# Hypothesis Testing and Model Comparison

Peter Hoff
Duke STA 610

# NELS data

```
nels[1:10,]

##    school enroll flp public urbanicity hwh   ses mscore
## 1    1011      5   3      1      urban   2 -0.23  52.11
## 2    1011      5   3      1      urban   0  0.69  57.65
## 3    1011      5   3      1      urban   4 -0.68  66.44
## 4    1011      5   3      1      urban   5 -0.89  44.68
## 5    1011      5   3      1      urban   3 -1.28  40.57
## 6    1011      5   3      1      urban   5 -0.93  35.04
## 7    1011      5   3      1      urban   1  0.36  50.71
## 8    1011      5   3      1      urban   4 -0.24  66.17
## 10   1011      5   3      1      urban   8 -1.07  46.17
## 11   1011      5   3      1      urban   2 -0.10  58.76
```

# Macro predictors

**flp**: percent category of students on the flp

flp=1 0-5% students on flp;

flp=2 5-30% students on flp;

flp=3 $>$ 30% students on flp.

```
table(tapply(nels$flp,nels$school,mean))

##
##   1   2   3
## 226 257 201
```

**enroll**: roughly the number of grade-10 students, in hundreds.

```
table(tapply(nels$enroll,nels$school,mean))

##
##   0   1   2   3   4   5
## 149 112 118  98 108  99
```

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# Macro predictors

`public`: public or private school.

```
table(tapply(nels$public,nels$school,mean))

##
##   0   1
## 168 516
```

`urbanicity`: rural, suburban or urban.

```
table(tapply(nels$urbanicity,nels$school,function(x){x[1]} ))

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##   1   2   3
## 125 324 235
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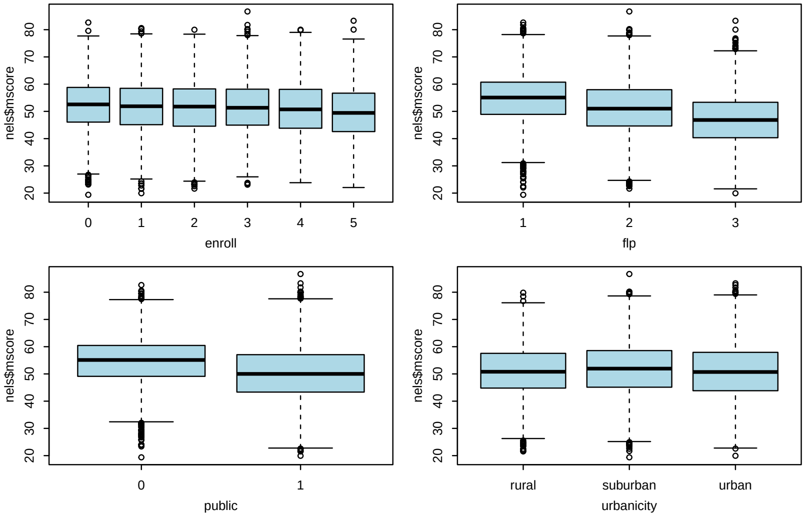
# Macro effects on `mscore`



# What is wrong with the following?

## Heterogeneity due to enroll:

```
anova(lm(mscore~as.factor(enroll),data=nels))

## Analysis of Variance Table
##
## Response: mscore
##                      Df  Sum Sq Mean Sq F value    Pr(>F)
## as.factor(enroll)     5    8660 1732.02   18.14 < 2.2e-16 ***
## Residuals         12968 1238175   95.48
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

## Heterogeneity due to urbanicity:

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## Analysis of Variance Table
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## Response: mscore
##                          Df  Sum Sq Mean Sq F value    Pr(>F)
## as.factor(urbanicity)     2    2652 1325.87  13.823 1.008e-06 ***
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```

# What is wrong with the following?

**Problem 1:** The analyses ignore grouping/assume independence.

**Problem 2:** Variables are not balanced across predictors:

```
table(nels$urbanicity,nels$enroll)

##
##               0    1    2    3    4    5
##   rural     959  449  369  264  215   93
##   suburban  922 1046 1215 1054  991  886
##   urban     790  659  772  590  782  918
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# "Controlling" for covariates

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anova(lm(mscore~as.factor(enroll) +
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                as.factor(public) +
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## as.factor(flp)            2  111662   55831 646.433 < 2.2e-16 ***
## as.factor(public)         1    3455    3455  39.998 2.626e-10 ***
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##                          Df  Sum Sq Mean Sq  F value    Pr(>F)
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## as.factor(public)         1   61162   61162 708.1572 < 2.2e-16 ***
## as.factor(flp)            2   61253   30627 354.6062 < 2.2e-16 ***
## as.factor(enroll)         5    2181     436   5.0493 0.0001261 ***
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# Model comparison

Often we are interested in evaluating the effects of a variable *after* accounting for effects of others.

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### model fits
fit.add<-lm(mscore~as.factor(enroll) +
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fit.menroll<-lm(mscore~as.factor(flp) +
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### evaluating enroll - not controlling for other effects
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### evaluating enroll - controlling for other effects
anova(fit.menroll,fit.add)

## Analysis of Variance Table
##
## Model 1: mscore ~ as.factor(flp) + as.factor(public) + as.factor(urbanicity)
## Model 2: mscore ~ as.factor(enroll) + as.factor(flp) + as.factor(public) +
##     as.factor(urbanicity)
##   Res.Df     RSS Df Sum of Sq      F    Pr(>F)
## 1  12968 1121768
## 2  12963 1119588  5    2180.5 5.0493 0.0001261 ***
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# Type III sums of squares

To evaluate effects *after controlling for others*,

- put in the term of interest last, or
- use type III sums of squares tests.

```
library(car)
Anova(fit.add,type=3)

## Anova Table (Type III tests)
##
## Response: mscore
##                        Sum Sq    Df    F value    Pr(>F)
## (Intercept)           3206322     1 37123.9724 < 2.2e-16 ***
## as.factor(enroll)        2181     5     5.0493 0.0001261 ***
## as.factor(flp)          57424     2   332.4354 < 2.2e-16 ***
## as.factor(public)        5121     1    59.2872 1.461e-14 ***
## as.factor(urbanicity)    3471     2    20.0932 1.937e-09 ***
## Residuals             1119588 12963
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```

# Model comparison

Alternatively, without the car package, you can use drop1:

```
drop1(fit.add,test="F")

## Single term deletions
##
## Model:
## mscore ~ as.factor(enroll) + as.factor(flp) + as.factor(public) +
##     as.factor(urbanicity)
##                        Df Sum of Sq     RSS   AIC  F value    Pr(>F)
## <none>                              1119588 57857
## as.factor(enroll)       5      2181 1121768 57872   5.0493 0.0001261 ***
## as.factor(flp)          2     57424 1177012 58502 332.4354 < 2.2e-16 ***
## as.factor(public)       1      5121 1124708 57914  59.2872 1.461e-14 ***
## as.factor(urbanicity)  2      3471 1123059 57893  20.0932 1.937e-09 ***
## ---
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# Questionable assumptions of macro $F$-tests

The ANOVA model above can be expressed as

$$y_{i,j} = \mu + a_{e(j)} + b_{f(j)} + c_{p(j)} + d_{u(j)} + \epsilon_{i,j}$$

$a_{e(j)} \in \{a_1, \dots, a_5\}$, $e(j)$ is enrollment category of $j$

$b_{f(j)} \in \{b_1, b_2, b_3\}$, $f(j)$ is flp category of $j$

etc.

The previous tests all assumed $\{\epsilon_{i,j}\} \sim$ *iid* $N(0, \sigma^2)$, and specifically,

$$\text{Cov}\left[\begin{pmatrix} \epsilon_{1,j} \\ \vdots \\ \epsilon_{n,j} \end{pmatrix}\right] = \begin{pmatrix} \sigma^2 & 0 & \cdots & 0 \\ 0 & \sigma^2 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & \sigma^2 \end{pmatrix}$$

Why, in general, might we question this assumption?

Why might responses within a school be more similar than across schools?

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# Attempted solution with fixed effects

To account for school heterogeneity, we could fit a school-specific intercept:

$$y_{i,j} = (\mu + a_j) + a_{e(j)} + b_{f(j)} + c_{p(j)} + d_{u(j)} + \epsilon_{i,j}$$

In the absence of macro effects, OLS/ANOVA was a reasonable approach:

$$y_{i,j} = \mu + a_j + \epsilon_{i,j}$$

- $\bar{y}_j$ provides an unbiased estimate of $\mu_j = \mu + a_j$
- $F$-test from ANOVA is a valid test of heterogeneity across groups.

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To account for school heterogeneity, we could fit a school-specific intercept:

$$y_{i,j} = (\mu + a_j) + a_{e(j)} + b_{f(j)} + c_{p(j)} + d_{u(j)} + \epsilon_{i,j}$$

In the absence of macro effects, OLS/ANOVA was a reasonable approach:

$$y_{i,j} = \mu + a_j + \epsilon_{i,j}$$

- $\bar{y}_j$ provides an unbiased estimate of $\mu_j = \mu + a_j$
- $F$-test from ANOVA is a valid test of heterogeneity across groups.

Could we use OLS/ANOVA in the presence of macro effects?

# Attempted solution with fixed effects

