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Recent methods for the restriction conjecture

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Abstract

We explore some of the recent methods used in the restriction theory of harmonic analysis. First, we develop some tools to discuss Tao's ε-removal theorem. Then, we add some further details on Guth's polynomial partitioning method for an improvement on the restriction conjecture in $\mathbb{R}^3$. We mainly focus on the analytical aspect of the paper.

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List of symbols

$\mathbb{R}$	–	set of real numbers.
$\mathbb{C}$	–	set of complex numbers.
$\mathbb{Z}$	–	set of integer numbers.
$\mathbb{S}^n$	–	unit n-sphere.
$\mathbb{B}^n$	–	unit n-ball.
$A \lesssim B$	–	there exists $C > 0$ such that $A \leq CB$.
$A \sim B$	–	there are $c, C > 0$ such that $cB \leq A \leq CB$.
E_S	–	extension operator on S.
R_S	–	restriction operator on S.
T_pM	–	tangent plane to M at a point p.
df_x	–	differential map of f at x.
$X \pitchfork Y$	–	the manifold X is transversal to Y.
$I_2(f, Z)$	–	intersection mod 2 of f with Z.
$\deg_2(f)$	–	degree mod 2 of f.
$W_2(f, z)$	–	winding number of f around z.
$\#$	–	cardinality of a finite set.
$Z(f)$	–	zero set of f.
L^p	–	p-Lebesgue integrable functions.
ℓ^p	–	p convergent series.
$L^{p,q}$	–	Lorentz (p, q)-space.
$\mathrm{Dim}\, X$	–	dimension of the space X.
∇	–	gradient function.
Γ	–	Gamma function.
$\fint_\Omega f$	–	average of f on Ω.
$\mathbf{H}_f$	–	Hessian matrix of f.
τ, θ	–	paraboloid caps.
$f = O(g)$	–	there is some constant $C > 0$ such that $\lvert f(x)\rvert \leq Cg(x)$.

Introduction

The Fourier transform is possibly the most *well-known* integral transformation, in the sense of popularity. However, when one asks questions about how continuous these functions are, much is yet to be discovered.

Let S be a compact, smooth hypersurface of $\mathbb{R}^n$. The Fourier transform of $f \in L^1(\mathbb{R}^n)$ is continuous, and hence, defined everywhere. That is, its restriction to S is well-defined in $L^1(\mathbb{R}^n)$. This is false for a function in $L^2(\mathbb{R}^n)$, as Plancharel's Theorem implies that the Fourier transform can be extended to a unitary operator from $L^2(\mathbb{R}^n)$ to itself, where its functions are *a priori* defined only almost everywhere. Thus, for any compact smooth hypersuface $S \subset \mathbb{R}^n$, we may find a function $f \in L^2(\mathbb{R}^n)$ for which the Fourier-restriction operator $R_S f = \hat{f}|_S$ is not well-defined.

The Hausdorff-Young inequality asserts that if $f \in L^p$, $1 < p < 2$, then $\hat{f} \in L^{p'}$, for $\frac{1}{p} + \frac{1}{p'} = 1$. Thus, for sets of positive measure, the restriction of the Fourier transform is well-defined. However, when dealing with sets of measure zero, the question cannot be immediately solved. One observation can be made; it is false if our manifold has zero Gaussian curvature. Indeed, if $\psi : \mathbb{R}^{n-1} \to \mathbb{R}$ is a bump function and we define

$$\begin{aligned} f : \mathbb{R}^n &\to \mathbb{R} \\ (x_1, \dots, x_n) &\mapsto \frac{\psi(x_2, \dots, x_n)}{1 + |x_1|}, \end{aligned}$$

then $f \in L^p$ for $p > 1$, but $\hat{f}(\xi)$ is infinite for all elements of $\{\xi \in \mathbb{R}^n : \xi_1 = 0\}$, which has zero Gaussian curvature.

The main theorem which is to be proved in this text is due to [5]. Suppose $S \subset \mathbb{R}^n$ is a smooth surface. We write E_S for the *extension operator*. If f is a functions $S \to \mathbb{C}$, then

$$E_S f(x) := \int_S e^{i\omega x} f(\omega) d\sigma(\omega),$$

where $d\sigma$ is the induced surface measure of S.

Theorem 0.1. *If $S \subset \mathbb{R}^3$ is a compact C^∞ surface (maybe with boundary) with positive second fundamental form, then for all $p > 3.25$,*

$$\|E_S f\|_{L^p(\mathbb{R}^3)} \leq C_{p,S} \|f\|_{L^\infty(S)}.$$

We will also define the *restriction operator*.

$$R_S f(\omega) \coloneqq \int_{\mathbb{R}^n} e^{-i\omega x} f(x)\, dx,$$

for $\omega \in E$. To make estimates of the type in Theorem 0.1, we denote by $E_S(p \to q)$ the statement

$$\|E_S f\|_{L^q} \lesssim_{p,q,S} \|f\|_{L^p}.$$

If, in addition, the estimate holds, we say that $E_S(p \to q)$ is true. Similarly, we define the statement $R_S(p \to q)$ as

$$\|R_S f\|_{L^p} \lesssim_{p,q,S} \|f\|_{L^q}.$$

The first thing to address is the range of exponents for which the conjecture can be true. Namely, why the estimate

$$\left\|\widehat{f d\sigma}\right\|_{L^{p'}(\mathbb{R}^n)} \leq \|f\|_{L^{q'}(S,d\sigma)}$$

fails for $p' \leq \frac{2n}{n-1}$. It can be proven (see, for example [9]) that the surface measure of the sphere, $d\sigma$, has the asymptotic behavior at infinity of

$$|\widehat{d\sigma}(\xi)| = O\big(\,|\xi|^{(1-n)/2}\,\big).$$

Hence, for the constant function $f = 1$, we have

$$\left\|\widehat{d\sigma}\right\|_{L^{p'}(\mathbb{R}^n)} \leq \|1\|_{L^{q'}(S,d\sigma)}.$$

