Microeconomics Prelim Exam
September 2007

- Please answer all questions.
- For each part of each question the maximum number of points that you can get from that part is indicated in square brackets.
- Good luck!

For the first three questions, simply identify the correct multiple choice answer.

QUESTION 1: [3.5 pts] Dr. Y has a preference relation over real number quantities of Turkish delight: $x \succeq y$ if $x \geq y + a$. His preferences are:

(a) complete if $a \leq 0$, transitive if $a \geq 0$, reflexive if $a \leq 0$
(b) complete if $a \geq 0$, transitive if $a \geq 0$, reflexive if $a \leq 0$
(c) complete if $a \leq 0$, transitive if $a \leq 0$, reflexive if $a \leq 0$
(d) complete if $a \geq 0$, transitive if $a \leq 0$, reflexive if $a \leq 0$
(e) complete if $a \leq 0$, transitive if $a \geq 0$, reflexive if $a \geq 0$
(f) complete if $a \geq 0$, transitive if $a \geq 0$, reflexive if $a \leq 0$

QUESTION 2: [3.5 pts] Assume Daisuke's price elasticity of demand for good X is 2 and the income elasticity of demand is (negative) 1. Assume the current price of X is \$2, and the income is \$40,000 a year. If the price level of X rises a nickel and the income rises \$1000 dollar a year, then Daisuke's demand changes by

(a) 1% (b) 0% (c) -1%
(d) -1.5% (e) -2% (f) -2.5%

QUESTION 3: [5 pts] Emre has increasing and strictly convex preferences over apples and bananas. He accepts buying two apples in exchange for three bananas, but is indifferent about this trade. After executing this trade:

(a) Emre must reject buying seven bananas for four apples.
(b) Emre must accept buying seven apples for eleven bananas.
(c) Emre might accept buying a banana for an apple.
(d) If Emre accepts a trade, then he will accept the same trade twice.
(e) Emre might accept buying three apples for five bananas.
(f) Emre must accept buying four bananas for three apples.
(Hint: Use a graph.)

QUESTION 4: Borger's Books — otherwise known as *The Firm* — is a criminal enterprise. It can steal x units at a cost of $c(x)$, where $c', c'' > 0$ for all x. *The Firm* is one of an ocean of like-minded firms, and so is a price taker, from a demand curve $X(p)$. In other words, the marginal revenue of each theft is a constant resale demand price p that is determined in partial equilibrium. *The Firm* knows that it eventually will get caught, and incur a penal loss $L > 0$. Suppose that free entry of potential Firms drives production to the point where net profits (i.e. minus the loss L) are zero.

(a) [4 pts] As the government raises the penal loss L, does theft x per Firm rise or fall, and does the price rise or fall?

(b) [4 pts.] What is the rate of change of the price in the loss L when theft level is $x = 100$?

QUESTION 5: [5 pts.] Lones is a strictly risk averse expected utility maximizer, with a smooth utility function u over wealth. He is pondering breaking the law, and stealing a prize worth x. If he does so, then he will be caught with chance p, and fined $f > x$ (but can retain his prize x). If the government raises the fine, and lowers the chance p, so as to hold fixed the expected penalty, does this make him more or less willing to engage in the crime?

QUESTION 6: A single indivisible object is sold through a second price auction. There are two bidders: $i = 1, 2$. Bidders submit their bids $b_i \geq 0$ simultaneously and independently. The highest bidder wins the object. She has to pay the other bidder's bid.

The value of the object is the same for both bidders, and it is some number v that is not known to either bidder. However, before the auction each bidder i privately observes a signal $s_i \in [0, 1]$, and the value of the object is given by: $v = s_1 + s_2$. The two signals are independent random variables, and they are uniformly distributed on the interval $[0, 1]$.

Bidder i's von Neumann Morgenstern utility is $v - b_j$ is she wins the object and pays b_j for it, and it is 0 if she does not win the object, and does not pay anything.

(a) Suppose bidder 2 chooses her bid according to the following strategy: $b_2 = 4s_2$. Bidder 1 knows this strategy of bidder 2, and takes it as given and fixed. Suppose bidder 1 has observed signal realization s_1, and contemplates placing some bid b_1, where $b_1 \leq 4$. Find bidder 1's expected utility as a function of b_1 and s_1. Then determine the optimal value of b_1 as a function of s_1.

(b) Does the strategy that you found in part (a), together with player 2's strategy $b_2 = 4s_2$ constitute a Bayesian Nash equilibrium of the auction?

QUESTION 7: Two car dealers $i = 1, 2$ sell identical cars. Each of them has a current stock of exactly one car. There are two car buyers, and they arrive sequentially: first buyer A arrives, possibly buys a car, and leaves; then buyer B arrives, possibly buys a car, and leaves. If a buyer buys a car and pays p for it, the buyer's utility is $1 - p$. If a buyer buys no car, the buyer's utility is 0. Dealers seek to maximize their revenues.

The dealers and the buyers play the following game: In period $t = 1$ both dealers quote simultaneously and independently a price $p_i^1 \geq 0$. Buyer A observes these offers and then accepts one of the two offers, or decides not to buy any car. In period $t = 2$ the dealers who did not sell a car in period 1, and therefore have one car left, quote a price p_i^2. Buyer B then observes these offers and accepts one of the offers, or decides not to buy any car. Then the game ends. The structure of the game and the payoffs are common knowledge among the car dealers and the buyers.

(a) Find a subgame-perfect equilibrium of this game. Proceed by backward induction. First determine Nash equilibria in pure strategies of the second stage games that may arise. Then examine the first stage game, assuming that firms correctly anticipate the equilibrium of the second stage game. Assume that buyers accept each dealer's offer with probability 1/2 if they are indifferent between the dealers' offers and weakly prefer buying to not buying. Assume that buyers, if indifferent between buying and not buying, choose to buy.

(b) Suppose it was commonly known that buyer B's utility, if he buys a car, is $0.8 - p$ (and 0 if he buys no car). How would your analysis change?

QUESTION 8: There are two countries, Türkiye and Japan. There are two consumers, Emre and Yusufcan, in Türkiye and there is one consumer, Daisuke, in Japan. There are two goods, x and y. Utility and endowments of consumers are summarized below:

Consumer	Utility	Endowment
Emre	$2\ln x + 2\ln y$	(6, 4)
Yusufcan	$\ln x + 3\ln y$	(8, 4)
Daisuke	$\ln x + \ln y$	(4, 16)

Price of good y is set to be 1 throughout, and write p instead of p_x.

(a) [2 pts.] Find individual demand functions.

(b) [2 pts.] First assume that there is no international trade. Find the equilibrium in each country.

(c) [3 pts.] Suppose consumers can trade internationally. Find the equilibrium.

(d) [3 pts.] Does free world trade make everybody better off? Is this consistent with the first fundamental theorem of welfare economics? Discuss.

QUESTION 9: An economy consists of two consumers, A, B, one good, x. There are two time periods, date 0 and date 1. There is only one state of nature in date 0, while in date 1 there are two, which are denoted states 1 and 2. Both consumers have identical log-linear utility functions:

$$U^i(x_0, x_1(1), x_1(2)) = \log x_0 + \log x_1(1) + \log x_1(2) \quad \text{for all } i = A, B$$

where the consumption of good x at date 0 is denoted by x_0, and $x_1(1)$ and $x_1(2)$ represent consumptions of good x at date 1 in state 1 and in state 2, respectively. (Subjective probabilities of states are directly embedded into the utility function.) Endowments are summarized in the following table.

	date 0	date 1	
		State 1	State 2
	w_0	$w_1(1)$	$w_1(2)$
A	1	2	ϵ
B	1	ϵ	2

Suppose that there is no state-contingent goods. Instead, there is a single asset which delivers one unit of the good in state 1 and $1+\delta$ units in state 2. Assume that $0 < \epsilon < 2$.

(a) [3 pts.] If $\delta = 0$, show that there is no equilibrium where trade takes place. (No calculation is needed.)

(b) [3 pts.] Under the same assumption, show that "no-trade" is the equilibrium.

(c) [3 pts.] If $\delta \neq 0$ (even if it is very small), show that no-trade is not an equilibrium.

(d) [3 pts.] Verify that an equilibrium in which trade takes place exists.

(e) [3 pts.] Is the $\delta \neq 0$ equilibrium strictly Pareto superior to the $\delta = 0$ equilibrium? Discuss.

QUESTION 10: Suppose two players (1 and 2) are deciding whether to invest in a project or not. There is a safe action (Not Invest) and there is a risky action (Invest). Payoffs are given as follows:

	Invest	Not Invest
Invest	θ, θ	$\theta - 2, 0$
Not Invest	$0, \theta - 2$	$0, 0$

(a) [3 pts.] Suppose $\theta \in \mathbb{R}$ is commonly known by both players. What are the pure-strategy equilibria of this game?

(b) [4 pts] Suppose there is incomplete information about θ. Players have a common prior about θ which is an (improper) uniform over the real line. Player $i \in \{1, 2\}$ observes a private signal $x_i = \theta + \varepsilon_i$. Suppose ε_1 and ε_2 are independently and normally distributed with mean 0 and standard deviation σ. Denote the cdf of standard normal by Φ. What probability does player i who receives signal x_i assign to his opponent receiving a signal less than $k \in \mathbb{R}$? (Hint: Sum of two independent normal random variables is also normal where the mean is the sum of the means and the variance is the sum of the variances.)

(c) [5 pts.] Show that there is an equilibrium in which players play Invest if and only if their private signals exceed a threshold k. What is the equilibrium value of k?

QUESTION 11: Consider the following cheap talk game between a sender and a receiver. The sender's type is uniformly distributed over the interval $[0,1]$. The message space for the sender is the same as the type space. The action space for the receiver is also the interval $[0,1]$. The receiver's payoff function is $U_R(t,a) = -(a-t)^2$ and the sender's payoff function is $U_S = -(a-(t+b))^2$ where $b > 0$, $a \in [0,1]$ is the action that gets implemented and $t \in [0,1]$ is the type of the sender.

Suppose that differently from the standard cheap talk game, the sender can implement action $a = 0$, but all other actions can only be implemented by the receiver.

The timing of this modified cheap talk game is as follows. First, the sender observes his type $t \in [0,1]$ and decides whether to implement action $a = 0$. If sender implements $a = 0$ then the game is over and the payoffs are realized. Otherwise, sender sends a message $m \in [0,1]$. The receiver observes this message (but not the type) and chooses an action $a \in [0,1]$,and the payoffs are realized.

(a) [6 pts.] Construct and describe an equilibrium in which (i) some types send messages (put differently, not all types implement $a = 0$) and (ii) all types that that send messages babble, i.e., they send random messages. For what values of b would such an equilibrium exist?

(b) [7 pts.] Suppose $b = 1/36$. Construct an equilibrium with three intervals such that types in the lowest interval implement $a = 0$, and types in the intermediate and high intervals send different messages.

ANSWERS TO SEPTEMBER 2007 MICROECONOMICS PRELIM QUESTIONS

ANSWER TO QUESTION 1:

Correct answer is (a)

Preferences are transitive if $a \geq 0$, since: $x \geq y + a$ and $y \geq z + a$ imply $x \geq z + 2a$. So $x \geq z + a$ if $z + 2a \geq z + a$, or $a \geq 0$.

Preferences are complete if $a \leq 0$, since: Completeness demands either $x \geq y + a$ or $y \geq x + a$ — neither of which need occur if $a > 0$.

Preferences are reflexive if $a \leq 0$, for $x \geq x + a$ true iff $a \leq 0$.

ANSWER TO QUESTION 2:

Correct answer is (f).

Price has risen 2.5%, while income has gone up 2.5%. Thus, demand changes by $-2 \cdot 2.5\% + 1 \cdot 2.5\% = -2.5\%$.

ANSWER TO QUESTION 3:

Correct answer is (c)

Put two points (A, B) and $(A + 2, B - 3)$ on a graph and draw a convex indifference curve that goes through these points. Now, observe that $(a), (b), (e)$ and (f) are false using the graph. Part (d) is precluded by strict convexity.

ANSWER TO QUESTION 4:

As a competitive firm, it sets $c'(x) = p$, the demand price. The latter will adjust in equilibrium, with entry or exit, to ensure that profits vanish: $xp - c(x) - L = 0$. Producer surplus therefore obeys $g(x) = xc'(x) - c(x) = L$ with slope $g'(x) = xc''(x) > 0$. When the loss is L, production is $f(L) = x$, the inverse function to the producer surplus function $g(x) = L$.

(a) Since $g' > 0$ implies $f' > 0$, production rises in L. Hence, the marginal cost rises, and price rises too, as it equals the marginal cost of production.

(b) In light of (a), the price is $h(L) = c'(f(L))$, namely, the optimal marginal cost at the optimum. Then $h'(L) = 1/f(L) = 1/x = 1/100$.

ANSWER TO QUESTION 5:

Observe that if we hold fixed $pf = \alpha > 0$, then $pu(x - f) + (1 - p)u(x) = (\alpha/f)u(x - f) + (1 - p)u(x)$ is a falling function of f, with slope if $u'' < 0$ on $[x, x - f]$:

$$\frac{\alpha}{f^2}\left[u(x) - u(x - f)\right] - \frac{\alpha}{f} f u'(x - f) = \frac{\alpha}{f^2}\left[u(x) - u(x - f) - f u'(x - f)\right] < 0$$

ANSWER TO QUESTION 6:

(a) Bidder 1's expected utility is given by:

$$\frac{b_1}{4}\left(s_1 + \frac{b_1}{8} - \frac{b_1}{2}\right)$$

where the first term is the probability with which bidder 1 wins the auction if she places bid b_1; the first two terms in the brackets are the expected value of the object to bidder 1 if she wins the object, and the last term in the brackets is bidder 1's expected payment if she wins the object. Maximizing the above function with respect to b_1 yields the optimal bid:

$$b_1 = \frac{4}{3}s_1.$$

(b) Bidder 1 has no incentive to deviate because, as the calculation in (a) shows, her bid is optimal in the interval $[0, 4]$, and placing a bid above 4 yields exactly the same expected utility as placing the bid 4, given that this is a second price auction. Therefore, bidder 1's strategy is a best response to bidder 2's strategy.

It remains to investigate bidder 2's incentives. If bidder 2 places a bid b_2 in the range of bidder 1's strategy, i.e. in the interval $[0, \frac{4}{3}]$, then her expected utility is:

$$\frac{3b_2}{4}\left(\frac{3b_2}{8} + s_2 - \frac{b_2}{2}\right)$$

Maximizing this with respect to b_2 yields the solution:

$$b_2 = 4s_2$$

For $s_2 \leq \frac{1}{3}$ this formula yields a bid that is inside of the interval to which we had restricted attention, i.e. the interval $[0, \frac{4}{3}]$. Bidder 2 could also bid outside of this interval, i.e. above $\frac{4}{3}$. But any bid above $\frac{4}{3}$ yields the same expected utility as the bid $\frac{4}{3}$, and therefore the bid that we calculated is optimal. For $s_2 \geq \frac{1}{3}$ the formula yields a bid that is outside of the interval to which we had restricted attention, i.e. the interval $[0, \frac{4}{3}]$. If bidder 2 had to choose only from this interval, because expected utility is quadratic in b_2, bidder 2 would thus choose the upper boundary $\frac{4}{3}$. But there is no harm in choosing any bid above this, because all bids above $\frac{4}{3}$ yield the same expected utility as the bid $\frac{4}{3}$. Thus, the strategy is optimal, and these two strategies do indeed constitute a Bayesian Nash equilibrium.

ANSWER TO QUESTION 7:

(a) Buyers should obviously accept the lowest price offer, provided that this offer is no higher than 1, and they should reject all other offers. If neither firm sells its car in the first stage, then the second stage is Bertrand competition, and the only Nash equilibrium is to set prices $p_i^2 = 0$ for $i = 1, 2$. If however, firm i sold its car in period 1, then the other firm $j \neq i$ is a monopolist in stage 2 and sets price $p_j^2 = 1$.

Now consider the first stage. We construct firm i's best response given firm j's choice p_j^1. If $p_j^1 > 1$, then firm i's best response is to choose $p_i^1 = 1$ and earn profit 1. If it were to sell a higher price, it would not sell anything in period 1, and enter into Bertrand competition in period 2, thus earning profits zero. If it were to set a lower price, its revenue would equal that lower price.

Suppose firm j chooses a price $p_j^1 \in [0,1]$. If firm i chooses a lower price, then it will earn p_i^1. If it chooses a higher price, it will earn 1. If it chooses the same price as firm j, then it will earn $\frac{1}{2}p_i^1 + \frac{1}{2}1$. Therefore, the best response is to choose any price higher than the price chosen by firm j, except in the case that firm j chooses $p_j = 1$. In that case the best response is to choose $p_i = 1$, or to choose $p_i > 1$.

These considerations make it evident that the Nash equilibria of the first stage are: $p_i^1 = 1$ and $p_j^1 \geq 1$ where $i \neq j$. To verify this one can also draw the graph of the best response correspondences described in the previous paragraph.

(b) The analysis of the second stage is the same as before, except that the monopolist firm now charges $p_j^2 = 0.8$. Consider the first stage, and construct again firm i's best response. The case $p_j^1 > 1$ is as before. So, assume $p_j^1 \in [0,1]$. If firm i chooses a price that is higher than firm j's price, it will earn 0.8. If it chooses a price that is lower than firm j's price, it will earn p_i^1. Finally, if it chooses the same price as firm j, then it will earn $\frac{1}{2}p_j^1 + \frac{1}{2}0.8$. Therefore, if $p_j < 0.8$, then firm i's best response is to choose any price higher than the price chosen by firm j. If $p_j = 0.8$ the best response is to choose any price that is higher than 0.8 or equal to 0.8. If $p_j > 0.8$, a best response does not exist because firm i seeks to undercut firm j by some arbitrarily small amount. These considerations make evident that the only Nash equilibrium of the first stage is: $p_1^1 = p_1^2 = 0.8$.

Answer to Question 8:

a) Individual demand functions are summarized in the following table:

	Türkiye		Japan
Demand	Emre	Yusufcan	Daisuke
x	$\frac{6p+4}{2p}$	$\frac{8p+4}{4p}$	$\frac{4p+16}{2p}$
y	$\frac{6p+4}{2}$	$\frac{3(8p+4)}{4}$	$\frac{4p+16}{2}$

b) Market clearing in *Türkiye*: $\frac{6p+4}{2p} + \frac{8p+4}{4p} = 14$, $p = \frac{1}{3}$.
Market clearing in *Japan*: $\frac{4p+16}{2p} = 4$, $p = 4$.

	Price	Demand x	Demand y
Türkiye	1/3		
Emre		9	3
Yusufcan		5	5
Japan	4		
Daisuke		4	16

c) Market clearing condition: $\frac{6p+4}{2p} + \frac{8p+4}{4p} + \frac{4p+16}{2p} = 18$, $p = 1$

	Price	Demand After Trade	
		x	y
International	1		
Emre		5	5
Yusufcan		3	9
Daisuke		10	10

d) The utilities are summarized in the following table. Notice that Emre becomes worse off. However, this is still consistent with the first fundamental theorem of welfare economics since everybody is better off comparing to the initial endowments.

	Utility		
	Before Trade		After Trade
Emre	6.5917	$>$	6.4378
Yusufcan	6.4378	$<$	7.6903
Daisuke	4.1589	$<$	4.6052

Answer to Question 9:

a) Since the asset pays one unit of good x in each state at date 1, both consumer want either to sell or buy it at the same time. This means that there is either excess supply or excess demand for the asset. Hence there is no equilibrium where trade takes place.

b) The price of x_0 is denoted by p. Note that there is no trade at date 1. The budget constraint at date 0 is

$$px_0 + q_a \leq p$$

where q_a denotes the quantity of the asset (the price of the asset is normalized to 1). Since the utility function is strictly concave, we can assume that $q_a = p(1 - x_0)$. Then consumer $A's$ maximization problem can be written as follows:

$$\max_{x_0} \log x_0 + \log(2 + p(1 - x_0)) + \log(\epsilon + p(1 - x_0))$$

F.O.C. for consumer A

$$\frac{1}{x_0^A} - \frac{p}{2 + p(1 - x_0^A)} - \frac{p}{\epsilon + p(1 - x_0^A)} = 0,$$

similarly, for consumer B

$$\frac{1}{x_0^B} - \frac{p}{\epsilon + p(1 - x_0^B)} - \frac{p}{2 + p(1 - x_0^B)} = 0.$$

To show that "no-trade" is the equilibrium, we need to find the equilibrium price. Since we know that $x_0^i = 1$ for all $i = A, B$, we can plug this in to the first order condition. The we have

$$\begin{aligned} \frac{1}{1} - \frac{p}{2} - \frac{p}{\epsilon} &= 0 \\ \text{or} & \\ p &= \frac{2+\epsilon}{2\epsilon}. \end{aligned}$$

Therefore, $\{(1,2,\epsilon);(1,2,\epsilon);\frac{2+\epsilon}{2\epsilon}\}$ is the only equilibrium.

c) Consumer $A's$ maximization problem can be written as follows:

$$\max_{x_0} \log x_0 + \log(2 + p(1-x_0)) + \log(\epsilon + p(1-x_0)(1+\delta))$$

F.O.C.

$$\frac{1}{x_0^A} - \frac{p}{2+p(1-x_0^A)} - \frac{p(1+\delta)}{\epsilon + p(1-x_0^A)(1+\delta)} = 0,$$

similarly

$$\frac{1}{x_0^B} - \frac{p}{\epsilon+p(1-x_0^B)} - \frac{p(1+\delta)}{2 + p(1-x_0^B)(1+\delta)} = 0,$$

To show that "no-trade" is not an equilibrium, we plug $x_0^A = x_0^B = 1$.The we have

$$1 - \frac{p}{2} - \frac{p(1+\delta)}{\epsilon} = 0 \qquad \text{(consumer A)}$$

and

$$1 - \frac{p}{\epsilon} - \frac{p(1+\delta)}{2} = 0 \qquad \text{(consumer B)}$$

It is routine to verify that there is no price which satisfies both equations at the same time unless either $\epsilon \in \{0,2\}$ or $\delta = 0$.

d) Note that $x_0^A(p^A) = 1$ and $x_0^B(p^A) > 1$ if $p^A = \frac{2\epsilon}{\epsilon+2(1+\delta)}$, and $x_0^A(p^B) < 1$ and $x_0^B(p^B) = 1$ if $p^B = \frac{2\epsilon}{2+\epsilon(1+\delta)}$. Hence, we have excess supply for the asset at p^A and excess demand for the asset at p^B. By continuity of the demand functions, there exists an equilibrium price p^* such that $p^A < p^* < p^B$.

e) Given that the initial endowment is always available, the $\delta \neq 0$ equilibrium is pareto superior to the $\delta = 0$ since the utility is strictly concave.