```
fit_ols<-lm(mscore~as.factor(school) +
                   as.factor(enroll) +
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                   as.factor(public) +
                   as.factor(urbanicity)  ,data=nels)
```

```
anova(fit_ols)

## Analysis of Variance Table
##
## Response: mscore
##                      Df Sum Sq Mean Sq F value    Pr(>F)
## as.factor(school)   683 342385  501.30  6.8118 < 2.2e-16 ***
## Residuals         12290 904450   73.59
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

School-specific fixed effects explain *all* heterogeneity in means across schools.

There is nothing left for the other factors to explain.

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# HLM solution

$$y_{i,j} = (\mu + a_j) + a_{e(j)} + b_{f(j)} + c_{p(j)} + d_{u(j)} + \epsilon_{i,j}$$
$$a_1, \ldots, a_m \sim \ iid\ N(0, \tau^2)$$

As we've discussed, the random intercept induces a covariance within schools, and the above model is *equivalent to*

$$y_{i,j} = \mu + a_{e(j)} + b_{f(j)} + c_{p(j)} + d_{u(j)} + \epsilon_{i,j}$$

where

$$\text{Cov}\left[\begin{pmatrix}\epsilon_{1,j}\\ \vdots \\ \epsilon_{n,j}\end{pmatrix}\right] = \begin{pmatrix}\sigma^2+\tau^2 & \tau^2 & \cdots & \tau^2\\ \tau^2 & \sigma^2+\tau^2 & \cdots & \tau^2\\ \vdots & & & \vdots\\ \tau^2 & \tau^2 & \cdots & \sigma^2+\tau^2\end{pmatrix}$$

$$\text{Cor}[y_{i,j}, y_{i,k}] = \frac{\tau^2}{\tau^2+\sigma^2}$$

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# Across school heterogeneity

```
fit0<-lmer( mscore ~ 1 + (1|school),data=nels)

fit0

## Linear mixed model fit by REML ['lmerMod']
## Formula: mscore ~ 1 + (1 | school)
##    Data: nels
## REML criterion at convergence: 93914.62
## Random effects:
##  Groups   Name        Std.Dev.
##  school   (Intercept) 4.866
##  Residual             8.585
## Number of obs: 12974, groups:  school, 684
## Fixed Effects:
## (Intercept)
##       50.94

s2.hat<-sigma(fit0)^2
t2.hat<-as.numeric(VarCorr(fit0)$school)

s2.hat

## [1] 73.70822

t2.hat

## [1] 23.6768

### ICC
t2.hat/(t2.hat+s2.hat)

## [1] 0.2431257
```

# Across school heterogeneity

```
fit1<-lmer( mscore ~ as.factor(enroll) + (1|school),data=nels)


s2.hat<-sigma(fit1)^2
t2.hat<-as.numeric(VarCorr(fit1)$school)

s2.hat

## [1] 73.71874

t2.hat

## [1] 23.3493

### ICC
t2.hat/(t2.hat+s2.hat)

## [1] 0.2405457
```

# Across school heterogeneity

```
fit2<-lmer( mscore ~ as.factor(enroll) +as.factor(flp) + (1|school),data=nels)


s2.hat<-sigma(fit2)^2
t2.hat<-as.numeric(VarCorr(fit2)$school)

s2.hat

## [1] 73.76314

t2.hat

## [1] 13.73191

### ICC
t2.hat/(t2.hat+s2.hat)

## [1] 0.156945
```

# Across school heterogeneity

```
fit3<-lmer( mscore ~ as.factor(enroll) +as.factor(flp) + as.factor(public) +
   (1|school),data=nels)


s2.hat<-sigma(fit3)^2
t2.hat<-as.numeric(VarCorr(fit3)$school)

s2.hat

## [1] 73.77206

t2.hat

## [1] 13.4839

### ICC
t2.hat/(t2.hat+s2.hat)

## [1] 0.1545327
```

# Across school heterogeneity

```
fit4<-lmer( mscore ~ as.factor(enroll) +as.factor(flp) + as.factor(public) +
  as.factor(urbanicity)  + (1|school),data=nels)


s2.hat<-sigma(fit4)^2
t2.hat<-as.numeric(VarCorr(fit4)$school)

s2.hat

## [1] 73.77562

t2.hat

## [1] 13.20577

### ICC
t2.hat/(t2.hat+s2.hat)

## [1] 0.151823
```

## Model selection and testing

**Notice:** As we add macro predictors,

- $\hat{\tau}^2$ decreases, $\hat{\sigma}^2$ remains roughly the same;
- the within-group correlation decreases.

**Questions:** For a given set of macro variables,

- Is there evidence of (strong) within class correlation?
  - If not, we can test for macro variables with ANOVA.
  - If so, how do we evaluate the effects of the macro variables?

**Goals:**

1. Develop tests of within-class correlation *in the presence of macro variables*
   *equivalently, test of excess across-school heterogeneity*
2. Develop tests of macro effects *in the presence of within-class correlation*
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# Testing for excess heterogeneity

Consier two competing models:

$M_0$: No excess heterogeneity

$$y_{i,j} = \beta^T \mathbf{x}_{i,j} + \epsilon_{i,j}$$
$$\{\epsilon_{i,j}\} \sim \; iid \; N(0, \sigma^2)$$

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## Model comparisons via tests

Suppose you would like a model selection procedure such that

if model $M_0$ were true,

you have a 95% chance of saying it is true.

If this is what you want, then a *level .05 hypothesis test* is for you.

$H_0$**:** No excess heterogeneity - model $M_0$ is true.

$H_1$**:** Excess heterogeneity - model $M_1$ is true.

**Objective:** A level $\alpha$ test of $H_0$ versus $H_1$.

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# Likelihood ratio tests

A popular tool for comparing nested models is the *likelihood ratio test (LRT)*:

$$\text{Reject } H_0 \text{ if } \Lambda(\mathbf{y}) = \frac{p(\mathbf{y}|\hat{\theta}_1)}{p(\mathbf{y}|\hat{\theta}_0)} \text{ is large.}$$

- $p(\mathbf{y}|\hat{\theta}_1)$ is the maximized prob density of data under $H_1$
- $p(\mathbf{y}|\hat{\theta}_0)$ is the maximized prob density of data under $H_0$
- $\Lambda(\mathbf{y})$ is the likelihood ratio statistic.

For a variety of reasons, the LRT is often expressed as

$$\text{Reject } H_0 \text{ if } \lambda(\mathbf{y}) = 2 \times \left(\log p(\mathbf{y}|\hat{\theta}_1) - \log p(\mathbf{y}|\hat{\theta}_0)\right) \text{ is large.}$$

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- $p(\mathbf{y}|\hat{\theta}_1)$ is the maximized prob density of data under $H_1$
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- $\Lambda(\mathbf{y})$ is the likelihood ratio statistic.

For a variety of reasons, the LRT is often expressed as

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# Example: NELS data

```
### model 0
fit0<-lm(mscore ~ as.factor(flp) , data=nels)
logLik(fit0)

## 'log Lik.' -47375.64 (df=4)

### model 1
fit1<-lmer(mscore ~ as.factor(flp) + (1|school), data=nels)
logLik(fit1)

## 'log Lik.' -46812.38 (df=5)

### log liklihood statistic
lrt.stat<- 2*( logLik(fit1) - logLik(fit0) )
lrt.stat

## 'log Lik.' 1126.509 (df=5)
```

The LRT statistic seems pretty big!

