Girl: I say there is one true soul mate for every person.
Boy: That's one busy soul mate. – **old, haggard joke**

When things get too complicated, it sometimes makes sense to stop and wonder: Have I asked the right question? – **Enrico Bombieri**

There's no sense in being precise when you don't even know what you're talking about – **John von Neumann**

Chapter 3

Making Precise Statements

1: Propositions and compound propositions.
2: Predicates: statements about infinitely many things.

Proofs are a mathematician's "rigorous" way of convincing you of something. A rigorous proof is worthless if the point being made is not clear. Here are some statements.

1. 2+2=4.
2. 2+2=5.
3. You may have cake OR candy.
4. IF pigs can fly THEN you get an A.
5. EVERY person has A soul mate.

Each statement above is either true T or false F (law of the excluded middle). The first is true and the second is false. We underlined potentially confusing words in the other statements. In the 3rd statement, can you have both cake and candy? In the 4th, clearly pigs can't fly. Does that mean you can't get an A? In the 5th statement, does every person have their own special soul mate (for me it's "Z" and for you it's "Joe"). Or, is there is a universal soul mate and all people share that same soul mate, whoever that may be?

5(a) There is a single soul mate that EVERY person shares.
5(b) EVERY person has their own special soul mate.

Big deal you say. Yes, when it comes to proof! To verify 5(b) we asked our friends $\{A, B, C, D, E, F\}$ from Chapter 1 to name a soul mate. Their replies are on the right. They all have some soul mate, so 5(b) is true. 5(a) is much harder to prove. The responses neither prove nor disprove 5(a).

A says Sue's their soul mate;
B says Joe's their soul mate;
C says Sue's their soul mate;
D's soul mate is a red Porshe;
E says Sue's their soul mate;
F says Sam's their soul mate.

Pop Quiz 3.1
(a) How would you verify 5(a) by asking questions to our 6 friends?
(b) Give two interpretations of "Every American has a dream."

Before seeking proof you must have the correct logical meaning of the claim. The building blocks of a claim are propositions. A proposition is true (T) or false (F). It may be hard (or impossible) to assign a truth value to a proposition, but the claim should be unambiguous. We use the letters $p, q, r, s, \dots$ for propositions.

Exercise 3.2
If you can, evaluate the truth value of these propositions (true T or false F):
(a) p: Porky the pig can fly. (b) q: You got an A. (c) r: Kilam is an American. (d) s: 4^2 is even.

3.1 Compound Propositions

We build more complex claims called compound propositions by connecting basic propositions using 4 basic connectors. We must give a precise meaning to these connectors to avoid ambiguity in claims involving them.

Connector	Symbol	An example in words
NOT	$\neg p$	IT IS NOT THE CASE THAT (Porky the pig can fly)
AND	$p \wedge q$	(Porky the pig can fly) AND (You got an A)
OR	$p \vee q$	(Porky the pig can fly) OR (You got an A)
IF... THEN...	$p \rightarrow q$	IF (Porky the pig can fly) THEN (You got an A)

A compound proposition is either true or false, just like a basic proposition.

Exercise 3.3

Here are three propositions p : It is raining. q : Kilam has his umbrella. r : It is cloudy.
Write out the compound propositions below and use common sense to assign true T or false F for each:

$$p \wedge r \qquad p \rightarrow q \qquad p \rightarrow r \qquad q \rightarrow r \qquad q \rightarrow p \qquad r \rightarrow p.$$

A mathematical claim is most often an "IF... THEN..." compound proposition. Proving a mathematical claim amounts to proving that the compound proposition is true. Therefore, we need to understand the meaning of compound propositions: When is a compound proposition true? Let us take each connector in turn.

Negation, NOT: IT IS NOT THE CASE THAT(Porky the pig can fly). A negation is true when the original proposition "Porky the pig can fly" is false. It's that simple.

> The negation $\neg p$ is true when p is false, and $\neg p$ is false when p is true.

Conjunction, AND: (Porky the pig can fly) AND (You got an A). AND is as expected from English.

> Both p and q must be true for $p \wedge q$ to be true. Otherwise $p \wedge q$ is false.

Since pigs cannot fly, the proposition "Porky the pig can fly" is F. We don't know whether you got an A or not in the course, but it does not matter. The conjunction is F.

Disjunction, OR: (You can have cake) OR (You can have ice-cream). OR is slightly different from common English, where you can have exactly one of cake and ice-cream (exclusive-OR). In mathematics, OR is true even if you have both cake and ice-cream. An OR is false only when both propositions are false.

> Both p and q must be false for $p \vee q$ to be false. Otherwise $p \vee q$ is true.

(Porky the pig can fly) OR (You got an A) is true if you got an A and false otherwise, as pigs can't fly.

Implication, IF ..., THEN: IF (Porky the pig can fly) THEN (You got an A). Intuition says you can't get an A because pigs can't fly. Wrong! Implication is subtle but important. Most theorems are implications, e.g. "IF n^2 is even, THEN n is even." Implication is also the basis for deductions. You find out that n^2 is even. You can now infer $n \neq 5$. Here is a more familiar example not involviing flying pigs.

$$\begin{aligned} p &: \text{it rained last night} \\ q &: \text{the grass is wet} \\ p \rightarrow q &: \text{IF (it rained last night) THEN (the grass is wet).} \end{aligned}$$

What does it mean for this common-sense implication to be true? What can you conclude? Not much. Did it rain last night? Is the grass wet? We don't know. So what use is an implication?

The morning news reports rain last night. Can we say anything about the grass? Here's what we know.

IF (it rained last night) THEN (the grass is wet)	T		$p \to q$	T
It rained last night (from the morning news)	T		p	T
Is the grass wet?	**YES!**	$\therefore$	q	T

($\therefore$ means "therefore.") On the right is the general pattern of this inference using a true implication (p and q are arbitrary). Instead of watching the news, you went out and found wet grass. Did it rain last night?

IF (it rained last night) THEN (the grass is wet)	T		$p \to q$	T
The grass is wet (from walking outside)	T		q	T
Did it rain last night?		$\therefore$	p	T or F

Wet grass does not mean rain – maybe sprinklers came on. The implication remains valid even if it didn't rain. Let's keep going. Suppose the morning news reports no rain last night:

IF (it rained last night) THEN (the grass is wet)	T		$p \to q$	T
It rained last night (from the morning news)	F		p	F
Is the grass wet?		$\therefore$	q	T or F

No surprise here. No rain last night does not mean dry grass (perhaps sprinklers came on). Lastly, suppose you went out and the grass is dry. We surely can conclude that it couldn't have rained last night.

IF (it rained last night) THEN (the grass is wet)	T		$p \to q$	T
The grass is wet (from walking outside)	F		q	F
Did it rain?	**NO!**	$\therefore$	p	F

Here is a summary of these inferences.

For a **true** implication $p \to q$: When p is true, you can conclude that q is true.
When q is true, you **cannot** conclude p is true.
When p is false, you **cannot** conclude q is false.
When q is false, you can conclude p is false.

Exercise 3.4

Use practical experience to determine T/F for each implication and then answer the related questions.

(a) T or F: IF it is cloudy THEN it is raining. It is cloudy. Is it raining?

(b) T or F: IF it is cloudy THEN you have your umbrella.
- (i) You have your umbrella. Is it cloudy?
- (ii) You do not have your umbrella. Is it cloudy?

(c) T or F: IF you study hard THEN you will get an A.
- (i) You studied hard. Did you get an A?
- (ii) You got an A. Did you study hard?
- (iii) You did not study hard. Did you get an A?
- (iv) You did not get an A. Did you study hard?