ANSWER TO QUESTION 10:

a) If $\theta < 0$, then there is a unique equilibrium in which both players play Not Invest. If $\theta > 2$, there is a unique equilibrium in which both players play Invest. If $0 \leq \theta \leq 2$, the there are two pure strategy equilibria. In the first one both players play Not Invest and in the second one both players play Invest.

b) Player i believes that θ is distributed with mean x_i and variance σ. Thus

$$Prob\{x_j \leq k\,|x_i\} = Prob\{\theta + \varepsilon_j \leq k\,|x_i\} = \Phi\left(\frac{k - x_i}{\sqrt{2}\sigma}\right).$$

c) Consider player i who receives signal $x_i = k$. Probability that this player assigns to his opponent receiving a signal above k is $1/2$. Therefore if this agent plays Invest his expected payoff is:

$$\begin{aligned} & \frac{1}{2}E\left(\theta\left|x_i = k\right.\right) + \frac{1}{2}\left(E\left(\theta\left|x_i = k\right.\right) - 2\right) \\ = \quad & k - 1. \end{aligned}$$

This agent will be indifferent between Invest and Not Invest if $k = 1$. To complete the construction you need to also check that if $x_i > 1$ then Invest is strictly preferred to Not Invest and if $x_i < 1$ Not Invest is strictly preferred to Invest. Thus it is an equilibrium for players to play Invest if and only if their private signals exceed 1.

Answer to Question 11:

a) Suppose all the types above x_1 send random messages. Then the receiver implements $(x_1 + 1)/2$ if he gets a message. Type x_1 must be indifferent between action 0 and action $(x_1 + 1)/2$ thus:

$$x_1 = \frac{1 - 4b}{3}.$$

Such an equilibrium exists for $b < 1/4$. In this equilibrium all types below $\frac{1-4b}{3}$ implement action $a = 0$ and types above $\frac{1-4b}{3}$ send random messages. Whenever the receiver receives a message, he implements $\frac{2-2b}{3}$.

b) Suppose types in $[0, x_1]$ implement $a = 0$, and types in $(x_1, x_2]$ and $(x_2, 1]$ send different messages. If the receiver receives the message from $(x_1, x_2]$ then he implements $(x_1 + x_2)/2$ and if he receives the message from $(x_2, 1]$ then he implements $(x_2 + 1)/2$. The idifference conditions are:

$$\begin{aligned} x_2 - x_1 &= 4b + 2x_1 \\ 1 - x_2 &= 4b + x_2 - x_1. \end{aligned}$$

Since, $x_1 + (4b + 2x_1) + (4b + 4b + 2x_1) = 1$, we have

$$\begin{aligned} x_1 &= \frac{1 - 12b}{5} = \frac{2}{15} \\ x_2 &= 4b + 3x_1 = \frac{23}{45}. \end{aligned}$$

Microeconomics Prelim Exam 2008
Friday, August 22, 9 AM - 2 PM.

- Please answer all questions.
- For each part of each question we have provided in square brackets [...] an indication of the maximum number of points that you can get for that part.
- Good luck!

Question 1: [8 points] Suppose that students in a class experiment are numbered 1-20. Odd students hold an ounce of gold and even students hold dollars. The student number indicates what the student is willing to pay for a single ounce of gold (if he holds dollars, being even). If he holds gold (odd), he is only willing to part with that ounce of gold for twice his student number. Suppose that after we run this experiment once, more students arrive, who are numbered 22, 24,. . . , 30, and handed money, but again willing to pay their number for gold. (There are no new gold holders.)

What happens to the market clearing price of gold, and quantity traded? Be as precise as possible (but you may only be able to state a range in one case).

Question 2: [6 points] Consider a variable intensity activity $x \geq 0$ that you can do. Its marginal cost is $x^3 - 6x^2 + 11x$ and its marginal benefit is 6. Assume that doing it at all incurs a fixed hassle cost of 3, at what levels are you willing to do it?

QUESTION 3: [7 points] Assume that the elasticity of labor supply in a market is currently -1 and the demand elasticity is -2. Assume that the current equilibrium price and quantity is \$50 and 1000. If a new sales tax is imposed that raises prices by \$1.50, what happens to the market quantity? State an answer of the form "it rises by about n" or "it falls by about n", where n is a natural number.

QUESTION 4: [4 points] My Bernoulli utility function over wealth x is $\log(x)$. There are two states of the world, with chance 1/3 and 2/3. We start at wealth 90 and fair market prices. If the price of contingent consumption rises 5% in one state, what happens to the consumption in the other state?

QUESTION 5: A group of $N(\geq 2)$ students $i = 1, 2, \ldots, N$ have to choose simultaneously and independently which contribution c_i to offer towards the restoration of the historic Lorch Hall. The offered contributions can be any nonnegative amount of Dollars: $c_i \geq 0$. The total amount required for the restoration is normalized to be 1. Thus, Lorch Hall is restored if the sum of offered contributions $S = \sum_{i=1}^{N} c_i$ satisfies $S \geq 1$, and not otherwise. If the sum S of contributions offered is not sufficient for the restoration of Lorch Hall, then nobody actually has to pay. If the sum of the contributions offered is at least one, then everybody has to make payments in proportion to their offered contributions, that is, student i's payment is: $p_i = c_i/S$. Each student's marginal value for a restored Lorch Hall is $v \geq 0$. If restoration takes place, agent i's utility is thus: $v - p_i$, and otherwise it is 0. The utility functions are common knowledge among the students.

(a) [6 points] Show that there is no weakly dominant strategies if $v > 0$, but that, if $v = 0$, each agent has a weakly dominant strategy. Explain why this weakly dominant strategy is not strictly dominant.

(b) [7 points] Suppose that all players choose the same level of offered contributions: $c_i = c^*$ for all $i = 1, 2, \ldots, N$. For which values of c^* is this a Nash equilibrium? Note that the answer will depend on the value of v. (Hint: consider first values of c^* such that $c^* \geq 1/N$. Then consider values of c^* such that $c^* < 1/N$.) Draw a graph that indicates as a function of v which values of c^* correspond to a Nash equilibrium. For which values v are there multiple equilibrium values c^*? For which values of v is there a unique equilibrium value c^*?

QUESTION 6: A seller and a buyer of a house have agreed on the price of 10 for the house. The buyer has to choose whether to sign (action "S") or not to sign ("NS") the final sales contract. The value of the house to the buyer depends on whether the neighboring house is inhabited by noisy students. The seller knows whether there are noisy students in the neighboring house, but the buyer does not. The buyer attaches probability 0.5 to each possibility. This probability is common knowledge.

The value of the house to the seller is zero, regardless of the neighbors. If the neighboring house is inhabited by noisy students, the value of the house to the buyer is $v = 5$. Otherwise, the value of the house to the buyer is $v = 20$. If the buyer signs the sales contract, then the buyer's von Neumann Morgenstern utility is $v - 10$. If the buyer does not sign the contract, then her von Neumann Morgenstern utility is 0.

Before the contract is signed, the seller chooses whether to provide papers (action "P") that show that the neighboring house is quiet, or not to provide such papers (action "NP"). The seller can produce false papers, but that is costly. If there are indeed no noisy students then the cost of providing the papers that show that the neighbors are quiet is $c = 0$. But if the neighbors are noisy students, then the cost of providing papers that pretend that they are not noisy is $c = \bar{c} > 0$. The seller's von Neumann Morgenstern utility is $p - c$, where p is the payment that he receives, i.e. 10 if the contract is signed, and 0 if the contract is not signed.

The sequence of moves is that the seller moves first and chooses P or NP. Then, having observed the seller's move, the buyer chooses S or NS. If the buyer chooses S, the house changes hands and the buyer pays 10. If the buyer chooses NS, the house remains with the seller and the buyer pays nothing.

(a) [3 points] Draw a game in extensive form (including an initial random move by nature) that represents this game of incomplete information.

(b) [4 points] For which values of $\bar{c}$, if any, does this game have a weak perfect Bayesian equilibrium in pure strategies such that the seller chooses P if and only if the neighboring house is quiet.

(c) [5 points] For which values of $\bar{c}$, if any, does this game have a weak perfect Bayesian equilibrium in pure strategies such that the seller chooses NP regardless of the neighbors.

QUESTION 7: Consider an Edgeworth box economy where both individuals have the same utility function

$$u_i(x_i, y_i) = x_i^{0.5}\ y_i^{0.5}$$

for $i = 1, 2$. Initial endowments are $(1, 0)$ for the first consumer and $(0, 1)$ for the second consumer.

(a) [2 points] Find the competitive equilibrium.

(b) [6 points] Now suppose that consumer 1 has to pay a tax of $t > 0$ for each unit of the first good that she consumes. Total tax revenue is distributed back equally among the agents. Find the competitive equilibrium. (Hint: Normalize $p_x = 1$, where p_x is the price of the first good before taxes. Obviously then, the first consumer pays $1 + t$ for each unit of the first good while the second consumer pays only 1.)

(c) [2 points] Show that the competitive equilibrium in part (b) is not Pareto efficient.

QUESTION 8: Consider the following pure exchange economy with two consumers, $i = 1, 2$, and two goods, x and y. For $i = 1, 2$, consumer i has the utility function $u^i(x, y)$ given by

$$u^i(x,y) = \begin{cases} 5, & \text{if } x \geq 1 \\ 0, & \text{if } x < 1 \text{ and } y \geq 1 \\ -1, & \text{if } x < 1 \text{ and } y < 1. \end{cases}$$

(a) [3 points] Does this utility function satisfy "local nonsatiation"?

(b) [4 points] Assume that the aggregate endowment for this economy is 1 unit of good x and 1 unit of good y. Describe as carefully as you can the set of Pareto optimal allocations (where there is no other allocation making at least one consumer strictly better off and all consumers at least as well off).

(c) [4 points] Suppose each consumer's endowment vector is $(1/2, 1/2)$. Does this economy have a competitive equilibrium? Explain your answer.

(d) [4 points] Is the conclusion of the second fundamental theorem of welfare economics true for this question? In other words, starting with any Pareto optimal allocation as the endowment, is that allocation a competitive equilibrium? Explain your answer.

QUESTION 9: Daisuke is opening a Turkish restaurant close to Lorch hall. He needs to hire *one* chef and there are two candidates: Emre and Yusufcan. The probability that the restaurant is successful depends on which of them is hired and whether the hired chef puts in effort or not. Those probabilities are given by the following matrix:

	Effort	No Effort
Yusufcan	p	q
Emre	$p-\varepsilon$	r

where $0 < r < q < p < 1$ and $\varepsilon > 0$. *Note that Yusufcan is better than Emre whether effort is exerted or not.* Other than that, they are completely identical. Daisuke is risk-neutral. On the other hand, Emre and Yusufcan are risk-averse: their payoffs are $v(w) - c$ with effort and $v(w)$ without effort, where w is the wage and $v(\cdot)$ is a concave function with $v' > 0$, $v'' < 0$, $v(-\infty) = -\infty$, $v(+\infty) = +\infty$ and $c > 0$. The wage can be positive or negative (interpreted as a fine). The reservation utilities of both Emre and Yusufcan are normalized to be zero.
Daisuke cannot observe whether the hired chef puts in effort or not because he cannot judge the quality of Turkish food, but he *does* observe if the restaurant is successful or not. Therefore, the wage payment can depend on whether the restaurant is successful or not. Let w_S and w_F be the wage payments when the restaurant is successful and it is not, respectively. The restaurant yields gross profits of S and F when it is successful and when it is a failure, respectively. Suppose that $S - F$ is large enough so it is always optimal for Daisuke to induce effort from the chef he hires.

(a) [4 points] Let (w_S^*, w_F^*) be the optimal wage scheme for Yusufcan. Provide the equations that characterize (w_S^*, w_F^*) and briefly explain why it must satisfy those equations.

(b) [6 points] Actually, Daisuke is strictly better off by hiring Emre instead of Yusufcan when ε is small enough even though Yusufcan is a better chef. Explain why this is the case by illustrating how the optimal wage scheme for Yusufcan can be modified to attain a strictly higher net profit with Emre. (Hint: What happens if the optimal wage scheme for Yusufcan is offered to Emre?)

QUESTION 10: Detroit Airport is auctioning off one arrival/departure slot. Only one company can win the slot. There are four airline companies: N (Northwest), D (Delta), U (United), and C (Continental), and two alliances: {Northwest, Delta} and {United, Continental}.

Company i's valuation for the slot, denoted by v_i, is known only to itself and valuations are independently and uniformly distributed over $[0, 1]$. All companies are risk-neutral. Unlike in the standard model, company i also gains a benefit of αv_i ($\alpha \in (0, 1)$) when its partner (Delta for Northwest, for example) wins the slot. Assume that the parameter α is commonly known.

For instance, Northwest's payoffs are:

- $v_N - b_N$ if Northwest wins and pays b_N,
- αv_N if Delta wins (and Northwest does not pay),
- 0 if United or Continental wins (and Northwest does not pay).

Although two companies in the same alliance have some common interest, they do not know each other's valuation, and make their bids independently.

(a) [2 points] Imagine you are the manager of Northwest (so you know the value of v_N but not other v's). Conditional upon v_N, what is the probability that Northwest has the highest valuation? What is the probability that Delta has the highest valuation?

(b) [2 points] If the slot is always allocated to the company with the highest valuation and the expected payment of Northwest when its valuation is v_N is $e(v_N)$, what is the expected payoff of Northwest conditional upon v_N?

(c) [6 points] Suppose that the allocation and the expected payment described in part (b) result from a Bayesian Nash equilibrium of some mechanism. Furthermore, assume $e(0) = 0$. Derive $e(\cdot)$.

(d) [5 points] Suppose the slot is sold via the first-price sealed-bid auction with no minimum bid. Let $b^f(\cdot)$ be the symmetric equilibrium bidding function that is strictly increasing. Using your result in part (c), derive an expression for b^f. Do the companies bid more or less compared to when there is no alliance (i.e. $\alpha = 0$)?

Hint: Be aware that a company pays only when it wins and it does not pay even when its partner wins.

Microeconomics Prelim Exam 2008
Friday, August 22, 9 AM - 2 PM.

ANSWERS

QUESTION 1: Initially, gold holders 1,3,5,...,19 have respective values 2,6,10,..., 38 for gold. Dollar holders 2,4,6,..,20 have respective values 2,4,6,...,20 for gold. Gold owners with values 2,6,10,14 sell to dollar holders with gold values 20, 18, 16,14,and the price is 14. Now, we get new dollar holders. Gold owners with values 2,6,10,14,18 sell to dollar holders with gold values 30,28,26,24,22. The price will lie in [18,22] clearly. In fact, it must keep dollar holder 20 from buying gold. So the price of gold is in [20,22]. Thus, the price of gold rises from 14 to the range of 20 to 22. Quantity traded rises from 4 to 5.
Observe that prices perform (at least) two roles in the market economy: they ensure those who should trade do trade, and ensure that those who should not trade do not trade.

Question 2: Solution: Only $x = 0$. Solving marginal cost equal marginal benefit, we get

$$0 = x^3 - 6x^2 + 11x - 6 = (x-1)(x-2)(x-3)$$

Checking these solutions, we find net benefits

$$-x^4/4 + 2x^3 - 11x^2/2 + 6x - 3$$

equal to -0.75 at both $x = 1$ and $x = 3$, and -1 at $x = 2$.

Question 3: Quantity rises by 30 units. To see why, let's imagine that the demand side pays the tax. With elasticities $\epsilon = -2$ and $\eta = -1$ (so we have a backward-bending supply curve!), the final quantity %age change equals the supply price %age change, while the demand %age change is half the quantity %age change (using standard approximations). Since the tax amounts to 3%, we have

$$P_S\% = Q\% = 0.5P_D\% = 0.5(P_S\% + 3\%)$$

The only solution is that the supply price %age change is 3%, and quantity %age change is 3%. This equals 30 units.

Question 4: Solution: No change. With log utility, the ultimate demand for state contingent consumption is Cobb-Douglas, and thus one consumes a constant fraction of one's wealth on each good.

QUESTION 5:

(a) If $v > 0$ no strategy is weakly dominant. Consider first strategies $c_i > 0$. If all other agents' contributions add up to at least 1, Lorch Hall will be restored, and agent i has to pay a positive amount, but had agent i chosen $c_i = 0$, Lorch Hall would also be restored, but agent i would have to pay nothing. Therefore, $c_i > 0$ is not weakly dominant. Now consider $c_i = 0$. Suppose that all other agents' contributions add up to $1 - \varepsilon$ where $0 < \varepsilon < v$. Then Lorch Hall will not be restored, and agent i's utility will be zero. By contrast, had agent i chosen $c_i = \varepsilon$, Lorch Hall would have been restored, and agent i's utility would have been $v - \varepsilon > 0$.

Now suppose that $v = 0$. Then $c_i = 0$ is a weakly dominant strategy. If agent i chooses $c_i = 0$, then agent i's utility is zero independent of what the other agents do. If agent i chooses $c_i > 0$, either agent i's utility is zero, if the sum of the other agents' offered contributions is less than $1 - c_i$, or it is $-\frac{c}{S}$ if the sum of the other agents contributions is more than $1 - c_i$. Thus, it is never greater, and sometimes lower than the utility from $c_i = 0$.

Strategy $c_i = 0$ is not strictly dominant because when compared to strategies c_i with $0 < c_i < 1$ the two strategies yield the same utility if the sum of the other agents' offered contributions is less than $1 - c_i$.

(b) Consider first the case: $c^* \geq 1/N$. Then Lorch Hall will be restored and each player will have to pay $1/N$. If $c^* > 1/N$, then each player could gain by lowering their offered contribution so that the sum of the offered contributions is exactly equal to 1. So, the only candidate for a Nash equilibrium is $c^* = 1/N$. Each player's utility is then $v - 1/N$. A player could deviate by lowering their contribution so that Lorch Hall is not restored. Then the deviating player's utility is 0. We have a Nash equilibrium if and only if $v - 1/N \geq 0 \Leftrightarrow v \geq 1/N$.

Consider next the case: $c^* < 1/N$. In that case Lorch Hall will not be restored, and every player's utility is zero. The only deviation by a single player that changes the outcome is that a player offers a contribution of at least $1 - (N-1)c^*$ so that Lorch Hall is restored. We need not consider the case that a player offers more than $1 - (N-1)c^*$, because this will not change the outcome that Lorch Hall is restored, and it might

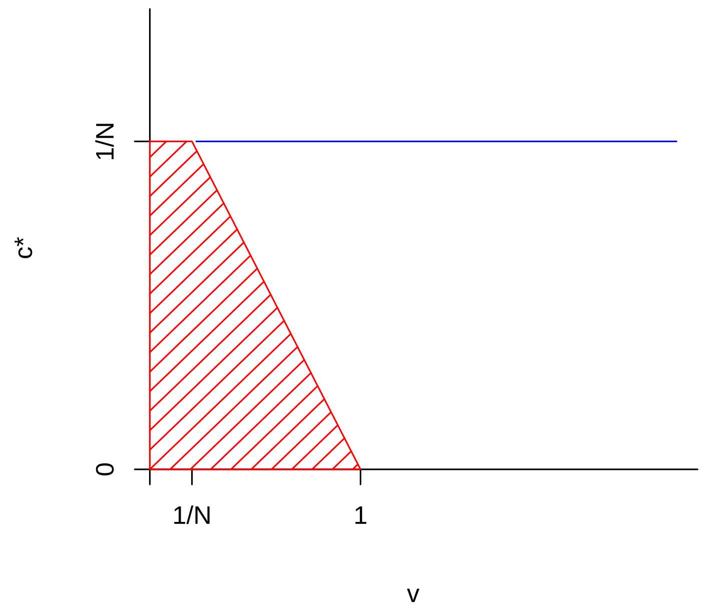


Figure 1: Equilibrium values of c^*

raise, and will never lower, the deviating player's payment. The deviating player's payoff if she offers exactly $1-(N-1)c^*$ is: $v-(1-(N-1)c^*)$. The deviation is not worthwhile and we have a Nash equilibrium if $v-(1-(N-1))c^* \leq 0 \Leftrightarrow (N-1)c^* \leq 1-v \Leftrightarrow c^* \leq (1-v)/(N-1)$.

Figure 1 shows for each value of v the corresponding equilibrium values of c^*. The red and shaded area indicates the equilibrium values of c^* such that $c^* < 1/N$.[1] The blue line corresponds to the equilibrium $c^* = 1/N$. There are multiple equilibria whenever $v \leq 1$. For $v > 1$ the only equilibrium is $c^* = 1/N$.

[1] Note that the upper boundary of the shaded area should not be included in the shaded area.

QUESTION 6:

(a) The extensive form is displayed in Figure 2 where for simplicity we write "c" instead of "$\bar{c}$."

