Exercise 26.1-1

Show that a maximum flow in G' has the same value as a maximum flow in G

Let G, G', x, u and v be given as in the the problem description.

What we want to argue here is that any flow in G is also feasible in G' and with the same flow value. Then we conclude that $|f^*_G| = |f^*_{G'}|$ where $|f^*_G|$ is the maximum flow value in the network G.

First observe that any flow $f(u, v)$ in G can be replaced with with a equivalent flow in G' where $f(u, v) = f(u, x) = f(x, v)$ since $c(u, v) = c(u, x) = c(x, v)$.

Let $p_{f^*_G}$ be the path the the maximum flow f^* takes in G.

If $(u, v) \notin p_{f^*_G}$

In this case the the same path is available in G' which would yield the same flow value. If introducing the new vertex x means that $p_{f^*_{G'}}$ is change such that $\{(u, x), (x, v)\} \subset p_{f^*_{G'}}$, then by our first observation above, the same flow value is possible in G, and since $(u, v) \notin p_{f^*_G}$ deviating cannot yield a higher flow value.

If $(u, v) \in p_{f^*_G}$

In this case we have already observed that replacing the $f(u, v)$ with $f(u, x) = f(x, v)$ will yield the same flow value. Hence the maximum flow value will be the same. If $f(u, x) \neq f(u, v)$ in G', then the same value value would be possible in G, and the flow value obtained by deviating cannot be larger than $|f^*_G|$ as that goes against the assumption that $|f^*_G|$ is the maximum flow value.

Hence we conclude that in all possible cases the maximum flow value will be the same, hence $|f^*_G| = |f^*_{G'}|$.

Exercise 26.1-4

Prove that the flows in a network form a convex set.

Let f_1, f_2, α be as given in the exercise description.

We want to prove that $f_\alpha \equiv \alpha f_1 + (1 - \alpha) f_2 \forall \alpha \in (0, 1)$ is a flow.

We start by showing the capacity constraint i.e.

$$\forall u, v \in V \text{ it holds that } 0 \leq f_\alpha(u, v) \leq c(u, v). \tag{1}$$

Since f_1 and f_2 are flows, the capacity constraint holds for both of them. Since the capacity function c is a property of the network, and not the individual flows, the capacity constraint for f_1 and f_2 are identical to (1) i.e they are bounded by the same upper and lower bounds.

Hence $\forall \alpha \in (0,1)$ and $\forall u, v \in V$ the following equations hold

$$\begin{aligned} f_\alpha(u,v) &= \alpha f_1(u,v) + (1-\alpha) f_2(u,v) \\ &\geq \alpha 0 + (1-\alpha)0 \\ &\geq 0 \end{aligned}$$

and

$$\begin{aligned} f_\alpha(u,v) &= \alpha f_1(u,v) + (1-\alpha) f_2(u,v) \\ &\leq \alpha c(u,v) + (1-\alpha) c(u,v) \\ &= c(u,v) \end{aligned}$$

Combining the two inequalities above yields that (1) holds.

We then want to show that the flow conservation property holds. i.e

$$\forall u \in V \setminus \{s,t\} \text{ it holds that } \sum_{v \in V} f_\alpha(v,u) = \sum_{v \in V} f_\alpha(u,v). \tag{2}$$

First note that since f_1 and f_2 are flows, the flow conservation property holds for both of them individually.

Pick a vertex u such that $u \in V \setminus \{s,t\}$ then

$$\begin{aligned} \sum_{v \in V} f_\alpha(v,u) &= \sum_{v \in V} (\alpha f_1(v,u) + (1-\alpha) f_2(v,u)) \\ &= \alpha \sum_{v \in V} f_1(v,u) + (1-\alpha) \sum_{v \in V} f_2(v,u) \\ &= \alpha \sum_{v \in V} f_1(u,v) + (1-\alpha) \sum_{v \in V} f_2(u,v) \\ &= \sum_{v \in V} (\alpha f_1(u,v) + (1-\alpha) f_2(u,v)) \\ &= \sum_{v \in V} f_\alpha(u,v) \end{aligned}$$

Since this holds for all $u \in V \setminus \{s,t\}$, we have now shown (2).

We have now shown the capacity constraint and the flow conservation porperty for f_α, hence f_α is a flow. So the flows in a network form a convex set.

Exercise 26.1-7

Show that a flow network G with vertex capacities can be transformed into a normal flow network G'. How many vertices and edges does G' have?

Let the G, G', v and l be defined as in the exercise.

First we describes how a network G_x where only one of the vertices, x, have a capacity constraint $l(x)$ and how it can be transformed into a new network without the vertex capacity constraint.

We define the set $E(x)$ of edges that have an edge going into the vertex with a capacity constraint. That is $E(x) = \{(v, x) \in E : l(x) > 0\}$.

To transform the network, we can augument G_x with a new vertex x' such that $G'_x = (V', E')$ where $V' = V \cup \{x'\}$. We define E' as $E' = E \setminus E(x) \cup E'(x) \cup \{(x', x)\}'$ where $E'(x)$ is the set of edges in $E(x)$ where the outgoing vertex have been changed to x'. So the set of edges E' in the new transformed network consist of all edges in the original network, that did not have an edge going into the vertex with a capacity constraint. And all the edges that did go into the vertex with a capacity constraint have been replaced with new edges that goes into the new vertex x'. And a new edge have been introduced that goes from the new vertex x' to the old vertex x. Hence in total one new vertex and one new edge have been introduced and none have been removed.

The capacity of the edges in the transformed network, where no vertex have a capacity is

$$c_{G'_x}(u, v) = \begin{cases} c_{G_x}(u, x), (u, v) = (u, x') \\ l(x), (u, v) = (x', x) \\ c_{G_x}(u, v), (u, v)\text{else} \end{cases}$$

That is, the capacity for an edge going into x' is the same as the capacity of going into x in the G_x network. The new edge have a capacity of $l(x)$ that is the capacity of the vertex in the G_x network. In all other cases the capacity in the transformed network is the same as in the original network.

We have now shown how a network with a single vertex capacity constraint can be transformed into a new network without a capacity constraint. This can easily be generalized to an arbitrary number of vertex capacity constraint. If x and x' is replaced with x_i and x'_i in the above, all of what we have shown still holds, and for each vertex with a capacity constraint, there will be one new vertex and one new edge. Hence if all of the vertices, including the source and sink, had a constraint there would be $|V|$ extra vertices and edges. So the total number of vertices in G' would be $2|V|$ and the total number of edges in G' would be $|E| + |V|$.

Exercise 26.2-2

What is the flow and cut

Let the cut be given as in the exercise, then the flow across the cut (and all other cuts) in figure 26.1(b) is

$$\begin{aligned} f(S,T) = f(\{s, v_2, v_4\}, \{v_1, v_3, t\}) &= \sum_{u \in S} \sum_{v \in T} f(u,v) - \sum_{u \in S} \sum_{v \in T} f(v,u) \\ &= 11 + 1 + 7 + 4 - 4 = 19 \end{aligned}$$

The capacity is

$$C(S,T) = \sum_{u \in S} \sum_{v \in T} c(u,v) = 16 + 4 + 7 + 4 = 31$$