(d) T or F: IF (you are hungry OR you are thirsty) THEN you will visit the cafeteria.
- (i) You are thirsty. Did you visit the cafeteria?
- (ii) You did visit the cafeteria. Are you hungry? Are you thirsty?
- (iii) You did not visit the cafeteria. Are you hungry? Are you thirsty?

A scientist collects data to verify an implication. One night it rained. The next morning the grass was bone dry. What do you think about our implication now? It cannot be valid, because the implication says that if it rained the grass must be wet. We have a falsifying scenario where it rained and the grass is dry – a counterexample. This is the only scenario in which the implication is false. Let us define implication,

$p \to q$ is false in only one case: when p is true and q is false. In all other cases $p \to q$ is true.

We have identified when an implication is true and when it's false. Now, back to our implication

IF (Porky the pig can fly) THEN (You got an A).

Pigs can't fly, so this is like the case where it did not rain last night. When it did not rain, the grass could be wet or not. Similarly, pigs can't fly, but you can get an A (or not) and the implication is still true. It is extremely important to understand implication. All the following are the implication $p \to q$ in English.

If it rained last night then the grass is wet.	IF p THEN q
It rained last night implies the grass is wet.	p IMPLIES q
It rained last night only if the grass is wet.	p ONLY IF q
The grass is wet if it rained last night.	q IF p
The grass is wet whenever it rains.	q WHENEVER p

3.2 Truth-Tables

We summarize what we know about our 4 connectors in a truth-table, shown below.

p	q	$\neg p$	$p \wedge q$	$p \vee q$	$p \to q$
F	F	T	F	F	T
F	T	T	F	T	T
T	F	F	F	T	F
T	T	F	T	T	T

On the left are the basic propositions p and q, and all their possible truth values. On the right are the compound propositions you are interested in. The truth-table tells you the truth-value of the compound propositions for all possible truth-values of the basic propositions – that is the mathematical "meaning" of the compound proposition. When intuition fails you, always use a truth-table to figure things out. An example will help clarify things, as well as introduce us to even more complex compound propositions.

Example 3.1. Let us solve Exercise 3.4(d). Certainly if you are hungry or thirsty, then you will visit the cafeteria. So, $(p \vee q) \to r$ is true for propositions p, q and r defined by

p : you are hungry $\qquad$ q : you are thirsty $\qquad$ r : you will visit the cafeteria,

Let us construct the truth-table for $(p \vee q) \to r$. First list the eight possible truth value combinations for the basic propositions p, q and r. A systematic way to list the possible truth values is to alternate T and F for r; then, alternate two T's and two F's for q; lastly, alternate four T's and four F's for p between. In the table below, as an intermediate step we show the truth-table for $p \vee q$ and use that to get the truth-table for $(p \vee q) \to r$. We highlighted the five rows with $(p \vee q) \to r$ true. We can restrict our analysis of the three cases in Exercise 3.4(d) to these highlighted rows since the compound proposition is known to be true.

(i) *You are thirsty:* q is true, (rows 4 and 8). In both cases r is true, i.e. you visit the cafeteria.

(ii) *You did visit the cafeteria:* r is true (rows 2, 4, 6 and 8). In some cases you are hungry, and in some you are thirsty. Are you hungry? We don't know. Are you thirsty? We don't know. You could be neither (row 2) and just wander into the cafeteria.

(iii) *You did not visit the cafeteria:* r is false (only row 1). Now p and q are both false. You are neither hungry nor thirsty.

	p	q	r	$(p \vee q)$	$(\boldsymbol{p \vee q}) \boldsymbol{\to r}$
1.	F	F	F	F	**T**
2.	F	F	T	F	**T**
3.	F	T	F	T	**F**
4.	F	T	T	T	**T**
5.	T	F	F	T	**F**
6.	T	F	T	T	**T**
7.	T	T	F	T	**F**
8.	T	T	T	T	**T**

Truth-tables makes these kinds of logical deductions mechanical. □

Equivalent Compound Propositions. English has many ways to say the same thing. Consider these statements using p = "it rained last night" and q = "the grass is wet".

(*i*) If it rained last night then the grass is wet.	$p \to q$
(*ii*) If the grass is not wet, then it did not rain last night.	$\neg q \to \neg p$
(*iii*) Either it did not rain last night or the grass is wet.	$\neg p \vee q$

These different statements are logically equivalent – they mean the same thing. To see this mathematically,

1. *Associative:* $(p \wedge q) \wedge r \stackrel{\text{eqv}}{\equiv} p \wedge (q \wedge r)$;
   $(p \vee q) \vee r \stackrel{\text{eqv}}{\equiv} p \vee (q \vee r)$.
2. *Commutative:* $p \wedge q \stackrel{\text{eqv}}{\equiv} q \wedge p$;
   $p \vee q \stackrel{\text{eqv}}{\equiv} q \vee p$.
3. *Negations:* $\neg(\neg p) \stackrel{\text{eqv}}{\equiv} p$;
   $\neg(p \wedge q) \stackrel{\text{eqv}}{\equiv} \neg p \vee \neg q$;
   $\neg(p \vee q) \stackrel{\text{eqv}}{\equiv} \neg p \wedge \neg q$.
4. *Distributive:* $p \vee (q \wedge r) \stackrel{\text{eqv}}{\equiv} (p \vee q) \wedge (p \vee r)$;
   $p \wedge (q \vee r) \stackrel{\text{eqv}}{\equiv} (p \wedge q) \vee (p \wedge r)$.
5. *Implication:* $p \rightarrow q \stackrel{\text{eqv}}{\equiv} \neg q \rightarrow \neg p$;
   $p \rightarrow q \stackrel{\text{eqv}}{\equiv} \neg p \vee q$.

Figure 3.1: Rules for manipulating logical connectors.

we use their truth-tables. You should verify the truth-table below (we added $q \rightarrow p$, the converse of $p \rightarrow q$).

		(i)	(ii)	(iii)	Converse of $p \rightarrow q$
p	q	$p \rightarrow q$	$\neg q \rightarrow \neg p$	$\neg p \vee q$	$q \rightarrow p$
F	F	T	T	T	T
F	T	T	T	T	F
T	F	F	F	F	T
T	T	T	T	T	T

Compound propositions with the same truth-table are logically equivalent (mean the same thing). We use $\stackrel{\text{eqv}}{\equiv}$ for logical equivalence. The first three implications are logically equivalent and mean the same thing.

$$p \rightarrow q \stackrel{\text{eqv}}{\equiv} \neg q \rightarrow \neg p \stackrel{\text{eqv}}{\equiv} \neg p \vee q$$

The last proposition, the converse $q \rightarrow p$, does not have the same truth-table as $p \rightarrow q$.

Order is very important: $p \rightarrow q$ and $q \rightarrow p$ **do not** mean the same thing.

"IF you're dead, THEN your eyes are shut" and "IF you're eyes are shut, THEN you're dead" are very different!

Rather than compute a massive truth table, it is often easier to show equivalence using the rules for manipulating logical connectors in Figure 3.1. These rules are similar to the set-operation rules in Figure 2.1 on page 17. Negation is like set-complement; AND is like set-intersection; and, OR is like set-union. For example, the negation of a conjunction, $\neg(p \wedge q) \stackrel{\text{eqv}}{\equiv} \neg p \vee \neg q$. The negation IT'S NOT(cold and rainy) means that either it is not cold or it is not rainy, exactly as the rule says. You may verify all the rules with truth-tables.