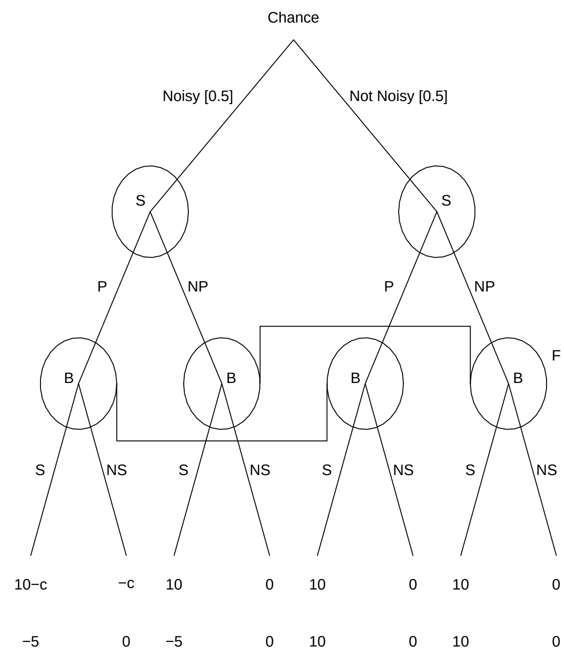


Figure 2

(b) In a weak perfect equilibrium as described in the question the buyer's beliefs after observing P have to be that with probability 1 the neighbors' house is quiet, and after observing NP the beliefs have to be that with probability 1 the neighbors' house is noisy (by Bayesian updating). Therefore, if the buyer observes P, the buyer will choose S, and if the buyer observes NP, the buyer will choose NP (by sequential rationality). When knowing that the neighbors are noisy, the seller will therefore get a payoff of $10 - \bar{c}$ from choosing P, and a payoff of 0 from choosing NP. Therefore, choosing NP is optimal if $10-\bar{c} \geq 0 \Leftrightarrow \bar{c} \geq 10$. On the other hand, when knowing that the neighbors are not noisy, the seller will get a payoff of 10 from choosing P, and a payoff of 0 from choosing NP. Therefore, it will always be optimal to choose P. Therefore, such a weak perfect Bayesian equilibrium exists if and only if $\bar{c} \geq 10$.

(c) In a weak perfect equilibrium as described in the question, the buyer's beliefs after observing NP have to be that the probability that the neighbors are noisy is 0.5. The buyer will therefore have expected utility of 2.5 from signing, and expected utility from 0 from not signing, and will therefore in equilibrium sign the contract. The buyer can have arbitrary beliefs when observing that the seller does present papers because Bayesian updating does not apply. For example, the buyer can believe in that case that with probability 1 that the neighbors are noisy. It will then be optimal for the buyer to choose NS. This implies that the seller, regardless of nature's move, will obtain payoff of 10 when not providing papers, and payoff of 0 when providing papers. Therefore, it will be optimal for the seller not to provide papers in both states of nature. Thus, a weak perfect Bayesian equilibrium of the type described in the question exists for all values of $\bar{c}$.

Question 7:

(a) The competitive equilibrium is $((1/2, 1/2), (1/2, 1/2); 1)$.

(b) Utility Maximization: For consumer 1, x_1^*, y_1^* solves

$$\max \quad u_1(x_1, y_1) \ \text{ subject to } (1+t)x_1 + p^* y_1 \leq 1 + T,$$

for consumer 2, x_2^*, y_2^* solves

$$\max \quad u_2(x_2, y_2) \ \text{ subject to } x_2 + p^* y_2 \leq p^* + T,$$

Market Clearing: For each good,

$$x_1^* + x_2^* = 1 \quad y_1^* + y_2^* = 1,$$

Balanced budget:

$$T + T = t x_1^*.$$

F.O.C's

$$\frac{y_1}{x_1} = \frac{1+t}{p^*} \qquad \frac{y_2}{x_2} = \frac{1}{p^*}.$$

Then we have

$$2(1+t)x_1 = 1 + \frac{T}{2}$$

by FOC and the budget of consumer 1. Given the zero budget equation,

$$x_1^* = \frac{2}{4+3t}.$$

Note that when $t = 0$, $x_1^* = \frac{1}{2}$.

To find out p^*, we use both budget equations and get $p^* = 1 + 2T$. Then by the zero budget equation,

$$p^* = \frac{4+5t}{4+3t}.$$

Again when $t = 0$, $p^* = 1$.

Then we have

$$x_2^* = \frac{2+3t}{4+3t}, \quad y_1^* = \frac{2+2t}{4+5t}, \quad y_2^* = \frac{2+3t}{4+5t}.$$

(c) It is easy to see that MRS are not equal.

Question 8:

(a) No. Take the point (1/2,1/2). There exists no point in $N_{1/4}(1/2, 1/2)$ strictly better than $(1/2, 1/2)$.

(b) $PO = \{(1, 0; 0, 1), (0, 1; 1, 0)\}$. Take $(1, 0; 0, 1)$. While the first consumer gets a utility of 5, the second one gets 0. The only way we make the second consumer better is to give him one unit of good x. If so, the first consumer will be strictly worse of. Therefore, $(1, 0; 0, 1)$ is a Pareto optimal allocation. Similarly, $(0, 1; 1, 0)$ is also PO.

Any interior point, $(x, y; 1 - x, 1 - y)$ with $0 < x, y < 1$, is pareto dominated by $(1, 0; 0, 1)$. We need to check boundaries. If either $x = 1$ or $y = 0$, then $(1, 0; 0, 1)$ Pareto dominates it. On the other hand, if either $x = 0$ or $y = 1$, then $(0, 1; 1, 0)$ Pareto dominates it.

(c) No. To see this, let $p_x = p$ and $p_y = 1$. When $p = 1$, both agents demand $(1, 0)$. Hence there is an excess demand for x. When $p < 1$, both agents demand at least one unit of x. Again, we have an excess demand for x. Similarly, when $p > 1$, both agents demand at least one unit of y, which is an excess demand for y. Therefore, there is no price to clear the market.

(d) Yes, all Pareto optimal allocations are competitive equilibria if they are initial endowment. For example, $[(1, 0; 0, 1), p]$ constitutes a competitive equilibrium when $p > 1$ and $(1, 0; 0, 1)$ is the endowment vector.

Question 9:

(a) Because of the participation constraint (PC) and the incentive compatibility (IC), we have

$$\begin{aligned} pv\left(w_S^*\right) + (1-p)\,v\left(w_F^*\right) - c &= 0 \\ pv\left(w_S^*\right) + (1-p)\,v\left(w_F^*\right) - c &= qv\left(w_S^*\right) + (1-q)\,v\left(w_F^*\right) \end{aligned}$$

The reasons why both conditions are binding are: If PC is not binding, w_F^* can be reduced without breaking IC. If IC is not binding, then Daisuke can make the difference between w_S^* and w_F^* smaller (but keeping the expected payment unchanged), which makes PC non-binding (because Yusufcan is risk averse). Therefore, Daisuke can reduce w_F^* furthermore. These change reduces the expected wage payment.

(b) If the optimal wage with Yusufcan is offered to Emre, then:

- The left-hand side of (PC) goes down *slightly* because p is now replaced with $p - \varepsilon$. To satisfy (PC), Daisuke needs to raise w_S^* *slightly*.
- The left-hand side of (IC) falls only *slightly* but the right-hand side of IC goes down *significantly* (because q is replaced with r) so IC is not binding. Therefore, as explained in the previous part, Daisuke can *significantly* reduce the expected wage payment without violating the constraints.

When ε is small, the first effect becomes negligible (but the second effect is not). Therefore, Daisuke can induce an effort from Emre with a lower expected wage payment so he prefers to hire Emre.

Question 10:

(a) Northwest has the highest valuation w.p. v_N^3. Delta has the highest valuation w.p. $(1 - v_N^3)/3$.

(b)

$$\begin{aligned} U(v_N) &= v_N^3 v_N + \frac{1 - v_N^3}{3}\alpha v_N - e(v_N) \\ &= \left(\frac{(3-\alpha)v_N^3 + \alpha}{3}\right) v_N - e(v_N) \end{aligned}$$

(c) By the envelope theorem type argument you must have seen many times,

$$\begin{aligned} e(v_N) &= \left(\frac{(3-\alpha)v_N^3 + \alpha}{3}\right) v_N - \int_0^{v_N} \left(\frac{(3-\alpha)v_N^3 + \alpha}{3}\right) ds \\ &= \frac{(3-\alpha)}{3} \cdot \frac{3}{4} v_N^4 \\ &= \left(\frac{3-\alpha}{4}\right) v_N^4 \end{aligned}$$

(d) Since it must be

$$e(v_N) = v_N^3 b^f(v_N)$$

the equilibrium bidding (symmetric and strictly increasing) must be

$$b^f(v_N) = \left(\frac{3-\alpha}{4}\right) v_N$$

Clearly, they bid less compared to the standard case.

Remark: The above argument only shows that if there is a symmetric equilibrium with a strictly increasing bidding function, it must be the one we derived. Although the question does not ask you to prove that it is indeed the equilibrium, I am providing a complete proof here for your curiosity.

Note

$$\left(\frac{(3-\alpha)v_N^3 + \alpha}{3}\right)$$

which appears in the formula of U, is nondecreasing in v_N. Therefore, no type has an incentive to mimic another type when all other players follow b^f. Furthermore, bidding above $b^f(1)$ is worse than bidding $b^f(1)$ because it does not change the winning probabilities (of itself and its partner) at all but increases the payment upon winning. Therefore, no type has an incentive to bid an amount that is not chosen by any type.

Hence, no type has an incentive to deviate from $b^f(\cdot)$. This proves that b^f is indeed an equilibrium.

Microeconomics Prelim Exam 2009
Friday, August 21, 9 AM - 2 PM.

- Please answer all questions.
- For each question we have provided in square brackets [...] an indication of the maximum number of points that you can get for that part.
- The exam consists of 100 total points.
- Good luck!

Part I

QUESTION 1: Suppose that one experiences a constant marginal benefit from producing gummy bears equal to p. They have *no fixed cost.* But the *marginal cost* for producing quantity q is a zig-zag line consisting of line segments (each with slope ± 1) joining the following pairs in (q, p) space:

$$(0, 2), (1, 1), (4, 4), (6, 2), \text{and then a line of slope 1 rising up and right forever}$$

(a) Carefully graph the marginal cost and marginal benefit curves.

(b) Carefully draw your *optimal production* q as a function of the price.

[7 points altogether for both parts of this question.]

QUESTION 2: Suppose that at all price levels, a competitive firm's profits always rise about 3% whenever the price rises 1% — with this approximation getting proportionately more accurate as the percentages shrink. How much do its costs rise when its quantity rises 1%? [7]

QUESTION 3: *The Firm* has two plants. Plant 1 has cost function $C_1(q) = q_1^2$ and plant 2 has higher cost $C_2(q) = 2q_2^2$. What is the *The Firm*'s supply curve for $q = q_1 + q_2$? [6]

QUESTION 4: If Lones has utility function $U(G, S) = G^2 + S^2$, where G is go-karting runs and S is stage diving jumps, what is his *demand function* for stage diving jumps? [5]

Part II

QUESTION 5: Consider the following two player game in normalform.

	L	C	R
T	1,1	0,2	-1,-2
M	2,0	0,0	-1,-2
B	-2,-1	-2,-1	-3,-3

(a) Find all Nash equilibria in pure or mixed strategies of this game. [5]

(b) Now suppose the game in (a) is played repeatedly in an infinite number of time periods: $t = 0, 1, 2, \ldots$. After each period, each player observes both players' choices in that period. Players maximize the present discounted value of their per period payoffs. Both players have the same discount factor δ where $0 < \delta < 1$. For which values of δ, if any, do the following strategies constitute a subgame-perfect Nash equilibrium of the infinitely repeated game? In period 0 player 1 chooses T and player 2 chooses L. In periods $t \geq 1$ player 1 chooses T and player 2 chooses L if in all previous periods player 1 chose T and player 2 chose L. If any other choices were made in the previous periods then player 1 chooses B and player 2 chooses L from then onwards for the rest of the game. [5]

(c) Consider again the repeated game described in part (b), but now suppose players choose the following strategies: In period 0 player 1 chooses T and player 2 chooses L. In periods $t \geq 1$ player 1 chooses T and player 2 chooses L if in all previous periods player 1 chose T and player 2 chose L. If in some period one of the two players deviates, then in the next two periods player 1 chooses B and player 2 chooses R. After this "punishment period" players return to the original strategies, and choose T and L, provided that no player deviated since the end of the punishment. If a player deviates when the players are supposed to play (T, L), then players punish this deviation again with two periods of (B, R). This continues indefinitely. If a player deviates from the punishment, that is if player 1 does not choose B in a period in which he is supposed to choose B, or if player 2 does not choose L in a period in which he is supposed to choose R, then players play (M, C) from then onwards for the rest of the game, independent of the remaining history of the game. For which values of δ, if any, do these strategies constitute a subgame-perfect Nash equilibrium?[5]

QUESTION 6: Three voters $i = 1, 2, 3$ play a voting game in which each of them has just two choices: vote for candidate A, or vote for candidate B. Voters choose simultaneously. The candidate who receives two votes wins. All voters have the same utility function: utility equals s if candidate A is elected, and utility equals 0 if candidate B is elected. Here, s is the weighted sum of three independent random variables: $s = s_1 + cs_2 + cs_3$. The random variables s_1, s_2 and s_3 all follow the uniform distribution on the interval $[-1, 1]$. The number c is a constant with $c > 0$. Each voter i observes the realization of s_i, but not the realization of the other random variables. The value of the constant c is common knowledge among the voters.

(a) For which values of c do the following strategies constitute a Bayesian Nash equilibrium: Each voter i votes for A if $s_i \geq 0$ and for B if $s_i < 0$. (Hint: For each voter i consider when this voter's vote will be decisive. Calculate the expected value of the other voters' signals, conditioning on the event that i's vote will be decisive. Then calculate the expected value of voting for A for voter i, conditioning again on the event that i's vote will be decisive.) [5]

(b) For which values of c do the following strategies constitute a Bayesian Nash equilibrium: Voter 1 votes for A if $s_1 \geq 0$, and for B if $s_1 < 0$. Voter 2 votes for A for all realizations of his signal. Voter 3 votes for B for all realizations of her signal. [5]

Part III

QUESTION 7: **(A Liquorice Market)** Let there be twenty students $1, 3, 5, 7, \ldots, 39$ who each hold a single liquorice bag, and twenty students $2, 4, 6, \ldots, 40$ with no liquorice bags. A student only needs one liquorice bag, and his student number is what that liquorice bag is worth to him (as a seller or buyer); additional liquorice bags are useless (too sweet). So odd numbered students are the potential sellers, and even numbered students are potential buyers.

(a) Find the market-clearing price and quantity traded range. Justify your answer.

(b) Into this mix walks Lones who simply loves liquorice. He is willing to pay 30 for his first liquorice bag, 27 for his second, 24 for his third, etc. Find the new market-clearing price range and quantity traded, and how many bags of liquorice Lones will buy. Justify your answer.

[10 points altogether for both parts of this question.]

Question 8: **(A Crime Market)** Consider the "implicit market" that arises in property crime. A *unit mass* of identical property owners each own one item of personal value $M > 0$. They can choose the "deterrence level" $\delta \in [0, 1]$ at effort cost $c\delta^2/2$, measured in units of money. With deterrence level $\delta \in [0, 1]$, an attempt to steal an item succeeds with chance $1 - \delta$. Any property owner faces a theft attempt with chance $\kappa \in [0, 1]$; this "crime rate" will be endogenously determined here.

There is a heterogeneous continuum of possible criminals, each of whom can enter and make a quantity $\lambda > 0$ of attempted thefts (assumed fixed). Criminals with "crime index" $i > 0$ value the legal penalty, and social stigma, and physical cost of committing the crimes at i. In other words, i is the fixed cost of committing a crime, independent of the number of the quantity of crimes.
Criminals value the good at bM, where $0 < b < 1$, since they must sell it at a loss. *The mass of individuals with crime indices below i is* μi.

Criminals and property owners alike are risk neutral, and care just about expected gains minus expected losses. Assume that crimes are random and independent across property owners, so that *if there is a mass* μ *of crimes then the crime rate is* $\lambda\mu$.

(a) Think of $p = 1 - \delta$ as a metaphorical price of crime to the property owners, namely its success chance. Derive the "supply and demand curves" for crime — in other words, the functions that relate the crime rate κ and crime price p. Plot them in κ-p space.

(b) Express the crime rate as a function of all exogenous variables. Verify any interiority assumptions you have made for property owners.

(c) What happens to the crime rate as the following change: the personal value M? the markdown $1 - b$? the deterrence cost parameter c? the criminal supply density μ?

(d) What should have a higher crime rate: money or equally-valued jewelry?

[15 points altogether for all four parts of this question.]

Part IV

QUESTION 9: Yusufcan has one ticket for a real Turkish dinner but he found he cannot make it, so he is giving it away to one of his **three** colleagues. Each colleague's valuation (type) for the ticket is independently and uniformly distributed over $[0, 1]$ and it is known only to himself. The colleagues are risk-neutral.

Yusufcan would like to give the ticket to one who values the ticket less so that such a colleague has a chance to learn the excellence of Turkish foods.

(a) Show that the following allocation is *not* implementable in Bayesian Nash equilibria:

> For any realization of valuations, the ticket is awarded to the *highest* type or the *lowest* type with equal probabilities. [4]

(b) Show that the following allocation *is* implementable in Bayesian Nash equilibria:

> For any realization of valuations, the ticket is awarded to the *highest* type or the *second highest* type with equal probabilities.

To implement the above allocation, what will be the expected payment of each type if type 0 never pays or receives money? [6]

(c) Consider the following auction:

> Each colleague submits a non-negative bid simultaneously and the winner is either the one making the *highest* bid or the *second* highest bid with equal probabilities. The winner pays his own bid.

Suppose this auction has a symmetric equilibrium with a strictly increasing bidding function $b(\cdot)$ (actually it does but you do not have to prove it). Derive $b(\cdot)$ by using the result obtained in (b). [3]

QUESTION 10: An artist offers to sell his painting to two potential buyers. The painting is equally likely to be of exceptionally good quality (A), moderately good quality (B), or just bad quality (C). The quality is known only to the artist. The valuation of each quality to each player is given in the following table:

	A	B	C
to the artist	$\$r_A$	\$11	\$0
to each buyer	\$31	\$15	\$5

(in thousands)

$r_A > 11$ is a non-integer

Buyers are risk neutral. Trade proceeds as follows: the two buyers make simultaneous offers to the artist, and the artist may then accept one of the two offers or reject them both. In the analysis below, confine attention to pure strategy perfect Bayesian equilibria.

(a) What are the equilibrium outcomes of this model for different possible values of r_A? Note that the buyers behave like Bertand competitors. (No explanation is needed.) [4]

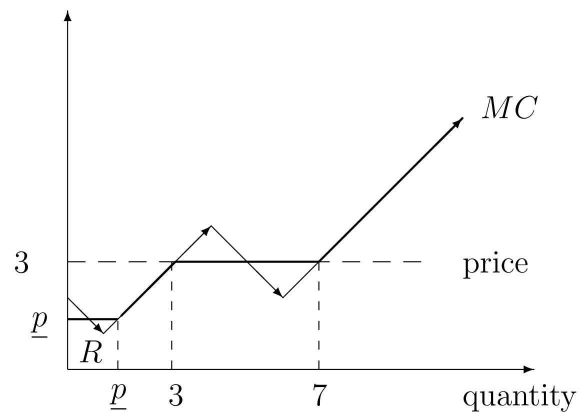
(b) Suppose that the artist can costlessly certify that his painting is type B if indeed it is a type B and that buyers can observe whether or not this certification has been obtained by the seller, prior to making their offers. What are the equilibrium outcomes of this model for different possible values of r_A (if any pure strategy equilibrium exists)?

Hint: Explore the two possibilities separately where type B reveals its type and where it does not. [8]

Microeconomics Prelim Exam 2009
Answers

Part I

QUESTION 1:



Local optimality demands that price equal marginal cost MC when MC is rising. Global optimality demands further that total benefits exceed total costs. The latter constraint binds in this case. At prices above $\underline{p} \in (1, 2)$, the quantity rises to a positive number (namely, at $\underline{p}$, we jump to the 45 degree line, where $q = p$). This key choke-off price ensures it so that the area of the indicated rectangle R equals the area beneath the zig-zag MC curve. In other words,

$$\underline{p}^2 = (2+1)/2 + (\underline{p}+1)(\underline{p}-1)/2 \quad \Rightarrow \quad \underline{p} = \sqrt{2}$$

Above this price, the supply curve must be as indicated, with a jump at price $p = 3$ from 3 up to 7 (namely, at $p = 3$, we jump to $q = p + 4$).

QUESTION 2: The premise implies that the profit function has the form $\pi(p) \equiv ap^3$ for some $a > 0$. Thus, the supply function is $q(p) = \pi'(q) = 3ap^2$, by Hotelling's Lemma. But profits are revenue minus costs, so that

$$ap^3 \equiv \pi(p) = pq(p) - c(q(p)) \equiv 3ap^3 - c(3p^2) \Rightarrow c(3p^2) \equiv 2ap^3 \Rightarrow c(q) \propto q^{3/2}$$

In other words, costs rise $3/2 = 1.5$ percent when quantity rises 1%.

QUESTION 3: It is best to equate $C_1'(q_1) = C'(q_2)$, so that $q_1 = 2q_2$, and thus $q_1 = 2q/3$ and $q_2 = q/3$. The cost function is $C(q) = C_1(2q/3) + C_2(q/3) = (2q/3)^2 + 2(q/3)^2 = 2q^2/3$. The supply curve is then $p = C'(q) = 4q/3$, or $q(p) = 3p/4$.