Explicitly, this reads as, after replacing the asymptotic estimate,

$$\left|\int_{\mathbb{R}^n} |\xi|^{p'(1-n)/2}\, d\xi\right|^{1/p'} \leq \left|\int_S d\sigma\right|^{1/q'},$$

and the left integral converges if and only if $p'(1-n)/2 < n \iff p' > 2n/(n-1)$.

In [9], it is also proven that the relation $q = p'(n-1)/(n+1)$ is best possible. The proof consists of constructing a function on an annulus around the origin by first defining its Fourier transform. These types of constructions are called *Knapp examples*, named after Anthony Knapp.

Another important property of L^p spaces to consider is that if $A \subset \mathbb{R}^n$ has finite measure, then $L^p(A) \subset L^q(A)$ for $p \geq q$. Then, if $E_S(p \to q)$ is true, then $E_S(r \to s)$ is true for all $r \geq p$ and $s \geq q$. This is a crucial fact to consider when reading proofs and results, as it is enough to prove estimates for a specific value and greater values will immediately follow.

This text will be divided into three main parts. The first part is topological, and deals with the proof of the Borsuk-Ulam theorem, which is used to prove the Stone-Tukey theorem, which is in turn used to prove the polynomial partitioning theorem which was used for the breakthrough in the restriction conjecture.

The second part consists of an introduction to Lorentz spaces and interpolation in these spaces. Then, we use this theory to discuss Tao's ε removal result, a powerful tool which allows one to convert sufficiently controlled local restriction estimates to global restriction estimates at the cost of losing an endpoint on the initial range of exponents. This result is used to obtain the range of $p > 3.25$ in Theorem 0.1. The third and last part focuses on the estimates and techniques used in [5] to prove Theorem 0.1. There, parabolic scaling is used to reduce the problem to proving the main estimate in surfaces which closely resemble a paraboloid. Then, on these specific surfaces, the estimate will be reduced into a broad point estimate, where we need to use a wave packet decomposition to obtain a sufficiently good estimate. Among these estimates, the choice of $p = 3.25$ as an endpoint will become apparent.

Part I

Topology

Chapter 1

The Borsuk-Ulam theorem

In this first chapter, we will state and prove the *Borsuk-Ulam theorem*. It is a topological result which deals with continuous maps from an n dimensional sphere to itself. This particular theorem will be explicitly used in the second chapter.

The majority of the theorems and results can be found in [4] with detailed proofs. Here, we only include the details missing, and assume those which were already proven in the aforementioned reference.

1.1 Transversality

In many contexts, transversality can be thought of as the angle in which two surfaces intersect. A more precise statement can be given.

Definition 1.1. Let $f : X \to Y$ be a smooth map. Let Z be a submanifold of Y. The map f is said to be *transversal* to Z, abbreviated as $f \pitchfork Z$ if

$$df_x(T_xX) + T_y(Z) = T_y(Y)$$

holds for all $x \in f^{-1}(Z)$, $y = f(x)$.

Proposition 1.2. *Let $f : X \to Y$ be a smooth map transversal to a submanifold Z in Y. Then $W = f^{-1}(Z)$ is a submanifold of X and $T_x(W)$ is the preimage of $T_{f(x)}(Z)$ under the linear map $df_x : T_x(X) \to T_{f(x)}(Y)$.*

Proposition 1.3. *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be a sequence of smooth maps of manifolds, and assume g is transversal to a submanifold W of Z. Then $f \pitchfork g^{-1}(W)$ if and only if $g \circ f \pitchfork W$.*

Proof. Assume $f \pitchfork g^{-1}(W)$. By Definition 1.1 we have that for every $x \in f^{-1}(Y)$,

$$T_{f(x)}(X) = df_x(T_xX) + T_{f(x)}(g^{-1}(W)).$$

Therefore, composing by $dg_{f(x)}$ from the left, we have

$$\begin{aligned} dg_{f(x)}(T_{f(x)}(Y)) &= dg_{f(x)} \circ [df_x(T_xX) + T_{f(x)}(g^{-1}(W))] \\ &= dg_{f(x)}(df_x(T_xX)) + dg_{f(x)}(T_{f(x)}(g^{-1}(W))) \\ &= d(g\circ f)_x(T_xX) + dg_{f(x)}(T_{f(x)}(g^{-1}(W))). \end{aligned}$$

By hypothesis, we also have $g \pitchfork W$, so for all $y \in g^{-1}(W)$,

$$T_{g(y)}(Z) = dg_y(T_yY) + T_{g(y)}(W).$$

In particular, for some $y = f(x)$. So

$$T_{g(f(x))}(Z) = dg_{f(x)}(T_{f(x)}Y) + T_{g(f(x))}(W).$$

Hence, adding $T_{g(f(x))}(W)$ to both sides of

$$dg_{f(x)}(T_{f(x)}(Y)) = d(g\circ f)_x(T_xX) + dg_{f(x)}(T_{f(x)}(g^{-1}(W))),$$

we obtain

$$T_{g(f(p))}(Z) = d(g\circ f)_x(T_xX) + dg_{f(x)}(T_{f(x)}(g^{-1}(W))) + T_{g(f(x))}(W)$$

Since $dg_{f(x)}(T_{f(x)}(g^{-1}(W))) \subset T_{g(f(x))}W$, we have

$$T_{g(f(p))}(Z) = d(g\circ f)_x(T_xX) + T_{g(f(x))}(W),$$

so $g\circ f \pitchfork W$.