Example 3.2. We use the rules in Figure 3.1 to derive equivalent propositions to $(q \wedge \neg r) \rightarrow \neg p$.

$$\begin{aligned}
\boldsymbol{(q \wedge \neg r) \rightarrow \neg p} &\stackrel{\text{eqv}}{\equiv} \neg(q \wedge \neg r) \vee \neg p && \text{(implication)}\\
&\stackrel{\text{eqv}}{\equiv} (\neg q \vee r) \vee \neg p && \text{(negation)}\\
&\stackrel{\text{eqv}}{\equiv} \neg q \vee (r \vee \neg p) && \text{(associative)}\\
&\stackrel{\text{eqv}}{\equiv} \neg q \vee (\neg p \vee r) && \text{(commutative)}\\
&\stackrel{\text{eqv}}{\equiv} (\neg q \vee \neg p) \vee r && \text{(associative)}\\
&\stackrel{\text{eqv}}{\equiv} \boldsymbol{(\neg p \vee \neg q) \vee r} && \text{(commutative)}\\
&\stackrel{\text{eqv}}{\equiv} \neg(p \wedge q) \vee r && \text{(negation)}\\
&\stackrel{\text{eqv}}{\equiv} \boldsymbol{(p \wedge q) \rightarrow r} && \text{(implication)}
\end{aligned}$$

All propositions in a derivation chain are equivalent. The ones in bold appear in Exercise 3.6. Many of our steps are trivial (e.g. commutativity). With experience you'll start omitting such trivial steps. □

Pop Quiz 3.5 [Compound Propositions and Programming]

Compound propositions are used to control the flow of a program Look at these snippets of C^{++} code.

`if(x > 0 || (y > 1 && x < y))` `Execute instructions A, B, C.`	`if(x > 0 || y > 1)` `Execute instructions A, B, C.`

Use truth-tables to show that both do the same thing. Which do you prefer and why?

Exercise 3.6

Use truth-tables or logical derivations to arrange these propositions into groups that are logically equivalent.

$\neg p \to q$	$\neg q \to p$	$p \vee q$	$\neg(p \wedge q)$	$\neg p \vee \neg q$
$(p \wedge q) \to r$	$(\neg p \vee \neg q) \vee r$	$(q \wedge \neg r) \to \neg p$	$p \vee (q \vee r)$	$\neg r \to (p \vee q)$

3.2.1 Proving an Implication

We will say more about proofs later. Here, we discuss the logic of proving a simple implication.

$$\text{IF } (n^2 \text{ is even}) \text{ THEN } (n \text{ is even}).$$

It may be confusing that n is unknown, unlike in Exercise 3.4 where everything was concrete. What is n? Is n^2 even? Is n even? We don't know and don't care. We only care about the truth of the implication. How can we know the truth of an implication without knowing anything about the basic propositions involved? This is what is subtle about proving an implication. Let us start by introducing the basic propositions, as usual.

p: n^2 is even
q: n is even

$p \to q$

p	q	$p \to q$
F	F	T
F	T	T
T	**F**	**F**
T	T	T

The shaded row in the truth-table is the only case in which $p \to q$ is false. To prove an implication, we must prove that the shaded row cannot happen. How can we do that without knowing n? Here is how. That row asserts that q is false and p is true. If q is false, n is odd, $n = 2k + 1$. Then p is also false as n^2 is odd:

$$n^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1. \qquad \leftarrow \text{odd}$$

p and q can be true or false, but p being true *and* q being false can't occur together. We don't know n, yet we know the shaded row in the truth-table is impossible. In all other rows of the truth-table, $p \to q$ is true, hence we have proved that $p \to q$ is true. That is, "if n^2 is even then n is even" is a true implication.

3.3 Quantifiers: Statements About Many Things

EVERY Person has A soul mate.

We need to deal with this complex sentence. Let's start simple. Here are other similar claims.

Kilam has **some** gray hair.
Everyone has **some** gray hair.
Any map can be colored with 4 colors with adjacent countries having different colors.
Every even integer $n > 2$ is the sum of 2 primes *(Goldbach, 1742)*.
Someone broke this faucet.
There exists a creature with blue eyes and blonde hair.
All cars have four wheels.

All the statements are well formed claims. They are either true or false. The complexity of these claims stems form their use of words/phrases called quantifiers:

EVERY; A; SOME; ANY; ALL; THERE EXISTS.

Quantifiers are attached to nouns, e.g. "EVERYone;" "SOME hair;" and, "THERE EXISTS a creature." The noun identifies some set and the quantifier indicates that we are making a statement about some of the members of the set. The quantifier tells us about how many objects in the set we are making a statement. Compare:

My Ford Escort has four wheels;
ALL cars have four wheels.

The first statement talks about a particular car. The second makes a general statement about cars. The quantifier ALL tells us which cars. Let's formally define the set of cars,

$$C = \{c \mid c \text{ is a car}\}.$$

It seems pedantic, so bear with us. A long winded way to say "ALL cars have four wheels" is

$$\text{for all cars } c \in C, \text{ car } c \text{ has four wheels.} \tag{3.1}$$

3.3.1 Predicates

We like precise, not long-winded. We use the symbol $\forall$ to stand for "for all". Let's shorten "car c has four wheels" to $P(c)$. Our statement from (3.1) in condensed form becomes

$$\forall c \in C : P(c).$$

Read aloud: "for all c in C, the statement $P(c)$ is true." In $P(c)$ is a new concept, the predicate. A predicate $P(c)$ looks like a function $f(x)$. Like a function, a predicate takes an input or inputs. In this case $P(c)$ takes input c. Unlike a typical function which outputs a value $f(x)$, a predicate outputs a statement $P(c)$. Just like a function, the input to a predicate can be any element within its domain, which is some set. In our example, the domain is the set of cars. An example of an input is $c =$ My Ford Escort and

$$P(\text{My Ford Escort}) = \text{"car 'My Ford Escort' has four wheels."}$$

It sounds odd and it's simpler to just say "My Ford Escort has four wheels". The statement is not that odd if you view 'My Ford Escort' as the label of a particular car. When we assert $P(c)$, we mean that the statement output by predicate P for input c is true.

You've used predicates before. $\forall x \in \mathbb{R},\ x^2 \geq 0$ is an informal version of $\forall x \in \mathbb{R} : Q(x)$, where the predicate $Q(x)$ returns the statement "the value x^2 is ≥ 0". The technical term is predicate. Don't be paralyzed, it is just notation. But, the concept is very important. A predicate defines a set of statements, one for each input. Each statement output by the predicate can be true or false. Let us compare a predicate to a function.

	Predicate	Function
	$P(c)$ = "car c has four wheels"	$f(x) = x^2$
Input	object $c \in C$	parameter $x \in \mathbb{R}$
Output	**statement** $P(c)$	**value** $f(x)$
Example	$P(\text{Jen's VW})$ = "car 'Jen's VW' has four wheels"	$f(5) = 25$
	$\forall c \in C : P(c)$	$\forall x \in \mathbb{R},\ f(x) \geq 0$
Meaning	For all $c \in C$, the statement $P(c)$ is true.	For all $x \in \mathbb{R}$, $f(x)$ is ≥ 0.

Pop Quiz 3.7

Consider the predicate $P(n)$ = "n is a perfect square".

(a) Give a domain for predicate P?
(b) Is the predicate P true or false?
(c) What is $P(4)$?
(d) Which are true: $P(4)$, $P(5)$, $P(9)$.

