> Good mathematics is not about how many answers you know. It's about how you behave when you don't know. – **unknown**
>
> The only way to learn math is to do math. – **Paul Halmos**
>
> Don't let anyone work harder than you. – **Serena Williams**
>
> To make a great dream come true, the first requirement is a great capacity to dream; the second is persistence. – **Cesar Chavez**

# Chapter 0

# Background and Pep Talk

1: The basics, the resources and putting yourself in the right mood.

Even "self-contained" books start somewhere and build. We cater to 2nd-year undergraduates in a mathematical, engineering or scientific discipline who have had one year of computer science (programming/data-structures) and one year of calculus. In short, we assume high-school mathematics (numbers, geometry, algebra, ...), some programming and some calculus. Here are some refresher questions. Answer them, perhaps with a little help from the solutions.

TO STUDY THIS BOOK YOU MUST NOT USE ELECTRONIC DEVICES UNLESS **EXPLICITLY** ASKED TO DO SO.

### Numbers and Sets.

1. What is the prime factorization of 252?
2. What is the minimum element in the set $\{8, 9, 3, 10, 19\}$?
3. What is the union of the sets $\{8, 9, 3, 10, 19\}$ and $\{3, 10, 1, 7\}$? What is the intersection?
4. Does this set of positive numbers have a minimum element:

   $$\{25, 97, 107, 100, 18, 33, 99, 27, 2014, 2200, 23, \ldots\}$$

   The set could be infinite. You only know that every number is positive.
5. Give examples of an integer, a rational number and a real number.
6. Let $k$ be a whole number (e.g. $k = 7$). Which of the following are divisible by 3:

   $$3k,\ 3k+1,\ 3k+2,\ 3k+3,\ 3k+4,\ 3k+5.$$

### Logarithms and Exponentials.

1. $\ln(2) \approx 0.693$; $\ln(3) \approx 1.098$. What is $\ln(12)$?
2. $2^{10} = 1024 \approx 1,000$. What is $2^{20}$?
3. How are $\ln(1 \times 2 \times 3 \times \cdots \times 10)$ and $(\ln 1 + \ln 2 + \ln 3 + \cdots + \ln 10)$ related?
4. How are $2^a/2^b$ and $2^{a-b}$ related? What is $2^0$?
5. Show $\log_2 100 = \log_2 10 \times \log_{10} 100$. More generally, show that $\log_\alpha x = \log_\alpha \beta \times \log_\beta x$.

### Sums and Products.

1. What are: (a) $1 + 2 + 3 + \cdots + 1000$ (b) $1 + 2 + 3 + \cdots + n$ (c) $1 + \frac{1}{7} + \frac{1}{7^2} + \frac{1}{7^3} + \frac{1}{7^4} + \cdots$?
2. What is $5!$? What is $n!$? What is $0!$?

3. $\sum$ is an invitation to add. $\sum_{i\geq 1}^{i\leq 10} f(i)$ asks you to add $f(i)$ for the whole numbers $i$ which satisfy the lower bound in the lower limit and the upper bound in the upper limit,

$$\sum_{i\geq 1}^{i\leq 10} f(i) = f(1) + f(2) + \cdots + f(10).$$

Similarly, $\prod$ is an invitation to multiply,

$$\prod_{i\geq 1}^{i\leq 10} f(i) = f(1) \times f(2) \times \cdots \times f(10).$$

We often simplify the notation even more and write $\sum\limits_{i=1}^{10} f(i)$ and $\prod\limits_{i=1}^{10} f(i)$.

What is $\sum\limits_{i=1}^{1000} i$? What is $\sum\limits_{k=1}^{1000} k$? What is $\sum\limits_{k=1}^{1000} i$? What is $\sum\limits_{|i-1|\leq 5} i$?

4. What is $1+2+3+\cdots+k$? What is $\sum\limits_{i=1}^{k} i$? What is $\sum\limits_{k=1}^{n} k$?

5. Write the next two quantities using factorials: $\sum\limits_{i=1}^{k} \ln(i)$; $\prod\limits_{i=1}^{k} i$.

6. "Empty" sums, e.g. $\sum\limits_{i\geq 3}^{i\leq 1} i$, are 0. "Empty" products, e.g. $2^0$ and $0!$, are 1. Here's why.

