are individually highly significant and have the expected signs.

Now we present regression results based on 500 bootstrapped samples, which are as shown in Table 23.10.

Let us see if the confidence intervals change if we use the percentile method instead of the normal distribution. The results are shown in Table 23.11.

Table 23.8 Empirical frequency distribution of the education coefficient.

```
. bootstrap b=_b[education],reps(500) saving(bs_b,replace):regress lwage education
(running regress on estimation sample)
(note: file bs_b.dta not found)
Bootstrap replications (500)
1 ---+--- 2 ---+--- 3 ---+--- 4 ---+--- 5
.................................................. 50
.................................................. 100
.................................................. 150
.................................................. 200
.................................................. 250
.................................................. 300
.................................................. 350
.................................................. 400
.................................................. 450
.................................................. 500

Linear regression                    Number of obs   =   1289
                                     Replications    =    500

command: regress lwage education
b: _b[education]
```

Observed	Bootstrap			Normal-based		
	Coef.	**Std. Err.**	**z**	**P>z**	**[95% Conf.Interval]**	
b	.0933176	.0053301	17.51	0.000	.0828708	.1037643

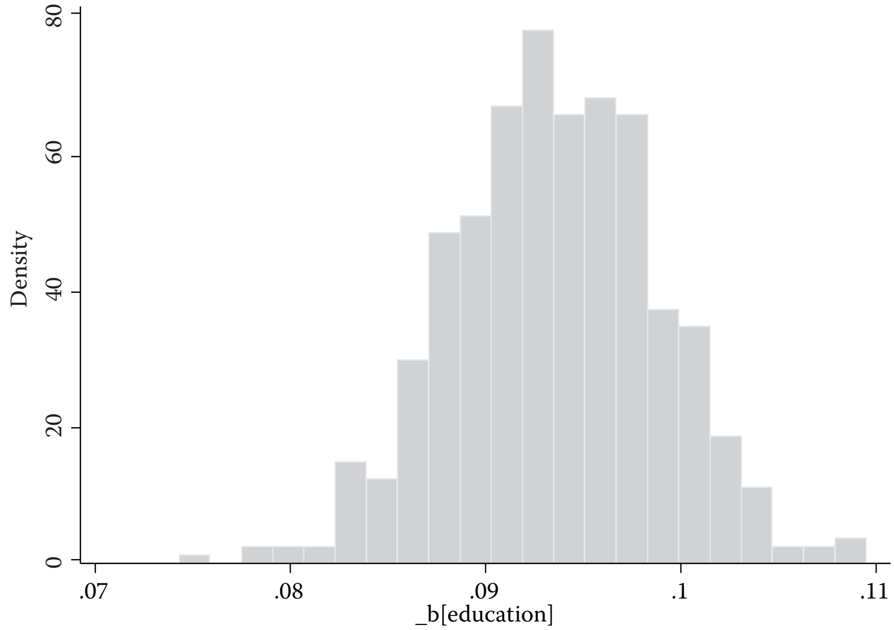
It does not seem that there is much difference in the normal-based confidence interval and the percentile confidence interval. Again, this may be because we have a large sample and the normality assumption may be appropriate.

23.6 Summary and conclusions

The primary objective of this chapter was to introduce the method of bootstrap sampling, which is becoming popular in applied econometrics. The classical linear regression model, the workhorse of econometrics, is based on the critical assumption that the regression error terms are independently and identically distributed (IID) as a normal distribution with zero mean and constant variance.

With this assumption, it is easy to show that the OLS estimators of the regression parameters are best linear biased estimators (BLUE) of their population values. With the normality assumption, it was easy to establish that the OLS estimators are themselves normally distributed. As a result, it was easy to establish confidence intervals and test hypotheses regarding the population values of the estimators.

In many situations the assumption that the error terms are IID may not be tenable. In that case, one has to resort to the large sample, or asymptotic, theory to derive the




```
. use bs_b,replace
(bootstrap: regress)
. histogram b
(bin=22, start=.074302, width=.0016001)
```

Figure 23.4 Empirical distribution of the education coefficient.