We use predicates to make multiple claims by specifying the inputs on which the predicate is true. A common case in mathematics is to assert the predicate is true for all inputs. This is what $\forall c \in C : P(c)$ claims, that all cars have four wheels. $\forall$ is the universal quantifier. Another common quantifier in mathematics is the existential quantifier, which claims that a predicate is true for at least one possible input, for example

There EXISTS a creature with blue eyes and blonde hair.

Define the set of creatures, $A = \{a \mid a \text{ is a creature}\}$, and the predicate

$$Q(a) = \text{"}a \text{ has blue eyes and blonde hair."}$$

We use the symbol $\exists$ to stand for "there exists." Our statement becomes

$$\exists a \in A : \ Q(a).$$

Read aloud, "There exists a creature a in the set of creatures A for which $Q(a)$ is true (i.e., a has blue eyes and blonde hair)." Words like "for all," "every," "any" and "all" are translated into mathematics using the universal quantifier $\forall$. Words like "some," "at least one" and "there exists" are translated into mathematics using the existential quantifier. The universal and existential quantifier are the main ones used in mathematics.

Exercise 3.8

Formulate predicates (and domains) and translate the following into "mathematics".

(a) Kilam has some gray hair.
(b) Any map can be 4-colored with adjacent countries having different colors.
(c) Every even integer $n > 2$ is the sum of 2 primes.
(d) There is no creature with blue eyes and blonde hair.

Universal and existential quantifiers are very different, especially when it comes to proof. To convince you that "Every car has four wheels," I must show you every car and count its wheels. It's much easier to convince you that "There is a creature with blue eyes and blonde hair." I simply find one creature with blue eyes and blonde hair. The same goes for proving a mathematical statement with the universal versus the existential quantifier. Think about what it takes to convince you of these three claims.

Claim 1. Every even integer $n \geq 2$ is the sum of two primes. [hard]
Claim 2. There is a triple of integers (a, b, c) for which $a^2 + b^2 = c^2$. [easy]
Claim 3. There is no triple of integers (a, b, c) for which $a^3 + b^3 = c^3$. [hard]

Combining quantifiers with connectors

A predicate produces a statement. That statement can be a compound statement formed using connectors. Consider the claim "There exists a creature with blue eyes and blonde hair." Define two predicates,

$$G(a) = \text{"}a \text{ has blue eyes;"} \qquad H(a) = \text{"}a \text{ has blonde hair."}$$

Our claim is "there exists a for which $G(a) \wedge H(a)$ is true".

$$\exists a \in A : (G(a) \wedge H(a)).$$

We are quantifying a compound statement involving predicates. Notice the use of parentheses to indicate the scope of the quantifier. When the context is clear, we often leave out the domain in our quantified statements. For example, if it's clear we are talking about creatures, then we simply write $\exists a : (G(a) \wedge H(a))$. We can also define the compound predicate $R(a) = G(a) \wedge H(a)$ and write $\exists a : R(a)$.

Exercise 3.9

Which expressions below are valid statements? For the valid ones, write an "English translation". (G and H are the same predicates defined above in the text.)

$$(\exists a : G(a)) \wedge (\exists a : H(a)); \qquad (\exists b : G(b)) \wedge (\exists c : H(c)); \qquad (\exists a : G(a)) \wedge H(c).$$

3.3.2 Negation

Consider the sentence "There is no creature with blue eyes and blonde hair." That is, "IT IS NOT THE CASE THAT(There is creature with blue eyes and blonde hair),"

$$\neg(\exists a : (G(a) \wedge H(a))).$$

An equivalent claim is: for all creatures a, a does not have blue eyes and blonde hair:

$$\forall a : \neg(G(a) \wedge H(a))).$$

Now consider the negation of "All cars have four wheels," which is

"IT IS NOT THE CASE THAT(all cars have four wheels)."

This means there is a car which does not have four wheels. Using the predicate $P(c)$ = "car c has four wheels,"

$$\neg(\forall c : P(c)) \overset{\text{eqv}}{\equiv} \exists c : \neg P(c).$$

In general, you can take the negation inside the quantifier, negating the predicate, provided you change the quantifier (existential to universal or universal to existential).

For any predicate $P(x)$, $\quad \neg(\forall x : P(x)) \overset{\text{eqv}}{\equiv} \exists x : \neg P(x);$
$\neg(\exists x : P(x)) \overset{\text{eqv}}{\equiv} \forall x : \neg P(x).$

3.3.3 Mixing Quantifiers

We finally get to the claim which began this section,

"EVERY person has A soul mate."

As we already mentioned, in English, this sentence is ambiguous. Does every person have the same soul mate or does every person have their own soul mate? To make things precise, let's define the set of people, $A = \{a \mid a \text{ is a person}\}$. We assume a soul mate is a person. Now, define a predicate for two inputs $a, b \in A$:

$$P(a, b) = \text{"Person } a \text{ has as a soul mate person } b\text{."}$$

Note, you can be my soul mate while I need not be yours. If we mean every person shares the same soul mate, then there exists this special soul mate who is soul mate to everyone. Here it is in mathematical notation:

$$\exists b : (\forall a : P(a, b)). \tag{3.2}$$

Read aloud: there is a soul mate b such that for every person a, $P(a, b)$ is true. We first identify this unique hard-working soul mate, then claim all people have said person as soul mate. If, instead, every person has their own soul mate, then for every person, <u>there exists</u> this personalized soul mate. In our notation,

$$\forall a : (\exists b : P(a, b)). \tag{3.3}$$

The mathematical meanings of (3.2) and (3.3) are now precise, and they are very different. The two statements only differ in the order of the quantifiers.

When quantifiers are mixed, the order in which they appear in the statement is important for the meaning, and the order generally cannot be switched.

Exercise 3.10

(a) For $P(a, b)$ write out the quantified proposition. Does changing the quantifier order preserve meaning?
   (i) $\forall a : (\forall b : P(a, b))$. Reversing order, is $\forall b : (\forall a : P(a, b))$ logically equivalent?
   (ii) $\exists a : (\exists b : P(a, b))$. Reversing order, is $\exists b : (\exists b : P(a, b))$ logically equivalent?

(b) Are the following valid predicates, $Q(a) = \exists b : P(a, b)$ and $R(b) = \forall a : P(a, b)$?
   Use Q and R to rewrite the statements in (3.2) and (3.3).

Exercise 3.11
In our haggard joke, the girl really meant "There is one [and only one] true soul mate for every person." Use the predicate $P(a,b)$, quantifiers and connectors to formulate this claim precisely.

3.3.4 Proofs with Quantifiers

Here are two claims, more on the mathematical side.

Claim 1. $\forall n > 2$: IF n is even, THEN n is a sum of two primes. *(Goldbach, 1742)*
Claim 2. $\exists (a,b,c) \in \mathbb{N}^3 : a^2 + b^2 = c^2$.

In claim 1, the domain of the quantifier is the natural numbers greater than 2. In claim 2, the domain is triples of natural numbers (a,b,c) and this domain is written as $\mathbb{N}^3$. Claim 1 (known as Goldbach's conjecture) has no known proof. One must show that the predicate is true for every value of $n > 2$:

"IF 3 is even, THEN 3 is the sum of two primes"	[true because 3 is not even.]
"IF 4 is even, THEN 4 is the sum of two primes"	[true because $4 = 2 + 2$.]
"IF 5 is even, THEN 5 is the sum of two primes"	[true because 5 is not even.]
"IF 6 is even, THEN 6 is the sum of two primes"	[true because $6 = 3 + 3$.]
"IF 7 is even, THEN 7 is the sum of two primes"	[true because 7 is not even.]
"IF 8 is even, THEN 8 is the sum of two primes"	[true because $8 = 5 + 3$.]
$\vdots$	$\vdots$

We have to show that an infinite number of IF ... THEN ... statements are true, one for every $n > 2$. You can imagine that this is going to be tough.

