A1 Given the recurrence relation $y_{r+1} = 0.75 - 0.75(r+4)y_r$ and that $y_0 = 0.1957 \pm 0.0001$, how accurately do we know y_7?

$$
\begin{array}{lcl}
0.1956 & \leqslant y_0 \leqslant & 0.1958 \\
0.1626 & \leqslant y_1 \leqslant & 0.1632 \\
0.13800 & \leqslant y_2 \leqslant & 0.14025 \\
0.118875 & \leqslant y_3 \leqslant & 0.12900 \\
0.07275 & \leqslant y_4 \leqslant & 0.12590625 \\
-0.0054375 & \leqslant y_5 \leqslant & 0.3135 \\
-1.366125 & \leqslant y_6 \leqslant & 0.786703125 \\
-5.150273438 & \leqslant y_7 \leqslant & 10.9959375
\end{array}
$$

i.e to four significant figures, all that we know about y_7 is that it lies between -5.150 and 11.00.

A2 This example is due to Wilkinson. Consider the polynomial given by $p(x) = (x-20)(x-19)(x-18)\ldots\ldots(x-2)(x-1)$, i.e. a polynomial with roots 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19 and 20.

What Wilkinson did was to change the coefficient of x^{19} from -210 to $-210 + 2^{-23} = -210 + 1.2\ 10^{-7}$. This change is an extremely small change and is in fact one that many computers would not be able to make.

Wilkinson then computed the roots of this perturbed polynomial and they are listed below:

$$
\begin{array}{l}
1.0000\,00000 \\
2.0000\,00000 \\
3.0000\,00000 \\
4.0000\,00000 \\
4.9999\,99928 \\
6.0000\,06944 \\
6.9996\,97234 \\
8.0072\,67603 \\
8.9172\,50249 \\
10.0952\,66145 \pm 0.6435\,00904i \\
11.7936\,33881 \pm 1.6523\,29728i \\
13.9923\,58137 \pm 2.5188\,30070i \\
16.7307\,37466 \pm 2.8126\,24894i \\
19.5024\,39400 \pm 1.9403\,30347i \\
20.8496\,08101
\end{array}
$$

It is clear that something has gone very badly wrong, but this alteration corresponds to storing the value 1/7 as 0.142857; which is what many computers do.

A3

$$x + 99y = 100$$
$$101x + 10\,000y = 2020$$

This system of simultaneous linear equations can be solved in four different 'orders' whereby each of the coefficients could be chosen as the 'pivot' for elimination. In other words, given the system

$$ax + by = e$$
$$cx + dy = f$$

we divide the first equation through by a, and rearrange to get

$$x = \frac{e - by}{a}$$

and we substitute this value of x into the second equation.

Use this procedure to solve the four systems below, using four figure arithmetic with rounding.

$$\begin{aligned} x + 99y &= 100 \\ 101x + 10000y &= 2020 \end{aligned} \qquad \begin{aligned} 10000y + 101x &= 2020 \\ 99y + x &= 100 \end{aligned}$$

$$\begin{aligned} 101x + 10000y &= 2020 \\ x + 99y &= 100 \end{aligned} \qquad \begin{aligned} 99y + x &= 100 \\ 10000y + 101x &= 2020 \end{aligned}$$

B1 Consider this system of simultaneous linear equations:

$$0.0003x + 0.1000y = 1 \qquad (1)$$
$$-0.8000x + 0.1249y = 0.6 \qquad (2)$$

Now divide (1) by 0.0003 giving

$$x + 333.3y = 3333 \qquad (3)$$

Multiplying (3) by 0.8000 gives

$$0.8000x + 266.6y = 2666 \qquad (4)$$

Adding (2) to (4) gives

$$266.7y = 2667 \qquad (5)$$

So we conclude that

$$y = 10 \qquad (6)$$

and substituting (6) into (3) we get

$$x = 0 \qquad (7)$$

Solve this system another way, and compare solutions. (Use four figure arithmetic with rounding throughout).

B2 How can we calculate $e^{-5.5}$?

We know that e^x has a power series expansion:

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$$

and so if we substitiute -5.5 for x and sum the terms we should be able to get as accurate a value as we like for $e^{-5.5}$.

$$\begin{aligned} e^x = \quad & 1.000 \\ & -5.500 \\ & +15.13 \\ & -27.30 \\ & +38.13 \\ & -41.94 \\ & +38.45 \\ & -30.21 \\ & +20.77 \\ & -12.69 \\ & +6.980 \\ & -3.490 \\ & +1.600 \\ & -0.6768 \\ & +0.2659 \\ & -0.09749 \\ & +0.03351 \\ & -0.01084 \\ & +0.003313 \\ & -0.0009599 \\ & +0.0002637 \\ & -0.00006907 \end{aligned}$$

after which addition of more terms does not affect the answer.