   You want the sum of the numbers in the set $\{3, 10, 1, 7\}$. You are lazy, but your two friends are not. You split the set into two disjoint subsets $\{3, 10\}$ and $\{1, 7\}$, give one subset to each friend, and request the subset-sums. You receive the subset-sums 13 and 8, and add these two numbers to get the full sum of 21. This simple procedure should work no matter how you split up your original set. Suppose you gave one friend all the numbers, and the other none of them. For the procedure to work, your friend with none of the numbers (who computes an empty sum) must return 0.

   To get the product instead of the sum, your friends would give you the subset-products 30 and 7, which you multiply to get 210. If one friend gets all the numbers, and the other none of them, the procedure still works if the friend with no numbers (who computes an empty product) returns 1.

**Algebra.**

1. What is $(1+2)^2$?
2. What is $(a+b)^2$? What about $(a+b)^3$?
3. What are the solutions to $x^2 - 5x - 6 = 0$?
4. What are the solutions to $e^{2x} - 5e^x - 6 = 0$?
5. What are $x$ and $y$ when $x + y = 2$ and $2x + 3y = 7$?
6. Use partial fractions to simplify the expressions $\dfrac{3x+11}{x^2 - x - 6}$ and $\dfrac{3x+11}{x^2+6x+9}$.

**Calculus.**

1. Which of these series converges:

$$\begin{aligned}
&1 + 2 + 2^2 + 2^3 + 2^4 + \cdots \\
&1 + \tfrac{1}{2} + (\tfrac{1}{2})^2 + (\tfrac{1}{2})^3 + (\tfrac{1}{2})^4 + \cdots \\
&1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 + \cdots \\
&1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \cdots \\
&1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{6} + \cdots
\end{aligned}$$

2. What are the derivatives of: $x^3$; $e^{2x}$; $2^x$; $\dfrac{1}{x}$; $\dfrac{1}{x^2}$; $\ln x$; $\log_2 x$; $\ln 2x$?

3. What are the indefinite integrals of: $x^3$; $e^{2x}$; $2^x$; $x^{-1}$; $x^{-2}$?
4. What is the limit as $x \to 0$ of the functions:
$$\frac{e^x - 1}{\sin(2x)}; \quad \frac{e^x - 1}{1+x}; \quad \frac{e^x - 1}{\sin(x^2)}; \quad \frac{e^x - 1}{x+x^2}; \quad \frac{e^x - 1}{e^{2x}-1}.$$
5. What is the limit as $x \to \infty$ of the functions: $\dfrac{e^x - 1}{e^{2x}-1}$; $\dfrac{e^x - 1}{x^3 + 2e^x}$; $\dfrac{e^x}{x^x}$.
6. What is the Taylor expansion of $f(x) = 1/(2 + \sin(x))$ around $x = \pi/2$?
7. What is $\int_0^T dx\ (1 + x^2)^{-1}$?
8. Define the function $f(t) = \int_0^t dx\ \sin(1 + x^2 e^x)$. What is $\frac{d}{dt} f(t)$?

## Setting Up Expectations

This book is by no means a complete coverage of discrete mathematics and computing. We chose some topics to cover, and within those topics we left out many advanced concepts to satisfy the bandwidth constraints of a course. To compensate, we have given a generous helping of quizzes, exercises and problems:

- pop quizzes ask you if you are still awake;
- exercises stretch your muscles within the current context;
- easier problems test your knowledge and provide practice with the concepts;
- harder problems guide you through some of the more advanced concepts.

There are several books which delve more deeply or more completely into discrete mathematics. As for the deeper books, we will say more in the epilogue. Here are some more complete books at this level:

- *Discrete Mathematics and its Applications,* by Rosen.
- *Discrete Mathematics with Applications,* by Epp.
- *Mathematics for Computer Science,* by Lehman, Leighton, and Meyer. (MIT **open** course.)

Analogous books for the theory of computation are

- *Introduction to the Theory of Computation,* by Sipser.
- *Elements of the Theory of Computation,* by Lewis and Papadimitriou.
- *Introduction to Automata Theory, Languages, and Computation,* by Hopcroft, Motwani, and Ullman.