Here is a proof of claim 2. Set $a = 3, b = 4, c = 5$ and observe that $3^2 + 4^2 = 5^2$. That's it! To prove an existential claim, you just need to find one instance where the predicate is true. Proving an existential claim is usually much easier than proving a universal claim.

Exercise 3.12
Which of the following claims are easier to disprove, and why?
(a) $\forall n \in \mathbb{N} : 2^{(2^n)} + 1$ is prime. ($2^{(2^n)} + 1$ is called a Fermat number.)
(b) $\exists (a,b,c) \in \mathbb{N}^3 : a^3 + b^3 = c^3$. (This is a special case of Fermat's last theorem.)
(To disprove a claim you show that the negation of the claim is true.)

3.4 Deduction Versus Induction

Here is a classic deductive argument. Someone tells you that p is true, where p is the strange claim:

p : IF (The earth is not flat) THEN (Pigs can fly)

With age comes the wisdom that the earth is round. Now, you can deductively conclude that pigs can fly. Deduction gives a hardcore proof that you can take to the bank. As long as p is true, the deduction is sound. The deduction is based on the knowledge that p is true and later the knowledge that the earth is round. How did we get this knowledge? In life it is a process called induction. With "enough" evidence for a claim you assert its truth. From the fanciful we move to the mundane.

ALL ravens are black.

Indeed, all 27 ravens I know are black. Every black raven I observe strengthens my belief in this claim and at some point I put it into my knowledge bank as a fact. That's induction and its very tricky (see Problem 3.58). Later, we will see a technique called mathematical induction which does indeed allow you to make a few observations and conclude something is true always, but one more ingredient will be added.

3.5 Problems

Problem 3.1. Determine T/F. If you think a statement is not a valid proposition, explain why.

(a) "2+7=10."
(b) "There are no wild killer bees in Alaska."
(c) "Miami is not in Florida."
(d) "Where is the train station?"
(e) "$2x > 5$."
(f) "$2^n < 100$."
(g) "There is a lot of pollution in Mumbai."
(h) "The answer to this question is F."

Problem 3.2. True or False: "The function f equals 5?" Explain.

Problem 3.3. True or False: "IF God exists, THEN the square of any real number is non-negative.". Explain

Problem 3.4. Define the propositions p = "Kilam is a CS major" and q = "Kilam is a hockey player". Use the connectors $\wedge, \vee, \neg$ to formulate these claims.

(a) Kilam is a hockey player and CS major.
(b) Kilam either plays hockey or is a CS major.
(c) Kilam plays hockey, but he is not a CS major.
(d) Kilam is neither plays hockey nor is a CS major.
(e) Kilam is a CS major or a hockey player, not both.
(f) Kilam is not a hockey player, but is a CS major.

Problem 3.5. What is the negation of these statements?

(a) Jan is rich and happy.
(b) If Kilam was born yesterday, then pigs fly.
(c) Niaz was born yesterday and pigs can't fly.
(d) Kilam's phone has at least 8GB of RAM.
(e) If Kilam is in pajamas, then all lights are off.
(f) Every student is a friend of another student.
(g) Some student is a friend of another student.
(h) All Kilam's friends are big and strong.

Problem 3.6. Kilam's has 2GB of RAM Liamsi has 4GB of RAM. Which propositions are true?

(a) IF Kilam has more RAM than Liamsi THEN pigs fly.
(b) IF Liamsi has more RAM than Kilam THEN pigs fly.
(c) Kilam has more RAM than Liamsi AND pigs fly.
(d) Kilam has more RAM than Liamsi OR pigs fly.
(e) Liamsi has more RAM than Kilam AND pigs fly.
(f) Liamsi has more RAM than Kilam OR pigs fly.

Problem 3.7. There are 3 spoons, 4 forks and 4 knives. How many utensils are:
(a) Forks or knives. (b) Forks and knives. (c) Neither Forks nor knives.

Problem 3.8. Rewrite each sentence in "IF..., THEN..." form.

(a) You pass the FOCS-final exam only if you studied this book for at least one week.
(b) Attending class is necessary for passing the course.
(c) For a quadrilateral to be square, it is sufficient that it have four equal angles.
(d) For a quadrilateral to be square, it is necessary that it have four equal sides.
(e) A natural number can't be an odd prime unless it is greater than 2.
(f) The giant flies come out whenever it is hot.
(g) All roads lead to Rome.

Problem 3.9. If the blind-spot indicator on the wing-mirror of a car lights up, there is car in your blind spot and it's not safe to switch lanes. The blind-spot indicator is not lit. Does it mean you can switch lanes or should you look first?

Problem 3.10. Ifar's parents always told him: "If you don't eat your peas, you can't have ice-cream."
Naturally, Ifar always ate his peas and eagerly expected his ice-cream to come. Are Ifar's parents obliged to give him ice-cream? What statement did Ifar think he heard, and is that logically equivalent to what his parents actually said.

Problem 3.11. What's the difference between these marketing slogans? Which is the more impressive claim?

"If you didn't buy your car from FOCS-Auto, then you paid too much."
"If you bought you car from FOCS-Auto, then you didn't pay too much."

Problem 3.12. Trolls are knights who are honest or knaves who are liars. Troll 1 says: "If we are brothers, then we are knaves." Troll 2 says: "We are brothers or knaves." (a) Can both trolls be knights? (b) Can both trolls be knaves?

Problem 3.13. If it rains on a day, it rains the next day. Today it didn't rain. On which days must there be no rain?
(a) Tommorrow. (b) All future days. (c) Yesterday. (d) All previous days.

Problem 3.14. For p = "You're sick", q = "You miss the final", r = "You pass FOCS". Translate into English:
(a) $q \to \neg r$. (b) $(p \to \neg r) \vee (q \to \neg r)$. (c) $(p \wedge q) \vee (\neg q \wedge r)$.

Problem 3.15. Here is a logic puzzle from a pshchology experiment studying how humans perform deductive analysis. You have before you the cards (only the top is visible). Each card has a number on one side and a letter on the other.

$\boxed{\mathbf{S}}$ $\boxed{\mathbf{5}}$ $\boxed{\mathbf{P}}$ $\boxed{\mathbf{T}}$ $\boxed{\mathbf{3}}$ $\boxed{\mathbf{4}}$

Rule: If a card has a P on it, then the other side must be a 5.

To verify that the rule is not been broken, which are the fewest cards that you need to turn over, and why?

Problem 3.16. Here is a logic puzzle from a bar setting.

Law: If you are drinking beer, then you must be 21 or older.

The bouncer sees 4 people A, B, C, D shown on the right. Which of the following must the bouncer check to ensure the bar abides by the law.

(a) A's age. (b) B's age. (c) C's drink. (d) D's drink.

> A is drinking a beer;
> B is drinking a coke;
> C is drinking something and looks under 21;
> D is drinking something and looks over 50.

Problem 3.17. Here's what we know about Kilam.

> 1: Kilam eats Italian or French each night.
> 2: He eats French or wears dress shoes.
> 3: Whenever he eats Italian and wears a coat, he does not wear a bow tie.
> 4: He never eats French unless he also wears a coat or dress shoes.
> 5: If he wears dress shoes, he wears a coat.

(a) Will Kilam ever be without a coat? (b) Today, Kilam in a bow tie. What else did he wear? What did he eat?