Table 23.9 Extended wage regression.

regress lwage education female nonwhite union exper,robust

Linear regression	Number of obs	=	1289
	F(5, 1283)	=	147.65
	Prob > F	=	0.0000
	R-squared	=	0.3457
	Root MSE	=	.47524

lwage	Coef.	Robust Std. Err.	t	P>t	[95% Conf. Interval]	
education	.0998703	.0051279	19.48	0.000	.0898103	.1099303
female	−.249154	.0266655	−9.34	0.000	−.3014668	−.1968413
nonwhite	−.1335351	.0348681	−3.83	0.000	−.2019398	−.0651304
union	.1802035	.0305296	5.90	0.000	.12031	.240097
exper	.0127601	.0012366	10.32	0.000	.010334	.0151861
_cons	.9055037	.0725482	12.48	0.000	.7631775	1.04783

Table 23.10 Extended wage regression using 500 bootstrap samples.

```
. bootstrap,reps(500):regress lwage education female nonwhite union exper
(running regress on estimation sample)
Bootstrap replications (500)
1 ---+--- 2 ---+--- 3 ---+--- 4 ---+--- 5
.................................................. 50
.................................................. 100
.................................................. 150
.................................................. 200
.................................................. 250
.................................................. 300
.................................................. 350
.................................................. 400
.................................................. 450
.................................................. 500

Linear regression                    Number of obs   =     1289
                                     Replications    =      500
                                     Wald chi2(5)    =   690.53
                                     Prob > chi2     =   0.0000
                                     R-squared       =   0.3457
                                     Adj R-squared   =   0.3431
                                     Root MSE        =   0.4752
```

	Observed	Bootstrap			Normal-based	
lwage	Coef.	Std. Err.	z	P>z	[95% Conf.	Interval]
education	.0998703	.0052614	18.98	0.000	.0895581	.1101825
female	–.249154	.0271258	–9.19	0.000	–.3023195	–.1959885
nonwhite	–.1335351	.0343974	–3.88	0.000	–.2009527	–.0661175
union	.1802035	.0289262	6.23	0.000	.1235092	.2368978
exper	.0127601	.001185	10.77	0.000	.0104375	.0150827
_cons	.9055037	.075458	12.00	0.000	.7576088	1.053399

statistical properties of the estimators, such as asymptotic normality. However, the results of the asymptotic theory are difficult to apply in finite samples.

In these situations, one can use bootstrap sampling to derive the sampling distributions of the OLS estimators, particularly the standard errors of the estimators. Without that, it is not possible to establish confidence intervals and test statistical hypothesis.

The key feature of bootstrap sampling is that we have a random sample from some population of interest and from that single sample we draw repeated samples, equal in size to the original sample, but with the important proviso that every time we draw a member of the sample we put it back before drawing another member

Table 23.11 Extended wage regression, percentile confidence interval.

estat bootstrap, percentile

Linear regression Number of obs = 1289

Replications = 500

	Observed	Bootstrap				
lnwage	**Coef.**	**Bias**	**Std. Err.**	**[95% Conf. Interval]**		
education	.08998041	−.0002734	.00520276	.0805101	.101246	(P)
female	−.25255478	.0025753	.02843964	−.3047748	−.1923659	(P)
nonwhite	−.16155255	−.0051782	.03690441	−.247693	−.0874146	(P)
union	.2450863	−.0004314	.03019861	.1867659	.3017699	(P)
_cons	1.2709204	.003357	.06985924	1.131758	1.403319	(P)

(P) percentile confidence interval

from the original sample. This is called sampling with replacement. We can draw as many bootstrap samples as we want. In practice, we may have to draw 500 or more bootstrap samples.

From the bootstrap samples, we can obtain the empirical frequency distributions (EFD) of the parameters of the model. From the EFD, we can estimate the mean value of the parameters and their standard errors and then engage in statistical inference – confidence intervals and hypothesis testing.

We considered four methods of establishing confidence intervals: (1) the normal approximation, (2) the percentile method, (3) the bias-corrected (BC) and accelerated bias-corrected (ABC) methods, and (4) the percentile t method. We discussed the pros and cons of each method and illustrated them with a simple example of 50 bootstrap samples from the standard normal distribution.

We also discussed the use of these methods in regression modeling. In particular, we estimated the wage regression model first introduced in Chapter 1, using 500 bootstrap samples, and showed how we interpret the results. Since the sample size in this example was fairly large (n = 1,289), the results of the bootstrap methods and the standard OLS method do not differ vastly. Of course, that may not be case in nonlinear parameter models, models with substantial number of outliers, models with heteroscedastic error terms, or models involving ratio variables.