Conversely, assume that $g \circ f \pitchfork W$. By Proposition 1.2, we have that $g \pitchfork W$ implies $T_{f(x)}(g^{-1}(W)) = (dg_{f(x)})^{-1}(T_{g(f(x))}W)$. Take $y \in T_{f(x)Y}$, then $dg_{f(x)}(y) \in T_{g(f(x))}Z$. Since $g \circ f \pitchfork W$, we have $u \in T_pX$ and $v \in T_{g(f(x))}W$ such that

$$dg_{f(x)}(y) = dg_{f(x)}df_x(u) + v,$$

so $y - df_x(u) \in (dg_{f(x)})^{-1}(T_{g(f(p))}W) = T_{f(x)}(g^{-1}(W))$. Hence, $y = df_x(u) + w$, for some $w \in T_{f(x)}g^{-1}(W)$. Thus, we have

$$T_{f(x)}Y = df_x(T_xX) + T_{f(x)}g^{-1}(W).$$

That is, $f \pitchfork g^{-1}(W)$. $\square$

1.2 Intersection theory mod 2

This section treats a topological invariant for intersecting manifold. Two submanifolds are said to have *complementary dimension* if $\dim X + \dim Z = \dim Y$. If $Y \pitchfork Z$, this dimension condition makes their intersection $X \cap Z$ a zero dimensional manifold (as these are assumed to be boundaryless). If we assume that both X and Z are closed and that at least one of them is compact, then $X \cap Z$ must be a finite set. We will denote the cardinality of the sets as $\#(X \cap Z)$.

Suppose X is any compact manifold, not necessarily inside Y, and $f : X \to Y$ is a smooth map transversal to the closed manifold Z in Y, where $\dim X + \dim Z = \dim Y$. Then $f^{-1}(Z)$ is a closed zero-dimensional submanifold of X, hence a finite set. Define the mod 2 *intersection number* of the map f with Z, $I_2(f, Z)$, to be the number of points in $f^{-1}(Z)$ mod 2. For an arbitrary smooth map $g : X \to Y$, select any map f that is homotopic to g and transversal to Z, and define $I_2(g, Z) = I_2(f, Z)$. We have the following theorem.

Theorem 1.4. *If $f_0, f_1 : X \to Y$ are homotopic and both transversal to Z, then $I_2(f_0, Z) = I_2(f_1, Z)$.*

Corollary 1.5. *If $g_0, g_1 : X \to Y$ are arbitrary homotopic maps, then we have $I_2(g_0, Z) = I_2(g_1, Z)$.*

Thus, if X is a compact submanifold of Y and Z is a closed submanifold of complementary dimension, we define $I_2(X, Z) = I_2(i, Z)$, where $i : X \hookrightarrow Y$ is the inclusion. When $X \pitchfork Z$, then $I_2(X, Z) = \#(X \cap Z)$ mod 2.

Theorem 1.6. *If $f : X \to Y$ is a smooth map of a compact manifold X into a connected manifold Y and $\dim X = \dim Y$, then $I_2(f, \{y\})$ is the same for all points $y \in Y$.*

This common value is called the mod 2 *degree of* f, denoted by $\deg_2(f)$. Note that calculating the mod 2 degree of f is easy. Just pick a regular value y for f and thus $\deg_2(f) = \#f^{-1}(y)$ mod 2.

Theorem 1.7. *Homotopic maps have the same* mod 2 *degree.*

Proposition 1.8. *Let $f : \mathbb{S}^1 \to \mathbb{S}^1$ be a smooth map. Prove that there exists a smooth map $g : \mathbb{R} \to \mathbb{R}$ such that $f(\cos t, \sin t) = (\cos g(t), \sin g(t))$, and satisfying $g(2\pi) = g(0) + 2\pi q$ for some integer q. Moreover,* $\deg_2(f) = q \mod 2$.

Proof. The map $r : \mathbb{R} \to \mathbb{S}^1$ given by $r(t) = (\cos t, \sin t)$ is a covering of $\mathbb{S}^1$. Let $h(t) = (f \circ r)(t) : [0, 2\pi] \to \mathbb{S}^1$. We now lift h to $\mathbb{R}$. Let $\tilde{h}$ denote this lift, then $\tilde{h} : [0, 2\pi] \to \mathbb{R}$ and $r \circ \tilde{h} = h = f \circ r$. Since r is a local diffeomorphism, $\tilde{h}$ is a smooth map. Furthermore,

$$\begin{aligned}(\cos g(2\pi), \sin g(2\pi)) &= f(\cos 2\pi, \sin 2\pi) \\ &= f(\cos 2\pi, \sin 2\pi) \\ &= (\cos g(0), \sin g(0)),\end{aligned}$$

and thus $g(2\pi) = g(0) + 2\pi q$, for an integer q. We then define $\tilde{h}(t + 2\pi) = \tilde{h}(t) + 2\pi q$, which is now an extension to $\mathbb{R}$. Let a be in the image of f. We now note that since g is periodic, we only need to check the cardinality of $f^{-1}(\{a\})$ in the range $[0, 2\pi)$. Thus, we need find every $t \in [0, 2\pi)$ such that $g(t) = g(0)$ mod 2π. Since $g(2\pi) = g(0) + 2\pi q$, the intermediate value theorem implies it must cross the points $g(0) + 2m\pi$ for $m \in \{0, 1, \ldots, q-1\}$. This proves $\deg_2(f) = q$. □

$$\begin{tikzcd}
& \mathbb{R} \arrow[d, "r"] \\
{[0,2\pi]} \arrow[ur, "\tilde{h}"] \arrow[r, "h"'] & \mathbb{S}^1
\end{tikzcd}$$

Figure 1.1: A diagram of the lifting map $\tilde{h}$.