QUESTION 4: His preferences are not convex, and indeed, his indifference curves are concave. So the solution is not interior, but instead is a corner. Easily, $S = I/P_S$ if $P_S < P_G$, and $S = 0$ otherwise.

Part II

QUESTION 5:

(a) Strategies B and R are strictly dominated. The remaining game has the following Nash equilibria: Player 1 plays M, and player 2 plays L or R where the probability of L can be any number between zero and one; player 2 plays C, and player 1 plays T or M, where the probability of T can be any number between zero and one.

(b) This is never a subgame-perfect equilibrium. In subgames in which the punishment is supposed to be played, player 1 has an incentive to deviate from B to M.

(c) We first check whether a player has an incentive to deviate if play is on the equilibrium path. A player who follows the equilibrium path receives payoff stream $1, 1, 1, \ldots$. A player who chooses the optimal single period deviation receives payoff stream: $2, -3, -3, 1, 1, 1 \ldots$. The present value of the former payoff stream is at least as large as the present value of the second payoff stream if:

$$\begin{aligned} 1 + \delta + \delta^2 &\geq 2 - 3\delta - 3\delta^2 \Leftrightarrow \\ 4\delta^2 + 4\delta - 1 &\geq 0 \Leftrightarrow \\ \delta &\geq -\frac{1}{2} + \sqrt{\frac{1}{2}}. \end{aligned}$$

Next, we check whether a player has an incentive to deviate off the equilibrium path. There are no incentives to deviate if players play indefinitely the one shot equilibrium (M, C). It remains to check for incentives to deviate in the punishment phase in which players play (B, R) twice. In this phase incentives to deviate are obviously larger in the first period than in the second period of the punishment. Therefore, we focus on the first period. A player who follows the punishment strategy receives payoff stream $-3, -3, 1, 1, \ldots$. A player who deviates receives the payoff stream: $-1, 0, 0, \ldots$. Following the equilibrium yields larger expected payoffs if:

$$\begin{aligned}
-3(1+\delta)+\delta^2\frac{1}{1-\delta} &\geq -1 \Leftrightarrow \\
-3(1+\delta)(1-\delta)+\delta^2 &\geq -1(1-\delta) \Leftrightarrow \\
4\delta^2-\delta-2 &\geq 0 \Leftrightarrow \\
\delta &\geq \frac{1}{8}+\sqrt{\frac{33}{64}}.
\end{aligned}$$

The conclusion is that these strategies form a subgame-perfect equilibrium if and only if both inequalities for δ hold. One can easily see that the second one is more demanding than the first one. The first one requires $4\delta^2$ to be at least $1-4\delta$. The second one requires $4\delta^2$ to be at least $2+\delta$. Clearly, the second requirement is more restrictive than the first. The strategies form a subgame-perfect equilibrium if and only if:

$$\delta \geq \frac{1}{8}+\sqrt{\frac{33}{64}} \approx 0.84.$$

QUESTION 6:

(a) We begin by considering voter 1's incentives. Voter 1's vote matters only if one of voters 2 and 3 votes for A and the other one votes for B. Conditional on that event, the expected value of voter 2 and 3's signals is +0.5 and -0.5. The conditional expected value of cs_1+cs_2 is therefore zero. Voter 1's conditional expected value from voting for A is therefore s_1. Therefore, voter 1's strategy is optimal for all c.

Next consider voter 2's incentives. (Voter 3's incentives are analogous.) Voter 2's vote matters only if voters 1 and 3 cast opposite votes. This occurs firstly if voter 1 votes for A and voter 3 votes for B, in which case the conditional expected value of s_1+cs_3 is: $(1-c)0.5$, or it occurs if voter 1 votes for B and voter 3 votes for A, in which case the conditional expected value of s_1+cs_3 is: $-(1-c)0.5$. Each of these two events occurs with probability 0.5. Therefore, the conditional expected value of s_1+cs_3 is zero. Voter 2's conditional expected value from voting for A is s_1, and therefore, voter 2's strategy is optimal for all c.

The conclusion is that these strategies form a Bayesian Nash equilibrium for all values of c.

(b) We begin by considering voter 1's incentives: voter 1 knows that his vote is decisive. The expected values of s_2 and s_3 is zero. Therefore, voter 1's expected value from voting for A is s_1. Therefore, voter 1's strategy is optimal for all values of c.

Next, we consider voter 2's incentives. Voter 2 knows that his vote is only relevant if voter 1 votes for A. The conditional expected value of voter 1's signal is then 0.5. The expected value of voter 3's signal is zero. Therefore, the conditional expected value from voting for A is: $0.5 + cs_2$. Voter 2's strategy is optimal if $0.5 + cs_2 \geq 0$ for all $s_2 \in [-1, 1]$. This is true if and only if it is true for $s_2 = -1$. Thus, voter 2's strategy is optimal if and only if $0.5 - c \geq 0 \Leftrightarrow c \leq 0.5$.

Finally, we consider voter 3's incentives. Voter 3 knows that her vote is relevant only if voter 1 votes for B. The conditional expected value of voter 1's signal is then -0.5. The expected value of voter 2's signal is zero. Therefore, the conditional expected value from voting for A is: $-0.5 + cs_3$. For voter 3's strategy to be optimal we need that $-0.5 + cs_3 \leq 0$ for all $s_3 \in [-1, 1]$. This is true if and only if it is true for $s_3 = 1$. Thus, voter 3's strategy is optimal if and only if $-0.5 + c \leq 0 \Leftrightarrow c \leq 0.5$.

The conclusion is that these strategies form a Bayesian Nash equilibrium if and only if $c \leq 0.5$.

Part III

Question 7:

(a) There are only 20 bags so they should be consumed by students 21-40. To support the outcome, the price should be between 20 and 22. But in order to keep the seller with value 21 from selling, we need $p \leq 21$. So price is in $[20, 21]$.

(b) With Lones, the buyers are Lones (with three liquorice bags, that he values at $30, 27, 24$) and the nine students who most like liquorice, namely $40, 38, 36, \ldots, 24$. The sellers are the twelve students who least like liquorice, valuing them at $1, 3, 5, \ldots, 23$, and the price is in $[23, 24]$. The price cannot be less than 23, or the student with valuation 23 refuses to sell! Also, the price cannot exceed 24, or the student with valuation 24 refuses to buy. Lones raised the price, and the amount traded.

Question 8: (This question is based on research that was outlined in class.)

(a) The optimization of property owners is

$$\min_{\delta \in [0,1]} \kappa M(1-\delta) + c\delta^2/2$$

The FOC is $\kappa = c\delta/M$. Thus, there is a unique interior solution $\delta^* < 1$ given by this FOC provided $0 < \kappa < c/M$. Thus, the FOC yields the "derived demand curve" for crime, so that $\kappa^D = c(1-p)/M$.

Turning to the supply side, potential criminals enter provided

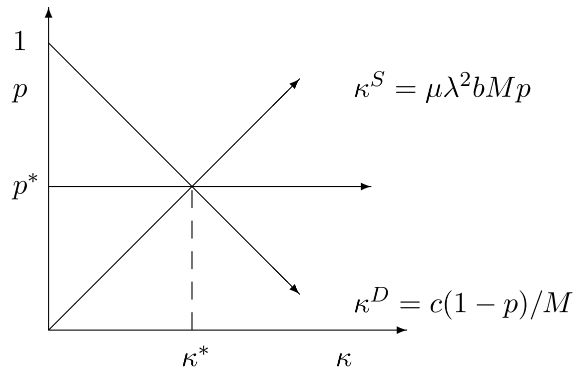
$$\lambda b M(1-\delta) \geq i$$

The mass of such individuals is $\kappa = \mu\lambda b M(1-\delta)$, and thus, since each such criminal commits λ crimes, the supply curve for crimes is $\kappa^S = \mu\lambda^2 b M p$, rising due to entry.

(b) Solving for equilibrium, we find $\kappa^* = c(1-p^*)/M = \mu\lambda^2 b M p^*$, and so:

$$p^* = \frac{c}{\mu\lambda^2 b M^2 + c} \qquad \text{and} \qquad \kappa^* = \frac{c\mu\lambda^2 b M}{\mu\lambda^2 b M^2 + c}$$

Supply and Demand Curves for Crime



Our regularity condition $\kappa^* < c/M$ is then automatically met since $\kappa^* > 0$ and

$$\kappa^* = \frac{c\mu\lambda^2 bM}{\mu\lambda^2 bM^2 + c} < c/M \Leftrightarrow \mu\lambda^2 bM^2 < \mu\lambda^2 bM^2 + c$$

(c) and (d) The crime rate

- falls in the markdown $1 - b \Rightarrow$ lower crime rate for jewelry than money
- is hill-shaped in the property value M, since the demand curve falls as the inverse of M and the supply curve rises linearly in M.
- rises in the verification cost parameter c, the criminal supply density μ, and the crimes per crook λ

Part IV

QUESTION 9:

(a) Type x wins the object with probability:

$$\frac{1}{2}x^2 + \frac{1}{2}(1-x)^2$$

It is easy to see that this is decreasing in x when $x < 1/2$. Therefore, it is not implementable in Bayesian Nash equilibria.

(b) Type x wins the object with probability:

$$\frac{1}{2}x^2 + \frac{1}{2}(2x(1-x)) = x - \frac{1}{2}x^2$$

Since this is strictly increasing in x ($x \in [0,1]$), it is implementable in Bayesian Nash equilibria with the following expected payment for each type:

$$\begin{aligned} E(x) &= \left(x - \frac{1}{2}x^2\right)x - \int_0^x \left(s - \frac{1}{2}s^2\right) ds \\ &= \frac{1}{2}x^2 - \frac{1}{3}x^3 \end{aligned}$$

(c) In the symmetric equilibrium with a strictly increasing bidding function, type x wins with probability

$$x - \frac{1}{2}x^2$$

Therefore, by the revenue equivalence theorem, his expected payment must be $x^2/2 - x^3/3$. Since he pays his bid only if he wins, his equilibrium bid $b(x)$ is given by

$$\left(x - \frac{1}{2}x^2\right)b(x) = \frac{1}{2}x^2 - \frac{1}{3}x^3$$

Therefore,

$$b(x) = \frac{\frac{1}{2}x - \frac{1}{3}x^2}{1 - \frac{1}{2}x}$$

(It is routine to conclude that the above b *indeed* constitutes an equilibrium by verifying that it is strictly increasing although the question does not ask you to do so.)

QUESTION 10:

(a) The expected value of buyers conditional on that the seller is willing to sell at p is given by:

$$\begin{aligned}
\$17 \text{ if } p &> r_A \\
\$10 \text{ if } p &\in (11, r_A) \\
\$5 \text{ if } p &< 11
\end{aligned}$$

Therefore, the equilibrium price is \$17 if $r_A < 17$ and \$5 if $r_A > 17$. All types are traded in the former case while only type C is traded in the latter case.

(b) If type B is excluded, the expected value to buyers conditional on that the seller is willing to sell at p is given by:

$$\begin{aligned}
\$18 \text{ if } p &> r_A \\
\$5 \text{ if } p &< r_A
\end{aligned}$$

Therefore, in any equilibrium where type B reveals its type, the equilibrium price without the certificate is \$18 if $r_A < 18$ and \$5 if $r_A > 18$.

First, let us check if there exists an equilibrium where type B reveals its type. By doing so, type B is sold at \$15 and the equilibrium price without the certification is given as above. Therefore, this can be an equilibrium when $r_A > 18$.

Next, let us investigate the case where type B does not reveal its type. In that case, the equilibrium price is the one derived in part (a) so type B has no incentive to deviate (by revealing its type) when $r_A < 17$. Therefore, if $r_A < 17$, it is an equilibrium where type B does not reveal its type and all types are traded at \$17.

There is no pure strategy equilibrium when $r_A \in (17, 18)$.

University of Wisconsin
Microeconomics Prelim Exam
Monday, August 7, 2012: 9AM - 2PM

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- Please return any unused portions of yellow tablets and question sheets.
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- Best wishes!

Part I

Suppose that a consumer's preference ordering on $\mathbb{R}^3_+$ can be represented by the utility function

$$u(x) = \alpha x_1 x_2 + \log(x_3)$$

where $\alpha > 0$.

The consumer has wealth w and faces (positive) prices p_1, p_2, p_3 for the three goods.

1. Suppose $\alpha = 2$, and $p_1 = 2, p_2 = 2, p_3 = 20, w = 5$. What is the optimal consumption plan?

2. For any given value of α, does there exist a budget constraint such that it is optimal to spend equal amounts on each of the three goods?

3. Is it possible that the consumer would choose to consume a smaller quantity of one of the goods when wealth increases?

Part II

Two customer service workers each choose effort levels $e_i \in [0, 10]$ to exert in helping a customer. The cost of effort is $c_1(e_1) = (e_1)^2/15$ for worker 1 and $c_2(e_2) = (e_2)^2/10$ for worker 2. The customer's satisfaction level equals the total effort that the workers exert, up to a maximum satisfaction level of 10. Each worker's payoff is the difference between the customer's satisfaction level and his own effort cost.

Consider the game G in which the workers simultaneously choose their effort.

1. Describe each worker's payoff function in G.
2. Describe each player's best response correspondence in G.
3. Compute all pure Nash equilibria of G.

Now we consider sequential-move versions of the interaction. In game Γ_2, first player 1 selects an effort level. Then player 2, after observing player 1's choice, chooses her own effort level.

4. Completely describe all pure strategy subgame perfect equilibria of Γ_2.

In Γ_3, the sequence of events starts with those in Γ_2. But after player 2 moves, player 1 observes her choice and decides whether to exert some additional effort, $a_1 \in [0, 10]$. His cost of effort in Γ_3 is given by $c_1(t_1) = (t_1)^2/15$, where $t_1 = e_1 + a_1$ is the sum of his initial effort choice and his additional effort choice.

5. What is a pure strategy for player 1 in Γ_3? What is a pure strategy for player 2? (You are free to state your answers in English rather than notation, but either way, your answers should be explicit.)
6. Completely describe all pure strategy subgame perfect equilibria of Γ_3.

Part III

1. You are traveling to Ottawa and stupidly you left the hotel choice until the last minute. Now you need to find a hotel to stay at for one night. You pay for a new travel service called www.fixedstigler.com. This service lets you pay a flat fee of \$100 for any hotel. If you spend nc dollars, you get a listing of n such hotels. *You can do this search only once!* Hotels vary in amenities, so that the value of any hotel is $100 + v$, where v is uniformly distributed on $[0, 100]$, independently across hotels. So the chance that any hotel yields you a *net value* or *surplus* of at least v is $1 - v/100$.

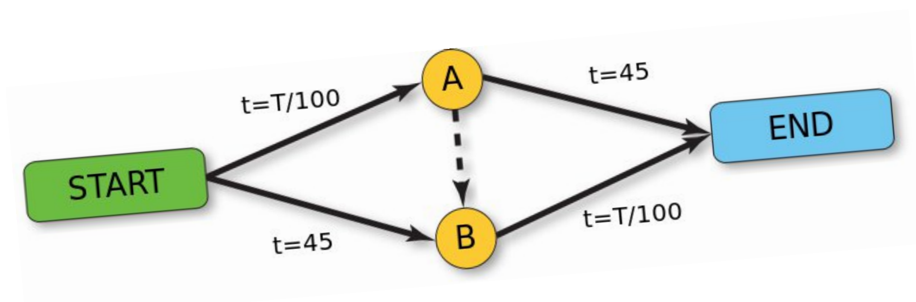
   Assume that your goal in choosing n is to maximize the surplus, net of costs. What is the demand curve $n(c)$? What is a simple approximation for $n(c)$?

2. A total of 4000 cars wish to travel from start to end. Assume first that no road joins A to B. Then there are two routes from start to end. The travel time in minutes on the road from Start to A is the number of travelers (T) divided by 100, and on the road from Start to B is a constant 45 minutes. Then these road times switch from A and B to end, as indicated.

   (a) What is the competitive equilibrium driving time from start to end?

   (b) Now assume that a *two-way* road joining A and B is built that takes 10 minutes. (Ignore the arrows on that road.) What is the competitive equilibrium driving time? Comment.

   (c) What driving choices would a social planner enforce who cared about total drive times of all drivers? How exactly would he do it?



Part IV

A firm can earn profits from two different activities undertaken by the worker. The firm's return π_1 from activity 1 is a *deterministic* function of the worker's effort on this activity. If the worker exerts high effort e_h on activity 1, then $\pi_1 = Y$; if he exerts low effort e_l, then $\pi_1 = Z$, where $Z < Y$. The firm's return π_2 from activity 2 is the following *stochastic* function of the worker's effort: If the worker exerts high effort e_h in activity 2, then $\pi_2 = X$ and $\pi_2 = 0$ with equal chances 0.5. If the worker puts in low effort e_l, then $\pi_2 = 0$ always.

The worker can exert high effort on at most one activity. Only three effort combinations are possible: effort e_l in both activities, effort e_h in activity 1 and e_l in activity 2, or effort e_h in activity 2 and e_l in activity 1. Exerting high effort is costly for the worker. Specifically, the worker's utility as a function of the contractually specified wage w is $\sqrt{w}$ if he puts low effort into both activities and $\sqrt{w} - g$ if he exerts high effort on one activity, where g is the utility effort cost. The worker's utility if he does not work for the firm is 0. Assume that $0.5X > Y - Z > g^2$. Thus, the expected marginal return of high effort in activity 2 exceeds the marginal return of high effort in activity 1. The firm is risk neutral.

1. Assume that the firm observes the worker's effort level in each activity. Characterize the contract (wages and effort levels) that the firm will offer.

2. Suppose that the firm observes both returns π_1 and π_2 but *not* the worker's effort choice. Find the profit-maximizing contracts for inducing each combination of efforts in both activities. Under what conditions on X, Y and Z is the effort combination chosen by the firm different from that in part (a)?

3. For each combination of efforts in parts (a) and (b), provide economic intuition why the wage schedules are the same/different. Verify whether the monotone likelihood ratio property holds and use this in your explanation. Interpret the risk sharing properties of the optimal contracts.

University of Wisconsin

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where $\alpha > 0$.
The consumer has wealth w and faces (positive) prices p_1, p_2, p_3 for the three goods.

1. Suppose $\alpha = 2$, and $p_1 = 2, p_2 = 2, p_3 = 20, w = 5$. What is the optimal consumption plan?
2. For any given value of α, does there exist a budget constraint such that it is optimal to spend equal amounts on each of the three goods?
3. Is it possible that the consumer would choose to consume a smaller quantity of one of the goods when wealth increases?

Solution Heuristics

Quasilinear utility means that the utility function is additively separable, and all goods except one have strictly decreasing marginal utility, while this one good has constant marginal utility. Once this good enters the optimal consumption plan, it absorbs all income effects, in the sense that increases in wealth are used to buy more of this good, with no changes in the quantities of the other good. If the price of this good is high, or wealth is low, there is a corner solution in which the marginal utility per dollar is the same for all but this one good, and this good is not consumed, because the marginal utility per dollar is too low.

In the problem here, something similar happens, but the analysis is a bit more complicated.

The marginal utility per dollar spent on goods 1 and 2 is increasing, starting from zero (if the money is equally divided between these two goods). Once these goods enter the optimal consumption plan, they absorb all income effects and then some: increases in wealth actually increase the marginal utility per dollar spent on these goods, with the result that the quantity of the other good decreases so as to raise marginal utility per dollar spend on that good to the new level. Thus good 3 is inferior (at higher wealth levels).

Buying only good 3 is always locally optimal, because marginal utility per dollar is positive, and it is zero for goods 1 and 2 at the corner.

Analysis

Since the marginal utility of good 3 is infinite at zero, x_3 must be positive at an optimum. Also, $x_1 > 0$ implies $x_2 > 0$, and vice versa, and at an interior solution, marginal utilities per dollar must be the same for all three goods. For the first two goods, this implies

$$\frac{x_1}{p_2} = \frac{x_2}{p_1}$$

so expenditure on these two good must be equal (and this is also true at a corner solution).

Let $y = w - p_3 x_3$ be the amount spent on goods 1 and 2. Then utility is

$$\begin{aligned} u &= \frac{\alpha y^2}{4 p_1 p_2} + \log\left(\frac{w-y}{p_3}\right) \\ &= \frac{1}{2}\frac{y^2}{Q} + \log(w-y) - \log(p_3) \end{aligned}$$

where $Q = 2\frac{p_1 p_2}{\alpha} > 0$.

Let

$$\phi(y) = \frac{1}{2}\frac{y^2}{Q} + \log(w-y)$$

The optimal solution is found by maximizing this with respect to y, and then setting

$$\begin{aligned} x_1 &= \frac{y^*}{2p_1} \\ x_2 &= \frac{y^*}{2p_2} \\ x_3 &= \frac{w-y^*}{p_3} \end{aligned}$$

The function ϕ has derivatives

$$\begin{aligned} \phi'(y) &= \frac{y}{Q} - \frac{1}{w-y} \\ \phi''(y) &= \frac{1}{Q} - \frac{1}{(w-y)^2} \end{aligned}$$

At an interior maximum the FOC can be written as

$$y(w-y) = Q$$

and $\phi''(y) \leq 0$ so

$$(w-y)^2 \leq Q$$

Taken together, these imply

$$(w-y)^2 \leq y(w-y)$$

and since $y < w$ this reduces to $w - y \leq y$, so at an interior optimum,

$$y^* \geq \frac{w}{2}$$

so the function $\phi(y)$ has at most two local maxima, including one at 0. Since $y(w-y) \leq \frac{w^2}{4}$, there is no interior maximum if $Q \geq \frac{w^2}{4}$. So the corner solution is optimal if wealth is low or Q is large (i.e. if the first two goods are expensive, or if they have low weight in the utility function).