# Example: NELS data

```r
### model 0
fit0<-lm(mscore ~ as.factor(flp) +
                  as.factor(enroll) +
                  as.factor(public) +
                  as.factor(urbanicity) , data=nels)
logLik(fit0)

## 'log Lik.' -47326.85 (df=12)

### model 1
fit1<-lmer(mscore ~ as.factor(flp) +
                    as.factor(enroll) +
                    as.factor(public) +
                    as.factor(urbanicity) + (1|school)   , data=nels)
logLik(fit1)

## 'log Lik.' -46797.45 (df=13)

### log liklihood statistic
lrt.stat<- 2*( logLik(fit1) - logLik(fit0) )
lrt.stat

## 'log Lik.' 1058.799 (df=13)
```

Still pretty big!

## Null distributions

How big is big? A level $\alpha$ test is one where we

$$\text{reject } H_0 \text{ if } \lambda(\mathbf{y}) = 2 \times \left( \log p(\mathbf{y}|\hat{\theta}_1) - \log p(\mathbf{y}|\hat{\theta}_0) \right) \text{ is bigger than } \lambda_\alpha$$

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# Null distribution example: $t$-test

If

$$y_{1,A}, \ldots, y_{n_A,A} \sim \textit{ iid } N(\mu, \sigma^2)$$
$$y_{1,B}, \ldots, y_{n_B,B} \sim \textit{ iid } N(\mu, \sigma^2)$$

then the distribution of the t-statistic

$$t(\mathbf{y}_A, \mathbf{y}_B) = \frac{\bar{y}_B - \bar{y}_A}{s_p\sqrt{1/n_A + 1/n_B}}$$

has a $t$-distribution.

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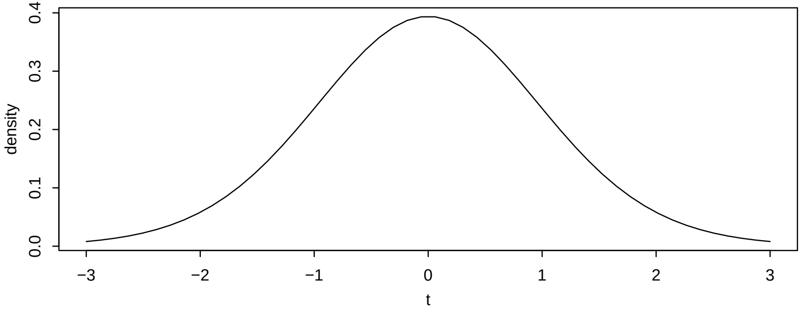
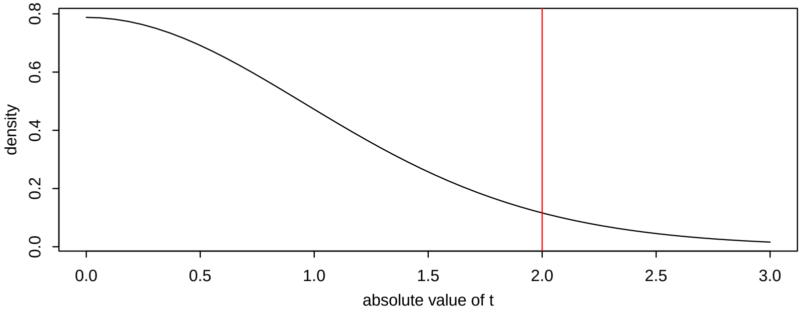


# Null distribution example: *t*-test

A typical t-test rejects if $|t(\mathbf{y}_A, \mathbf{y}_B)| > 2$.




$$\Pr(|t(\mathbf{y}_A, \mathbf{y}_B)| > 2) \approx 0.05$$

- 2 is the critical value of the test;
- 0.05 is the (approximate) level of the test.

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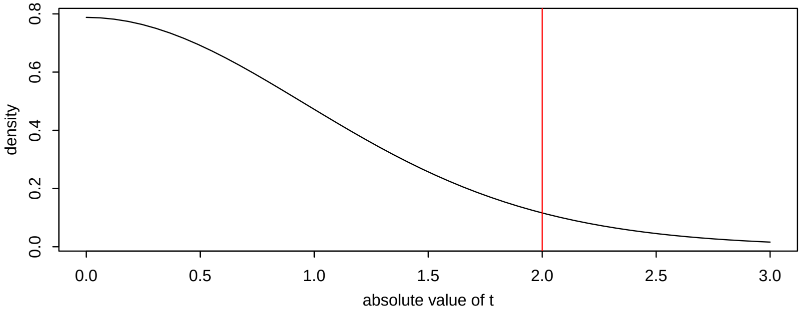


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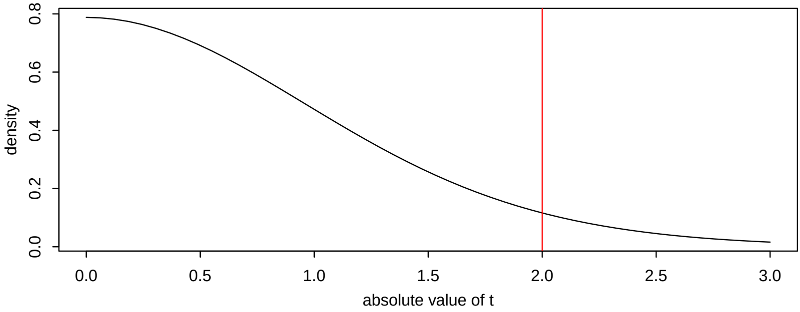


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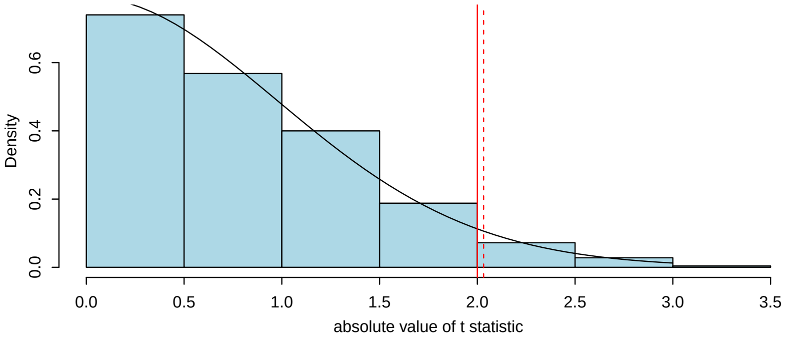
# Null distribution example: $t$-test empirical validation

```
n<-20 ; ATSTAT<-NULL

for(i in 1:S)
{
  yA<-rnorm(n)
  yB<-rnorm(n)
  ATSTAT<-c(ATSTAT, abs(t.test(yA,yB,pooled=TRUE)$stat))
}
```

# Null distribution example: $t$-test empirical validation




```
quantile(ATSTAT,probs=.95)

##      95%
## 2.032179

qt(.975,2*(n-1))

## [1] 2.024394
```

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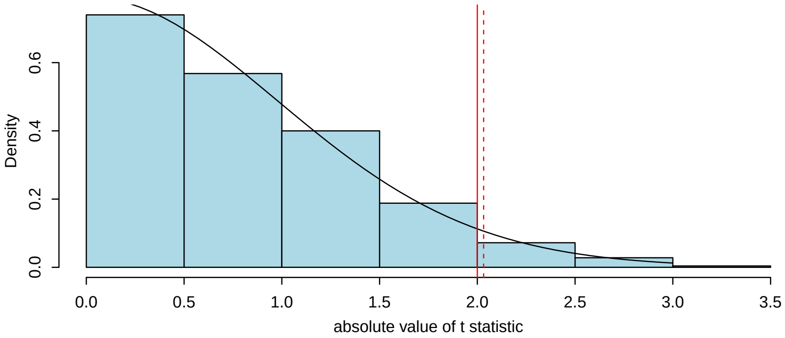


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# Null distribution for LRT

**LRT:**

$$\text{Reject } H_0 \text{ if } \lambda(\mathbf{y}) = 2 \times \left( \log p(\mathbf{y}|\hat{\theta}_1) - \log p(\mathbf{y}|\hat{\theta}_0) \right) \text{ is greater than } c,$$

where $c$ is the value such that

$$\Pr(\lambda(\mathbf{y}) > c | H_0) = 0.05.$$

To figure out what $c$ is, we need the distribution of $\lambda(\mathbf{y})$ when $H_0$ is true. That is, we need to know the *null distribution*.

