A Finite Difference Time Domain Method for Two Dimensional Electromagnetic Wave Propagation In Lossy Dielectric Media With Cyclic and Perfectly Matched Layer Boundary Conditions

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Abstract

In this work, the foundations behind the one- and two-dimensional finite-difference time-domain (FDTD) method in the context of electromagnetic wave propagation are presented. Two different boundary conditions, cyclic and perfectly matched layer (PML), are presented and discussed. This work includes implementations of the method using both boundary conditions, written in standard Python 3.

Keywords: Electromagnetic wave simulation, finite-difference time-domain, perfectly matched layer

1 Introduction

The goal of the finite-difference time-domain (FDTD) method is to simulate the propagation of electromagnetic waves in complex geometry. For instance, propagation through several different materials with different dielectric constants, or electrical conductivities. Problems such as these are often impossible to solve analytically, and so numerical approaches, such as this one, are required.

This work is an overview of the derivation of the necessary expressions used in the FDTD method as well as two types of boundary conditions. Implementations of the method for both boundary conditions with results are presented and discussed.

2 Maxwell's Equations for Electromagnetic Waves

The differiential form of Maxwell's equations for the electric field, **E**, and magnetic field, **B**, in vacuum, are [?]

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}, \tag{1}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{2}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{3}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}. \tag{4}$$

Where

- ρ is the free electric charge density
- $\mathbf{J}$ is the free current density
- ϵ_0 is the permittivity of free space
- μ_0 is the permeability of free space
- c is the speed of light, $c \equiv \frac{1}{\sqrt{\epsilon_0 \mu_0}}$

In the context of electromagnetic wave propagation, only Eqs. **??** and **??** are of interest. Rewriting these two equations in terms of the magnetic field strength, $\mathbf{H} = \mathbf{B}/\mu_0$, and subsituting $c \equiv \frac{1}{\sqrt{\epsilon_0 \mu_0}}$ yields

$$\frac{\partial \mathbf{H}}{\partial t} = -\frac{1}{\mu_0}\nabla \times \mathbf{E}, \tag{5}$$

$$\epsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \nabla \times \mathbf{H} - \mathbf{J}. \tag{6}$$

In a dielectric medium, we introduce the relative permittivity, ϵ_r, which is material-specific. This allows us to write the permittivity as $\epsilon = \epsilon_r \epsilon_0$. We could also introduce the relative permeability, but in this work we will assume all materials are nonmagnetic, meaning $\mu_r = 1$ and so $\mu = \mu_0$. We also introduce the electrical conductivity $\sigma \equiv J/E$, so that we may write the free current density as $\mathbf{J} = \sigma \mathbf{E}$. The electrical conductivity serves as a measure of how propagating waves are attenuated in the medium. So Eqs. **??** and **??** become

$$\frac{\partial \mathbf{H}}{\partial t} = -\frac{1}{\mu_0}\nabla \times \mathbf{E}, \tag{7}$$

$$\frac{\partial \mathbf{E}}{\partial t} = \frac{1}{\epsilon}\nabla \times \mathbf{H} - \frac{\sigma}{\epsilon}\mathbf{E}. \tag{8}$$

A more general form of these equations uses the electric displacement, **D**, which can be written as $\mathbf{D}(\omega) = \epsilon_0 \epsilon_r^*(\omega)\mathbf{E}(\omega)$, where ω is the frequency of the wave and ϵ_r^* is the frequency dependent dielectric constant of the media [?]. If we assume ϵ_r^* is of the form

$$\epsilon_r^*(\omega) = \epsilon_r + \frac{\sigma}{i\omega\epsilon_0}, \tag{9}$$

then we see that the electric displacement is

$$D(\omega) = \epsilon_0\epsilon_r E(\omega) + \frac{\sigma}{i\omega}E(\omega). \tag{10}$$

Transforming this back to the time domain requires an identity which is derived in Appendix **??**. In the time domain, the above equation is

$$D(t) = \epsilon_0\epsilon_r E(t) + \sigma\int_0^t E(\tau)d\tau. \tag{11}$$

These equations govern the propagation of electromagnetic waves in a general dielectric media with permittivity ϵ and conductivity σ.