If $z = w - y = p_3 x_3$, the interior maximum is where the rectangular hyperbola $yz = Q$ intersects the budget line $y + z = w$; if these don't intersect there is no interior maximum (so the corner solution is optimal), and if they do intersect, there are two (symmetric) intersections, and the relevant one has $y \geq z$. An increase in income moves the budget line out without affecting the hyperbola, so the intersection moves down along the hyperbola to the right, meaning that y increases and z decreases.

Answers

1. *Suppose $\alpha = 2$, and $p_1 = 2, p_2 = 2, p_3 = 20, w = 5$. What is the optimal consumption plan?*
   At any interior optimum,

   $$\begin{aligned} \phi(y) &= \frac{1}{2}\frac{y^2}{Q} + \log(w-y) \\ &= \frac{1}{2}\frac{y}{w-y} + \log(w-y) \end{aligned}$$

   Here $Q = 4$, so the FOC is solved by setting $y = 4$, with $\phi(4) = 2$, and $\phi(0) = \log(5)$, so the interior solution is optimal ($2 > \log(5)$ because $e^2 > \left(1 + 1 + \frac{1}{2}\right)^2 > 5$).

2. *For any given value of α, does there exist a budget constraint such that it is optimal to spend equal amounts on each of the three goods?*

No. This would imply

$$\begin{aligned} Q &= y\,(w-y) \\ &= \frac{2w^2}{9} \end{aligned}$$

Then

$$\begin{aligned} \phi\,(y) &= \frac{1}{2}\frac{\frac{4}{9}w^2}{\frac{2}{9}w^2} + \log\,(w) - \log\,(3) \\ &= 1 + \phi\,(0) - \log\,(3) \end{aligned}$$

So setting $y = 0$ gives higher utility (since $\log\,(3) > 1$).

3. *Is it possible that the consumer would choose to consume a smaller quantity of one of the goods when wealth increases?*
   Yes. For example, in the answer to the first part, $x_3 = \frac{1}{20}$. But if w is reduced from 5 to 3, then $Q > \frac{w^2}{4}$, so the corner solution is optimal, meaning that $x_3 = \frac{3}{20}$.

Part II

Two customer service workers each choose effort levels $e_i \in [0, 10]$ to exert in helping a customer. The cost of effort is $c_1(e_1) = (e_1)^2/15$ for worker 1 and $c_2(e_2) = (e_2)^2/10$ for worker 2. The customer's satisfaction level equals the total effort that the workers exert, up to a maximum satisfaction level of 10. Each worker's payoff is the difference between the customer's satisfaction level and his own effort cost.

Consider the game G in which the workers simultaneously choose their effort.

1. Describe each worker's payoff function in G.
2. Describe each player's best response correspondence in G.
3. Compute all pure Nash equilibria of G.

Now we consider sequential-move versions of the interaction. In game Γ_2, first player 1 selects an effort level. Then player 2, after observing player 1's choice, chooses her own effort level.

4. Completely describe all pure strategy subgame perfect equilibria of Γ_2.

In Γ_3, the sequence of events starts with those in Γ_2. But after player 2 moves, player 1 observes her choice and decides whether to exert some additional effort, $a_1 \in [0, 10]$. His cost of effort in Γ_3 is given by $c_1(t_1) = (t_1)^2/15$, where $t_1 = e_1 + a_1$ is the sum of his initial effort choice and his additional effort choice.

5. What is a pure strategy for player 1 in Γ_3? What is a pure strategy for player 2? (You are free to state your answers in English rather than notation, but either way, your answers should be explicit.)
6. Completely describe all pure strategy subgame perfect equilibria of Γ_3.

Solution

1. The players' payoff functions are

$$u_1(e_1, e_2) = \begin{cases} e_1 + e_2 - (e_1)^2/15 & \text{if } e_1 + e_2 \leq 10, \\ 10 - (e_1)^2/15 & \text{if } e_1 + e_2 > 10, \end{cases}$$

$$u_2(e_1, e_2) = \begin{cases} e_1 + e_2 - (e_2)^2/10 & \text{if } e_1 + e_2 \leq 10, \\ 10 - (e_2)^2/10 & \text{if } e_1 + e_2 > 10, \end{cases}$$

2. For fixed e_2, the first case of $u_1(e_1, e_2)$ above is maximized when $1-2e_1/15 = 0$, or equivalently when $e_1 = 7\frac{1}{2}$. But if $e_2 > 2\frac{1}{2}$, player 1 would prefer to choose $e_1 = 10 - e_2$, which generates the same benefit at a lower cost. Thus

$$b_1(e_2) = \begin{cases} 7\frac{1}{2} & \text{if } e_2 \leq 2\frac{1}{2}, \\ 10 - e_2 & \text{if } e_2 > 2\frac{1}{2}. \end{cases} \tag{1}$$

Similarly,

$$b_2(e_1) = \begin{cases} 5 & \text{if } e_1 \leq 5, \\ 10 - e_1 & \text{if } e_1 > 5. \end{cases} \tag{2}$$

3. Draw the graphs of the two best response correspondences. The Nash equilibria are their intersection, namely, all pairs $(x, 10 - x)$ with $x \in [5, 7\frac{1}{2}]$.

4. We use backward induction. When player 2 moves, she will play a best response to player 1's initial action as described by (2). So player 1 seeks to maximize

$$u_1(e_1, b_2(e_1)) = \begin{cases} e_1 + 5 - (e_1)^2/15 & \text{if } e_1 \leq 5, \\ 10 - (e_1)^2/15 & \text{if } e_1 > 5, \end{cases}$$

This is maximized when $e_1 = 5$. Thus the unique subgame perfect equilibrium is $s_1 = 5, s_2(e_1) = b_2(e_1)$, where the function b_2 is defined in (2).

5. A pure strategy for player 1 specifies his initial effort level, as well as his additional effort level as a function of both his initial effort level and player 2's effort level. A pure strategy for player 2 specifies her effort level as a function of player 1's initial effort level.

6. We use backward induction. At the end of the game, as in (1), it is optimal for player 1 to target the sum of his efforts $t_1 = e_1 + a_1 = 7\frac{1}{2}$ if $e_2 \leq 2\frac{1}{2}$, and to target the total effort $e_1 + e_2 + a_1 = 10$ if $e_2 > 2\frac{1}{2}$. But if player 1's initial effort exceeds these requirements, he should make no additional effort. All told, player 1's strategy at the end of the game in a subgame perfect equilibrium is

$$\beta_1(e_1, e_2) = \begin{cases} 7\frac{1}{2} - e_1 & \text{if } e_2 \leq 2\frac{1}{2} \text{ and } e_1 \leq 7\frac{1}{2}, \\ 10 - e_1 - e_2 & \text{if } e_2 > 2\frac{1}{2} \text{ and } e_1 \leq 10 - e_2, \\ 0 & \text{if } e_2 \leq 2\frac{1}{2} \text{ and } e_1 > 7\frac{1}{2}, \text{or } e_2 > 2\frac{1}{2} \text{ and } e_1 > 10 - e_2. \end{cases}$$

In the middle of the game, player 2 realizes that if she chooses $e_2 \geq 2\frac{1}{2}$, player 1's additional effort will bring the total effort up to at least 10. Thus player 2 should never choose an effort level higher than $2\frac{1}{2}$. If player 1 chose an effort $e_1 \leq 7\frac{1}{2}$ in period 1, then this is exactly what player 2 should do. But if player 1 chose a higher effort level, then player 2 need only exert enough effort to bring the total effort up to 10. Thus, player 2's strategy in any subgame perfect equilibrium is described by

$$\beta_2(e_1) = \begin{cases} 2\frac{1}{2} & \text{if } e_1 \leq 7\frac{1}{2}, \\ 10 - e_1 & \text{if } e_1 > 7\frac{1}{2}. \end{cases}$$

In the initial period, player 1 should never choose an effort higher than $7\frac{1}{2}$, since choosing $7\frac{1}{2}$ is enough to guarantee that the total effort will be 10. But any $e_1 \in [0, 7\frac{1}{2}]$ will cause player 2 to choose $e_2 = 2\frac{1}{2}$, which player 1 will follow by choosing $a_1 = 7\frac{1}{2} - e_1$. Thus any $e_1 \in [0, 7\frac{1}{2}]$ can be chosen initially in subgame perfect equilibrium.

Part III

1. You are traveling to Ottawa and stupidly you left the hotel choice until the last minute. Now you need to find a hotel to stay at for one night. You pay for a new travel service called www.fixedstigler.com. This service lets you pay a flat fee of \$100 for any hotel. If you spend nc dollars, you get a listing of n such hotels. Hotels vary in amenities, so that the value of any hotel is $100 + v$, where v is uniformly distributed on $[0, 100]$, independently across hotels. So the chance that any hotel yields you a *net value* or *surplus* of at least v is $1 - v/100$.

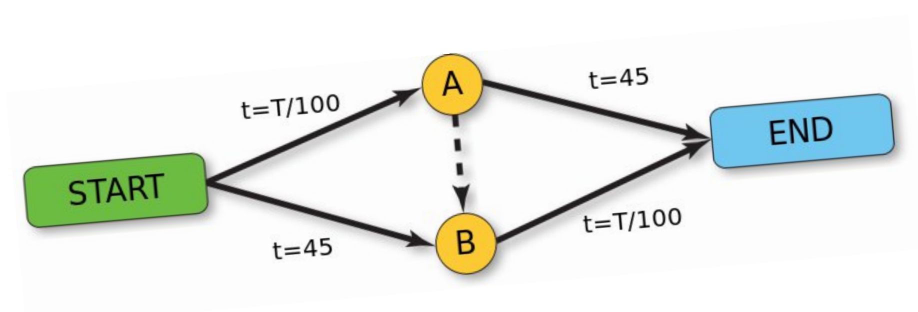
   Assume that your goal in choosing n is to maximize the surplus, net of costs. What is the demand curve $n(c)$? What is a simple approximation for $n(c)$?

2. A total of 4000 cars wish to travel from start to end. Assume first that no road joins A to B. Then there are two routes from start to end. The travel time in minutes on the road from Start to A is the number of travelers (T) divided by 100, and on the road from Start to B is a constant 45 minutes. Then these road times switch from A and B to end, as indicated.

   (a) What is the competitive equilibrium driving time from start to end?

   (b) Now assume that a *two-way* road joining A and B is built that takes 10 minutes. (Ignore the arrows on that road.) What is the competitive equilibrium driving time? Comment.

   (c) What driving choices would a social planner enforce who cared about total drive times of all drivers? How exactly would he do it?



Solution

1. One will choose the highest net value v. Now, the cdf of the maximum of these net values is the chance $F(v) = (v/100)^n$ that all n searches yield a net value below v. Thus, the expected maximum is the area over the survivor:

$$\int_0^{100} 1 - F(v)dv = \int_0^{100} 1 - (v/100)^n dv = 100 - \frac{100^{n+1}}{(n+1)100^n} = 100 - \frac{100}{n+1}$$

The marginal benefit is the differenced form of this net benefit. Since the marginal cost is c, the FOC is discrete. One chooses n if

$$\frac{100}{n(c)[n(c)-1]} \geq c > \frac{100}{n(c)[n(c)+1]} \qquad \Rightarrow n(c) \approx 10/\sqrt{c}$$

This implicit equation is the simplest precise formula we have, apart from using the quadratic formula.

2. (*a*) In equilibrium, travel times will equalize. Assume that m take the A route, and n take the B route. Then we find $m + n = 4000$ and

$$m/100 + 45 = n/100 + 45$$

Thus, $m = n = 2000$, and the total time is 65 minutes.

(*b*) At the above equilibrium, when arriving at point A, a driver is tempted to switch, because $45 > 10 + 2000/100$. In fact, let the numbers of drivers on the A route be initially m and then M, and on the B route, initially n and then N.

$$45 = 10 + N/100 \quad \Rightarrow \quad N = 3500$$

So $M = 500$. Likewise, $m = 3500$ and $n = 500$. The total time is now $45 + 35 = 80$. *A new option has hurt drivers!* This is Braess's paradox.

(*c*) The social planner would choose m, M to minimize

$$\mathcal{L} = m(m/100)+45(4000-m)+(m-M)10+45M+(4000-M)^2/100+\lambda(m-M)$$

$$\Rightarrow \text{FOC's:} \qquad m/50 - 45 + 10 - \lambda = 0 = (M - 4000)/50 + 45 - 10 + \lambda$$

Thus, $m = 1750 - 50\lambda$ and $M = 2250 + 50\lambda$ and if $m \gtreqless M$, as $\lambda \gtreqless 0$ by the saddle point property, given the complementary slackness condition $\lambda(m - M) = 0$. The only solution is $m = M$, and $\lambda = -5$. That is, the planner should employ the road tax 5, measured in minutes. (The negative sign reflects the fact that the tax only need applies going one direction.) The planner could just impose a mandatory five minute wait before traversing the AB road.

Part IV

A firm can earn profits from two different activities undertaken by the worker. The firm's return π_1 from activity 1 is a *deterministic* function of the worker's effort on this activity. If the worker exerts high effort e_h on activity 1, then $\pi_1 = Y$; if he exerts low effort e_l, then $\pi_1 = Z$, where $Z < Y$. The firm's return π_2 from activity 2 is the following *stochastic* function of the worker's effort: If the worker exerts high effort e_h in activity 2, then $\pi_2 = X$ and $\pi_2 = 0$ with equal chances 0.5. If the worker puts in low effort e_l, then $\pi_2 = 0$ always.

The worker can exert high effort on at most one activity. Only three effort combinations are possible: effort e_l in both activities, effort e_h in activity 1 and e_l in activity 2, or effort e_h in activity 2 and e_l in activity 1. Exerting high effort is costly for the worker. Specifically, the worker's utility as a function of the contractually specified wage w is $\sqrt{w}$ if he puts low effort into both activities and $\sqrt{w} - g$ if he exerts high effort on one activity, where g is the utility effort cost. The worker's utility if he does not work for the firm is 0. Assume that $0.5X > Y - Z > g^2$. Thus, the expected marginal return of high effort in activity 2 exceeds the marginal return of high effort in activity 1. The firm is risk neutral.

1. Assume that the firm observes the worker's effort level in each activity. Characterize the contract (wages and effort levels) that the firm will offer.

2. Suppose that the firm observes both returns π_1 and π_2 but *not* the worker's effort choice. Find the profit-maximizing contracts for inducing each combination of efforts in both activities. Under what conditions on X, Y and Z is the effort combination chosen by the firm different from that in part (a)?

3. For each combination of efforts in parts (a) and (b), provide economic intuition why the wage schedules are the same/different. Verify whether the monotone likelihood ratio property holds and use this in your explanation. Interpret the risk sharing properties of the optimal contracts.

Solution

(1) The first-best contract: The firm's optimization involves choosing a combination of efforts in both activities that maximizes expected profit subject to the participation/individual rationality (IR) constraint for the worker,

$$\sqrt{w} - g \geq 0 \quad (IR).$$

To induce low effort e_l in both activities, the firm will pay the worker 0. To induce high effort e_h in one of the activities, the firm will pay the worker g^2 so that the participation constraint is binding. To determine in which activity, if any, the firm will induce e_h, let's compare the firm's payoffs: The firm's profit from e_h in activity 1 is $Y - g^2$ and its profit from e_h in activity 2 is $0.5X + Z - g^2$. The profit from low effort in both activities is Z. So by the assumptions, the profit maximizing choice for the firm is high effort in activity 2.

(2) The second-best contract: If the firm chooses to induce the worker to choose low effort e_l in both activities, the profit maximizing wage is constant across return realizations and, to ensure that the IR binds, the constant wage is 0. In this case, the firm's profit is Z.

If the firm chooses to induce the worker to choose high effort e_h in activity 1, since effort in activity is effectively observable, the firm can induce e_h by offering a wage contract in which the worker is paid $w = g^2$ if $\pi_2 = Y$ and $w = 0$ otherwise. This yields profit equal to $Y - g^2$. Given that $Y - Z > g^2$, this combination of efforts is preferred by the firm to low effort in both activities.

If the firm chooses to induce the worker to choose high effort e_h in activity 2, the firm's optimization now involves choosing the wages to maximize expected profit subject to the individual rationality constraint (IR) and incentive constraint (IC) for the worker. Let w_X and w_0 be the wage paid to the worker, conditional on the firm observing returns $\pi_2 = X$ and $\pi_2 = 0$, respectively. The participation constraint is

$$0.5\sqrt{w_X} + 0.5\sqrt{w_0} - g \geq 0 \ (IR)$$

and the incentive constraint is

$$0.5\sqrt{w_X} + 0.5\sqrt{w_0} - g \geq \sqrt{w_0} \ (IC).$$

Since both constraints bind at the optimum,

$$\begin{aligned} 0.5\sqrt{w_X} + 0.5\sqrt{w_0} - g &= 0, \\ 0.5\sqrt{w_X} + 0.5\sqrt{w_0} - g &= \sqrt{w_0}, \end{aligned}$$

this gives $w_0 = 0$. Substituting w_0 into the IR constraint, we have $0.5\sqrt{w_X} = g$, or $w_X = 4g^2$. Hence, the firm's expected profit from this combination of effort levels is $0.5(X - 4g^2) + Z$.

The firm will offer a contract that induces a different combination of effort levels from the first-best if

$$0.5(X - 4g^2) + Z > Y - g^2,$$

or $0.5X - Y + Z > g^2$.

(3) In the first-best, the optimal contract then equalizes the ratios of marginal utilities of the firm and the worker across states (return realizations). Given the risk neutrality of the firm and the risk aversion of the worker, the firm fully insures the worker against the risk by offering a constant wage schedule. The desired combination of effort levels can be induced with a wage schedule conditional on effort levels directly and an incentive constraint does not enter the firm's optimization.

When effort is not observable, the wage schedule cannot condition on it directly. Hence, if the firm chooses to induce high effort in activity 2, in which return is a stochastic function of effort, both participation and incentive constraints affect the wage contract, which is no longer constant across states. The monotone likelihood ratio property holds (in activity 2, the likelihood ratio of high-to-low output is higher conditional on high effort than on low effort, $\frac{0.5}{0.5} > \frac{0}{1}$) and dictates that the worker should be paid more in higher-return state. Thus, the profit maximizing contract that induces high effort in activity 2 uses the correlation between effort and output. The worker now bears risk, as a result of a trade-off between risk sharing and incentives: optimal risk sharing recommends that the wage does not vary too much across states, whereas incentive provision recommends that the wage does depend on the state. If the firm chooses to induce high effort in activity 1, in which return is a deterministic function of effort, there is no risk to share and the incentives are taken care of by participation constraint.

Microeconomic Theory I, a collection of additional problems

This is a collection that contains extra problems suitable for the course. You are not expected to solve these for the class: they are not done in the excercise session, nor are there ready solutions available. The idea is just to make these available in case someone needs additional excercise material. Note that the course book also contains a lot of suitable excercises.

1. Let $\succeq$ be a complete and transitive preference relation on some set X. Define the binary relations $\succ$ and $\sim$ as follows:

$$\begin{aligned} x &\succ y \Longleftrightarrow x \succeq y \text{ but not } y \succeq x, \\ x &\sim y \Longleftrightarrow x \succeq y \text{ and } y \succeq x. \end{aligned}$$

   Show that:

   (a) $\succ$ is negatively transitive, irreflexive, asymmetric, and transitive.

   (b) $\sim$ is reflexive, transitive, and symmetric.

   (c) if $x \sim y \succeq z$, then $x \succeq z$.

   (d) if $x \succ y \succeq z$, then $x \succ z$.

2. Given a choice structure $(\mathcal{B}, C(\cdot))$, we say that a complete and transitive preference relation $\succeq$ rationalizes $C(\cdot)$ if

$$C(B) = C^*(B, \succeq)$$

   for all $B \in \mathcal{B}$, where

$$C^*(B, \succeq) = \{x \in B : x \succeq y \text{ for all } y \in B\}.$$

   (a) Let $X = \{x, y, z\}$ and let $\mathcal{B}$ contain all subsets of X. Give an example of a choice rule and a preference relation on X that rationalizes the resulting choice structure.

   (b) Take an arbitrary finite X, and let $\mathcal{B}$ contain all the two-element subsets of X. Is it possible to find a choice rule such that the resulting choice structure can be rationalized by several preference relations?

   (c) Give an example of a choice structure that can be rationalized by several preference relations.

3. Consider the following statements and determine if they are true or false. Give a precise explanation for your answer.

   (a) If $\succsim$ is represented by a continuous function, then $\succsim$ is continuous.

   (b) A continuous preference can be represented by a noncontinuous function.

(c) Let the choice set be $X = \mathbb{N} \times \mathbb{N}$, i.e. each alternative is a vector of two natural numbers. Define lexicographic preferences on that set as follows:

$$(x_1, x_2) \succeq (y_1, y_2) \Leftrightarrow \{x_1 > y_1\} \text{ or } \{x_1 = y_1 \text{ and } x_2 \geq y_2\}.$$

This preference relation has a utility representation.

4. Let us start with strict binary relation $\succ$ defined on $X \times X$, and let us see how far we get without the assumption of negative transitivity. So, in this excercise, we only require that $\succ$ must be *asymmetric* and *transitive*. Assume throughout that X is finite.

   (a) If $\succ$ is asymmetric and transitive, is it necessarily acyclic? Either provide a proof or counterexample. ($\succ$ is cyclic if for some positive n there exist points $x_1, ..., x_n \in X$ such that $x_1 \succ x_2 \succ ... \succ x_n \succ x_1$. If no such cycle exists, $\succ$ is acyclic)

   (b) Define $\sim$ as $x \sim y \Leftrightarrow (x \nsucc y \text{ and } y \nsucc x)$. Show that if $\succ$ is asymmetric and transitive, then $\sim$ is reflexive and symmetric. Show by counterexample that $\sim$ does not need to be transitive, though.