Although a very useful empirical tool, we discussed the pros and cons of bootstrapping. Bootstrapping may not be appropriate if the original sample is not a random sample.[28] Other situations where bootstrapping might fail are: (1) if the sample size is too small, (2) distributions with infinite moments, (3) estimating extreme values, and (4) survey sampling.

In parametric problems, like regression analysis, we have several diagnostic tools to check on the assumptions underlying the analysis. Since non-parametric sampling, such as bootstrapping, involves minimal assumptions, it may not be easy to assess the conditions in which bootstrapping may work or fail. However, some diagnostic tools have been developed for assessing non-parametric bootstrap. Efron (1992), for

28 There are methods of getting around this problem. The interested reader may consult the references.

example, introduced the concept of a **jackknife-after-bootstrap** measure to assess the non-parametric bootstrap.

The basic idea behind jackknifing is to take repeated subsamples of the original sample, eliminating observations one at a time, and see the effect on the original bootstrap calculations. Of course, this is computationally very intensive, for we would be repeating bootstrap resampling *n* times, *n* being the original sample size.

Research on bootstrapping is evolving and the reader is urged to surf the Internet for the latest developments in the field.

Since the objective of this chapter was to give the bare essentials of bootstrapping, we have not touched on many aspects of this topic, such as bootstrap vs. asymptotic tests, the (statistical) power of bootstrap tests vs. exact tests, the power of bootstrap hypothesis tests, and the like. The interested reader may consult the references.[29]

29 In particular, see Davidson and MacKinnon, *op cit.*, Chapter 4.

APPENDIX

Table 23.1 Original (Column 1) and 25 bootstrap samples.

B1	B2	B3	B4	B5	B6	B7	B8	B9
0.9814844	0.131797	0.1683419	−1.1760251	0.7957837	0.1683419	0.7957837	−0.3024571	−1.1792789
−1.2874404	0.1683419	0.3816332	0.0121017	−0.3024571	2.7132431	0.131797	0.7957837	2.7132431
−1.3740123	0.3782315	−0.6775813	0.0121017	−1.3740123	−0.5357866	0.9814844	0.0121017	0.7786763
0.3816332	1.6295739	0.9814844	0.7786763	0.0121017	−0.9645436	0.5667842	0.7957837	0.7786763
2.7132431	0.131797	0.9814844	−0.5357866	−0.9645436	0.7957837	0.9814844	−0.5357866	−0.5357866
−0.3024571	−0.6775813	−0.3024571	0.0121017	0.1683419	−0.3024571	0.5667842	0.3782315	0.1323746
0.131797	−0.7249892	0.5667842	−1.3740123	1.6295739	0.7786763	−0.9645436	−1.1792789	0.382755
0.382755	−1.1760251	−1.1760251	−0.5357866	−0.5357866	−1.3740123	0.5667842	−1.1760251	1.3200602
1.6295739	−0.6775813	0.0121017	−1.1760251	0.5667842	0.7957837	0.9814844	−0.7249892	0.0121017
0.5667842	−0.2267193	−0.6775813	−1.3740123	1.8808503	−0.3024571	0.7786763	0.3782315	−0.2267193
0.1683419	0.131797	0.7786763	−1.3740123	1.6295739	1.8808503	0.0121017	2.7132431	0.1683419
−1.1760251	2.7132431	−0.5357866	0.9814844	1.6295739	−1.1760251	−1.2874404	−0.2267193	−0.3024571
1.3200602	1.8808503	0.1683419	−1.1792789	0.1323746	−1.2874404	−1.2874404	0.0121017	0.9814844
−0.5357866	0.9814844	0.3816332	0.7786763	0.5667842	−0.5357866	0.3782315	1.6295739	0.131797
0.7957837	−1.1760251	2.7132431	2.7132431	0.131797	0.3816332	1.3200602	−0.5357866	0.7786763
1.8808503	−0.7249892	0.3816332	−1.3740123	2.7132431	−0.7249892	−0.9645436	1.6295739	0.9814844
0.1323746	−0.3024571	−0.2267193	0.3782315	0.9814844	0.3816332	1.6295739	1.6295739	0.9814844
−0.7249892	0.1683419	0.0121017	0.1323746	−0.6775813	1.8808503	0.382755	−0.2267193	−0.2267193
−0.2267193	−0.7249892	−1.2874404	−0.7249892	−0.3024571	0.3816332	0.9814844	−0.5357866	−1.1760251
0.3782315	−0.5357866	1.3200602	−1.1760251	−0.7249892	−0.5357866	1.6295739	1.3200602	0.0121017
−1.1792789	0.382755	2.7132431	1.3200602	−0.6775813	−0.9645436	−1.1760251	1.3200602	2.7132431
−0.9645436	−0.9645436	−0.5357866	−0.7249892	−1.2874404	−0.9645436	−1.1792789	−0.3024571	0.5667842
0.7786763	1.3200602	0.5667842	0.5667842	0.7957837	−0.9645436	0.1683419	−1.3740123	0.131797
−0.6775813	0.3782315	−0.7249892	0.382755	0.7786763	0.0121017	0.9814844	0.1323746	0.1683419
0.0121017	−1.1760251	0.3782315	0.7957837	0.1683419	0.131797	−1.3740123	−0.3024571	−1.1792789
M=0.152	0.052	0.254	0.154	0.309	−0.013	0.224	0.213	0.356
SD=1.049	1.015	0.997	1.054	1.049	1.061	0.979	1.034	0.984
SEM=0.21	0.203	0.199	0.211	0.21	0.212	0.196	0.207	0.197

