Causality

Alex & Pier

August 6, 2021

Factorization of the joint density

$$f(x_1, \dots, x_p) = \prod_{j=1}^{p} f(x_j | x_{pa(j)}) \tag{1}$$

Local Markov Property

$$X_S \perp\!\!\!\perp X_{nondesc(S) \backslash pa(S)} | X_{pa(S)} \tag{2}$$

Global Markov Property

$$\text{A and B are d-sep by S} \implies X_A \perp\!\!\!\perp X_B | X_S \text{ in P} \tag{3}$$

Faithfulness

$$X_A \perp\!\!\!\perp X_B | X_S \text{ in P} \implies \text{A and B are d-sep by S} \tag{4}$$

Total average causal effect

$$ACE(x, x') = E(y|do(X = x)) - E(y|do(X = x')) \tag{5}$$

Average causal effect on the treated

$$ATT = E(Y(x_1) - Y(x_0)|X = 1) \tag{6}$$

Causal Bayesian Network

$$p(x_{V \backslash W} \mid do(x_W = x'_W)) = \prod_{i \in V \backslash W} p(x_i | x_{pa(i)})|_{x_W = x'_W} \tag{7}$$

Total Causal Effect

- There is a total causal effect from X to Y
- There are x' and x'' such that $p(y|do(X = x')) \neq p(y|do(X = x''))$
- There is x' such that $p(y|do(X = x')) \neq p(y)$

Reweighting

$$p(x_{V \backslash \{i\}} | do(x'_i)) = \prod_{j \in V \backslash \{i\}} p(x_j | x_{pa(j)})|_{x'_i} = \frac{p(x_v)}{p(x_i | x_{pa(i)})}|_{x'_i} \tag{8}$$

Parent Adjustment Criteria

$$p(x_k | do(x_i)) = \int_{x_{pa(i)}} p(x_k | x_i, x_{pa(i)}) p(x_{pa(i)}) dx_{pa(i)} \tag{9}$$

Backdoor Criteria

- $Z \cap$ desc(i)$= \emptyset$
- Z blocks all backdoor paths from i to k in G

$$p(x_k | do(x_i)) = \int_{x_z} p(x_k | x_i, x_z) p(x_z) dx_z \tag{10}$$

Simplification for multivariate Gaussian

$$E(X_k | do(x_i = x'_i + 1)) - E(X_k | do(x_i = x'_i)) = \gamma \tag{11}$$

where γ is the coefficient of X_i in the linear regression of X_k on X_i and X_z

Adjustment Criteria

- Z blocks all paths between i and k that are not directed from i to k
- Z does not contains descendants of $r \neq i$ on a directed path from i to k
- Necessary and sufficient for identifying total causal effect **via adjustment**

Optimal Valid Adjustment

- **Causal nodes** cn(i,k): nodes $r \neq i$ on a directed path from i to k
- **Forbidden nodes** forb(i,k): descendants of causal nodes and node i
- **Optimal adjustment**: pa(cn(i,k)) \ forb(i,k)

Controlled Direct Effect

$$E(X_k|do(x_i), do(pa(k)\backslash i)) - E(X_k|do(x_i+1), do(pa(k)\backslash i)) \tag{12}$$

in a linear SEM, CDE does not depend on the level of pa(k) \ i

---

Frontdoor Criteria

- M blocks all directed paths from i to k
- There are no unblocked backdoor paths from i to M
- i blocks all backdoor paths from M to k

$$p(x_k|do(x_i')) = \int_{x_M} p(x_M|x_i') \int_{x_i} p(x_k|x_i, x_M) p(x_i) dx_i dx_M \tag{13}$$

Instrumental Variables

- X, Y, I observed, U unobserved
- Assume faithfulness
- I affects X
- $I \perp U$
- I affects Y only through X

Conditional Instrumental Variables

- X, Y, I, S observed, U unobserved
- $I \not\perp X|S$
- I affects Y only through X once we control for S
- S does not contain descendants of X

Transport Formula

$$P(y|do(t)) = \sum_z P(y|do(t), z) P(z) \tag{14}$$

Inverse Probability Weighting

$$\hat{E}(Y|do(X=1)) = \frac{1}{n} \sum_i Y_i \mathbb{1}\{X_i = 1\} w_i \tag{15}$$

where $w_i = \dfrac{1}{P(X=1|Z_i)}$

Intervention stable

- Intervention stable set: $Y \perp I | X_S$ for all I active in $\mathcal{E}$
- Plausible causal predictors: $Y^e|X_S^e = x = Y^f|X_S^f$
- intervention stable $\Longrightarrow$ plausible causal predictor

---