   (c) Show that if $\succ$ is asymmetric and transitive, then there exists a function $u : X \to \mathbb{R}$ such that $x \succ y$ implies $u(x) > u(y)$.

   (d) Now, let us not assume anything of $\succ$ except that it is a binary relation, and that there is a function $u : X \to \mathbb{R}$ such that $x \succ y$ implies $u(x) > u(y)$. Does this mean that $\succ$ must necessarily be irreflexive? Asymmetric? Transitive? Negatively transitive? Acyclic?

5. Let us define Independence of Irrelevant Alternatives (IIA) as follows: The choice structure $(\mathcal{B}, C(\cdot))$ satisfies IIA if, for all $A, B \in \mathcal{B}$, the following holds:

$$\text{if } A \subseteq B \text{ and } C(B) \cap A \neq \emptyset, \text{ then } C(A) = C(B) \cap A.$$

   (a) Show that Weak Axiom of Revealed Preferences (WARP) implies IIA (i.e., whenever a choice structure satisfies WARP, it also satisfies IIA).

   (b) Can you find a choice structure that satisfies IIA but not WARP? (hint: take X with four elements, and use two subsets of X to violate WARP...)

   (c) Show that if the choice structure $(\mathcal{B}, C(\cdot))$ contains all subsets of X that contain two elements, then IIA and WARP are equivalent.

6. Let $X = \{x_1, ..., x_n\}$ and let S be an arbitrary subset of X. Consider a decision maker, who uses the following sequential choice procedure to choose an element from any given set $A \subseteq X$. The decision maker examines sequentially the alternatives in A in the order of the indexes, (i.e., if

$x_i, x_j \in A$ and if $i < j$, then x_i is examined before x_j), and as soon as she confronts an element that is a member of set S, she stops and chooses that element. If none of the elements of A is a member of set S, then the decision maker chooses the last element of A. The set S is thus the set of "satisfactory" elements. This sequential procedure is called *satisficing procedure* (due to Herbert Simon).

(a) Define formally the choice function for all possible subsets of X.

(b) Does the resulting choice structure satisfy the weak axiom of revealed preferences?

(c) Can you write a utility representation for the revealed preference relation implied by this choice structure?

7. MWG 1.B.1, 1.B.2., and 1.D.4.

8. A strict preference relation, $\succ$, is said to be negatively transitive on X if for all $x, y, z \in X$ with $x \succ y$, either $x \succ z$, or $z \succ y$, or both hold. If there is no pair of alternatives, $x, y \in X$ such that $x \succ y$ and $y \succ x$, then $\succ$ is said to be asymmetric. Derive the weak preference relation, $\succeq$, from $\succ$ by:
$$x \succeq y \;\Leftrightarrow \neg\,(y \succ x)\,,$$
and indifference, $\sim$, by:
$$x \sim y \;\Leftrightarrow \neg\,(y \succ x) \text{ and } \neg\,(x \succ y)\,.$$
Show that if $\succ$ is asymmetric and negatively transitive, then $\succeq$ is complete and transitive. Also show that $\sim$ is reflexive, symmetric and transitive.

9. Consider preference relation R. Let $I(x) \equiv \{y \,|y \in X, yIx\}$. Show that the set $\{I(x) \,|x \in X\}$ is a partition of X, i.e.,

(a) $\forall x, y$, either $I(x) = I(y)$ or $I(x) \cap I(y) = \varnothing$.

(b) For every $x \in X$, $\exists y \in X$ such that $x \in I(y)$.

10. Let $\mathbb{X}$ be a set, and let $\mathbb{P}$ be a family of subsets of $\mathbb{X}$ with the property that for every $P \in \mathbb{P}$ and $Q \in \mathbb{P}$ either $P \subset Q$ or $Q \subset P$. We then say that $\mathbb{P}$ is a family of nested subsets. Given such $\mathbb{P}$, define a binary relation $\prec$ on $\mathbb{X}$ by
$$x \prec y \Longleftrightarrow \text{there is a } P \in \mathbb{P} \text{ with } y \in P,\; x \notin P.$$
Define
$$x \succeq y \Longleftrightarrow x \nprec y.$$

(a) Show that $\succeq$ is a rational preference relation.

(b) Show that for any rational preference relation, there is a corresponding family $\mathbb{P}$ that defines the preferences.

11. Throughout this exercise we consider for a given choice set X the following binary relations on $X \times X$:

$$\begin{aligned} xRy \text{ is "}x &\succeq y\text{"} \\ xPy \text{ is "}x &\succ y\text{"} \\ xIy \text{ is "}x &\sim y\text{".} \end{aligned}$$

If we want to say "not xRy ", we write $x\tilde{R}y$.

Asymmetry: For no x and y, we have both xPy and yPx.

Negative Transitivity: the following holds for all $y \in X$,

$$(xPz) \Longrightarrow (xPy \text{ or } yPz). \tag{1}$$

A binary relation P on a set X is called a preference relation if it is asymmetric and negatively transitive. Explain the sense in which this formalization is the same as R.

12. Assume that $X = \mathbb{R}^n$ and that P is a preference relation. Suppose that P satisfies the Weak Monotonicity Axiom:

$$(x_i \geq y_i, \forall i) \Longrightarrow (xRy).$$

Moreover, suppose that P satisfies the Local Non-Satiation Axiom: $\forall x$ and scalars $\delta > 0$, $\exists y$ such that

$$\begin{aligned} &1.\ \|y - x\| < \delta \text{ and} \\ &2. \qquad yPx. \end{aligned} \tag{2}$$

Your job is to show that for all $(x, z) \in X$,

$$(z_i > x_i, \forall i) \Longrightarrow (zPx).$$

13. Consider the Weak Axiom (WA) as defined as MWG. Show that the WA is equivalent to the following property:

$$\begin{gathered} [B \in \mathcal{B}, B' \in \mathcal{B}, (x,y) \in B, (x,y) \in B', x \in C(B), y \in C(B')] \\ \Longrightarrow \\ [x \in C(B') \ \textit{and} \ y \in C(B)] \end{gathered}$$

(You need to demonstrate that WA implies the property and that the property implies WA.)

14. Consider the following choice function induced by a complete and transitive preference relation $\succeq$ on X.

$$\text{For all } A \subset X,\ x \in C^{-}(A; \succeq) \Longleftrightarrow\ y \succeq x \text{ for all } y \in A.$$

Show that $C^{-}(A; \succeq)$ is a legitimate choice function satisfying the Weak Axiom. Consider the welfare interpretation of this choice function.

15. (Nonstandard dice)

    The choice set X consists of all possible (fair) dice with a number between 1 and 6 attached to each face. An example of such a die would be $x = (2, 2, 4, 5, 6, 6)$. An agent prefers (weakly) die x to die y if and only if the probability that x wins over y in a single roll of the dice is at least as large as the probability that y wins over x. Is this preference relation complete and transitive?

16. Consider an economy with N individuals. Let X be the set of alternatives available in this economy. For each pair $(x, y) \in X \times X$, define the variable d_i for each $i \in \{1, ..., N\}$ as follows:

$$d_i = \begin{cases} 1 \text{ if } x \succ y, \\ 0 \text{ if } x \sim y, \\ -1 \text{ if } y \succ x. \end{cases}$$

    A social choice function is a function $f : \{-1, 0, 1\}^N \to \{-1, 0, 1\}$ (with the same interpretation as above). Let $d = (d_1, ..., d_N)$. The majority decision rule is defined as follows:

$$f(d_1, ..., d_N) = \begin{cases} 1 \text{ if } \Sigma_{i=1}^N d_i > 0, \\ 0 \text{ if } \Sigma_{i=1}^N d_i = 0, \\ -1 \text{ if } \Sigma_{i=1}^N d_i < 0. \end{cases}$$

    Let $n^+(d) = \#\{i \text{ such that } d_i = 1\}$ and $n_-(d) = \#\{i \text{ such that } d_i = -1\}$. A social choice function is said to be *anonymous* if $f(d) = f(d')$ whenever $n^+(d) = n^+(d')$ and $n_-(d) = n_-(d)$. In other words, the rule treats all individuals in the same manner. A social choice function is *neutral* if $f(-d) = -f(d)$. A social choice function is *responsive* if we $f(d) \geq 0$ and $d' > d$ imply that $f(d') = 1$.

    (a) Show that the majority rule is anonymous, neutral and responsive.
    (b) Show that whenever f is anonymous and neutral, $n^+(d) = n_-(d)$ implies that $f(d) = 0$.
    (c) Prove that whenever f is anonymous, neutral and responsive, it is given by the majority rule.

17. MWG 1.D.3 and 1.D.4.

18. Show that a strongly monotone preference relation is monotone, and a monotone preference relation is locally nonsatiated.

19. (Harder) Consider an economy with a countable infinity of generations. Each generation can obtain a utility $x_i \in \{0, 1\}$. An outcome in this economy is then a sequence $x = (x_1, x_2, ...)$ with $x_i \in \{0, 1\}$ for each i. Thus $X = \{0, 1\}^{\mathbb{N}}$ Suppose that a social planner for the economy has preferences that satisfy two properties:

i) Pareto-principle: For all $x, y \in X$, $x \geq y \Rightarrow x \succ y$.

ii) Intergenerational Equity: For all $x, y \in X$, if $\exists i, j$ such that $x_i = y_j$ and $y_i = x_j$ and $x_k = y_k$ for $k \notin \{i, j\}$, then $x \sim y$.

(It can be shown that such rational preferences do exist). Show that the planner's preferences cannot have a utility representation. Hint: Find a way to use the argument that we had in class for the non-existence of a representation for lexicographic preferences.

(a) Assume that $x(p, w)$ is homogenous of degree one with respect to w and satisfies Walras' law. Show that $\varepsilon_{lw}(p, w) = 1$ for all $l = 1, ..., L$. What is the wealth expansion path like?

(b) Suppose that in addition to what is assumed above, $x(p, w)$ is homogenous of degree zero, and $\varepsilon_{lk} = 0$ whenever $l \neq k$. Show that this implies that for all l, $x_l(p, w) = \alpha_l w / p_l$, where $\alpha_l > 0$ is a constant independent of (p, w).

(c) Show that elasticity of demand of l with respect to price of k can be written as $\varepsilon_{lk}(p, w) = d \ln(x_l(p, w)) / d \ln(p_k)$.

20. MWG 2.E.4, 2.F.5, 2.F.14

21. (Schmeidler, 1971, Harder)

    Let $X = R^n_+$. A preference relation $\succeq$ is continuous on X if the sets $R_x = \{y \in X \mid y \succeq x\}$ and $Q_x = \{y \in X \mid x \succeq y\}$ are closed for all $x \in X$ and $B_x = \{y \in X \mid y \succ x\}$ and $W_x = \{y \in X \mid x \succ y\}$ are open for all $x \in X$. The preference relation is non-trivial if there are x', $x'' \in X$ with $x' \succ x''$. Show that a nontrivial continuous and transitive preference relation is negatively transitive. (Note that $\succeq$ is not assumed to be complete, so the proof is not trivial.)

    (Hint: In the notation of this problem, negative transitivity means that for all $x \succ y$, $B_y \cup W_x = X$. To show this, observe that $B_y \cup W_x \subset R_y \cup Q_x$. Show the reverse inclusion and conclude that since $B_y \cup W_x$ is both open and closed and nonempty by nontriviality, $B_y \cup W_x = X$.)

22. (Continuation)

    Show that $\succeq$ is complete.

    (Hint: assume by way of contradiction that $v, w \in X$ are incomparable. Because of negative transitivity and nontriviality, $v \succ x''$ or $x' \succ v$ (or both). Suppose the first is the case (without loss of generality). Use negative transitivity to conclude that either $w \succ x''$ or $v \succ w$. By incomparability of v, w, the former must be true. Therefore $w \succ x''$ and $v \succ x''$. Complete the proof by showing that $W_v \cap W_w = Q_v \cap Q_w$, which is a contradiction since the first intersection is nonempty and open (and does not contain all of X), and the second intersection is closed and X is a connected set.)

Moral of the exercise: Assumptions that are imposed for technical reasons may have unexpected behavioral implications.

(a) Assuming that the Walrasian demand function $x(p,w)$ satisfies Walras' law, derive the following formulas:

$$\begin{aligned} \Sigma_{l=1}^{L} b_l(p,w)\,\varepsilon_{lw}(p,w) &= 1, \\ \Sigma_{l=1}^{L} b_l(p,w)\,\varepsilon_{lk}(p,w) + b_k(p,w) &= 0, \end{aligned}$$

where

$$\begin{aligned} b_l(p,w) &= p_l x_l(p,w)/w, \\ \varepsilon_{lk}(p,w) &= \frac{\partial x_l(p,w)}{\partial p_k}\frac{p_k}{x_l(p,w)}, \\ \varepsilon_{lw}(p,w) &= \frac{\partial x_l(p,w)}{\partial w}\frac{w}{x_l(p,w)}. \end{aligned}$$

(b) Assuming that Walrasian demand function $x(p,w)$ is homogeneous of degree zero, derive the following formula:

$$\Sigma_{k=1}^{L}\varepsilon_{lk}(p,w) + \varepsilon_{lw}(p,w) = 0.$$

23. Recall the two definitions of Weak Axiom of Revealed Preferences (the first written in terms of a general choice structure and the second in terms of Walrasian demand function):

- $(\mathcal{B}, C(\cdot))$ satisfies the weak axiom of revealed preference (WA) if the following property holds:

$$\begin{aligned} \text{If } x,y &\in B \text{ and } x \in C(B), \text{ then for all } B' \text{ such that } x,y \in B' \\ \text{and } y &\in C(B'), \text{ we have } x \in C(B') \end{aligned}$$

- $x(p,w)$ satisfies WA if for any (p,w) and (p',w'), the following holds:

$$\text{If } p\cdot x(p',w') \leq w \text{ and } x(p',w') \neq x(p,w), \text{ then } p'\cdot x(p,w) > w'.$$

(a) Show that for a single valued Walrasian demand function $x(p,w)$, the two definitions conincide.

(b) Suppose now that $x(p,w)$ may be multivalued. From the first definition, develop the generalization of the second for Walrasian demand correspondences.

(c) Show that if demand correspondence $x(p,w)$ satisfies the axiom you developed in (b), and in addition satisfies Walras' law, then it satisfies the following property:

$$\text{If } x \in x(p,w),\ x' \in x(p',w'), \text{ and } p\cdot x' < w, \text{ then } p'\cdot x > w'.$$

24. Consider for starters three intermediate microeconomics type questions.

    (a) In order to aid the poor, the Government introduces a scheme whereby the first 1kg of butter a family buys is subsidized and the remaining amounts are taxed. Consider a family which consumes butter and is made neither better off nor worse off as a result of this scheme. Is it correct to state that the total amountof tax this family pays cannot exceed the subsidy it receives? Explain your answer.

    (b) A consumer buys one unit of a good when its price is £2 and two units when its price is £1. Is it correct to state that he would rather pay £2.80 for two units of the good than go without it altogether? Explain your answer.

    (c) You can only adjust your consumption of x_2 in the long run, but x_1 is flexible in the short run. Is it true that if x_1 is normal, then the demand for x_1 is more elastic in the long run than in the short run? Explain your answer.

25. Given a consumer with income w, draw the budget sets corresponding to the following cases.

    (a) There are two goods, $l = 1, 2$. Good $l = 1$ is indivisible and can only be bought in integer units, i.e. $x_1 \in \{0, 1, 2, ...\}$. Good $l = 2$ is divisible. The unit prices are given by p_l, $l = 1, 2$.

    (b) There are two goods, $l = 1, 2$. The total price for the purchase of x_l units of good l is given by

    $$p_l\left(x_l\right) = \begin{cases} 0 \text{ if } x_l = 0 \\ a_l + b_l x_l \text{ if } x_l > 0 \end{cases},$$

    where a_l and b_l, $l = 1, 2$, are positive constants.

    (c) There are two goods, $l = 1, 2$. Those can be bought separately with prices p_l, $l = 1, 2$. In addition, one can buy the two goods in a bundle that contains equal amounts of both goods. The price of a bundle that contains quantity x of both goods is given by px, where $p < p_1 + p_2$ and $p > \max\left(p_1, p_2\right)$.

26. The consumption choice of a consumer for three goods across four different price-income combinations is given in the table below:

p_1	p_2	p_3	w	x_1	x_2	x_3
1	1	1	20	10	5	5
3	1	1	20	3	5	6
1	2	2	25	13	3	3
1	1	2	20	15	3	1

where p_i denotes price of good i, w denotes income, and $x_i := x_i\left(p, w\right)$ denotes consumption of good i, $i = 1, 2, 3$. Check whether the demand is consistent with i) Walras' law, ii) weak axiom of revealed preferences.

27. There are two goods and two time periods. The prices of the two goods in period one are given by vector $p_1 = (2, 2)$ while the prices for period two are given by $p_2 = (8, 4)$. A consumer consumes only these two goods. Her consumption bundle in period one is given by $q_1 = (3, 3)$ and for period 2 by $q_2 = (4, x)$. Answer the following questions. It is probably helpful to draw figures!

    (a) Over which range of values of x would you conclude that the consumption choices across the time periods are consistent with weak axiom of revealed preferences (WA)?

    (b) Given that consumption satisfies WA, when is consumption bundle in period 1 revealed preferred to that in period 2? (and vice versa)

    (c) Given that consumption satisfies WA, when is good 1 an inferior good (for some price vector)? How about good 2?

28. There are L commodities, and the Walrasian demand is given by

$$x_k(p, w) = \frac{w}{\left(\sum_{l=1}^{L} p_l\right)}.$$

    (a) Is this demand function homogenous of degree zero in (p, w)?

    (b) Does this demand satisfy Walras' law?

    (c) Does this demand satisfy the weak axiom of revealed preferences?

    (d) Derive the Slutsky substitution matrix for this demand and interprete your finding.

29. Cost-of-living index numbers seek to reduce the comparison between two price vectors p^1 and p^2 to a single scalar. Define this index number as a ratio of the minimum expenditure needed to reach a reference utility level at the two sets of prices.

    (a) Suppose the two price vectors p^1 and p^2 are observed. Show that if the underlying preferences are homothetic, then there exists utility-based price index depending only on prices.

    (b) Show that the Laspereys price index forms an upper bound and the Paasche price index the lower bound for the utility-based index derived above (assume two goods for this case).

    (c) What is the true cost-of-living index when preferences are Cobb-Douglas.

30. Assume $L = 2$ and consider the Cobb-Douglas preferences:

$$u(x_1, x_2) = \alpha \ln x_1 + (1 - \alpha) \ln x_2.$$

(a) Derive the Walrasian demand and verify that it is homogeneous of degree zero and that Walras' law holds.

(b) Verify that the indirect utility function is homogeneous of degree zero, strictly increasing in w, nonincreasing in p_1 and p_2, and quasiconvex.

31. A consumer is interested in buying apples x and oranges y. The prices are given by p_x and p_y respectively when the fruits are purchased separately. Suppose that the fruits can also be purchased in bundles where one apple and one orange are sold at a joint price p. Suppose furthermore that $p > \max\{p_x, p_y\}$ and $p < p_x + p_y$ so that the pair is cheaper than its individual components.

    (a) Suppose that the consumer has wealth w available. Draw the budget set in the (x, y) plane.

    (b) Assume that the consumer has a diffenrentiable strictly increasing and quasiconcave utility function $u(x, y)$. Give the first order conditions for the optimal consumption bundle and argue that they are also sufficient conditions.

    (c) For the case of $u(x, y) = x^{.3}y^{.7}$, find the Walrasian demand.

32. The government finances public expenditure of magnitude g by collecting taxes. In this question, you are invited to think about the optimal ways of collecting taxes.

    (a) Suppose that there are goods x and y. The government can finance g by choosing either a tax on income t_w or by taxing consumption of good x by rate t_x. The government budget constraint for the two cases reads: $t_w w = g$ and $t_x x(p_x, p_y, t_x) = g$. Show that the consumer prefers an income tax in this case.

    (b) Suppose now that there is no exogenous income in the model and good y is now interpreted as leisure. Assume that he consumer has an initial endowment y^e of leisure that she may sell to buy the other good. Hence the budget constraint is now

$$p_x x(p_x, p_y) = p_y (y^e - y(p_x, p_y)), \text{ or}$$

$$p_x x(p) + p_y y(p) = p_y y^e.$$

    This last equation gives a way in which all problems with income resulting from sales of endowments should be thought of. First sell the endowment at market prices and then purchase the desired amounts of the goods wit the proceeds. Compare now the effect of taxes on x and y as in the previous part.

33. (Becker) Consider the aggregate demand in a model where individual consumers behave in a random manner (and thus do not satisfy any of the

rationality criteria that we had for individual choice). To be more specific, assume that a consumer with wealth w facing prices p picks a consumption vector at random from the budget set $B(p,w) = \{x : p \cdot x = w\}$ according to the uniform distribution. Suppose furthermore that there are a continuum of such consumers (and assume that you can apply the law of large numbers for this setting, i.e. the distribution of realized choices in the population coincides with the distribution of a single consumer's choice).
(i) Denote the individual (random) demand by $x^i(p,w)$. Compute the average demand

$$\overline{x}(p,w) = \int x^i(p,w)\,di.$$

(ii) Does this average demand satisfy weak axiom of revealed preference?
(iii) Can you find a utility function such that $x^i(p,w)$ is the walrasian demand function for that utility function?

34. Let P be a preference relation defined on a finite choice set X with N elements.

    (a) Show that the elements of X can be ranked in order of preference.

    (b) By induction on N, show that there exists a utility function that represents P. (the induction hypothesis is that the representation exists for sets of size $N-1$. Then, consider putting back the eliminated element and look if you can attach a utility number to it. Go through all the cases and complete the induction step).