3 Formulation of FDTD in 1D

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A Inverse Fourier Transform of $E(\omega)/i\omega$

First, we must develop some Fourier transforms of useful functions. Starting with the constant function, $x(t) = a$. Instead of directly evaluating the transform, we instead look at the inverse transform of $X(\omega)$,

$$F^{-1}(X(\omega)) = x(t) = a = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{i\omega t}d\omega.$$

Clearly, then $X(\omega)$ must be $2\pi a\delta(\omega)$. So,

$$a \rightleftharpoons 2\pi a\delta(\omega). \tag{12}$$

Now, we examine the signum function,

$$sgn(t) = \begin{cases} +1 & \text{if } t > 0 \\ 0 & \text{if } t = 0 \\ -1 & \text{if } t < 0 \end{cases}. \tag{13}$$

Which has the property

$$\frac{d}{dt}sgn(t) = 2\delta(t). \tag{14}$$

Taking the Fourier transform of this function follows, using integration by parts

$$\begin{aligned} F(sgn(t)) &= \int_{-\infty}^{\infty} sgn(t)e^{-i\omega t}dt \\ &= \left[\frac{-1}{i\omega}sgn(t)e^{-i\omega t}\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\frac{-2}{i\omega}\delta(t)e^{-i\omega t}dt \\ &= \frac{2}{i\omega}. \end{aligned}$$

So,

$$sgn(t) \rightleftharpoons \frac{2}{i\omega}. \tag{15}$$

The last function we need is the Heaviside step function,

$$H(t) = \begin{cases} 1 & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}. \tag{16}$$

Noticing that $H(x) = \frac{sgn(t)+1}{2}$ we take the Fourier transform of H,

$$\begin{aligned} F(H(t)) &= F\left(\frac{sgn(t)+1}{2}\right) \\ &= F\left(\frac{sgn(t)}{2}\right) + F\left(\frac{1}{2}\right) \\ &= \frac{1}{i\omega} + \pi\delta(\omega). \end{aligned}$$

So,

$$H(t) \rightleftharpoons \frac{1}{i\omega} + \pi\delta(\omega). \tag{17}$$

Now, consider a general function, $f(t)$ which can be written as the integral from $-\infty$ to t of some other function, $g(t)$,

$$f(t) = \int_{-\infty}^{t} g(\tau)d\tau. \tag{18}$$

This can be expressed as a convolution integral with the Heaviside function,

$$f(t) = \int_{-\infty}^{t} g(\tau)d\tau = \int_{-\infty}^{\infty} g(\tau)H(t-\tau)d\tau = (g * u)(t). \tag{19}$$

Fourier's convolution theorem states that for any two functions, $\theta(t)$ and $\phi(t)$, whose Fourier transforms exist,

$$F((\theta * \phi)(t)) = \Theta(\omega)\Phi(\omega). \tag{20}$$

Applying this to our function, $f(t)$, yields,

$$F(f(t)) = G(\omega)\left[\frac{1}{i\omega} + \pi\delta(\omega)\right]. \tag{21}$$

So,

$$\int_{-\infty}^{t} g(\tau)d\tau \rightleftharpoons G(\omega)\left[\frac{1}{i\omega} + \pi\delta(\omega)\right]. \tag{22}$$

Finally, we can use this identity to find $E(t)$, given $E(\omega)$. Let $G(\omega) = E(\omega)$ in the above equation. Then,

$$\int_{-\infty}^{t} E(\tau)d\tau \rightleftharpoons E(\omega)\left[\frac{1}{i\omega} + \pi\delta(\omega)\right]. \tag{23}$$

We may assume that $E(t)$ is zero for $t < 0$ and $E(\omega = 0) = 0$, in which case the above equation reduces to the desired identity,

$$\int_{0}^{t} E(\tau)d\tau \rightleftharpoons \frac{E(\omega)}{i\omega}. \tag{24}$$