B10	B11	B12	B13	B14	B15	B16	B17	B18
0.1323746	1.6295739	0.3782315	1.8808503	0.131797	0.1323746	0.382755	0.9814844	0.131797
−0.2267193	1.8808503	1.8808503	−0.9645436	0.9814844	−1.3740123	0.7786763	−0.7249892	0.7786763
1.6295739	−1.1760251	0.7957837	0.7786763	−0.6775813	−0.3024571	0.7957837	−0.5357866	1.8808503
0.0121017	0.7957837	−1.1760251	0.1683419	0.382755	−0.7249892	−1.1792789	−1.1760251	0.7957837
2.7132431	0.3782315	0.382755	−0.5357866	1.3200602	1.6295739	2.7132431	1.3200602	−0.3024571
1.3200602	−0.5357866	0.131797	−0.2267193	0.131797	−0.6775813	0.382755	0.1683419	−0.6775813
0.5667842	0.9814844	−0.6775813	−0.2267193	−0.3024571	−0.7249892	0.3816332	−0.2267193	−1.1760251
0.7957837	−0.3024571	0.1323746	0.1683419	−0.6775813	0.7786763	−1.3740123	0.1323746	0.3816332
−1.2874404	−0.6775813	1.3200602	−1.1760251	0.0121017	0.7786763	−1.3740123	−1.3740123	−0.6775813
0.5667842	0.131797	1.8808503	0.9814844	0.382755	−1.1760251	0.0121017	0.1323746	2.7132431
1.8808503	−1.3740123	−0.2267193	0.7786763	0.9814844	−1.2874404	−0.7249892	0.0121017	0.7957837
−0.5357866	−0.9645436	0.1323746	−1.1792789	−0.9645436	1.8808503	2.7132431	−0.9645436	−0.3024571
0.1323746	0.7957837	−0.3024571	−1.3740123	0.131797	0.1683419	−0.5357866	−1.1792789	0.7786763
0.0121017	0.382755	−0.2267193	0.131797	1.6295739	0.7957837	−0.5357866	−0.3024571	−1.1760251
−1.1760251	0.131797	−0.5357866	−0.2267193	−0.6775813	0.7957837	−0.3024571	0.382755	0.1323746
0.0121017	−0.5357866	0.1323746	0.0121017	−0.2267193	−1.1792789	−0.6775813	−1.3740123	0.1683419
−0.9645436	0.3816332	1.3200602	−0.9645436	1.6295739	0.3782315	0.7957837	−1.1760251	0.7786763
0.1683419	−0.3024571	−0.9645436	1.3200602	0.0121017	−0.3024571	0.7786763	0.382755	0.9814844
1.3200602	0.3782315	0.382755	1.8808503	0.7957837	1.3200602	−0.7249892	0.131797	0.7957837
0.131797	−1.3740123	1.8808503	2.7132431	2.7132431	−1.1760251	0.1323746	0.1683419	0.3816332
−1.3740123	0.7786763	0.3782315	−0.6775813	0.0121017	0.131797	1.6295739	0.382755	0.131797
0.1323746	−0.2267193	0.3816332	−1.2874404	−1.1792789	−1.1792789	0.3816332	0.7957837	0.382755
−0.9645436	1.8808503	0.1683419	0.1683419	0.1683419	−1.2874404	0.0121017	0.131797	0.5667842
−1.2874404	−0.5357866	−0.9645436	0.1683419	0.0121017	−0.7249892	0.0121017	−0.7249892	0.131797
−1.3740123	1.6295739	0.131797	0.131797	−0.2267193	0.7786763	0.5667842	0.7786763	0.3816332
M=0.093	0.166	0.269	0.098	0.26	−0.102	0.202	−0.154	0.351
SD=1.097	0.974	0.865	1.067	0.903	0.999	1.063	0.771	0.851
SEM=0.219	0.195	0.173	0.213	0.181	0.2	0.213	0.154	0.17