35. Consider the following model of decision making. Let $X, Y \subset \mathbb{R}$ and suppose that there is are two well defined (sub-utility) functions $u, v : (X,Y) \to \mathbb{R}$. The preference relation of the decision maker is parametrized by a real number $\sigma > 0$ and given by:

$$i)\;\; (x,y) \succ (x',y') \;\text{ if } u(x,y) > u(x',y') + \sigma, \;\text{ or}$$
$$|u(x,y) - u(x',y')| < \sigma \text{ and } v(x,y) > v(x',y')$$
$$ii)\;\; (x,y) \sim (x',y') \;\text{ otherwise.}$$

    In other words, the decision maker uses u as the primary criterion in her decision making, but cannot distinguish between outcomes that yield utilities that are less than σ apart from each other. Is the above preference relation complete and transitive?

36. Consider the following statements and determine if they are true or false.

    (a) If $\succsim$ is represented by a continuous function, then $\succsim$ is continuous.

    (b) A continuous preference can be represented by a noncontinuous function.

    (c) Let $X = \mathbb{R}$ and $U(x) =$ [the largest integer n such that $x \geq n$]. The underlying preference relation is continuous.

(d) If both U and V represent $\succsim$, then there is a strictly monotonic function $f : \mathbb{R} \to \mathbb{R}$ such that $V(x) = f(U(x))$

37. Consider a standard utility maximizing consumer who consumes three goods per year. The Walrasian consumption bundle in year t is $x^t = (x_1^t, x_2^t, x_3^t)$, prices are $p^t = (p_1^t, p_2^t, p_3^t)$, and income is I^t.

    (a) Consider first three years, $t = 1, 2, 3$, with prices

    $$\begin{aligned} p^1 &= (4, 6, 4) \\ p^2 &= (2, 4, 6) \\ p^3 &= (6, 8, 2). \end{aligned}$$

    Income is constant in all years $I^1 = I^2 = I^3 = I$. The utility in years 2 and 3 is the same. Is the utility level in year 1 below or above the level in years 2 and 3?

    (b) In year $t = 4$, prices and income are the same as in $t = 3$ but the country in which the consumer lives joins a monetary union. As a result all monetary units are multiplied by 1/5,94573. What is the utility level in $t = 4$?

    (c) In year $t = 5$, the consumer has a new wage contract: the income is linked to prices such that whenever prices increase, the consumer's utility remains unchanged. Suppose now that one price has increased as period $t = 5$ arrives. Show that the demand of that good decreases.

38. The consumer has the following demands for three goods:

p_1	p_2	p_3	w	x_1	x_2	x_3
1	1	1	20	10	5	5
3	1	1	20	3	5	6
1	2	2	25	13	3	3
1	1	2	20	15	3	1

where p_i is the price, w is income, and x_i is the demand for good $i = 1, 2, 3$. Are these demands consistent with maximization of a utility function that represents locally nonsatiated preferences?

39. (Harder)

    Suppose that $\succeq$ is a rational preference relation on X. A subset $Z \subset X$ is called order dense if for all $x, y \in X$ such that $x \succ y$, there is a $z \in Z$ such that $x \succeq z \succeq y$. Show that if X has a countable order dense subset, then $\succeq$ has a utility representation.

40. The government considers various subsidy programs to induce low-income families to live in better quality housing than they would otherwise live in. Three plans are:

i) Income subsidy: provide additional income I to a family that can be spent in any way.

ii) Price subsidy: pay a fixed percentage p of a family's rent.

iii) Voucher: pay an amount s toward a family's rent, provided the rent is at least R.

Suppose commodities can be separated into housing (in amount x) and a general commodity repsresenting all other goods (in amount z). Suppose a family has utility function

$$u(x, z) = z^{4/5} x^{1/5}$$

and a monthly income of €500. Suppose the prices are $p = (p_x, p_z) = (1, 1)$. The government wishes this family to live in a house that rents for at least €150.

(a) Without subsidy, how much will the family spend on rent?

(b) Under each of the plans above, how much subsidy would be required to induce the family to live in a €150 house?

41. The following questions do not require answers in essay form; couple of well chosen sentences or lines of formalism are enough.

    (a) Explain the implications of the Walras Law and homogeneity restriction for elasticities of the Walrasian demand.

    (b) Show the equivalence of convex preferences and quasiconcavity of the utility representation.

    (c) Consider a firm with a well-defined profit function. Suppose first that price vector p is given. Suppose then the same p is the expected price vector. As a manager of this firm, which case do you like more?

    (d) Decreasing returns to scale imply decreasing marginal products. Is this true?

    (e) Giffen good is a good for which the demand goes up when the price goes up. Show that good that is Giffen must be inferior.

42. Consumer has utility function $u(x_1,x_2) = x_1^\rho + x_2^\rho$, wealth w, and faces prices $p_1 \neq p_2$.

    (a) Write down the UMP and solve the Walrasian demands.

    (b) Are the underlying preferences are homothetic?

    (c) Let $\rho = 1$. Calculate now the indirect utility and verify the Roy's identity. What is expenditure function?

43. Consider the following functions.

(a)
$$e(p,u) = u \cdot \min\{p_1^r, p_2^q\}.$$

For what values of r and q is this a legitimate expenditure function?

(b)
$$v(p,w) = \alpha \ln w + \beta \ln p_1 + \gamma \ln p_2.$$

For what values of α, β and γ is this a legitimate indirect utility finction? (Hint: you may want to use Roy's identity here).

(c)
$$\pi(p) = \left(p_1^\alpha + p_2^\beta\right)^\gamma.$$

For what values of α, β and γ is this a legitimate profit finction? (Hint: Indicate what should be checked and how you would check it, do not get stuck on calculations for too long.)

44. Assume that a competitive firm producing a single output from K inputs has a homothetic technology and therefore its cost function $c(w,q)$ can be written in the form

$$c(w,q) = h(q)\,c(w),$$

where q is the level of production and w is the vector of input prices. Denote the output price by p.

(a) How can you find the optimal production level for the output and the optimal input demands?

(b) Suppose that the output price increases. What happens to the input demands?

(c) Suppose that one of the input prices rises. What happens to the input demands and the output?

45. For the following statements, either prove the claim or provide a counterexample.

(a) A compensated price change results in an increase in the consumer's welfare.

(b) If all factors have decreasing marginal product, then the production function cannot have increasing returns to scale.

(c) Consider a consumer with wealth w. Every risk averse consumer prefers certain prices $(\alpha p_1 + (1-\alpha)p_1', \alpha p_2 + (1-\alpha)p_2')$ to the random prices of (p_1, p_2) with probability α and (p_1', p_2') with probability $(1-\alpha)$.

(d) Suppose that the demand functions of all buyers satisfy Walras' law. All competitive equilibria in such an exchange economy are Pareto efficient.

46. Analyze the Walrasian demands for the following cases.

   (a) The indirect utility function is $v(p_1, p_2, w) = \left(wp_1^{-\alpha}p_2^{-1-\alpha}\right)^{\gamma}$, where $\gamma > 0$ and $0 < \alpha < 1$.

   (b) The expenditure function is given by $e(p_1, p_2, u) = u \min\{p_1, p_2\}$

   (c) Show that the expenditure function in part b. is concave in prices.

47. A consumer divides her total working time endowment L between domestic work l and working outside the house y. She gets utility from l (cleaner house, better food etc.) and consumption c. The prices of the consumption good are normalized to 1 and the wage rate per unit of time worked outside of house is w. No wages are paid for domestic work. Assume that the utility function of the consumer takes the form $u(c, l) = cl$. Notice that the consumer gets no utility from leisure, or in other words, you may take $y = L - l$.

   (a) Set up the consumer's optimization problem and analyze the optimal choices l, y and c as functions of w.

   (b) Assume next that the consumer may buy domestic help l_d at wages $w_d < w$. Suppose that the preferences take the form $u(c, l, l_d) = c(l + l_d)$. Solve the optimal c and l, y and also the optimal purchases of l_d in this case. How does the consumer's labor supply outside the house change in comparison to your result in part a)?

   (c) Suppose next that the government decides to subsidize the hiring of domestic help by paying an amount sl_d if l_d units of help are purchased. What happens to optimal c and l_d in comparison to part b)?

48. MWG 3.C.2, 3C.6.

49. MWG 3.D.5.

50. A consumer in a three-commodity environment (x, y, z) behaves as follows.

   i) When prices are $p_x = 1, p_y = 1$ and $p_z = 1$ the consumer buys $x = 1, y = 2$ and $z = 3$;

   ii) When prices are $p_x = 4, p_y = 6$ and $p_z = 4$ the consumer buys $x = 3, y = 2$ and $z = 1$.

   Does the consumer maximize a strictly quasi-concave utility function?

51. Consider the following problem faced by a well traveled international trade economist. S/he can buy liquor of different varieties, $x = (x_1, x_2, ..., x_n)$ either at home at prices $p = (p_1, p_2, ..., p_n) \gg 0$ or (the same varieties) in duty free shops $y = (y_1, y_2, ..., y_n)$ at prices $q = (q_1, ..., q_n)$ which are uniformly lower than at home: $q \ll p$. The utility depends on total consumption $x + y = z = (x_1 + y_1, ..., x_n + y_n)$ and $U(z_1, ..., z_n) = z_1^{\alpha_1} \cdot$

$\cdot \cdot z_n^{\alpha_n}$. The travelling economist faces an overall wealth constraint w. In addition, there are import restrictions limiting the number of bottles allowed for import to $K : \sum_i y_i \leq K$.

(a) Draw the constraint set for $n = 2$ (in the (z_1, z_2)- space) for the case where reselling the liquor at home is prohibited.

(b) Find the first order conditions for the problem. Be especially careful with the complementary slackness constraints and find a characterization for optimal consumptions.

(c) Suppose next that the import restriction pertains to the value of the imported goods, i.e. $qy \leq V$ and not to the volume. Draw the constraint set for $n = 2$ and then redo the analysis. Again you should arrive at a description of optimal decisions.

(d) How would your analysis above change if resales were allowed?

52. Show that:

(a) A continuous preference relation $\succsim$ is homothetic if there exists a representation $u(x)$ that is homogeneous of degree one.

(b) A continuous $\succsim$ on $\mathbb{R} \times \mathbb{R}_+^{L-1}$ is quasilinear with respect to the first commodity if it admits a utility function of the form $u(x) = x_1 + \phi(x_2, ..., x_L)$.

(Both of the statements actually hold "if and only if", so feel free to try also "only if" part if you have time).

53. Let $h(p, u)$ denote the Hicksian demand function. Show that if the preferences are convex, then $h(p, u)$ is a convex set, and if the preferences are strictly convex, then $h(p, u)$ is single-valued.

54. Denote by $x_i(p, w_i) = [x_{1i}(p, w_i), ..., x_{Li}(p, w_i)]$ the Walrasian demand of consumer i with income w_i. Let the indirect utlity function of consumer i be given by the Gorman form:

$$v_i(p, w_i) = a_i(p) + b(p) w_i,$$

where $a_i(p)$ can be different for each consumer but $b(p)$ is the same for all consumers.

(a) Show that:

$$\frac{\partial x_{li}(p, w_i)}{\partial w_i} = \frac{\partial x_{lj}(p, w_j)}{\partial w_j}$$

for any $l = 1, ..., L$, and for any two individuals i and j with arbitrary wealth levels w_i and w_j (assuming $x_{li}(p, w_i) > 0$ and $x_{lj}(p, w_i) > 0$). Hint: use Roy's identity.

(b) How is this finding related to aggregate demand and aggregate income (read Section 4.B of MWG)?

(c) Show that the corresponding expenditure functions are of the form

$$e_i(p, u_i) = c(p)\, u_i + d_i(p)$$

for some $c(p)$ and $d_i(p)$.

55. Consider the following utility function.

$$U(x_1, x_2) = \begin{cases} x_1 x_2, \text{ if } x_1 x_2 < 4, \\ 4, \text{ if } 4 \leq x_1 x_2 \leq 8, \\ x_1 x_2, \text{ if } 8 < x_1 x_2. \end{cases}$$

Show that the corresponding preferences are convex. Then show that the implied preferences could not be represented by a concave utility function. Are the corresponding preferences continuous?

56. MWG 3.G.10 and 3.G.11.

57. (Harder)

The exercise above makes use of thick indifference curves. Can you find an example of a convex preference ordering that has thin indifference curves, and no concave utility representation? (A possible way to formalize thin indifference curves might be to require that for all $x \in X$, $cl\{y \in X : y \succ x\} \cup cl\{y \in X : y \prec x\} = X$. Here we write clA for the smallest closed set containing A. This definition will be of no use to you in coming up with the example. It is best to just work with 2- dimensional pictures.)

58. A firm's workers have identical and well-behaved preferences over leisure and income. They are paid \$10/hour for the first 40 hours and \$15/hour for each additional hour. They choose to work 50 hours per week . Management proposes to replace the current "time and a half for overtime" pay schedule with a constant wage rate of \$11/hour. The workers claim that this will reduce their income, management claims that this will make workers better off. Who is right?

59. Fred's preferences between pairs of shoes (S), and all other goods, (D), are represented by the utility function

$$u(S, D) = SD.$$

Because of government regulations, left shoes L and right shoes R are *not* sold together. They are sold at prices p_L and p_R respectively. Fred has w dollars of money and the price of D is 1.

a) Set up Fred's optimization problem and derive expressions for his optimal purchases and his maximal level of utility as functions of the exogenous variables.

b) Fred's one-legged uncle has died and left him all his shoes. They wear the same size. Thus Fred now enters the market with w dollars and L_0 left shoes. Do the analysis of part a) for this case.

60. Ginger and Fred are married. Ginger makes all the decisions about how to spend their family income w and divide the purchases. She does this in such a way that she maximizes her utility function subject to the constraint that Fred's utility is at least u_f. This is what Fred would get after divorce. Suppose that the family demand vector $x(p,w)$ is observable to outsider, but not how the goods are divided within the family $(x(p,w) = x_f(p,w) + x_g(p,w))$. Show that $x(p,w)$ satisfies the Slutsky equation.

    (Hint: Formulate the UMP and EMP for Ginger and show that the solution to one is a solution to the other.)

61. Suppose that the expenditure function of a consumer is of Gorman polar form:
$$e(p,u) = a(p) + ub(p).$$
    Derive the demands for each good and calculate also the income shares that each good receives. Can you find an economic interpretation for your results.

62. Preferences are said to be additively separable if they can be represented by a utility function of the form;
$$u(x) = \Sigma_{i=1}^{L} u_i(x_i).$$
    (a) If $u_i''(x_i < 0)$ for all i and for all $x_i \geq 0$, show that all goods are normal.

    (b) Show also that
$$\frac{\partial x_i(p,w)/\partial p_k}{\partial x_j(p,w)/\partial p_k} = \frac{\partial x_i(p,w)/\partial w}{\partial x_j(p,w)/\partial w}.$$

63. Consider the normalized price vector, $\mathbf{q} = \frac{1}{w}\,\mathbf{p} \in R^L$ and write the indirect utility function as a function of $\mathbf{q}$ only, i.e. $v(\mathbf{p},w) = v^*(\mathbf{q})$. Preferences are said to display indirect additivity if
$$v^*(\mathbf{q}) = f(\Sigma_l v_l(q_l)).$$
    Show that indirect additivity implies that for all distinct i,j,k, the price elasticity of good i with respect to price k is the same as the price elasticity of good j w.r.t. the price of good k.

64. Use the two problems above to show that if a preference relation admits both an additive utility function and if the preferences are also indirectly additive, then the preferences are homothetic.

65. Suppose that $f(z_1, ..., z_L)$ is a monotonic and strictly quasiconcave production function that is also homogenous of degree one. Show that f is also concave.

66. A real valued function $f : \mathbb{R}^L_+ \to \mathbb{R}$ is called superadditive if for all z^1, z^2,

$$f\left(z^1 + z^2\right) \geq f\left(z^1\right) + f\left(z^2\right).$$

a) Show that every cost function is superadditive in input prices.

b) Using this fact, show that the cost function is nondecreasing in input prices.

67. Let $c(w, q)$ be the cost function of a single-output technology with production function f and that $z(w, q)$ is the corresponding conditional factor demand correspondence.

    (a) Show that if f is homogeneous of degree one, then c and z are homogenous of degree one in q.

    (b) Show that if f is concave, then c is convex function of q.

68. Let $c(w, q)$ be the cost function of a decreasing-scale technology. Let p be the output price, and let $\pi(p)$ be the maximum profit. Define a new technology that replicates the original one by factor α. By this, we mean a technology that can produce q at cost $\alpha \cdot c(w, q/\alpha)$, where $c(\cdot)$ is the same function as above. The scaling factor α can be thought of as the capital associated with a certain degree of decreasing returns (by investing in high α one gets a technology with less decreasing returns). Assume that there is a price (or rent) r at which one can invest in α so that the total cost of producing q at scale α is $r \cdot \alpha + \alpha \cdot c(w, q/\alpha)$. Express the first-order conditions for optimal levels of q and α for a given r. What is the value of r that makes $\alpha = 1$ the optimal scale? (Express it in terms of $\pi(p)$).

69. Two firms produce food using labor, one with production function $q = \sqrt{l}$ and the other with $q = \beta\sqrt{l}$. Denote the wage by w and price of food by p.

    (a) Find each firm's supply function $q(w, p)$.

    (b) Find the social production function (i.e., for each level of labor input $\bar{l}$, find the maximum production possible, splitting the input between the firms) and the corresponding supply function.

    (c) What is the social cost function? (i.e., the minimum total cost of producing a given output level)

    (d) Compare the social supply function to the sum of the individual supply functions.

70. (John Panzar)

    Joe is an "empire builder". That is, his goal is to produce and sell as much output as possible. However, his stockholders impose the constraint that he not lose money. He operates using the production function $y = f(x)$, and faces parametric prices p and w for his output and (vector of) inputs. The production function is with positive marginal products.

    (a) Set up Joe's problem and state the first order necessary conditions

    (b) Is the nonnegativity constraint on profit binding? Why or why not?

    (c) Interpret the Lagrangian multiplier. What is its sign?

    (d) Show that Joe's supply curve slopes up.

    (e) Show that Joe's output is decreasing function of all input prices.

71. Consider a firm which has n independent divisions. Each division uses a common factor x and an individual factor z_i.The question deals with the issue of allocating the cost of the common factor to divisions. The production function of division i is given by

    $$q_i = f_i(z_i, x).$$

    Assume that this function can be inverted to yield the common factor requirement

    $$x_i = g_i(q_i, z_i).$$

    In words, division i requires x_i units of the common factor to convert z_i units of private factor into q_i units of its output. Let w be the vector of private input prices, v the price of the common input and p the vector of output prices. The cost function of the firm is obtained from

    $$\begin{aligned} c(q, w, v) &= \min_{z,x} \ w \cdot z + vx \\ \text{s.t. } x_i &\geq g_i(q_i, z_i) \text{ for all } i. \end{aligned}$$

    The proposal is to allocate the price of the common resource according to the Lagrange multipliers in the above minimization problem, i.e. each division is to buy the common resource at price λ_i.

    (a) Show that these prices cover the cost, i.e.

    $$\Sigma_{i=1}^n \lambda_i = v.$$

    (b) Based on these prices, consider the individual cost minimization problems of the divisions and show that the individual cost functions sum to the common cost function.

    (c) Suppose that

    $$f_i(z_i, x) = \min\{z_i, x\} \text{ for all } i.$$

    Calculate the total cost function and determine the cost allocation.

72. Let C be a *finite* set of consequences and let $\mathcal{L}$ be the corresponding set of all simple lotteries. Let be a rational preference relation on $\mathcal{L}$ that satisfies the independence axiom. Show that there are best and worst lotteries in $\mathcal{L}$ (that is, there are $L^- \in \mathcal{L}$ and $L^+ \in \mathcal{L}$ s.t. $L^+ \succsim L \succsim L^-$ for all $L \in \mathcal{L}$).

73. A consumer has a Bernoulli utility function of the form $u(x) = \frac{-1}{x}$ for $x \geq 0$. Suppose she is given a bet with a possible gain x_1 and a possible loss of x_2 with probabilities p and $(1-p)$ respectively.

    (a) At what initial wealth level x_0 is she indifferent between accepting the bet or not.

    (b) Suppose that $x_1 = x_2 < x_0$. For each level of initial wealth, calculate the probability with which the individual accepts the bet. Based on this evidence, would you guess that the individual has increasing or decreasing absolute risk aversion?

    (c) Verify or disprove your guess in b. by computing the coefficient of absolute risk aversion.

74. Prove that if a risk averse decision maker rejects a fixed favorable bet at all levels of wealth, then the Bernoulli utility of the decision maker is bounded from above.

75. Consider a strictly risk-averse decision maker with a Bernoully utility function $u(x)$. The initial wealth is w, and there is some probability π that she will lose an amount L. The decision maker can purchase insurance that will pay her q in the event of loss. The price per unit of money insured is p, so the total amount she has to pay for the insurance of coverage q is pq.

    (a) Formulate the expected utility maximization problem, where the decision variable is the insurance coverage q.

    (b) Derive the first order condition for the optimal choice of insurance coverage.

    (c) Derive the optimal insurance coverage for the actuarily fair insurance, that is, for the case $p = \pi$.

    (d) Assume that $p > \pi$. How does the optimal insurance coverage depend on the absolute risk aversion of the agent?

76. Excercise 6.B.5 in MWG

77. Excercise 6.C.2 in MWG

78. Consider a decision maker who likes money, is risk-averse, obeys the expected utility theory and adheres to this doctrine. Show the following: if such a decision maker, at all wealth levels, turns down the bet that yields a win of \$11 with probability 1/2 and a loss of \$10 with probability 1/2, then s/he must reject a lottery that yields an arbitrarily large gain with

probability 1/2 and a loss of \$100 with probability 1/2, no matter what the intial wealth level. What is going on here? To get a better idea, read Rabin, M.: "Risk Aversion and Expected-Utility Theory: A Calibration Theorem", Econometrica, Vol. 68, No. 5 (Sep., 2000), pp. 1281-1292.

