Constant for

$\|A-B\|_F^2 \; \underbrace{-\|\Lambda_A - \Lambda_B\|_F^2}_{\text{min of perm}}$ and $-\|A-B\|_F^2 \; \underbrace{+\|\Lambda_A-\Lambda_B\|_F^2}_{\text{max of perm}}$.

The $O(\Theta^2)$ term is

$$C = \left(tr(K^2\Lambda_B\Lambda_A) - tr(K\Lambda_A K\Lambda_B)\right)$$

(1) Exact computation of the constant is possible if we choose a (Λ_A, Q_A) and (Λ_B, Q_B) ordering. This in turn decides

$$Rot = (Q_A)^T Q_B$$

Then using Rodrigues-formula

$Rot = \exp(\Theta \underline{\underline{K}})$ with $Rot^T = Rot^{-1}$, $\det(Rot) = 1$

and choosing $\|\vec{k}\| = 1$ will force $\|\underline{\underline{K}}\|_F^2 = 2$.

The ordering of (Λ_A, Q_A), (Λ_B, Q_B) will determine Rot and this in turn will determine Θ and $\underline{\underline{K}}$.

Question :-

How can we estimate C?

(2) We would prefer not to determine C exactly since this will require first finding Q_A and Q_B which defeats the point of using the metric in the first place.

What do we know about K and K^2?

$$K = \begin{pmatrix} 0 & -k_3 & k_2 \\ k_3 & 0 & -k_1 \\ -k_2 & k_1 & 0 \end{pmatrix} = -K^T$$

$$K^2 = \begin{pmatrix} -(k_3^2+k_2^2) & k_2 k_1 & k_3 k_1 \\ k_2 k_1 & -(k_3^2+k_1^2) & k_3 k_2 \\ k_3 k_1 & k_3 k_2 & -(k_2^2+k_1^2) \end{pmatrix} = (K^2)^T$$

$$\underbrace{K^2 \Lambda_A \Lambda_B}_{\substack{\shortparallel \\ \Lambda_B \Lambda_A}} = \begin{pmatrix} -\Lambda_{11}^B (k_3^2+k_2^2) & \Lambda_{22}^B k_2 k_1 & \Lambda_{33}^B k_3 k_1 \\ \Lambda_{11}^B k_2 k_1 & -\Lambda_{22}^B (k_3^2+k_1^2) & \Lambda_{33}^B k_3 k_2 \\ \Lambda_{11}^B k_3 k_1 & \Lambda_{22}^B k_3 k_2 & -\Lambda_{33}^B (k_2^2+k_1^2) \end{pmatrix} \Lambda_A$$

$$= \begin{pmatrix} -\Lambda_{11}^B \Lambda_{11}^A (k_3^2+k_2^2) & \Lambda_{22}^A \Lambda_{22}^B k_2 k_1 & \Lambda_{33}^A \Lambda_{33}^B k_3 k_1 \\ \Lambda_{11}^A \Lambda_{11}^B k_2 k_1 & -\Lambda_{22}^A \Lambda_{22}^B (k_3^2+k_1^2) & \Lambda_{33}^A \Lambda_{33}^B k_3 k_2 \\ \Lambda_{11}^A \Lambda_{11}^B k_3 k_1 & \Lambda_{22}^A \Lambda_{22}^B k_3 k_2 & -\Lambda_{33}^A \Lambda_{33}^B (k_2^2+k_1^2) \end{pmatrix}$$

$$\mathrm{tr}(K^2 \Lambda_B \Lambda_A)$$

$$= -\Lambda_{11}^B \Lambda_{11}^A \underbrace{(k_3^2+k_2^2)}_{1-k_1^2} - \Lambda_{22}^A \Lambda_{22}^B \underbrace{(k_3^2+k_1^2)}_{1-k_2^2} + \Lambda_{33}^A \Lambda_{33}^B \underbrace{(k_2^2+k_1^2)}_{1-k_3^2}$$

$$= -\Lambda_{11}^A \Lambda_{11}^B - \Lambda_{22}^A \Lambda_{22}^B - \Lambda_{33}^A \Lambda_{33}^B$$
$$+ \left(\Lambda_{11}^A \Lambda_{11}^B k_1^2 + \Lambda_{22}^A \Lambda_{22}^B k_2^2 + \Lambda_{33}^A \Lambda_{33}^B k_3^2 \right)$$

$$K\Lambda_A = \begin{pmatrix} 0 & -k_3 & k_2 \\ k_3 & 0 & -k_1 \\ -k_2 & k_1 & 0 \end{pmatrix} \begin{pmatrix} \Lambda_{11}^A & & \\ & \Lambda_{22}^A & \\ & & \Lambda_{33}^A \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -k_3 \Lambda_{22}^A & k_2 \Lambda_{33}^A \\ k_3 \Lambda_{11}^A & 0 & -k_1 \Lambda_{33}^A \\ -k_2 \Lambda_{11}^A & k_1 \Lambda_{22}^A & 0 \end{pmatrix}$$