Problem 3.18 (Converse, Contrapositive). The converse of an implication $p \to q$ is $q \to p$; the contrapositive is $\neg q \to \neg p$. For each implication, give the converse and contrapositive.

(a) If a/b and b/c are in $\mathbb{Z}$, then $a/c \in \mathbb{Z}$.
(b) $x^2 = 1 \to x = 1$.
(c) If $x^2 = x + 1$, then $x = (1 \pm \sqrt{5})/2$.
(d) If $p > 2$ is prime, then p is odd.
(e) $ab = 0 \to a = 0$ or $b = 0$.
(f) If $n \in \mathbb{N}$ ends in 3, then 3 divides n.
(g) For $n \in \mathbb{Z}$, $3n = 9 \to n^2 = 9$.
(h) For $n \in \mathbb{Z}$, $n^2 > 9 \to n > 3$.
(i) For $n \in \mathbb{N}$, $n^2 > 9 \to n > 3$.
(j) If $n \in \mathbb{N}$ is odd then $n^2 + n - 2$ is even.
(k) Every connected graph G has 32 vertices.
(l) Honk if you love FOCS.

In each case, if you can, determine true or false for the converse and contrapositive (both can be true or false).

Problem 3.19. Mathematically formulate the usual meaning of each sentence using p, q, r.

(a) p : "you will succeed at this job"; q : "you know Java"; r : "you know Python".
Sentence: IF you know Java or Python, you will succeed at this job."

(b) p : "you buy a lunch entree"; q : "you can have soup"; r : "you can have salad".
Sentence: IF you buy a lunch entree, you can have soup or salad."

(c) p : "you may enter the US"; q : "you have a job"; r : "you have a green-card".
Sentence: IF you have a job or green-card, you may enter the US."

Problem 3.20 (DNF). Use $\neg, \wedge, \vee$ to give compound propositions with these truth-tables. *[Hints: Consider only rows which are* T *and use* OR *of* AND*'s.]*

(a)

q	r	
T	T	F
T	F	T
F	T	F
F	F	F

(b)

q	r	
T	T	F
T	F	T
F	T	F
F	F	T

(c)

p	q	r	
T	T	T	F
T	T	F	F
T	F	T	F
T	F	F	F
F	T	T	F
F	T	F	T
F	F	T	F
F	F	F	F

(d)

p	q	r	
T	T	T	F
T	T	F	T
T	F	T	F
T	F	F	F
F	T	T	T
F	T	F	T
F	F	T	F
F	F	F	F

(AND-OR-NOT formulas use only $\neg, \wedge, \vee$. Any truth-table can be realized by an AND-OR-NOT formula. Even more, one can construct an OR of AND's, the disjunctive normal form (DNF).)

Problem 3.21. Give pseudocode for a program that takes the input $n \in \mathbb{N}$ and outputs all the possible truth values (rows in the truth table) for the statements $p_1, p_2, \ldots, p_n$. The correct output for $n = 1$ and $n = 2$ are shown. We suggest you either use recursion or a while loop.

p_1
F
T

p_1	p_2
F	F
F	T
T	F
T	T

Problem 3.22. How many rows are in the truth table of $\neg(p \vee q) \wedge \neg r$? Give the truth table.

Problem 3.23.

(a) Give the truth-table for these compound propositions.

$$p \wedge \neg p; \quad p \vee \neg p; \quad p \to (p \vee q); \quad ((p \to q) \wedge (\neg q)) \to \neg p.$$

(b) How many rows are in the truth-table of the proposition $(p \vee q) \to (r \to s)$.

(c) Show that $(p \to q) \vee p$ is ALWAYS true. This is called a tautology.

Problem 3.24. Let $q \to p$ be F and $q \to r$ be T. Answer T/F: (a) $p \vee q$ (b) $p \to q$ (c) $p \wedge q \wedge r$.

Problem 3.25. Given the information, answer the question true, false or I don't know.

(a) IF you ace the quiz and final, THEN you get an A. You aced the final. Did you get an A?
(b) IF you ace the quiz or final, THEN you get an A. You aced the final. Did you get an A?
(c) IF you ace the quiz and final, THEN you get an A. You got an A. Did you ace the final?
(d) IF you ace the quiz or final, THEN you get an A. You got an A. Did you ace the final?
(e) IF you ace the quiz and final, THEN you get an A. You got a B. Did you ace the final?
(f) IF you ace the quiz or final, THEN you get an A. You got a B. Did you ace the final?

Problem 3.26. Given the information, answer the question true, false or I don't know.

(a) IF it rains, THEN Kilam brings an umbrella. It did not rain. Did Kilam bring an umbrella?
(b) EVERYONE who eats apples is healthy. Kilam is healthy. Does Kilam eat apples?
(c) EVERYONE who eats apples is healthy. Kilam is not healthy. Does Kilam eat apples?
(d) EVERYONE who eats apples is healthy. Kilam eats apples. Is Kilam healthy?
(e) You can have cake OR ice-cream. You had cake. Can you have ice-cream?
(f) Lights are turned on in the night. Lights are off. Is it day?
(g) Lights are turned on in the night. It is day. Are the lights on?
(h) IF you are a singer, THEN you don't eat cheese. You don't eat cheese. Are you a singer?

Problem 3.27. It rains on Tuesdays. When it rains, Kilam does not run. When it's dry, Kilam either runs or goes to work early. When Kilam runs, he must eat breakfast. All people have coffee with breakfast. Answer T/F/I don't know.

(a) Today is Wednesday, so Kilam had coffee.
(b) Today, Kilam went to work early, so it did not rain.
(c) Today, Kilam did not go for a run, so either it is Tuesday or Kilam went early to work.
(d) On Friday, Kilam did not go to work early, so he must have had coffee.
(e) On Friday, Kilam did not go to work early, so either it rained or Kilam had coffee.

Problem 3.28. On your back bumper is a sticker saying "Honk if you love FOCS." What can you conclude about the driver in the car behind you if: (a) You hear honking? (b) You don't hear any honking?

Problem 3.29. For each pair of implications, determine which one is likely to be true in practice.

(a) If I was in the rain, then my hair is wet.
If my hair is wet, then I was in the rain.

(b) If you have a CS-degree, then you took FOCS.
If you took FOCS, then you have a CS-degree.

Problem 3.30. Sherrif Suzie and Big Mike debate the implication

If $n \in \mathbb{N}$ is odd, then $n^2 + 4$ is prime.

Suzie tries to convince Mike it's false by giving a counterexample (right).

(a) Why is Sherrif Suzie only trying odd n to find a counterexample?
(b) Does the conversation convince you that the implication is true? If not, why not?
(c) Find a counterexample to show that the implication is false.

SS: Perhaps 1 is a counterexample.
BM: Nope: $1^2 + 4 = 5$, which is prime.
SS: What about 3?
BM: Nope: $3^2 + 4 = 13$, which is prime.
SS: Let's try 5?
BM: No again: $5^2 + 4 = 29$, a prime.
SS: You win. The implication seems true.
BM: Phew! I'm tired. Let's have a drink.

Problem 3.31. Use truth tables to determine logical equivalence of compound statements.

(a) Are $(p \to q) \to r$ and $p \to (q \to r)$ (b) $(p \wedge \neg q) \vee q$ and $p \vee q$.

Problem 3.32. Use truth-tables to verify the rules for derivations in Figure 3.1 on page 29. Now use the rules in Figure 3.1 to show logical equivalence $\neg((p \wedge q) \vee r) \overset{\text{eqv}}{\equiv} (\neg p \wedge \neg r) \vee (\neg q \wedge \neg r)$.