79. Consider the savings and consumption model analyzed in lectures. There are two periods, $t = 0, 1$. The decision maker has a strictly concave separable Bernoulli utility function

$$u(c_0, c_1) = u_0(c_0) + \delta u_1(c_1),$$

where c_t denotes consumption in period t. Assume that the consumer receives a certain income w_0 in period 0 and a random income $\widetilde{w_1}$ in period 1. The only means for transfering wealth between periods for the consumer is by either borrowing or lending at a risk free rate r.

    (a) Set up the consumer's intertemporal budget constraint and characterize the solution to the savings problem through first order conditions (are these also sufficient conditions?).

    (b) Consider the changes in optimal savings resulting from changes in interest rate r. Can you find an income and a substitution effect in your expression for $\frac{ds}{dr}$?

    (c) Show that when the Arrow-Pratt coefficient of relative risk aversion is less than unity, savings increase in interest rate.

80. Consider the model of the previous exercise. Assume that $u_i(c_i) = \alpha + \beta c_i + (\gamma - c_i)^2$.

    (a) What is the range for possible consumptions where utitlity is increasing in consumption?

    (b) Assume that all the possible realizations from $\widetilde{w_1}$ lie in the range found in part a. Does the demand for savings depend on the riskiness of the distribution of $\widetilde{w_1}$?

81. Consider an economy where all agents face an independent risk to lose 100 with probability p. N agents decide to create a mutual agreement where the aggregate loss in the pool is equally split among its members.

    (a) Describe the change in the lotteries facing individuals in the pool when N is changed from 2 to 3.

    (b) Show that the risk with $N = 3$ is smaller in the sense of second order stochastic dominance that the risk with $N = 2$.

82. (Harder) In models of oligopolistic competition, it is typical that the profit of a firm lagging behind the leader in the industry in terms of the quality of its product has a profit function that is first convex and then concave

in any improvements to its own quality. R&D investments within a firm result normally in random improvements in the quality A possible way of modeling the R&D activity is by considering the choice of various types of projects, i.e. various distributions over quality improvements. With this as a motivation, consider the following decision model. The decision maker has a Bernoulli utility function $u(x)$ defined for $x \geq 0$, and there is an x_0 such that $u''(x) \geq 0$ for all $x \leq x_0$, and $u''(x) \leq 0$ for all $x \geq x_0$, and furthermore $u'(x) > 0$ for all x and $\lim_{x\to\infty} u'(x) = 0$. The decision maker chooses amongst all possible random distributions on $\mathbb{R}_+$.

(a) For each possible expected value μ of the random variable $\widetilde{x}$, show that the expected utility maximizing distribution has at most two points in its support, and characterize the optimal distributions.

(b) Suppose that here is a cost of increasing μ. More specifically, let $c(\mu)$ denote the cost, and assume that $c, c', c'' \geq 0$. Find the optimal distribution.

83. MWG 6.C.20.

84. Consider a decision maker (DM) having initial wealth W which can be allocated between a risk-free and risky asset. Let $(W - \alpha, \alpha)$ denote DM's portfolio where α is the amount invested in the risky asset. Risk-free rate of return is $r > 0$ and risky rate of return is a random variable x whose cumulative distribution function (CDF) is either $F_1(x)$ or $F_2(x)$. DM's Bernoulli utility function is $u(x)$.

(a) Write DM's final wealth conditional on the realized return.

(b) Pick one of the CDFs and write the first-order condition for the optimal portfolio.

(c) Suppose DM prefers $F_1(x)$ over $F_2(x)$ whenever $F_1(x)$ second-order stochastically dominates $F_2(x)$. Based on this, what can you tell about $u(x)$?

(d) Consider now a situation where DM does not know for sure which one of the CDFs applies but has only (objective) probabilities p_1 for $F_1(x)$, and p_2 for $F_2(x)$. Of course, $p_1 + p_2 = 1$. Write the DM's porfolio choice problem.

(e) Let $p_1 = 1$ in the previous item and let $\alpha = \alpha_1 > 0$ be the optimal choice. Moving to objective assessments where $p_2 > 0$ changes the portfolio. Describe how this change depends on the assumpions on $u(x)$. You can assume that the distributions have the same mean and that $F_1(x)$ second-order stochastically dominates $F_2(x)$.

(f) Let again $p_1 = 1$ and assume that DM sells the portfolio before the return is realized. Can you approximate the amount of money in excess of W that DM needs to achieve at the least the same utility as with the optimal portfolio? Assume that the portfolio risk is small.

85. Consider a risk-neutral agent who owns a call option on an underlying asset whose value at the excercise date is $\widetilde{x}$. The call option gives to the agent the option to purchase the asset at a prespecified price p, called excercise price. Show that the agent likes an increase in risk in the value of the asset. How would your answer change if the agent owns $n > 1$ call options with exercise prices $p_1, ..., p_n$?

86. A rational preference relation $\succeq$ satisfies betweenness if for all $p, q \in \mathcal{L}$ and all $\alpha \in (0,1)$, we have

$$p \succ q \Rightarrow p \succ \alpha p + (1-\alpha) q \succ q.$$

Show that betweenness is equivalent to the following condition. For all $p, q \in \mathcal{L}$ and all $\alpha \in (0,1)$, we have

$$p \sim q \Rightarrow p \sim \alpha p + (1-\alpha) q \sim q.$$

In other words, betweenness is equivalent to linearity of indifference curves in the Machina triangle.

87. Consider the following model of criminal behavior due to Becker. There is a continuum population of individuals considering whether to commit a crime. The resulting benefit from crime to an individual i is b_i. Assume that b is distributed in population according to the continuously differentiable strictly positive density function (on the entire real line) $g(b)$ and denote the corresponding cdf. by $G(b)$. If the individual commits a crime, then she will be caught with probability π and in this case she must pay a fine F.

   (a) Show first that there is a unique cutoff level b^* such that individual i commits the crime if and only if $b_i > b^*$.

   (b) Show next that b^* is increasing in π and F.

   (c) Suppose next that the individual has labor income w in case that no crime is committed. If caught in a crime, the individual must go to jail for fraction f of her total labor time. Then we have $F = fw$. Show that if the coefficient of relative risk aversion is larger than 1, then b^* is increasing in w.

88. An investor has initial wealth w_0 that she must divide between two assets. Denote the total investment into asset i by w_i for $i \in \{1, 2\}$. There are two possible states H and L for the economy. Assume that the economy will be in state H with probability p and in L with probability $(1-p)$.The first asset pays a gross return R^H if the state of the economy is H and a gross return 0 if the state is L. I.e. in state H, an investment of w_1 in asset 1 results in a return $w_1 R^H$ and in state L, the return is 0. The second asset pays R in both states. Denote the fraction of initial wealth that is invested in asset i by α_i for $i \in \{1, 2\}$ so that $w_i = \alpha_i w_0$.

(a) For an arbitrary investment portfolio $\{\alpha_1, \alpha_2\}$, determine the final wealth of the investor.

(b) Assume the investor has a Bernoulli utility function $u(w)$ where w denotes the final wealth. Assume that $u'(w) > 0$ and $u''(w) < 0$. Write down the expected utility resulting from an arbitrary portfolio $\{\alpha_1, \alpha_2\}$ and characterize the first order conditions for the optimal portfolio. Argue also that second order conditions are also satisfied at the point that satisfies the first order conditions.

(c) Consider now a third asset that pays 0 in H and R^L per unit invested in L. Assume further that the utility function takes the form $u(w) = \ln w$.For what values of p, R^H, R^L, R will the investor invest in assets 1 and 3 only?

(d) (Slightly harder) Can you generalize your answer in c. to the case where there are N states and N risky assets such that asset i pays a return of R^i in state i and nothing in the other states.

89. Consider an agent living two periods, $t = 1, 2$. At $t = 1$ the agent owns an asset of size $s_1 > 0$. The size of this asset is fully known at $t = 1$ and it is the only source of income (consumption) for this agent. Let c_t denote consumption, $t = 1, 2$. Then, the asset available for consumption at $t = 2$ is what is saved from $t = 1$ plus a random term z, that is, $c_2 = s_1 - c_1 + z \geq 0$. Random term z is distributed according to some cumulative distribution function $F(z)$ on $[-k, k]$, where $s_1 \geq k$, and has mean zero. That is, $E\{z\} = \int_{-k}^{k} z dF(z) = 0$. The consumer's Bernoulli utility function $u(c_t)$ for consumption per period is increasing, strictly concave, and three times differentiable. That is, the consumer gets utility $u(c_1)$ from the first period and $u(c_2)$ from the second period (no discounting). The decision problem is to maximize expected utility achievable from the initial asset s_1 over the two periods. Assume that $u'(c_t) \to \infty$ as $c_t \to 0$, so that optimal consumptions will be positive for both periods.

(a) Write the consumer's expected utility as a function of her consumption at $t = 1$.

(b) Derive the first-order condition for optimal c_1.

(c) Consider then another agent who is facing exactly the same problem but with a different Bernoulli utility function. Fix c_1 for both agents and consider the following statement: the second agent is willing to trade her remaining random asset for a nonrandom asset of a given size, whereas our original agent does not accept this offer. What can you tell about the agents' attitudes to risk? There is also a link between their utility functions. What is it?

90. (MWG 6.C.5) Consider a consumer with utility function $u(\cdot) : \mathbb{R}^L_+ \to \mathbb{R}$ defined over bundles of L goods, just like in lecture 3.

(a) Argue that concavity of $u(\cdot)$ can be interpreted as the decision maker exhibiting risk aversion with respect to lotteries whose outcomes are bundles of the L commodities.

(b) Suppose that a Bernoulli utility function $\widetilde{u}(w)$ for wealth is derived from the utility maximization problem defined over bundles of consumption for each given wealth level w, keeping the price vector p for commodities fixed. Show that, if the utility function u for the commodities exhibits risk aversion, then so does the derived Bernoulli utility function for wealth. Interpret.

(c) Can you find an example that shows that the converse of (b) does not necessarily hold? (take $L = 2$, and look for a nonconcave function u such that the derived Bernoulli utility function on wealth exhibits risk aversion)

91. Consider an economy with one representative agent and two dates, $t = 0, 1$. The economy has an exogenous consumption process in the following sense: in the absence of savings, period $t = 1$ consumption, $\tilde{c}_1$, is distributed according to some cumulative distribution function $F(c)$. Thus, $E\{\tilde{c}_1\} = \int c dF(c)$. Period $t = 0$ consumption c_0 is given. The agent has separate attitudes towards time and risk, so we seek to formulate preferences such that the two can disentangled. The utility over time is given by

$$U(c_0, \tilde{c}_1) = u(c_0) + \beta u(c(v, F))$$

where $\beta \in (0, 1)$ is the discount factor, u is a weakly concave function, and $c(v, F)$ is the certainty equivalent consumption for period $t = 1$ using another weakly concave function v. That is, $v(c(v, F)) = E\{v(\tilde{c}_1)\}$. Note first that if $u = v$ we have the usual time-separable expected-utility objective. Second, given $c(v, F)$, all uncertainty has been removed from calculations using $U(c_0, \tilde{c}_1)$, so the concavity of u relates to consumption smoothing only. Third, concavity of v measures risk aversion only.

(a) Suppose the agent can invest and sacrify consumption at $t = 0$ to achieve a sure benefit $(1 + r)$ per unit invested at $t = 1$. Now, if you can find a rate of return r that makes the agent just indifferent between investing and not investing, given c_0 and the expected $\tilde{c}_1$, you have found the socially efficient discount rate for this economy. Do this and discuss how it depends on consumption smoothing and risk aversion.

(b) Consider now the effect of increasing uncertainty on the discount rate. To obtain a benchmark compute the socially efficient discount rate under the assumption that $E\{\tilde{c}_1\}$ is the period $t = 1$ consumption for sure. Denote this by r^c. Then, calculate the true socially efficient discount rate under the assumption that $\tilde{c}_1$ is uncertain. Does the uncertainty reduce the discount rate? Assume time separable preferences, that is, $v = u$.

(c) The same problem as above, but assume now $v \neq u$. Show that the socially efficient discount rate falls with uncertainty about future income levels if the agent is decreasingly absolute risk averse.

92. Joe and Mary must divide a 12-ounce cake and a 16-ounce pitcher of milk between them. Joe always consumes these in a ratio of one to one on a weight basis, while Mary always consumes two ounces of milk for every ounce of cake; they each have Leontief utility functions. They both would prefer as much of their ideal combination as possible.

    (a) Draw an Edgeworth box for this situation and indicate the Pareto efficient divisions of cake and milk.

    (b) What is the one point where equilibrium prices for cake and milk exist that are both positive? Is the ratio of these prices unique? If so, what is it?

    (c) What are the equilibrium prices at other Pareto-efficient points?

93. A country has a large number of competitive firms and produces only two commodities: watches and cheese. Watches are sold at the world price of \$40 each, and cheese is sold at a world price of \$16 per round. Both products are produced at constant returns to scale, and they require inputs of capital and labor as follows: one watch requires 0.2 units of capital and 1.0 units of labor, and one round of cheese requires 0.1 units of capital and 0.1 units of labor. The country has a yearly supply of 2000 units of capital and 6000 units of labor. The residents consume many goods, including the two that they produce. The other goods consumed are purchased on the world market. Find the amounts of watches and cheese produced and the equilibrium prices of capital and labor. How do these answers depend on the country's consumption preferences?

94. (Edgeworth Boxes) Consider a two person, two good exchange economy. The utility functions, u^h, and initial endowments, ω^h are as specified below. For each case, find the set of Pareto optimal allocations and the Walrasian equlibria and illustrate them in an Edgeworth box.

    (a) $u^1\left(x^1, y^1\right) = \ln x^1 + \ln y^1;\ u^2\left(x^2, y^2\right) = x^2 y^2,\ \omega^1 = \omega^2 = (.5, .5)\,.$

    (b) $u^h\left(x^h, y^h\right) = \max\left\{x^h, y^h\right\},\ \omega^h = (1, 1)$ for $h = 1, 2$.

    (c) $u^1\left(x^1, y^1\right) = \min\left\{x^1, y^1\right\};\ u^2\left(x^2, y^2\right) = \min\left\{2x^2, \frac{1}{2}y^2\right\},\ \omega^1 = (1, 2)\,, .\omega^2 = (2, 1)\,.$

    (d) $u^1\left(x^1, y^1\right) = x^1 + y^1;\ u^2\left(x^2, y^2\right) = 2x^2 + y^2,\ \omega^1 = \omega^2 = (.5, .5)\,.$

95. It is impossible to consume an ounce of a chili without the accompaniment of between1/2 and 2 pints of beer. The exact quantity consumed is a matter of individual taste. Assume that chili can be consumed in any nonnegative amount

(a) Draw this individual's consumption set.

(b) Consider a two-person exchange economy in which each of the agents hasconsumption sets as in part (a). Draw the Edgeworth box for this economy.

(c) Show by example that there are economies of the sort in part (b) in which utility is strictly increasing yet there are equilibria in which one of the goods has price zero. What must happen for this to be true?

(d) Can the result in (c) occur with the 'normal' consumption set?

96. (Gorman aggregation) Consider an exchange economy with H households having the same differentiable, strictly increasing and strictly quasi-concave utility functions but (possibly) different endowments. Assume that the common utility function is homogeneous of degree 1 and that the aggregate endowment of all goods is positive.

(a) Find the core this economy.

(b) Find the Walrasian equilibria for this economy.

(c) How would your answer to questions (a) and (b) change if u was homogeneous of degree $k > 0$? (Ignore corner solutions for this problem!)

97. (Free Entry and Returns to Scale) The aggregate production set Y satisfies "free entry" if $Y + Y \subset Y$. A technology has constant returns to scale (CRS) if $y \in Y \;\Rightarrow\; \alpha y \in Y$ for $\alpha > 0$.

(a) Give an example that shows that CRS does not necessarily imply free entry.

(b) Show that if Y is convex, CRS implies free entry.

(c) Give an example that satisfies free entry but does not satisfy CRS.

(d) Show that if Y satisfies free entry and $0 \in Y_j$ for all j, then in a competitive equilibrium, profits for all firms are zero.

98. (Robinson Crusoe) Consider an economy with 3 goods (x, y, z), 1 firm and 1 consumer. (The consumer owns the firm.) The firm has the following technology: $(z, -x, -y) \in Y$ if there exists a $\lambda \in [0, 1]$ such that

$$z \leq \lambda \min\{x, 2y\} + (1 - \lambda) \min\{2x, y\} \text{ and } x, y, z \geq 0.$$

The household's preferences are given by

$$u(x, y, z) = xyz.$$

The consumer's initial endowment is $(1, 1, 0)$.

(a) Draw the isoquants of the firm's technology.

(b) Can you find a competitive equilibrium for this economy?

99. The city of Beverly Hills considers displaying a statue on Rodeo Drive. The statue has already been commissioned and therefore has to be paid no matter whether it will be displayed or not. Let v_h describe the value that resident h assigns to the display of the statue. Inhabitants of Beverly Hills have no problem assigning monetary values to art and hence should be interpreted as the amount of money that i would be willing to pay if the statue were displayed. (Note that this could be positive or negative!) Thus, the utility of individual h when she has m_h units of money is $v_h + m_h$ if the statue is displayed otherwise it is m_h.As usual, m_h can be any real number.

    (a) Characterize Pareto efficient outcomes. What decision rule should the city use when deciding whether or not to display the statue?

    (b) Extend the commodity space so that this situation can be described within the competitive model. (The city of Beverly Hills should be modeled as a firm; define the appropriate consumption sets, budget sets, etc.)

    (c) Find the competitive equilibrium.

100. Consider an economy with specialized labor. There are I worker/consumers and each i produces a separate good y_i.Hence there are I separate goods, and we denote individual consumption vectors by $x^i = \left(x_1^i, ..., x_I^i\right)$ and allocations by $x = \left(x^1, ..., x^I\right)$. Assume in the beginning that labor is inelastically supplied and as a result, we can take y_i as an exogenously fixed endowment and analyze the economy as an exchange economy.

    (a) Define Pareto-optimal allocations and competitive equilibrium for this economy.

    (b) Assume that all consumers have the same utility functions

    $$u^i\left(x^i\right) = u\left(x^i\right) = \Sigma_{l=1}^I \alpha_l \ln x_l.$$

    Characterize the set of Pareto optimal allocations for this economy.

    (c) Compute the general equilibrium prices and allocations for this economy.

    (d) Assume next that the workers derive utility from leisure. Let L denote the labor endowment of the workers in the economy and denote the labor supply of i by l^i. Assume furthermore that

    $$y_i = f\left(l^i\right).$$

    The utility functions of the worker/consumers are given by

    $$u^i\left(x^i, l^i\right) = \Sigma_{k=1}^I \ln x_k + \ln\left(L - l^i\right).$$

When chosing the labor supply, each worker considers how much the resulting output will be worth in an exchange market as in question 1. Compute the Pareto-optimal allocations and general equilibrium allocations for this economy. Does the first welfare theorem hold for this economy?

101. (General equilibrium for infinitely many commodities) Suppose there are two consumers in an exchange economy that exists over an infinite number of time periods. Let x_i be consumption at time i, and suppose consumers 1 and 2 have the following intertemporal utility functions:

$$\begin{aligned} u_1(x_0, x_1, ...) &= \sum_{i=0}^{\infty} \alpha^i \ln x_i \\ u_2(x_0, x_1, ...) &= \sum_{i=0}^{\infty} \beta^i \ln x_i. \end{aligned}$$

Suppose the endowments vectors for both consumers are $\omega = (1, 1, ...)$. Solve for the equilibrium prices $p_1, p_2, ...$, taking $p_0 = 1$, and determine the demand functions for the first consumer at these prices. Interpret the prices by comparing the prices of commodities at adjacent time periods (i.e. consider ratio p_i/p_{i+1}).

102. Consider the following economy: There are two periods, two states in the second period, and two consumers. There is one physical commodity. The endowments in period 0 is 2 for both consumers. Consumer 1 has an endowment of 1 in both states in period 1 and consumer 2 has an endowment of 2 in both states in period 1. By x_0 we denote the consumption in period 0. By x_{1s} we denote the consumption in period 1 in state s.m The utility functions of the consumers are given by:

$$u^i(x_0, x_{11}, x_{12}) = x_0 + x_{11} + x_{12} - \frac{1}{4}x_{11}^2 - \frac{1}{4}x_{12}^2.$$

There is a single asset that pays α units of the good in state 1 and $(1 - \alpha)$ units in state 2. The assets are assumed to be in zero net supply. Hence an agent who buys one unit of the asset in period 0 gets a payment of α from the seller of the asset in state 1 in period 1 and $(1 - \alpha)$ in state 2. Denote the price of the asset by q in period 0. Hence the seller of one unit of the asset gets q units of the consumption good in period 0.

(a) What are the Pareto-optimal allocations in this economy?

(b) What are the asset market equilibria in this economy? Can they be Pareto ranked in terms of α?

103. Consider an exchange economy with two goods. Consumers A and B have utility function

$$u^i(x_1^i, x_2^i) = \frac{(x_1^i)^{1-\gamma} - 1}{1 - \gamma} + x_2^i,$$

and endowments $w_1^i > 0$ and $w_2^i > 0$, where $\gamma > 1$ and $i = A, B$.

(a) Solve the demand of both goods for consumer A and B as a function of prices, p_1 and p_2.

(b) Find an equilibrium price vector.

(c) Analyze how the equilibrium depends on agents' endowments, and also on the preference parameter γ.