Exercises part.1 - Gradient & Conjugate Gradient algorithms

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1. Implement the algorithms

Gradient algorithm

- At first step $x_0 = 0$, $r_0 = b$
- At each step k:
  - compute $t = Ar_{k-1}$
  - compute $\alpha = \frac{(r_{k-1}, r_{k-1})}{(r_{k-1}, t)}$
  - update $x : x_k = x_{k-1} + \alpha r_{k-1}$
  - update $r : r_k = r_{k-1} - \alpha t$
- Iterate until $\hat{r} = \sqrt{\frac{(r_k, r_k)}{(b, b)}} < \hat{r}_{targ}$

Conjugate Gradient algorithm

- At first step $x_0 = 0$, $r_0 = p_0 = b$
- At each step k:
  - compute $t = Ap_{k-1}$
  - compute $\alpha = \frac{(r_{k-1}, r_{k-1})}{(p_{k-1}, t)}$
  - update $x : x_k = x_{k-1} + \alpha p_{k-1}$
  - update $r : r_k = r_{k-1} - \alpha t$
  - compute $\beta = \frac{(r_k, r_k)}{(r_{k-1}, r_{k-1})}$
  - update $p_k = r_k + \beta p_{k-1}$
- Iterate until $\hat{r} = \sqrt{\frac{(r_k, r_k)}{(b, b)}} < \hat{r}_{targ}$

2. Solve the linear system

$$\begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix} x = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

and verify:

   (a) the solution obtained with the numerical algorithm with the one derived solving explicitly the system of equations

   (b) that convergence is achieved after 2 iterations, as required by the algorithm

3. Using the provided routine that generate random symmetric definite matrices with fixed condition number, verify:

   (a) the scaling of the number of iteration required to solve the system of equation at a fixed precision $\hat{r}_{targ}$ and matrix condition number with the size of the problem

   (b) that with fixed residue and fixed (large) matrix size, the number of iterations scales linearly with the square root of the condition number

4. (optional) check that for a non-positive definite matrix the algorithm does not converge

5. (optional) verify that the functional

$$F(x) = \frac{1}{2} x^t A x - bx$$

is monotonously minimized during the iterations (while this is not true for $\hat{r}$)