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## Null distribution for LRT

Statistical folklore says the following: If

- $M_0$ is *nested* in $M_1$ ($M_0$ is a special case of $M_1$), and
- $M_0$ is true, then

$$\lambda(\mathbf{y}) \dot{\sim} \chi^2_d$$

where $d$ is the difference in the number of parameters between $M_1$ and $M_0$.

```
qchisq(.95,1)

## [1] 3.841459
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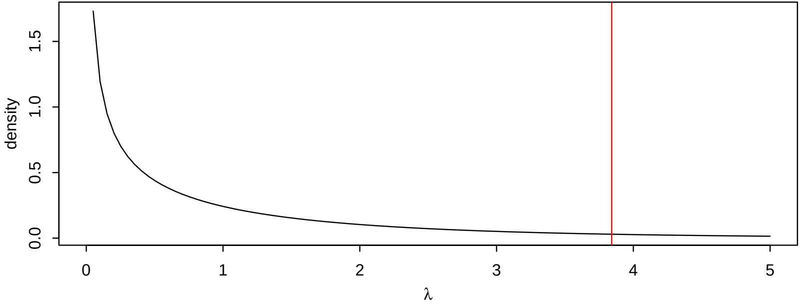


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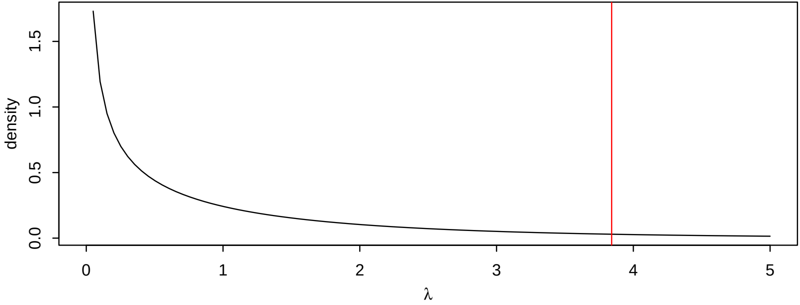


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# Null distribution for LRT: Fixed effects

$M_0$**:** No fixed effect of $x_{i,j}$

$$y_{i,j} = \beta_0 + a_j + \epsilon_{i,j}$$
$$a_j \sim \; N(0, \tau^2)$$

$M_1$**:** Yes fixed effect of $x_{i,j}$

$$y_{i,j} = \beta_0 + \beta_1 x_{i,j} + a_j + \epsilon_{i,j}$$
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**Distribution of LRT:** The change in the number of parameters is $d = 1$.

Presumably,

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The $\dot{\sim}$ means "approximately distributed as."

The approximation improves as sample size increases.

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# Null distribution for LRT: Empirical evaluation

```
m<-20 ; n<-10
beta0<-1 ; beta1<-0

g<-rep(1:m,times=rep(n,m))

LAMBDA.H0<-NULL
for(s in 1:S)
{
  a<-rnorm(m)
  x<-rnorm(m*n)

  y<-a[g] + beta0 + beta1*x + rnorm(m*n)

  fit0<-lmer(y ~ 1 + (1|g), REML=FALSE )
  fit1<-lmer(y ~ x + (1|g), REML=FALSE )

  lambda<-2*( logLik(fit1) - logLik(fit0) )

  LAMBDA.H0<-c(LAMBDA.H0,lambda)
}
```

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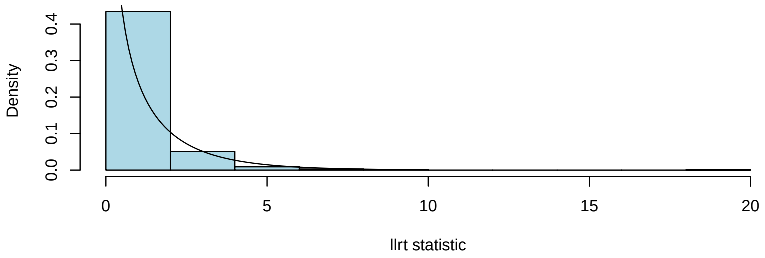


```
quantile(LAMBDA.H0,.95)

##      95%
## 3.258501

qchisq(.95,1)

## [1] 3.841459
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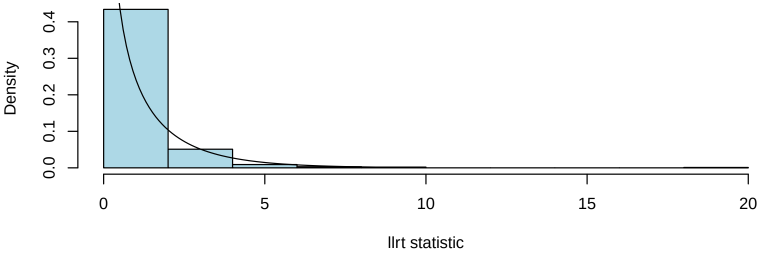


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# LRT for HLM

$M_0$**:**

$$\mathbf{y}_j = \mathbf{X}_j\boldsymbol{\beta} + \boldsymbol{\epsilon}_j \ , \quad \text{Cov}\left[\begin{pmatrix}\epsilon_{1,j}\\ \vdots \\ \epsilon_{n,j}\end{pmatrix}\right] = \begin{pmatrix}\sigma^2 & 0 & \cdots & 0\\ 0 & \sigma^2 & \cdots & 0\\ \vdots & & & \vdots\\ 0 & 0 & \cdots & \sigma^2\end{pmatrix}$$

$M_1$**:**

$$\mathbf{y}_j = \mathbf{X}_j\beta + \epsilon_j \ , \quad \text{Cov}\left[\begin{pmatrix}\epsilon_{1,j}\\ \vdots \\ \epsilon_{n,j}\end{pmatrix}\right] = \begin{pmatrix}\sigma^2+\tau^2 & \tau^2 & \cdots & \tau^2\\ \tau^2 & \sigma^2+\tau^2 & \cdots & \tau^2\\ \vdots & & & \vdots\\ \tau^2 & \tau^2 & \cdots & \sigma^2+\tau^2\end{pmatrix}$$

**Q:** What is the difference in the number of parameters?

**A:** $d = 1$

# LRT for HLM

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**A:** $d = 1$

# LRT for HLM

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**Q:** What is the difference in the number of parameters?

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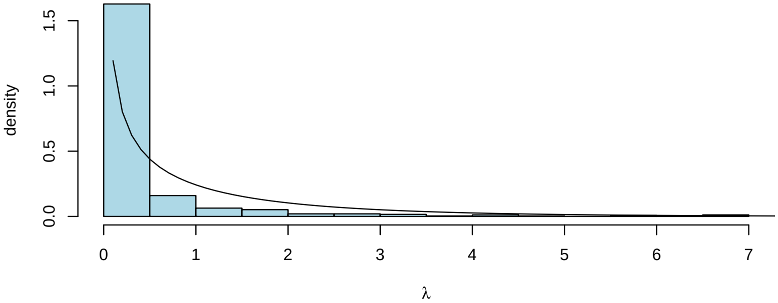
# Simulation study

```
m<-20 ; n<-10
beta0<-1 ; beta1<-1

g<-rep(1:m,times=rep(n,m))

LAMBDA.H0<-NULL
for(s in 1:S)
{
  x<-rnorm(m*n)

  y<-beta0 + beta1*x + rnorm(m*n)

  fit0<-lm(y ~ x )

  fit1<-lmer(y ~ x + (1|g), REML=FALSE)

  lambda<-2*( logLik(fit1) - logLik(fit0) )

  LAMBDA.H0<-c(LAMBDA.H0,lambda)
}
```

# Simulation study




```r
mean( LAMBDA.H0>= qchisq(.95,1) )

## [1] 0.02
```

# Simulation study

```r
zapsmall(LAMBDA.H0[1:20])
```

```
##  [1] 0.000000 0.891508 0.497324 0.177651 0.000000 0.417878 0.000000 0.000000
##  [9] 0.000138 0.040075 0.000000 4.920390 0.000000 0.000000 0.387080 0.000000
## [17] 0.000000 0.000000 0.281322 0.052502
```

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mean( zapsmall(LAMBDA.H0[1:20]) == 0 )
```

```
## [1] 0.5
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# Mixture null distributions

**What is going on?** Suppose we are fitting $M_1$ in the simple HNM:

$$y_{i,j} = \mu + a_j + \epsilon_{i,j}$$
$$a_j \sim N(0, \tau^2)$$

Recall,

$$\mathrm{E}[MSE] = \sigma^2$$
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If $M_0$ is in fact true, then $\tau^2 = 0$ and

$$\mathrm{E}[MSE] = \sigma^2$$
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If we are fitting $M_1$, then sometimes (due to sampling variability)

$$MSE > MSG$$
$$(MSG - MSE)/n < 0 \;\Rightarrow\; \text{use } \hat{\tau}^2 = 0 \text{ in practice.}$$

In these cases (roughly speaking),

- the MLE $\hat{\tau}^2$ is zero.
- the best $M_0$ fit is the same as the best $M_1$ fit.