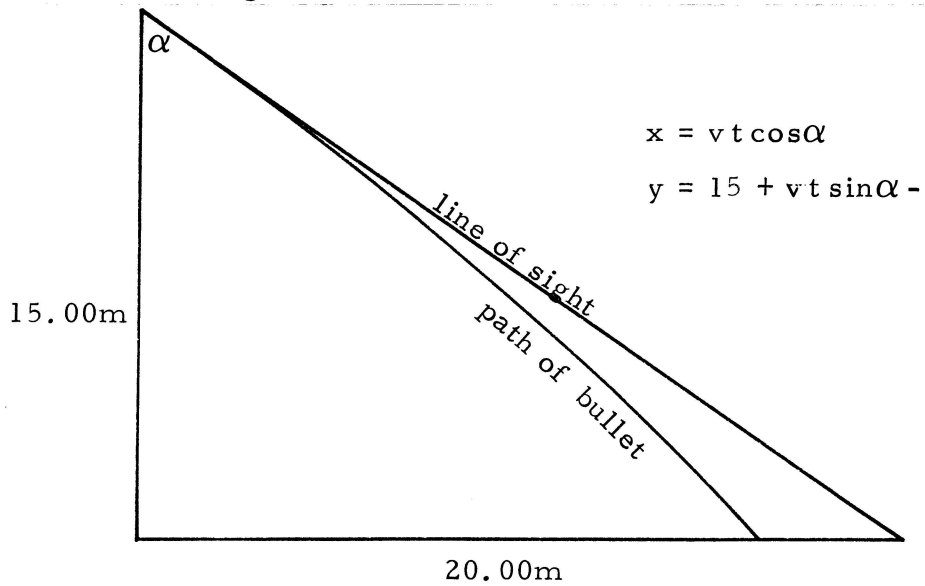
The answer that we get after adding these using four figure arithmetic is 0.4469. This answer does not have even one significant figure, since the true value is around 0.004087.

We can salvage something by noting that $e^{-5.5} = \frac{1}{e^{5.5}}$ and so with the data above:

$$e^{-5.5} = \frac{1}{e^{5.5}} = \frac{1}{244.4} = 0.004092.$$

This compares very favourably with the true value, being accurate to three significant figures, which is all we can reasonably expect if we are only using four figure arithmetic. (In fact, using five figure arithmetic in the original example does not help: why not ?).

B3 Consider the following situation; we wish to find the 'shortfall' of a bullet fired at an angle to the horizontal.



Eliminating t and setting y to zero we get

$$\left(\frac{-g}{2v^2\cos^2\alpha}\right)x^2 + x\tan\alpha + 15 = 0$$

This is a quadratic of the form $ax^2 + bx + c = 0$, and since a is negative, we want the smaller root.

We use

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

[and we assume $v = 626.0\ ms^{-1}$.]

and we can perform our arithmetic either with chopping or rounding.

<u>Rounding</u>: $a = -0.00001956$
$b = -0.7500$
$c = 15.00$

$$\text{so } x = \frac{0.7500 - \sqrt{(0.5625 + 0.001174)}}{-0.00003912} = \frac{0.7500 - \sqrt{(0.5637)}}{-0.00003912}$$

$$= \frac{-0.0008}{-0.00003912} = 20.45 \ (!)$$

<u>Chopping</u>: $a = -0.00001954$
$b = -0.7500$
$c = 15.00$

$$\text{so } x = \frac{0.7500 - 0.7507}{-0.00003908} = 17.91$$

<u>Reformulation</u>: since x is close to 20.00 then we can assume that ax^2 is very close to $(-0.00001956)(20.00)^2 = -0.007824$, and so our equation reduces to $-0.007824 - 0.75x + 15 = 0$, from which we conclude that x is approximately 19.99.

B4 How accurately can you find the roots of $x^2 - 18x + 1 = 0$ if you are restricted to using four figure arithmetic with rounding?
Devise an implementation of your algorithm suitable for use on a computer.

C1 Given that $A = \begin{bmatrix} 9999 & 9998 \\ 10000 & 9999 \end{bmatrix}$

and that $B = \begin{bmatrix} 9999.9999 & -9997.0001 \\ -10001 & 9998 \end{bmatrix}$

since when we compute AB it is very close to the identity matrix, we might assume that these two matrices are almost inverses, but when we compute BA we find $BA = \begin{bmatrix} 19998.0001 & 19995.0003 \\ -19999 & -19994 \end{bmatrix}$

C2 Solve

$$0.5x - 0.4y = 0.32$$
$$-0.4x - 0.5y = -0.22 \quad \text{to two decimal places}$$

using the iteration

$$x^{(r+1)} = \begin{bmatrix} 0.5 & 0.4 \\ 0.4 & 0.5 \end{bmatrix} x^{(r)} + \begin{bmatrix} 0.32 \\ -0.22 \end{bmatrix}$$

One would assume that when the difference between successive iterates was less than 0.005, then we could stop, but comparison of $x^{(7)}$ with $x^{(8)}$ shows this assumption to be unjustified.
In fact $x^{(7)} = \begin{pmatrix} 0.75217 \\ 0.15217 \end{pmatrix}$ and $x^{(8)} = \begin{pmatrix} 0.75695 \\ 0.15695 \end{pmatrix}$ while $x = \begin{pmatrix} 0.8 \\ 0.2 \end{pmatrix}$

C3 Solve $x^3 - 6x^2 + 9x - 4 = 0$ using the Newton-Raphson iteration:

$$x_{r+1} = x_r - \frac{f(x)}{f'(x)}$$

i.e. with $f(x) = x^3 - 6x^2 + 9x - 4$

and $f'(x) = 3x^2 - 12x + 9$

A your problem is ill-conditioned (inherent instability),
B your method is unstable (induced instability),
C your stopping criteria are inadequate.