Theorem 1.9. *If $X = \partial W$ and $f : X \to Y$ may be extended to all of W, then* $\deg_2(f) = 0$.

Proposition 1.10. *If the* mod 2 *degree of $p/|p| : \partial W \to \mathbb{S}^1$ is non-zero, then the function p has a zero inside W.*

1.3 Winding numbers

Let X be a compact, connected manifold and a smooth map $f : X \to \mathbb{R}^n$. Suppose that $\dim X = n - 1$. We wish to study how f wraps X around $\mathbb{R}^n$, so take any $z \in \mathbb{R}^n$ not lying in the image $f(X)$. To see how $f(x)$ winds around z, we need to know how many times the unit vector

$$u(x) = \frac{f(x) - z}{|f(x) - z|},$$

which indicates the direction from z to $f(x)$, points in a given direction. Then, section 1.2 would tell us that $u : W \to \mathbb{S}^{n-1}$ hits almost every direction vector the same number of times mod 2, namely, $\deg_2(u)$ times. We then define the *mod 2 winding number of f* around z to be $W_2(f, z) = \deg_2(u)$.

Theorem 1.11. *Suppose that X is the boundary of D, a compact manifold with boundary, and let $F : D \to \mathbb{R}^n$ be a smooth map extending f; that is, $\partial F = f$. Suppose that z is a regular value of F that does not belong to the image of f. Then $F^{-1}(z)$ is a finite set, and $W_2(f, z) = \#F^{-1}(z) \bmod 2$. That is, f winds X around z as often as F hits z, mod 2.*

1.4 The Borsuk-Ulam theorem

The Borsuk-Ulam theorem, whose formulation is attributed to Steinslaw Ulam, was first proved by Karol Borsuk in 1933 in [1].

Lemma 1.12 (Sard's Theorem)**.** *For any smooth map f of a manifold X with boundary into a boundaryless manifold Y, almost every point of Y is a regular value of both $f : X \to Y$ and $\partial f : \partial X \to Y$.*

Proposition 1.13. *Any map $f : \mathbb{S}^1 \to \mathbb{S}^1$ mapping antipodal points to antipodal points has* $\deg_2(f) = 1$.

Proof. By Proposition 1.8, there exists $g : \mathbb{R} \to \mathbb{R}$ such that $f(\cos t, \sin t) = (\cos g(t), \sin g(t))$ with $g(t + 2\pi) = g(t) + 2\pi q$ and $\deg_2(f) = q \bmod 2$. We have

$$\begin{aligned}(\cos g(t+\pi), \sin g(t+\pi)) &= f(\cos(t+\pi), \sin(t+\pi)) \\ &= f(-\cos t, -\sin t) \\ &= -f(\cos t, \sin t) \\ &= -(\cos g(t), \sin g(t)).\end{aligned}$$

Thus $\cos g(t) = \cos g(t+\pi)$ and $\sin g(t) = -\sin g(t+\pi)$. Hence, there is an integer k such that $g(t+\pi) = g(t) + k\pi$. Thus,

$$g(t) + 2\pi q = g(t + 2\pi) = g(t+\pi) + k\pi = g(t) + 2k\pi,$$

which proves $q = k$. From $f(1, 0) = (-1, 0)$ we have $\cos g(0) = -1$, so $g(0) = (2m + 1)\pi$ for $m \in \mathbb{Z}$. Similarly, $f(-1, 0) = (1, 0)$ implies $g(\pi) = 2l\pi$ for $l \in \mathbb{Z}$. Therefore, $g(\pi) = g(0)+q\pi$ implies $2l\pi = (2m+1)\pi+q\pi$, so that $q = 2(l+m)+1$ is an odd integer, and thus $\deg_2(f) = q \bmod 2 = 1$. $\square$

Theorem 1.14 (**Borsuk-Ulam**). *Let $f : \mathbb{S}^k \to \mathbb{R}^{k+1}$ be a smooth map whose image does not contain the origin, and suppose that f satisfies the symmetry condition*

$$f(-x) = -f(x) \qquad \textit{for all } x \in \mathbb{S}^k,$$

then $W_2(f, 0) = 1$.

Proof. We prove it by induction on k. Proposition 1.13 proves the case $k = 1$. Assume the statement holds for $k-1$, and let $f : \mathbb{S}^k \to \mathbb{R}^{k+1}\setminus\{0\}$ be symmetric. Consider $\mathbb{S}^{k-1}$ as an equator of $\mathbb{S}^k$, embedded by $(x_1, \ldots x_{,k}) \mapsto (x_1, \ldots, x_k, 0)$. The idea of the proof is to compute $W_2(f, 0)$ by counting how often f intersects a line ℓ in $\mathbb{R}^{k+1}$. By choosing ℓ disjoint from the image of the equator, we can use the inductive hypothesis to show that the equator winds around ℓ an odd number of times. Once we determine the behavior of f in the equator, we can calculate its intersection with ℓ.