$$\max_{\mu,\sigma^2,\tau^2} \log p(\mathbf{y}|\mu,\sigma^2,\tau^2) = \max_{\mu,\sigma^2} \log p(\mathbf{y}|\mu,\sigma^2,\tau^2 = 0)$$

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# Example dataset

```
set.seed(2)
y<-1 + rnorm(m*n)

anova(lm(y~as.factor(g)) )

## Analysis of Variance Table
##
## Response: y
##               Df  Sum Sq Mean Sq F value Pr(>F)
## as.factor(g)  19  14.745 0.77606  0.6503 0.8629
## Residuals    180 214.812 1.19340

MSE<-anova(lm(y~as.factor(g)) )[2,3]
MSG<-anova(lm(y~as.factor(g)) )[1,3]

MSE

## [1] 1.193401

MSG

## [1] 0.7760613

MSG-MSE

## [1] -0.4173393
```

# Example dataset

```
fit0<-lm(y ~ 1 )
fit1<-lmer(y ~ 1 + (1|g), REML=FALSE)
```

```
fit0
```

```
##
## Call:
## lm(formula = y ~ 1)
##
## Coefficients:
## (Intercept)
##      0.9993
```

```
fit1
```

```
## Linear mixed model fit by maximum likelihood  ['lmerMod']
## Formula: y ~ 1 + (1 | g)
##       AIC       BIC    logLik  deviance  df.resid
##  601.1424  611.0374 -297.5712  595.1424       197
## Random effects:
##  Groups   Name        Std.Dev.
##  g        (Intercept) 0.000
##  Residual             1.071
## Number of obs: 200, groups:  g, 20
## Fixed Effects:
## (Intercept)
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## optimizer (nloptwrap) convergence code: 0 (OK) ; 0 optimizer warnings; 1 lme4 warnings
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```
2*( logLik(fit1) - logLik(fit0) )
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## 'log Lik.' -2.273737e-13 (df=3)
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## The (asymptotic) null distribution

It turns out that *under $M_0$*,

$$\Pr(\lambda(\mathbf{y}) = 0) = \frac{1}{2}$$

The values that are *not* equal to zero are distributed as $\chi^2_1$:

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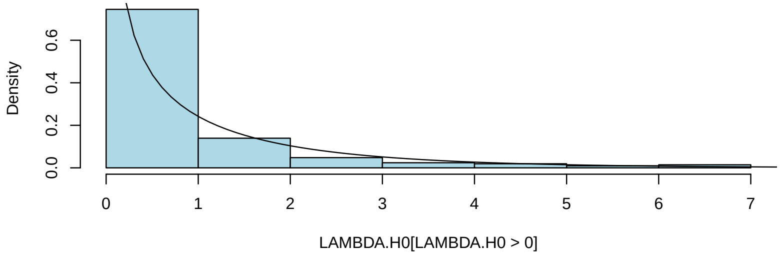
This means that under $M_0$, $\lambda(\mathbf{y})$ has a *mixture distribution*

# The empirical null distribution

```
LAMBDA.H0<-zapsmall(LAMBDA.H0)
mean(LAMBDA.H0==0)

## [1] 0.584

hist(LAMBDA.H0[LAMBDA.H0>0],col="lightblue",prob=TRUE,main="")
lines(xs,dchisq(xs,1),type="l")
```



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We can represent the distribution of $\lambda(\mathbf{y})$ as follows:

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## Computing a $p$-value

Recall, a *p-value* is the probability under the null of getting a test statistic equal to or larger than the observed test statistic.

For a given observed value $\lambda_{obs}$,

$$p - \text{value} = \Pr(\lambda(\mathbf{y}) \geq \lambda_{obs} | H_0)$$

How do we compute this for a given value $\lambda_{obs}$?

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$$\begin{aligned}\Pr(\lambda(\mathbf{y}) \geq \lambda_{obs}) &= \Pr(\lambda(\mathbf{y}) = X_0 \text{ and } X_0 \geq \lambda_{obs}) + \Pr(\lambda(\mathbf{y}) = X_1 \text{ and } X_1 \geq \lambda_{obs}) \\ &= \tfrac{1}{2}0 + \tfrac{1}{2}\Pr(X_1 \geq \lambda_{obs}) \\ &= \tfrac{1}{2}\Pr(\chi^2_1 \geq \lambda_{obs}),\end{aligned}$$

which is 1/2 the $p$-value that would be obtained using the $\chi^2_1$ null distribution.

**Folklore:** "The $p$-value for testing …the random intercept variance is half this [$\chi^2_1$] tail value."

(true if $\lambda_{obs} \neq 0$).

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which is 1/2 the $p$-value that would be obtained using the $\chi^2_1$ null distribution.

**Folklore:** "The $p$-value for testing ... the random intercept variance is half this [$\chi^2_1$] tail value."

(true if $\lambda_{obs} \neq 0$).

## Computing a $p$-value

**Case 1:** $\lambda_{obs} = 0$.

$$\Pr(\lambda(\mathbf{y}) \geq 0) = 1$$

as $X_0$ and $X_1$ are $\geq 0$.

**Case 2:** $\lambda_{obs} > 0$.

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Recall one of our original questions:

*Can the heterogeneity across schools be ascribed to known macro covariates?*

**Model fits:**

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fit0<-lm(mscore ~ as.factor(enroll) + as.factor(flp) + as.factor(urbanicity) +
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```

**Hypothesis test:**

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### LRT statistic
lambda<-2*(logLik(fit1)-logLik(fit0))

lambda

## 'log Lik.' 696.8672 (df=14)

### p-value
.5*(1-pchisq(c(lambda),1) )

## [1] 0
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- `pchisq(lambda,1)` is the probability of being smaller than `lambda`
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# Summary of testing

$$y_{i,j} = \boldsymbol{\beta}^T x_{i,j} + a_j + \epsilon_{i,j}$$
$$a_j \sim N(0, \tau^2)$$

For models consisting of

- fixed effects, and
- a single random intercept,

**Tests involving $\beta$** : Testing components of $\beta$ equal zero can be obtained with the usual $LRT$.

- Null distribution: $\lambda_0 \sim \chi^2_d$,
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# Testing examples

```
fit.full<-lmer(mscore~
   as.factor(enroll) + as.factor(flp) + as.factor(urbanicity) +
   hwh + ses +
  (1|school) , data=nels,REML=FALSE)

fit.full

## Linear mixed model fit by maximum likelihood  ['lmerMod']
## Formula: mscore ~ as.factor(enroll) + as.factor(flp) + as.factor(urbanicity) +
##     hwh + ses + (1 | school)
##    Data: nels
##       AIC       BIC    logLik  deviance  df.resid
##  92408.36  92512.95 -46190.18  92380.36     12960
## Random effects:
##  Groups   Name        Std.Dev.
##  school   (Intercept) 2.969
##  Residual             8.243
## Number of obs: 12974, groups:  school, 684
## Fixed Effects:
##                   (Intercept)             as.factor(enroll)1
##                      52.82676                        0.54442
##            as.factor(enroll)2             as.factor(enroll)3
##                       0.61973                        0.61739
##            as.factor(enroll)4             as.factor(enroll)5
##                       0.52867                        0.16135
##               as.factor(flp)2                as.factor(flp)3
##                      -2.09257                       -4.84231
## as.factor(urbanicity)suburban     as.factor(urbanicity)urban
##                      -0.05113                       -0.86587
##                           hwh                            ses
##                       0.01354                        4.13467
```

# Testing examples

```
fit.menr<-lmer(mscore~
   as.factor(flp) + as.factor(urbanicity) +
   hwh + ses +
  (1|school) , data=nels,REML=FALSE)
```

```
fit.mflp<-lmer(mscore~
   as.factor(enroll) + as.factor(urbanicity) +
   hwh + ses +
  (1|school) , data=nels,REML=FALSE)
```

```
fit.murb<-lmer(mscore~
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# Testing examples

## Compute the LRT statistic:

```
lambda<-2*(logLik(fit.full) - logLik(fit.menr))

lambda

## 'log Lik.' 3.204099 (df=14)
```

## Calculate *d*:

```
table(nels$enroll)

##
##    0    1    2    3    4    5
## 2671 2154 2356 1908 1988 1897


attr( logLik(fit.full),"df")

## [1] 14

attr( logLik(fit.menr),"df")

## [1] 9

d<- attr( logLik(fit.full),"df") - attr( logLik(fit.menr),"df")

d

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# Testing examples

**Compute the *p*-value:**