B19	B20	B21	B22	B23	B24	B25	B26
−0.9645436	−0.7249892	−0.5357866	−1.2874404	0.1323746	0.7786763	0.382755	−0.3024571
0.382755	0.1683419	1.6295739	0.0121017	1.6295739	0.382755	−1.1792789	0.1323746
0.382755	1.8808503	0.3782315	0.9814844	−0.2267193	−0.7249892	0.131797	0.382755
0.7957837	−0.6775813	0.3816332	−1.2874404	1.6295739	−0.9645436	1.8808503	1.8808503
0.1323746	−0.3024571	−1.3740123	−1.2874404	0.382755	−1.1760251	−0.9645436	−0.9645436
−0.6775813	−0.3024571	−1.2874404	1.3200602	−0.2267193	0.382755	0.3816332	0.0121017
0.5667842	0.7786763	0.3782315	−0.2267193	1.3200602	−0.3024571	−0.5357866	1.3200602
0.3782315	2.7132431	0.3782315	−0.9645436	1.8808503	−1.1792789	−1.1760251	0.7957837
0.7786763	−0.9645436	−1.2874404	0.5667842	1.8808503	−1.3740123	−1.3740123	0.1323746
2.7132431	0.0121017	0.7786763	−0.3024571	−0.9645436	−0.5357866	−0.5357866	0.7957837
−1.3740123	−1.1760251	−1.1792789	−1.3740123	0.1323746	0.1323746	0.7786763	2.7132431
0.5667842	0.7786763	0.3816332	−0.7249892	0.131797	0.3782315	−0.6775813	−0.3024571
−0.9645436	−1.1760251	−0.3024571	0.3816332	1.6295739	−0.6775813	2.7132431	0.5667842
1.8808503	−0.2267193	0.0121017	0.1323746	−0.5357866	0.1683419	0.3782315	0.1683419
−1.2874404	−0.5357866	0.9814844	2.7132431	1.3200602	0.0121017	0.3782315	0.7957837
−1.2874404	0.131797	2.7132431	2.7132431	0.131797	−1.3740123	−1.3740123	1.6295739
0.9814844	1.3200602	0.9814844	0.382755	0.9814844	−0.3024571	0.1683419	0.3816332
0.131797	0.3782315	−1.3740123	−0.5357866	0.7957837	0.382755	0.0121017	0.1683419
0.0121017	−0.6775813	−1.1760251	−0.9645436	0.382755	0.1683419	−0.5357866	0.3782315
0.7957837	−0.5357866	0.1323746	−0.6775813	0.1683419	0.3816332	0.1683419	0.382755
0.7786763	−1.1760251	2.7132431	0.3816332	−1.3740123	−1.1792789	−0.3024571	−0.3024571
−0.5357866	−0.2267193	−1.1760251	1.8808503	−0.2267193	0.7786763	1.6295739	0.3816332
0.1323746	1.6295739	−0.3024571	0.131797	0.3782315	0.1323746	0.7957837	−0.3024571
−0.9645436	0.9814844	−1.2874404	−1.1760251	0.7957837	0.5667842	−1.1792789	−0.7249892
−0.7249892	0.382755	0.7957837	−1.2874404	0.1323746	1.3200602	−1.3740123	0.1683419
M=0.105	0.098	0.54	−0.02	0.491	−0.153	−0.056	0.411
SD=1.01	1.02	1.2	1.266	0.879	0.759	1.071	0.815
SEM=0.202	0.204	0.24	0.241	0.176	0.152	0.214	0.163