$$K\Lambda_B = \begin{pmatrix} 0 & -k_3 \Lambda_{22}^B & k_2 \Lambda_{33}^B \\ k_3 \Lambda_{11}^B & 0 & -k_1 \Lambda_{33}^B \\ -k_2 \Lambda_{11}^B & k_1 \Lambda_{22}^B & 0 \end{pmatrix}$$

$$K\Lambda_A K\Lambda_B = \begin{pmatrix} 0 & -k_3 \Lambda_{22}^A & k_2 \Lambda_{33}^A \\ k_3 \Lambda_{11}^A & 0 & -k_1 \Lambda_{33}^A \\ -k_2 \Lambda_{11}^A & k_1 \Lambda_{22}^A & 0 \end{pmatrix} \begin{pmatrix} 0 & -k_3 \Lambda_{22}^B & k_2 \Lambda_{33}^B \\ k_3 \Lambda_{11}^B & 0 & -k_1 \Lambda_{33}^B \\ -k_2 \Lambda_{11}^B & k_1 \Lambda_{22}^B & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -k_3^2 \Lambda_{22}^A \Lambda_{11}^B - k_2^2 \Lambda_{33}^A \Lambda_{11}^B, & k_1 k_2 \Lambda_{33}^A \Lambda_{22}^B, & +k_3 k_1 \Lambda_{22}^A \Lambda_{33}^B \\ k_1 k_2 \Lambda_{33}^A \Lambda_{11}^B, & -k_3^2 \Lambda_{11}^A \Lambda_{22}^B - k_1^2 \Lambda_{33}^A \Lambda_{22}^B, & k_3 k_2 \Lambda_{11}^A \Lambda_{33}^B \\ k_1 k_3 \Lambda_{22}^A \Lambda_{11}^B, & k_3 k_2 \Lambda_{11}^A \Lambda_{22}^B, & -k_2^2 \Lambda_{11}^A \Lambda_{33}^B - k_1^2 \Lambda_{22}^A \Lambda_{33}^B \end{pmatrix}$$

$$\Rightarrow tr(K\Lambda_A K\Lambda_B)$$

$$= -k_3^2 \Lambda_{22}^A \Lambda_{11}^B - k_2^2 \Lambda_{33}^A \Lambda_{11}^B - k_3^2 \Lambda_{11}^A \Lambda_{22}^B - k_1^2 \Lambda_{33}^A \Lambda_{22}^B$$

$$- k_2^2 \Lambda_{11}^A \Lambda_{33}^B - k_1^2 \Lambda_{22}^A \Lambda_{33}^B$$

$$tr(K^2 \Lambda_B \Lambda_A) = -\Lambda_{11}^A \Lambda_{11}^B (k_3^2 + k_2^2) - \Lambda_{22}^A \Lambda_{22}^B (k_3^2 + k_1^2)$$

$$- \Lambda_{33}^A \Lambda_{33}^B (k_2^2 + k_1^2)$$

Thus

$$tr(K^2 \Lambda_B \Lambda_A) - tr(K \Lambda_A K \Lambda_B)$$

$$= -\Lambda_{11}^A \Lambda_{11}^B (k_3^2 + k_2^2) - \Lambda_{22}^A \Lambda_{22}^B (k_3^2 + k_1^2)$$
$$- \Lambda_{33}^A \Lambda_{33}^B (k_2^2 + k_1^2)$$
$$+ k_3^2 \Lambda_{22}^A \Lambda_{11}^B + k_2^2 \Lambda_{33}^A \Lambda_{11}^B + k_3^2 \Lambda_{11}^A \Lambda_{22}^B + k_1^2 \Lambda_{33}^A \Lambda_{22}^B$$
$$+ k_2^2 \Lambda_{11}^A \Lambda_{33}^B + k_1^2 \Lambda_{22}^A \Lambda_{33}^B$$

Making the approximation $k_1^2 = k_2^2 = k_3^2 = 1$

then $tr(K^2 \Lambda_B \Lambda_A) - tr(K \Lambda_A K \Lambda_B)$

$$= -2\Lambda_{11}^A \Lambda_{11}^B - \Lambda_{22}^A \Lambda_{22}^B - \Lambda_{33}^A \Lambda_{33}^B + \Lambda_{11}^A \Lambda_{22}^B + \Lambda_{33}^A \Lambda_{22}^B + \Lambda_{11}^A \Lambda_{33}^B$$
$$+ \Lambda_{22}^A \Lambda_{11}^B + \Lambda_{33}^A \Lambda_{11}^B + \Lambda_{11}^A \Lambda_{22}^B$$
$$+ \Lambda_{22}^A \Lambda_{33}^B$$