```
(1-pchisq(c(lambda),d))

## [1] 0.668553
```

**This is mostly automated in R:**

```
anova(fit.full,fit.menr)

## Data: nels
## Models:
## fit.menr: mscore ~ as.factor(flp) + as.factor(urbanicity) + hwh + ses + (1 | school)
## fit.full: mscore ~ as.factor(enroll) + as.factor(flp) + as.factor(urbanicity) + hwh + se
##          npar   AIC   BIC logLik deviance  Chisq Df Pr(>Chisq)
## fit.menr    9 92402 92469 -46192    92384
## fit.full   14 92408 92513 -46190    92380 3.2041  5     0.6686
```

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## fit.full   14 92408 92513 -46190    92380 3.2041  5     0.6686
```

# Testing other factors

```
anova(fit.full,fit.mflp)

## Data: nels
## Models:
## fit.mflp: mscore ~ as.factor(enroll) + as.factor(urbanicity) + hwh + ses + (1 | school)
## fit.full: mscore ~ as.factor(enroll) + as.factor(flp) + as.factor(urbanicity) + hwh + se
##          npar   AIC   BIC logLik deviance  Chisq Df Pr(>Chisq)
## fit.mflp   12 92564 92654 -46270    92540
## fit.full   14 92408 92513 -46190    92380 159.58  2  < 2.2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

```
anova(fit.full,fit.murb)

## Data: nels
## Models:
## fit.murb: mscore ~ as.factor(enroll) + as.factor(flp) + hwh + ses + (1 | school)
## fit.full: mscore ~ as.factor(enroll) + as.factor(flp) + as.factor(urbanicity) + hwh + se
##          npar   AIC   BIC logLik deviance  Chisq Df Pr(>Chisq)
## fit.murb   12 92412 92502 -46194    92388
## fit.full   14 92408 92513 -46190    92380 7.7808  2    0.02044 *
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
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## Data: nels
## Models:
## fit.murb: mscore ~ as.factor(enroll) + as.factor(flp) + hwh + ses + (1 | school)
## fit.full: mscore ~ as.factor(enroll) + as.factor(flp) + as.factor(urbanicity) + hwh + se
##          npar   AIC   BIC logLik deviance  Chisq Df Pr(>Chisq)
## fit.murb   12 92412 92502 -46194    92388
## fit.full   14 92408 92513 -46190    92380 7.7808  2    0.02044 *
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

# Testing examples

```
fit.mhwh<-lmer(mscore~
   as.factor(enroll) + as.factor(flp) + as.factor(urbanicity) +
   ses +
  (1|school) , data=nels,REML=FALSE)
```

```
fit.mses<-lmer(mscore~
   as.factor(enroll) + as.factor(flp) + as.factor(urbanicity) +
   hwh +
  (1|school) , data=nels,REML=FALSE)
```

# Testing examples

```
fit.mhwh<-lmer(mscore~
   as.factor(enroll) + as.factor(flp) + as.factor(urbanicity) +
   ses +
  (1|school) , data=nels,REML=FALSE)
```

```
fit.mses<-lmer(mscore~
   as.factor(enroll) + as.factor(flp) + as.factor(urbanicity) +
   hwh +
  (1|school) , data=nels,REML=FALSE)
```

# Testing examples

```
anova(fit.full,fit.mhwh)

## Data: nels
## Models:
## fit.mhwh: mscore ~ as.factor(enroll) + as.factor(flp) + as.factor(urbanicity) + ses + (1
## fit.full: mscore ~ as.factor(enroll) + as.factor(flp) + as.factor(urbanicity) + hwh + se
##          npar   AIC   BIC logLik deviance  Chisq Df Pr(>Chisq)
## fit.mhwh   13 92407 92504 -46190    92381
## fit.full   14 92408 92513 -46190    92380 0.3107  1     0.5772
```

```
anova(fit.full,fit.mses)

## Data: nels
## Models:
## fit.mses: mscore ~ as.factor(enroll) + as.factor(flp) + as.factor(urbanicity) + hwh + (1
## fit.full: mscore ~ as.factor(enroll) + as.factor(flp) + as.factor(urbanicity) + hwh + se
##          npar   AIC   BIC logLik deviance Chisq Df Pr(>Chisq)
## fit.mses   13 93634 93731 -46804    93608
## fit.full   14 92408 92513 -46190    92380  1228  1  < 2.2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

# Testing examples

```
anova(fit.full,fit.mhwh)

## Data: nels
## Models:
## fit.mhwh: mscore ~ as.factor(enroll) + as.factor(flp) + as.factor(urbanicity) + ses + (1
## fit.full: mscore ~ as.factor(enroll) + as.factor(flp) + as.factor(urbanicity) + hwh + se
##          npar   AIC   BIC logLik deviance  Chisq Df Pr(>Chisq)
## fit.mhwh   13 92407 92504 -46190    92381
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```

```
anova(fit.full,fit.mses)

## Data: nels
## Models:
## fit.mses: mscore ~ as.factor(enroll) + as.factor(flp) + as.factor(urbanicity) + hwh + (1
## fit.full: mscore ~ as.factor(enroll) + as.factor(flp) + as.factor(urbanicity) + hwh + se
##          npar   AIC   BIC logLik deviance Chisq Df Pr(>Chisq)
## fit.mses   13 93634 93731 -46804    93608
## fit.full   14 92408 92513 -46190    92380  1228  1  < 2.2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

# Testing examples

```
summary(fit.full)$coef

##                                    Estimate Std. Error     t value
## (Intercept)                     52.82676162  0.4309192 122.5908794
## as.factor(enroll)1               0.54442472  0.4569472   1.1914390
## as.factor(enroll)2               0.61973124  0.4541606   1.3645642
## as.factor(enroll)3               0.61738849  0.4828518   1.2786293
## as.factor(enroll)4               0.52866612  0.4891502   1.0807849
## as.factor(enroll)5               0.16135353  0.4932025   0.3271547
## as.factor(flp)2                 -2.09257387  0.3497278  -5.9834361
## as.factor(flp)3                 -4.84231161  0.3677904 -13.1659532
## as.factor(urbanicity)suburban -0.05113111  0.3932499  -0.1300219
## as.factor(urbanicity)urban    -0.86587407  0.4204572  -2.0593634
## hwh                              0.01353902  0.0242850   0.5575056
## ses                              4.13466985  0.1142795  36.1803310

2*(1-pnorm(.5575))

## [1] 0.5771859

2*(1-pnorm(36.1803))

## [1] 0
```

# Testing examples

Now that you know where the numbers come from,

```
drop1(fit.full,test="Chisq")

## Single term deletions
##
## Model:
## mscore ~ as.factor(enroll) + as.factor(flp) + as.factor(urbanicity) +
##     hwh + ses + (1 | school)
##                       npar   AIC     LRT Pr(Chi)
## <none>                     92408
## as.factor(enroll)        5 92402    3.20 0.66855
## as.factor(flp)           2 92564  159.58 < 2e-16 ***
## as.factor(urbanicity)    2 92412    7.78 0.02044 *
## hwh                      1 92407    0.31 0.57725
## ses                      1 93634 1228.01 < 2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

# Summary of tests so far

$$y_{i,j} = \boldsymbol{\beta}^T x_{i,j} + a_j + \epsilon_{i,j}$$
$$a_j \sim \ N(0, \tau^2)$$