Denote the restriction of f to the equator $\mathbb{S}^{n-1}$ as g. After choosing a suitable line ℓ, by Sard's Theorem we may choose a unit vector $\vec{a}$ that is a regular value to the functions

$$\frac{g}{|g|} : \mathbb{S}^{n-1} \longrightarrow \mathbb{S}^n \quad \text{and} \quad \frac{f}{|f|} : \mathbb{S}^n \longrightarrow \mathbb{S}^n.$$

By symmetry, $-u$ must also be a regular value for both maps. By dimensional comparison, regularity for $g/\,|g|$ means that $g/\,|g|$ never hits $\vec{a}$ or $-\vec{a}$. Therefore, $\ell = \mathbb{R}\cdot\vec{a}$ never intersects g. The regularity of $f/\,|f|$ is equivalent to the condition $f \pitchfork \ell$ by Proposition 1.3.

We now calculate $W_2(f, 0)$ by definition:

$$W_2(f, 0) = \deg_2\left(\frac{f}{|f|}\right) = \#\left(\frac{f}{|f|}\right)^{-1}(\vec{a}) \bmod 2.$$

And $f/|f|$ hits $+\vec{a}$ the same number of times as it hits $-\vec{a}$ by symmetry. Thus

$$\#\left(\frac{f}{|f|}\right)^{-1}(\vec{a}) = \frac{1}{2}\#f^{-1}(\ell).$$

We restrict our calculation to only the upper hemisphere. Let f_+ denote the restriction of f to the upper hemisphere S^k_+, the points where $x_{k+1} \geq 0$. By symmetry, plus the fact that $f(\text{equator})$ does not intersect ℓ, we know that

$$\#f_+^{-1}(\ell) = \frac{1}{2}\#f^{-1}(\ell).$$

We conclude that $W_2(f, 0) = \#f_+^{-1}(\ell) \bmod 2$.

The reason why this latter expression for $W_2(f, 0)$ is obtained is because $\partial S^{k+1}_+ = S^k$, where we may apply the inductive hypothesis. More work is needed, though, as the map $g : \mathbb{S}^{k-1} \to \mathbb{R}^{k+1}$ does not have the correct dimensions. Let $V = \ell^\perp$ be the orthogonal complement of ℓ, and let $\pi : \mathbb{R}^{k+1} \to V$ be an orthogonal projection. Since g is symmetric and π is linear, the composite $\pi \circ g : \mathbb{S}^{k+1} \to V$ is symmetric; moreover $\pi \circ g$ is never zero, for g intersects $\pi^{-1}(0) = \ell$. Identify the $k-$ dimensional vector space V with $\mathbb{R}^k$ and invoke the inductive hypothesis: $W_2(\pi \circ g, 0) = 1$.

Now, since $f_+ \pitchfork \ell$,

$$\pi \circ f_+ : \mathbb{S}^k \to V$$

is transversal to 0 by Proposition 1.3. Now, by Theorem 1.11 we have

$$W_2(\pi \circ g, 0) = \#(\pi \circ f_+)^{-1}(0).$$

But

$$(\pi \circ f_+)^{-1}(0) = f_+^{-1}(\ell),$$

so

$$W_2(f, 0) = \#f_+^{-1}(\ell) = W_2(\pi \circ g, 0) = 1 \mod 2.$$

□

Chapter 2

Sandwich theorems

Suppose we have a piece of ham between two identical pieces of bread. A sensible way to cut all three pieces in half in a single swing would be to do so diagonally. Very roughly, one ends up with two pieces that have the same volume. However, this is not a trivial matter when the three pieces are not aligned. That is, how can one make a single knife swing to cut all three pieces in half, given that these are anywhere in our kitchen. Although this is nearly impossible to do in real-life, we can prove that it is possible to find a plane in $\mathbb{R}^3$ which bisects each of the pieces. More generally, for any three arbitrary solids in $\mathbb{R}^3$, we can find a plane which bisects them each. This is known as the ham sandwich theorem, whose formulation is attributed to Hugo Steinhaus. The first known proof is credited to Stefan Banach, which relied on a reduction to the Borsuk-Ulam theorem.

In this second chapter, we will use the Borsuk-Ulam theorem proved in chapter 1 to deduce various *generalized sandwich theorems*. As explained above, one can prove a cutting statement in $\mathbb{R}^3$. However, one can generalize this statement to certain vector spaces.

2.1 The Stone-Tukey theorem

In 1942, Arthur H. Stone and John W. Tukey published [11], where a generalization of the ham-sandwich theorem is proven. Here, we present a slightly modified version which fits our purposes.

We will denote by $Z(f) \coloneqq \{x : f(x) = 0\}$, the zero set of a function f.