Problem 3.33. Show that $(p \rightarrow q) \vee (q \rightarrow p)$ is always true for arbitrary statements p, q.

Problem 3.34 (Satisfiability). We list four clauses using the propositions p, q, r, s.

$$(\overline{p} \vee q) \qquad (\overline{q} \vee r) \qquad (\overline{r} \vee s) \qquad (\overline{s} \vee q)$$

Give a truth assignment (T/F) to each of p, q, r, s so that every clause is true, i.e. satisfied.

Problem 3.35. Assign T/F to p, q, r, s to make the compound proposition true (if possible).
(a) $(p \rightarrow q) \wedge (q \leftrightarrow \neg p)$. (b) $(p \vee \neg r) \wedge (r \wedge s) \wedge (\neg s \wedge \neg p) \wedge (p \wedge (q \rightarrow r))$. (c) $(p \leftrightarrow q) \rightarrow \neg(p \rightarrow q)$.

Problem 3.36. Using statements p, q, r, s, we list eight clauses. Each clause is an OR of 3 terms, where a term is a statement or its negation. A proposition appears at most once in a clause.

$$(p \vee q \vee r) \quad (\overline{q} \vee r \vee s) \quad (\overline{p} \vee q \vee s) \quad (\overline{p} \vee q \vee r) \quad (p \vee r \vee \overline{s}) \quad (\overline{p} \vee \overline{q} \vee s) \quad (\overline{p} \vee \overline{r} \vee s) \quad (\overline{p} \vee q \vee \overline{s})$$

(a) Give a truth assignment (T or F) to each of p, q, r, s so that every clause is true (satisfied).
(b) Construct eight clauses for which no truth assignment to p, q, r, s can satisfy all eight clauses.
(c) Show that it is always possible to satisfy at least 7 of the 8 clauses, no matter how many variables there are, as long as there are exactly three terms in each clause.

Problem 3.37. For the domain of all students, use the predicates $S(x)$ = "x is a student", $I(x)$ = "x is a smart" and $F(x, y)$ = "x is a friend of y" to formulate the following statements.

(a) Kilam is a student.
(b) All students are smart.
(c) No student is a friend of Kilam.
(d) Every student is a friend of some other student.
(e) There is a student who is a friend of every other student.
(f) All smart students have a friend.

Problem 3.38. What is the negation of each statement. You are being asked to "translate" a negation like IT IS NOT THE CASE THAT(Kilam is a student) into "normal" English.

(a) Kilam is a student.
(b) All students are smart.
(c) No student is a friend of Kilam.
(d) Every student is a friend of some other student.
(e) There is a student who is a friend of every other student.
(f) All smart students have a friend.

Problem 3.39. Use predicates and connectors to precisely state each interpretation of "Every American has a dream."
(a) There is a single dream, the "American dream," and every American has that same special dream.
(b) Every American has their own personal dream (possibly a different one for each person).
(c) Every American has one (and only one) personal dream (possibly a different one for each person).

Problem 3.40. Give the negation of each claim in sensible English. Start with IT IS NOT THE CASE THAT($\cdot$) and then take the negation inside the quantifiers.
(a) There is a constant C for which $n^3 \leq Cn^2$ for all $n \in \mathbb{N}$.
(b) For some $x > 0$, there is a constant C for which, for all $n \in \mathbb{N}$, $n^{2+x} \leq Cn^2$.

Problem 3.41. Give the negation of each claim. Simplify your statement so that the negation, $\neg$, is not to the left of any quantifier. Determine which of the original statement or the negation is T. (Can both be T? Can neither be T?)
(a) $\forall x \in \mathbb{Z} : (\exists y \in \mathbb{Z} : x+2y = 3)$. (b) $\exists x > 0 : (\forall y > 0 : xy < x)$. (c) $\exists (x, y) \in \mathbb{Z}^2 : (x+y = 13) \wedge (xy = 36)$.

Problem 3.42. On the Isle-of-FOCS are Saints who always tell the truth and Sinners who always lie.
(a) Two people, A and B, made these statements. Which (if any) of them must be Saints?
A: "Exactly one of us is lying." B: "At least one of us is telling the truth."
(b) Three people, A, B, C, made these statements. Which (if any) of them must be Saints?
A: "Exactly one of us speaks the truth." B: "We are all lying." C: "The other two are lying."

Problem 3.43. For $x \in \{1, 2, 3, 4, 5\}$ and $y \in \{1, 2, 3\}$, determine T/F with short justifications.
(a) $\exists x : x+3 = 10$ (b) $\forall y : y+3 \leq 7$ (c) $\exists x : (\forall y : x^2 < y+1)$ (d) $\forall x : (\exists y : x^2+y^2 < 12)$

Problem 3.44. For $x, y \in \mathbb{Z}$, determine T/F with short justifications.
(a) $\forall x : (\exists y : x = 5/y)$ (b) $\forall x : (\exists y : y^4 - x < 16)$ (c) $\forall x : (\forall y : \log_2 x \neq y^3)$

Problem 3.45. Let $P(x, h)$ = "Person x has hair h" and $M(h)$ = "Hair h is grey". Formulate:
(a) Kilam has some grey hair.
(b) Someone has all grey hair.
(c) Nobody is bald.
(d) Kilam does not have all grey hair.

Problem 3.46. Formulate the appropriate predicates, identify the domain of the predicate and give the "mathematical" version of the following statements.

(a) Every person has at most one job.
(b) Kilam has some grey hair.
(c) Everyone has some grey hair.
(d) Everyone is a friend of someone.
(e) All professors consider their students as a friend.
(f) No matter what integer you choose, there is always an integer that is larger.
(g) Every natural number has a prime factorization.
(h) Two courses which have the same student cannot have exam times that overlap.
(i) No student has won a TV game-show.
(j) There is a soul-mate for everyone.
(k) 15 is a multiple of 3. (In your predicate you must define what a multiple is.)
(l) 15 is not a multiple of 4.
(m) 16 is a perfect square. (In your predicate you must define what a perfect square is.)
(n) Every student in FOCS has taken a course in calculus and a course in programming.
(o) Every CS-major who graduates has taken FOCS.
(p) Between any two rational numbers there is another rational number.
(q) Between any rational number and larger irrational number is another irrational number.
(r) Between any rational number and larger irrational number is another rational number.

Problem 3.47. Use quantifiers to precisely formulate the associative laws for multiplication and addition and the distributive law for multiplication over addition.

Problem 3.48. For the predicates $F(x)$="x is a freshman" and $M(x)$="x is a math major', translate into English:
(a) $\exists x : M(x)$ (b) $\neg\exists x : F(x)$ (c) $\forall x : (M(x) \rightarrow \neg F(x))$ (d) $\neg\exists x : (M(x) \wedge \neg F(x))$

Problem 3.49. What is the difference between $\forall x : (\neg\exists y : P(x) \rightarrow Q(y))$ and $\neg\exists y : (\forall x : P(x) \rightarrow Q(y))$?

Problem 3.50. P and Q are predicates. Are these pairs of statements equivalent. Explain.
(a) $\forall x : (\neg\exists y : P(x) \rightarrow Q(y))$ and $\neg\exists x : (\exists y : P(x) \rightarrow Q(y))$.
(b) $\forall x : (\neg\exists y : P(x) \rightarrow Q(y))$ and $\neg\exists y : (\forall x : P(x) \rightarrow Q(y))$.