**Fixed effects:**

enrollment : no strong evidence of effect

flp : decreasing scores with increasing flp

urban : urban schools have lower scores than others

hwh : no strong evidence of an effect *on average across schools*

ses : strong evidence of a positive effect *on average across schools*

**Random effects:** Strong evidence of excess across-school heterogeneity in mean score.

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# ANOVA comparison

Compare to tests that don't account for across-group heterogeneity:

```
### model fit
fit.afull<-lm(mscore~
   as.factor(enroll) + as.factor(flp) + as.factor(urbanicity) +
   hwh + ses,
   data=nels )


### factor evaluation
drop1(fit.afull,test="F")

## Single term deletions
##
## Model:
## mscore ~ as.factor(enroll) + as.factor(flp) + as.factor(urbanicity) +
##     hwh + ses
##                       Df Sum of Sq     RSS   AIC   F value    Pr(>F)
## <none>                              991486 56283
## as.factor(enroll)      5       377  991863 56278    0.9863    0.4243
## as.factor(flp)         2     28135 1019621 56642  183.9096 < 2.2e-16 ***
## as.factor(urbanicity)  2      1516  993002 56298    9.9107 5.002e-05 ***
## hwh                    1       167  991653 56283    2.1819    0.1397
## ses                    1    132644 1124130 57910 1734.0918 < 2.2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

## Testing for heterogeneous slopes

**General two-level HLM:**

$$y_{i,j} = \boldsymbol{\beta}^T x_{i,j} + \boldsymbol{a}_j^T z_{i,j} + \epsilon_{i,j}$$
$$\boldsymbol{a}_j \sim N(0, \Psi)$$

For example, maybe

$$\begin{pmatrix} z_{i,j,1} \\ z_{i,j,2} \end{pmatrix} = \begin{pmatrix} 1 \\ ses_{i,j} \end{pmatrix} \quad \Psi = \begin{pmatrix} \psi_1^2 & \psi_{1,2} \\ \psi_{2,1} & \psi_2^2 \end{pmatrix}$$

We would like to be able to test

$$H_0 : \psi_2^2 = 0 \text{ (no heterogeneity in slope with } ses\text{)},$$

in the presence of heterogeneity in intercept.

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$$
\begin{aligned}
y_{i,j} &= \boldsymbol{\beta}^T x_{i,j} + \boldsymbol{a}_j^T z_{i,j} + \epsilon_{i,j} \\
\boldsymbol{a}_j &\sim \; N(0, \Psi)
\end{aligned}
$$

For example, maybe

$$
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We would like to be able to test

$$H_0 : \psi_2^2 = 0 \;\; \text{(no heterogeneity in slope with } ses\text{)},$$

in the presence of heterogeneity in intercept.

# Testing for heterogeneous slopes

$$H_0 : \psi_2^2 = 0 \;\; \text{(no heterogeneity in slope with } \textit{ses}\text{)}$$

If the variance of something is zero, its covariance with anything else is zero.

This means that under $H_0 : \psi_2^2 = 0$,

$$\Psi = \begin{pmatrix} \psi_1^2 \end{pmatrix}$$

while under $H_1 : \psi_2^2 \neq 0$,

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The difference in the number of parameters is $d = 2$.

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$$\mathbf{\Psi} = \begin{pmatrix} \psi_1^2 & \psi_{1,2} \\ \psi_{2,1} & \psi_2^2 \end{pmatrix}$$

The difference in the number of parameters is $d = 2$.

# NELS data

```r
fit.r1<-lmer(
   mscore~
        as.factor(flp) + as.factor(urbanicity) +
        ses +
       (ses | school) , data=nels,REML=FALSE)
```

```r
summary(fit.r1)$coef
```

```
##                                    Estimate Std. Error    t value
## (Intercept)                     53.13668593  0.3943076 134.759485
## as.factor(flp)2                 -2.02135574  0.3342738  -6.047006
## as.factor(flp)3                 -4.81780351  0.3612673 -13.335840
## as.factor(urbanicity)suburban    0.05675027  0.3803280   0.149214
## as.factor(urbanicity)urban      -0.80937534  0.4049585  -1.998663
## ses                              4.12877819  0.1255087  32.896343
```

```r
VarCorr(fit.r1)
```

```
##  Groups   Name        Std.Dev. Corr
##  school   (Intercept) 2.9673
##           ses         1.2712   -0.005
##  Residual             8.2008
```

# NELS data

```
fit.r1<-lmer(
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## as.factor(urbanicity)suburban  0.05675027  0.3803280   0.149214
## as.factor(urbanicity)urban    -0.80937534  0.4049585  -1.998663
## ses                            4.12877819  0.1255087  32.896343
```

```
VarCorr(fit.r1)

##  Groups   Name        Std.Dev. Corr
##  school   (Intercept) 2.9673
##           ses         1.2712   -0.005
##  Residual             8.2008
```

# NELS data

```
fit.r0<-lmer(
   mscore~
        as.factor(flp) + as.factor(urbanicity) +
        ses +
       (1 | school) , data=nels,REML=FALSE)
```

```
summary(fit.r0)$coef
```

```
##                                 Estimate Std. Error     t value
## (Intercept)                  53.12042202  0.3928410 135.2211600
## as.factor(flp)2              -2.00043931  0.3324308  -6.0176108
## as.factor(flp)3              -4.77163280  0.3596303 -13.2681609
## as.factor(urbanicity)suburban  0.06620705  0.3792811   0.1745593
## as.factor(urbanicity)urban   -0.78129077  0.4032054  -1.9376990
## ses                           4.13800015  0.1141748  36.2426730
```

```
VarCorr(fit.r0)
```

```
##  Groups   Name        Std.Dev.
##  school   (Intercept) 2.9760
##  Residual             8.2437
```

# NELS data

```r
fit.r0<-lmer(
   mscore~
        as.factor(flp) + as.factor(urbanicity) +
        ses +
       (1 | school) , data=nels,REML=FALSE)
```

```r
summary(fit.r0)$coef

##                                  Estimate Std. Error     t value
## (Intercept)                   53.12042202  0.3928410 135.2211600
## as.factor(flp)2               -2.00043931  0.3324308  -6.0176108
## as.factor(flp)3               -4.77163280  0.3596303 -13.2681609
## as.factor(urbanicity)suburban  0.06620705  0.3792811   0.1745593
## as.factor(urbanicity)urban    -0.78129077  0.4032054  -1.9376990
## ses                            4.13800015  0.1141748  36.2426730
```

```r
VarCorr(fit.r0)

##  Groups   Name        Std.Dev.
##  school   (Intercept) 2.9760
##  Residual             8.2437
```

# NELS data

```
fit.r0<-lmer(
   mscore~
        as.factor(flp) + as.factor(urbanicity) +
        ses +
       (1 | school) , data=nels,REML=FALSE)
```

```
summary(fit.r0)$coef

##                                    Estimate Std. Error     t value
## (Intercept)                     53.12042202  0.3928410 135.2211600
## as.factor(flp)2                 -2.00043931  0.3324308  -6.0176108
## as.factor(flp)3                 -4.77163280  0.3596303 -13.2681609
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## 'log Lik.' -46185.14 (df=10)
```

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## 'log Lik.' -46191.93 (df=8)
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```r
lambda<-2*c( logLik(fit.r1) - logLik(fit.r0) )

lambda

## [1] 13.58696
```

What do we compare `lambda` to?

What types of values would we expect under $H_0$?