Theorem 2.1 (Stone-Tukey)**.** *Suppose V is a vector space of continuous functions on $\mathbb{R}^n$. Suppose that for each non-zero element $f \in V$, the set $Z(f) \subset \mathbb{R}^n$ has measure zero.*

Let $W_1, \dots, W_N$ be L^1-function on $\mathbb{R}^n$, and suppose that $N < \operatorname{Dim} V$. Then there exists a non-zero function $v \in V$ so that for each W_j, $j = 1, 2, \dots, N$,

$$\int_{\{v<0\}} W_j = \int_{\{v>0\}} W_j.$$

Proof. We may assume that $\operatorname{Dim} V = N+1$, so we have $V \cong \mathbb{R}^{N+1}$, and therefore $\mathbb{S}^N$ can be identified with a subset $A \subset V \backslash \{0\}$. Abusing notation, we write $\mathbb{S}^N \subset V\backslash\{0\}$. We now define a function $F : V\backslash\{0\} \to \mathbb{R}^{N+1}$ component-wise by

$$F_i(p) = \int_{\{p<0\}} W_i - \int_{\{p>0\}} W_i$$

for $i = 1, \ldots, N$. We make the observation that F in fact maps antipodal points to antipodal points.

$$F_i(-p) = \int_{\{-p<0\}} W_i - \int_{\{-p>0\}} W_i = \int_{\{p>0\}} W_i - \int_{\{p<0\}} W_i = -F_i(p).$$

That is, $F(-p) = -F(p)$. We now prove that F is in fact a continuous function. Suppose $p_k \to p$ in $V\backslash\{0\}$. We want to prove $F(p_k) \to F(p)$ in $\mathbb{R}^{N+1}$. Let $A_k \subset \mathbb{R}^n$ be the set of points where the sign of p_k is not equal to the sign of p. Let $A_k^{\pm} \subset A_k$ denote the sets where $p > 0$ and $p_k < 0$, $p < 0$ and p_k, respectively.

Then we have

$$\begin{aligned}
|F_i(p_k) - F_i(p)| &= \left| \int_{\{p_k>0\}} W_i + \int_{\{p<0\}} W_i - \left(\int_{\{p_k<0\}} W_i + \int_{\{p>0\}} W_i \right) \right| \\
&\leq \int_{A_k^-} |W_i| + \int_{A_k^+} |W_i| \\
&= \int_{A_k} |W_i| .
\end{aligned}$$

Since $p_k \to p$ point-wise, we must have $\bigcap_{k_0} \bigcup_{k=k_0} A_k \subset p^{-1}(0)$. By the dominated convergence theorem,

$$\lim_{k_0 \to \infty} \int_{\cup_k \geq k_0 A_k} |W_i| \leq \int_{Z(p)} |W_j| = 0.$$

Hence, $F_i(v_k) \to F_i(v)$ as $k_0 \to \infty$, so F_i is continuous. By the Borsuk-Ulam theorem, there exists $q \in V\backslash\{0\}$ such that $F_i(q) = 0$ for each i and equivalently,

$$\int_{\{q>0\}} W_i = \int_{\{q<0\}} W_i.$$

□

We now state a result on the Lebesgue measure of the zero set of a polynomial.

Lemma 2.2. *If $P \in \mathbb{R}[x_1, \ldots, x_n]$ is not identically zero, then $|Z(P)| = 0$.*

The proof of this lemma can be found in [7]. Henceforth, we assume that V is the space of polynomials in $\mathbb{R}^n$ of degree at most D, whose dimension is

$\binom{D+n}{n}$. Indeed, for $n = 1$, this has dimension $D+1 = \binom{D+1}{1}$. Now, a polynomial in $\mathbb{R}^n$ of degree at most D can be written as an element of $(\mathbb{R}[x_1, \ldots, x_{n-1}])[x_n]$. That is, a polynomial of variable x_n with coefficients in $\mathbb{R}[x_1, \ldots, x_{n-1}]$. The degree of this vector space is then, by our induction hypothesis,

$$\sum_{i=1}^{D+1} \binom{D+n-i}{n-1} = \binom{D+n}{n}.$$

We may also point out that $\binom{D+n}{n} \sim_n D^n$.

A corollary to Theorem 2.1 comes in the form of a polynomial performing the slicing of the solids. More precisely,

Corollary 2.3. *If $W_1, \ldots, W_N$ are L^1-functions on $\mathbb{R}^n$, then there exists a non-zero polynomial P of degree at most $C_n N^{1/n}$, so that for each W_j,*

$$\int_{\{P<0\}} W_j = \int_{\{P>0\}} W_j.$$

Remark 2.4. The degree restriction comes directly from Theorem 2.1. That is, $N < \dim V$, and $\dim V \sim_n D^n$, so $N^{1/n} < D$.

2.2 Polynomial partitioning theorem

We will now use these results to conclude that one may use a polynomial to partition a single object into $\sim D^n$ pieces.

Theorem 2.5. *Suppose that $W \geq 0$ is a non-zero L^1-function on $\mathbb{R}^n$. Then for each D there is a non-zero polynomial P of degree at most D so that $\mathbb{R}^n \backslash Z(P)$ is a union of $\sim D^n$ disjoint cells O_i, and the integrals $\int_{O_i} W$ are all equal.*

Proof. By Corollary 2.3, there exists a non-zero polynomial P_1 of degree at most C_n, so that

$$\int_{\{P_1>0\}} W = \int_{\{P_1<0\}} W = 2^{-1} \int W.$$

That is, we now have two regions. Defining $W_+ := \chi_{\{P_1>0\}} W$ and $W_- := \chi_{\{P_1<0\}} W$ now gives us two L^1 functions. Using Corollary 2.3 again gives a non-zero polynomial P_2 which bisects the regions given by W_+ and W_-. That is, for $j = +, -$,

$$\int_{\{P_2>0\}} W_j = \int_{\{P_2<0\}} W_j = 2^{-2} \int W.$$

If we now define $W_{+,+} = W_+ \chi_{\{P_2>0\}}$, $W_{+,-} = W_+ \chi_{\{P_2<0\}}$, $W_{-,+} = W_- \chi_{\{P_2>0\}}$ and $W_{-,-} = W_- \chi_{\{P_2<0\}}$, we may once again use Corollary 2.3 to find a polynomial P_3 such that