Problem 3.51. Let $P(x)$ and $Q(y)$ be predicates. Verify that these quantified compound propositions are equivalent. (To show that quantified predicates are equivalent, show that when one is true, the other is true and *vice versa.*)

$$\forall x : (\neg\exists y : P(x) \rightarrow Q(y)) \quad \text{and} \quad \neg\exists x : (\exists y : P(x) \rightarrow Q(y)).$$

Problem 3.52. Let $P(x)$ and $Q(x)$ be arbitrary predicates. Prove or disprove:
(a) $\forall x : (P(x) \wedge Q(x)) \overset{\text{eqv}}{\equiv} (\forall x : P(x)) \wedge (\forall x : Q(x))$ (b) $\forall x : (P(x) \vee Q(x)) \overset{\text{eqv}}{\equiv} (\forall x : P(x)) \vee (\forall x : Q(x))$

Problem 3.53. Suppose P and Q are predicates taking an input whose domain is D which happens to be an empty domain. If you can, determine the truth values of:
(a) $\forall x : P(x)$ (b) $\exists y : P(y)$ (c) $(\forall x : P(x)) \vee (\exists y : P(y))$ (d) $(\forall x : P(x)) \vee P(y)$

Problem 3.54. Determine if each quantified compound statement is always true. If no, give a counterexample (specify the predicate and domain). If yes, explain why. (a) $(\exists x : P(x)) \rightarrow (\forall x : P(x))$ (b) $(\forall x : P(x)) \rightarrow (\exists x : P(x))$.

Problem 3.55. x and y are integers. Answer true or false, explaining your reasoning.
(a) $\forall x : (\exists y : 2x - y = 0)$ (c) $\exists x : (\forall y : 2x - y = 0)$ (e) $\exists x : (\forall y : x2^y = 0)$
(b) $\forall y : (\exists x : 2x - y = 0)$ (d) $\exists y : (\forall x : 2x - y = 0)$ (f) $\exists y : (\forall x : x2^y = 0)$

Problem 3.56. In which (if any) of the domains $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ are these claims T. (x and y can have different domains.)
(a) $\exists x : x^2 = 4$ (b) $\exists x : x^2 = 2$ (c) $\forall x : (\exists y : x^2 = y)$ (d) $\forall y : (\exists x : x^2 = y)$

Problem 3.57. True or false. A counterexample to "ALL ravens are black" must be: (a) Non-raven. (b) Non-black.

Problem 3.58 (Hempel's Paradox). Do you believe in induction? Consider the claim "ALL ravens are black."
(a) You observe a black raven. Does that strengthen your belief that "ALL ravens are black?" Is it a proof?
(b) You observe a white sock. Does that strengthen your belief that "ALL non-black things are not ravens?"
(c) Show that "ALL ravens are black." is logically equivalent to "ALL non-black things are not ravens." So, does inductive logic suggest that observing a white sock strengthens your belief that "ALL ravens are black?" Hmm...

Problem 3.59 (Closure). A set $\mathcal{S}$ is closed under an operation if performing that operation on elements of $\mathcal{S}$ returns an element in $\mathcal{S}$. Here are five examples of closure.

$$\begin{array}{rcl} \mathcal{S} \text{ is closed under addition} & \to & \forall(x,y) \in \mathcal{S}^2 : x+y \in \mathcal{S}. \\ \mathcal{S} \text{ is closed under subtraction} & \to & \forall(x,y) \in \mathcal{S}^2 : x-y \in \mathcal{S}. \\ \mathcal{S} \text{ is closed under multiplication} & \to & \forall(x,y) \in \mathcal{S}^2 : xy \in \mathcal{S}. \\ \mathcal{S} \text{ is closed under division} & \to & \forall(x,y \neq 0) \in \mathcal{S}^2 : x/y \in \mathcal{S}. \\ \mathcal{S} \text{ is closed under exponentiation} & \to & \forall(x,y) \in \mathcal{S}^2 : x^y \in \mathcal{S}. \end{array}$$

Which of the five operations are the following sets closed under? (a) $\mathbb{N}$. (b) $\mathbb{Z}$. (c) $\mathbb{Q}$. (d) $\mathbb{R}$.

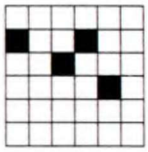
Problem 3.60. Compute the number of positive divisors of the following integers:

$$6, 8, 12, 15, 18, 30 \qquad\qquad 4, 9, 16, 25, 36$$

State a precise conjecture that relates a property of the number of divisors of n to a property of n (proof not needed). (You may define convenient notation, for example let $\phi(n)$ be the number of positive divisors of n).

Problem 3.61. Use dominos to tile an 8×8 chessboard with two opposite-color squares removed. Tinker. Formulate a precise conjecture about whether the board with missing squares can be tiled. You don't have to prove your conjecture.

Problem 3.62. In the Ebola model, a square is infected if at least two (non-diagonal) neighbors are infected.



(a) An initial infection is shown. Show the final state of the grid, i.e., who is ultimately infected.
(b) Are there 5 initial infections that can infect the whole 6×6 square. What about with 6 initial infections? Also try the 4×4 and 5×5 grids.
(c) For the $n \times n$ grid, $n \in \mathbb{N}$, formulate a conjecture for the fewest initial infections required to infect the whole square. You do not have to prove your claim, but be **precise** in your statement.
(d) **[Hard]** Can you think of a way to justify your conjecture?

Problem 3.63. For n numbers $x_1, \ldots, x_n$, define the average $\mu = (\sum_{i=1}^n x_i)/n$ and the average of squares $s^2 = (\sum_{i=1}^n x_i^2)/n$. Tinker with some numbers and $n = 2, 3$, computing μ^2 and s^2. Make a conjecture that relates μ^2 and s^2. You don't have to prove your conjecture.

Problem 3.64 (Josephus Problem). From n children, a winner is to be picked by standing the children in a circle and then proceeding around the circle removing every other child until one remains (the winner). (Variants of this method are popular: 1 potato, 2 potato, 3 potato, 4; 5 potato, 6 potato, 7 potato, more and Eeny, meeny, miny, moe.) Number the children $1, \ldots, n$, in order as they stand in the circle (the process starts at child 1, and child 2 is the first to be removed). Let $J(n)$ be the winner when there are n children.

(a) For $n = 8$, in what order are the children removed? Who is the winner (what is $J(8)$).
(b) Compute $J(2), J(4), J(8), J(16), J(32)$. Can you guess $J(64)$?
(c) Compute $J(3), J(6), J(12), J(24), J(48)$. Can you guess $J(96)$?
(d) Formulate a conjecture for $J(2^k)$ for $k \geq 0$. You don't have to prove it.
(e) Formulate a conjecture for $J(q2^k)$ in terms of $J(q)$, for $k \geq 0$. You don't have to prove it.
(f) **[Hard]** Tinker like crazy and formulate a conjecture for $J(n)$. You don't have to prove it.

(In one version of the legend, Jewish historian Flavius Josephus and 40 other Jews are trapped in a cave by Romans. Instead of surrender, they stood in a circle, picking every seventh person to die at the hand of the next person picked (the last man standing commits suicide). Josephus determined where he and a friend should stand to be the last two men standing, at which point they promptly surrendered. Can you figure out where they stood?)

Problem 3.65 (Internet task). The function $f : A \mapsto B$ maps A to B. Lookup definitions of 1-to-1 (injection), onto (surjection), invertible (bijection) and give examples of each type of function when A and B are given by:

(a) $A = \{1, 2\}$, $B = \{a, b\}$ (b) $A = \{1, 2\}$, $B = \{a, b, c\}$ (c) $A = \{1, 2, 3\}$, $B = \{a, b\}$.

(If you think it cant be done for specific cases, explain why.)