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## Null distribution

**Speculation 1:** Maybe under $H_0$, $\lambda \sim \frac{1}{2}(\{0\} + \chi^2_1)$.

**Speculation 2:** Maybe under $H_0$, $\lambda \sim \chi^2_2$, as $d = 2$.

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# Null distribution

```
m<-30 ; n<-10
beta0<-1 ; beta1<-1
g<-rep(1:m,times=rep(n,m))

LAMBDA.H0<-NULL
for(s in 1:S)
{
  a<-rnorm(m) # random effects

  x<-rnorm(m*n) # covariates

  y<-beta0 + a[g] + beta1*x + rnorm(m*n)  #simulated under null

  fit0<-lmer(y ~ x + (1|g), REML=FALSE )

  fit1<-lmer(y ~ x + (x|g), REML=FALSE)

  lambda<-2*( logLik(fit1) - logLik(fit0) )

  LAMBDA.H0<-c(LAMBDA.H0,lambda)
}

## Warning in checkConv(attr(opt, "derivs"), opt$par, ctrl = control$checkConv, :
Model failed to converge with max|grad| = 0.00448623 (tol = 0.002, component 1)
## Warning in checkConv(attr(opt, "derivs"), opt$par, ctrl = control$checkConv, :
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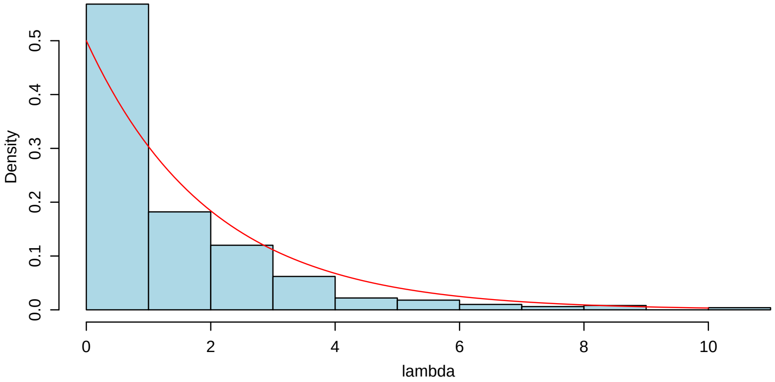
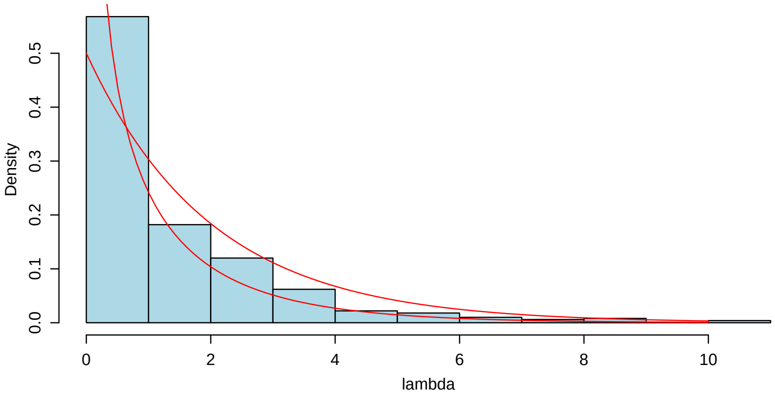
# Null distribution

Compare to a $\chi^2_2$ distribution:



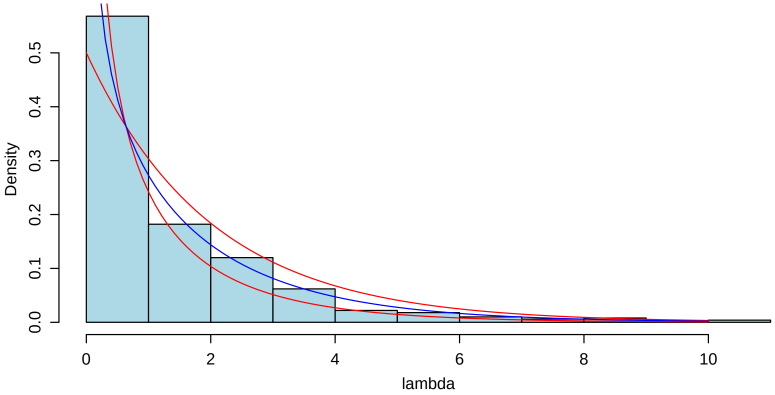
# Null distribution

Compare to a $\chi^2_1$ distribution:



# Null distribution

Here is the theoretical, asymptotic null distribution: $\lambda \sim \frac{1}{2}(\chi^2_1 + \chi^2_2)$



## Mixture distributions

We can represent the distribution of $\lambda(\mathbf{y})$ as follows:

$$\lambda(\mathbf{y}) = \begin{cases} X_1 & \text{with probabilty } 1/2 \\ X_2 & \text{with probabilty } 1/2 \end{cases}$$

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# Computing the p-value

$$\begin{aligned}\Pr(\lambda(\mathbf{y}) \geq \lambda_{obs}) &= \Pr(\lambda(\mathbf{y}) = X_1 \text{ and } X_1 \geq \lambda_{obs}) + \Pr(\lambda(\mathbf{y}) = X_2 \text{ and } X_2 \geq \lambda_{obs}) \\ &= \tfrac{1}{2}\Pr(X_1 \geq \lambda_{obs}) + \tfrac{1}{2}\Pr(X_1 \geq \lambda_{obs}) \\ &= \tfrac{1}{2}\left(\Pr(\chi_1^2 \geq \lambda_{obs}) + \Pr(\chi_2^2 \geq \lambda_{obs})\right)\end{aligned}$$

which is a 50-50 average between the naive $p$-value (based on a $\chi_2^2$ distribution), and one based on a reduced degrees of freedom.

***p*-value**: The $p$-value can be obtained with `pchisq` as before:

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# The general result

$$y_{i,j} = \boldsymbol{\beta}^T \mathbf{x}_{i,j} + \mathbf{a}_j^T \mathbf{z}_{i,j} + \epsilon_{i,j}$$

If $\mathbf{a}_j \in \mathbb{R}^p$, then

$$\text{Cov}[\mathbf{a}_j] = \Psi = \begin{pmatrix} \psi_1^2 & \psi_{12} & \cdots & \psi_{1p} \\ \psi_{21} & \psi_2^2 & \cdots & \psi_{2p} \\ \vdots & & & \vdots \\ \psi_{p1} & \psi_{p2} & \cdots & \psi_p^2 \end{pmatrix}$$

Consider testing to compare the following models:

$M_1$**:** Full model

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**Question:** What is the change in number of parameters?

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$M_1$ $p$ random effects coefficients

$M_0$ $p-1$ random effects coefficients

**Null distribution:** Under $M_0$, the LRT statistic has is distributed as

$$\lambda(\mathbf{y}) = \begin{cases} X_{p-1} & \text{with probabilty } 1/2 \\ X_p & \text{with probabilty } 1/2 \end{cases}$$

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**CAREFUL:** Some authors say $\lambda | M_0 \sim \frac{1}{2}(\chi^2_{p} + \chi^2_{p+1})$.
This is because they are not counting the intercept.

## The null distribution in the general case

Shorthand for this is

$$\lambda | M_0 \sim \frac{1}{2}(\chi^2_{p-1} + \chi^2_{p}).$$

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# Check with previous results:

**Single random effect:**

$$M_0 \;:\; y_{i,j} = \boldsymbol{\beta}^T \mathbf{x}_{i,j} + \epsilon_{i,j}$$
$$M_1 \;:\; y_{i,j} = \boldsymbol{\beta}^T \mathbf{x}_{i,j} + b_{1,j} + \epsilon_{i,j}$$
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**Two random effects:**

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## Effects on $p$-values and critical values

**Naive critical value:**

- $p$ random effects implies $d = p$.
- The naive 0.05 critical value is $\lambda_c$=qchisq(.95,p)

**Actual $p$-value:** Suppose you observed a test statistic equal to $\lambda_c$:

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p<-1:20
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pval<-.5*( (1-pchisq(lc.naive,p-1)) + (1-pchisq(lc.naive,p)) )
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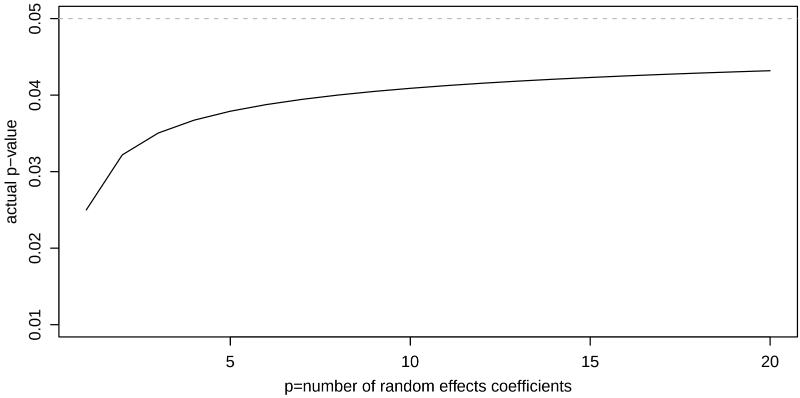

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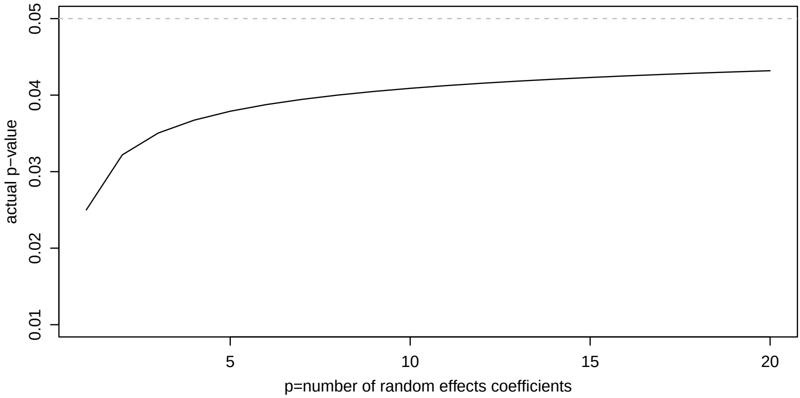

# Summary of testing

**LRT:** The LRT can be used to compare nested models:

- models with and without various fixed effects;
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**LRT:** The LRT statistic can be compared to a null distribution:

- $\chi^2_d$ for testing if $d$ fixed effects are zero.
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**Consequences of ignoring the mixture null distribution:**

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- The naive $p$-value will underrepresent evidence against the null.
- From a decision-theory perspective, if your naive $p$-value is lower than your type I error, then it doesn't matter.

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If you are concerned about the validity for your sample size, then simulate!

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