$$\int_{\{P_3>0\}} W_{j_1,j_2} = \int_{\{P_3<0\}} W_{j_1,j_2} = 2^{-3} \int W$$

for $j_1, j_2 = +, -$. Inductively, we may find non-zero polynomial $P_1, \ldots, P_s$ such that

$$\int_{\{P_k>0\}} W_{j_1,\ldots,j_{k-1}} = \int_{\{P_k<0\}} W_{j_1,\ldots,j_{k-1}} = 2^{-s}\int W,$$

for $j_1, \ldots, j_{k-1} = +, -$, and $W_{j_1,\ldots,j_{k-1}} = \chi_{\{j_{k-1}P_{k-1}>0\}} W_{j_1,\ldots,j_{k-2}}$, for $k = 1, \ldots, s$. Define $P = \prod_{k=1}^{s} P_k$, then the sign of P depends on all of its factors. The region in which it is positive is the union of the regions in which we have an even number of $-$ signs. That is,

$$\{P > 0\} = \bigcup_{j_\ell} \bigcap_{\ell=1}^{s-1} \{j_\ell P_\ell > 0\}$$

such that $\prod j_\ell = +$. And similarly for $\{P < 0\}$. This polynomial then splits $\mathbb{R}^n \backslash Z(P)$ into 2^s cells O_i and

$$\int_{O_i} W = 2^{-s} \int W.$$

The restriction of Corollary 2.3 on the degree gives $\deg P_k \lesssim_n 2^{(k-1)/n}$, and thus $\deg P \leq C_n 2^{s/n}$. Choosing s such that $C_n 2^{s/n} \in [D/2, D]$ (the lower bound is to guarantee we do not lose a full power of 2, and thus $\deg P \leq D$). Then we have that the number of cells O_i is $2^s \sim_n D^n$. $\square$

Remark 2.6. Although the construction of polynomials is simple through the use of Stone-Tukey, which is proved using the Borsuk-Ulam theorem, we lack a constructive/algorithmic argument which would ease the explicit definition of such a polynomial.

2.3 Non-singular polynomial partitioning

A non-singular polynomial on $\mathbb{R}^n$ is a polynomial P with 0 as a regular value. That is, for all $x \in Z(P)$ we have $\nabla P(x) \neq 0$. These polynomials are in fact dense in $\mathbb{P}_D(\mathbb{R}^n)$, the space of polynomials in $\mathbb{R}^n$ of degree at most D.

Lemma 2.7. *Non-singular polynomials are dense in $\mathbb{P}_D(\mathbb{R}^n)$ for any n, D. Moreover, the singular polynomials have measure zero.*

That is, we may approximate singular polynomials with non-singular polynomials. We can use this to prove a slightly imperfect bisection on the solids.

Corollary 2.8. *Suppose $W_1, \ldots, W_N \geq 0$ are non-zero functions in $L^1(\mathbb{R}^n)$. Then for any $\delta > 0$, there is a non-singular polynomial P so that for each W_j,*

$$(1-\delta)\int_{\{P<0\}} W_j \leq \int_{\{P>0\}} W_j \leq (1+\delta)\int_{\{P>0\}} W_j.$$

Proof. Let P_0 be a non-zero polynomial obtained via Corollary 2.3, that is, with $\int_{\{P_0>0\}} W_i \int_{\{P_0<0\}} W_i$. By Theorem 2.5, there is a sequence of non-singular polynomials $P_k \to P_0$. Using the same continuity argument as in the proof of Theorem 2.1, we have $\lim_{k\to\infty} \int_{P_k>0} W_j = \int_{P_0>0} W_j$, so for large k, P_k satisfies the desired inequality. $\square$

Finally, we can use Corollary 2.8 instead of Corollary 2.3 in the proof of Theorem 2.5 to obtain the following.

Corollary 2.9. *Let W be a non-negative L^1 function on $\mathbb{R}^n$. Then for any D, there is a non-zero polynomial P of degree at most D so that $\mathbb{R}^n \backslash Z(P)$ is a disjoint union of $\sim D^n$ cells O_i, and the integrals $\int_{O_i} W$ agree up to a factor of 2. Moreover, the polynomial P is a product of non-singular polynomials.*

Part II

Function spaces

Chapter 3

Lorentz spaces

Throughout the this chapter, we will often use $L^p = L^p(X, \mu)$ for simplicity. We will use the explicit notation when distinction is necessary. We begin by recalling some basic definitions and results of measure theory.