The internet is an endless resource for enrichment. Combined with the above books which have many solved exercises, there is no shortage of practice problems. Mathematics is like any sport. You have to train.

**Final Exam.** You relaxed all winter and ran the spring Boston Marathon. You got destroyed. The next year, you put in some intense 5-hour workouts three days before the race. You got destroyed. The third year you wisened up and got a coach 3 months ahead of the race. You took notes which helped fine-tune your 5-hour workouts three days before the race. You still got destroyed. Finally on the 4th attempt you tried something new. You got the coach as before and incorporated the coach's teachings into a 3 hour work-out every day. You were getting fitter. Three days before the race, you ramped up your training. Wow! You finished in the top-10% of runners in the race. 🙂 *Your* Boston Marathon is the final exam in the course.

## Pep Talk

Society won't object if you say "Math isn't for me," and yet it's as ridiculous as saying "Running isn't for me," or "English isn't for me." If you move to USA, to get around you learn English. Anyone can do it. If you wish to precisely model "stuff", you learn math. Anyone can do it, enough said. Yes, math is hard. But so is learning English. Math is for anyone who will put in the effort. You *must* put the effort.

You also can learn to "think", to be creative and solve problems. Solving a problem starts with strategic considerations, the foremost being to prepare the psychology. Two classics on that front are

*How to Solve It: A New Aspect of Mathematical Method* by George Polya;
*The Art and Craft of Problem Solving*, by Paul Zeitz.

To get into the right state of mind, listen well to the story of Polya's little mouse.

> "A mouse tries to escape from an old fashioned cage. After many futile attempts bouncing back-and-forth, thumping his body against the cage bars, he finally finds one place where the bars are slightly wider apart. The mouse, bruised and battered escapes through this small opening, and to his elation, finds freedom." – Polya

Try, try and try again. Vary the trials that you may find the rare favorable path to a discovery. The story of the mouse is for everyone. When you hit an interesting problem just outside the reach of your familiar realm, don't throw your hands in the air out of submission because a solution doesn't magically appear.

*The solution to an interesting problem is not obvious.*

That is the law. When you don't see the solution immediately, don't be discouraged. Realize that such situations are faced by every explorer entering the unknown. All that Columbus needed was a curiosity, perseverance and a little good luck. Columbus marched forward despite adversity. A mathematics problem can throw much adversity your way. Don't fret; don't give up. Be patient; persevere. Be that mouse, trying this and that. Eventually a small opening will appear and you may then walk through. Most importantly, realize that these are the rules for everyone, from the amateur to the professional mathematician.

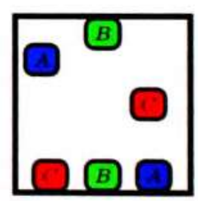
It may be true that there is an innateness to mathematical creativity, just as not everyone is a Mozart. Alas, some may appear better at mathematics than others. They may be better trained, or just better. In all cases, it doesn't matter. Leave them to their business, and you focus on yours. To warm up to the importance of perseverance, try not to turn another page until you solve this visual connection puzzle, which we have lifted directly from *The Art and Craft of Problem Solving*.

**Pop Quiz 0.1 [Connection puzzle]**

Connect tiles of the same letter with wires that don't cross, enter tiles, or exit the box (you may bend wires). If you think it can't be done, why not?

Don't be too quick to dismiss either conclusion, or to peek at the answer. Patience. Try this and that. Fiddle around. Make sure you understand the challenge.

To solve such problems, "You need brains and good luck. But, you must also sit tight and wait till you get a bright idea." – Polya.




Ask this puzzle of your best mathematician friends. It will drive home the fact that everyone has to think about a problem. Everyone has to try one or two things that fail, then one or two things that partially succeed. Through perseverance, these failures and partial successes are what ultimately shine light on the way out.

Our primary goal is to impart the knowledge and tools of discrete mathematics to you. But, you are the one who must use these tools to solve problems. And, to solve problems, you must prepare yourself mentally for a tough road. Don't be afraid to experiment. Don't be afraid to crash your CPU or corrupt your operating system – you can always perform a fresh install of linux. Put yourself in the right frame of mind.

**Be the mouse!**

Let us begin.