3.1 The distribution function

Definition 3.1. For a measurable function f on (X, μ), the *distribution function* of f is the function d_f defined on $[0, \infty)$ as follows:

$$d_f(\alpha) = \mu(\{x \in X : |f(x)| > \alpha\}). \tag{3.1}$$

Remark 3.2. We may also write

$$d_f(\alpha) = \int_X \chi_{\{x:|f(x)|>\alpha\}} d\mu(x).$$

Proposition 3.3. *For $f \in L^p(X, \mu)$, $0 < p < \infty$, we have*

$$\|f\|_{L^p}^p = p \int_0^\infty \alpha^{p-1} d_f(\alpha) d\alpha. \tag{3.2}$$

Proof. We start from the right side of the equation to prove.

$$\begin{aligned} p\int_0^\infty \alpha^{p-1} d_f(\alpha) d\alpha &= p\int_0^\infty \alpha^{p-1} \int_X \chi_{\{x:|f(x)|>\alpha\}} d\mu(x)\, d\alpha \\ &= \int_X \int_0^{|f(x)|} p\alpha_{p-1} d\alpha\, d\mu(x) \\ &= \int_X |f(x)|^p\, d\mu(x) = \|f\|_{L^p}^p . \end{aligned}$$

□

3.2 Decreasing rearrangements

Definition 3.4. Let f be a complex-valued function defined on X. The *decreasing rearrangement* of f is the function f^* defined on $[0, \infty)$ by

$$f^*(t) = \inf\{s > 0 : d_f(s) \leq t\}. \tag{3.3}$$

We define $\inf \varnothing = \infty$, so that $f^*(t) = \infty$ whenever $d_f(\alpha) > t$ for all $\alpha \geq 0$. We give an example to make these definitions more reasonable.

Example 3.5. Suppose we are given the function $f(x) = \sum_{i=1}^{N} a_i \chi_{A_i}(x)$, where the sets $E_j \subset X$ are pairwise disjoint and $a_1 > a_2 > \cdots > a_N > 0$. If $\alpha \geq a_1$, then $d_f(\alpha) = 0$. For $a_{i+1} \leq \alpha < a_i$, then we have $|f(x)| > \alpha$ whenever $x \in E_1 \cup \cdots \cup E_i$. Hence, setting $B_j = \sum_{i=1}^{j} \mu(A_i)$, we have that

$$d_f(\alpha) = \sum_{i=1}^{N} B_i \chi_{[a_{i+1}, a_i)}(\alpha),$$

for $a_{N+1} = B_0 = 0$. Now note that for $0 = B_0 \leq t < B_1$, the smallest s for which $d_f(s) \leq t$ is a_1. For $B_j \leq t < B_{j+1}$, the smallest s would then be a_{j+1}. For the particular case of $N = 3$, see Figure 3.1-Figure 3.3

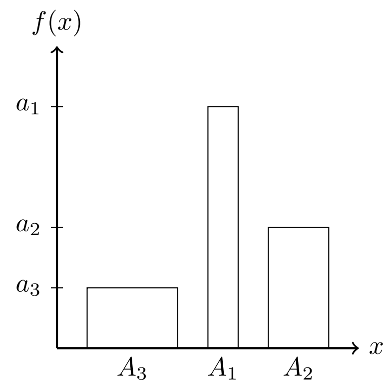

Figure 3.1: The graph of $f = \sum_{i=1}^{3} a_i \chi_{A_i}$.

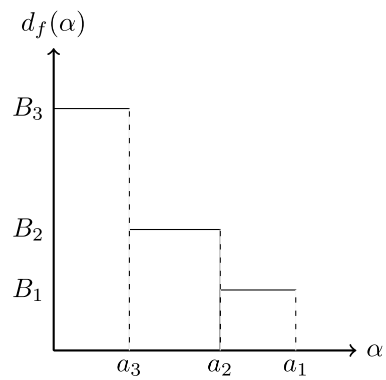

Figure 3.2: The graph of d_f, where $B_j = \sum_{i=1}^{j} \mu(A_i)$.

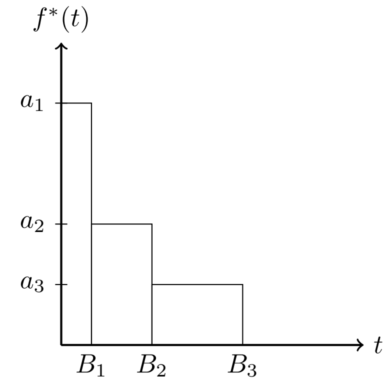

Figure 3.3: The graph of $f^*(t)$.

Proposition 3.6. *For f, g $\mu-$measurable, $k \in \mathbb{C}$, and $0 \leq t, s, t_1, t_2 < \infty$ we have*

(1) $f^*(d_f(\alpha)) \leq \alpha$ *whenever* $\alpha > 0$.

(2) $d_f(f^*(t)) \leq t$.

(3) $f^*(t) > s$ *if and only if* $t < d_f(s)$; *that is,* $\{t > 0 : f^*(t) > s\} = [0, d_f(s))$.

(4) $|g| \leq |f|$ μ*-a.e. implies that* $g^* \leq f^*$ *and* $|f|^* = f^*$.

(5) $(kf)^* = |k| f^*$.

(6) $(f+g)^*(t_1 + t_2) \leq f^*(t_1) + g^*(t_2)$.

(7) f^* *is right continuous on* $[0, \infty)$.

(8) $d_f = d_{f^*}$.

(9) $(|f|^p)^* = (f^*)^p$ *when* $0 < p < \infty$.

(10) $\displaystyle\int_X |f|^p \, d\mu = \int_0^\infty f^*(t)^p dt$ *when* $0 < p < \infty$.

3.3 Lorentz spaces

Definition 3.7. Given a measurable function f on (X, μ) and $0 < p, q \leq \infty$, define

$$\|f\|_{L^{p,q}} = \begin{cases} \left(\displaystyle\int_0^\infty \left(t^{1/p} f^*(t) \right)^q \frac{dt}{t} \right)^{1/q} & \text{if } q < \infty, \\ \sup\limits_{t>0} t^{1/p} f^*(t) & \text{if } q = \infty. \end{cases}$$

The set set of all f such that $\|f\|_{L^{p,q}} < \infty$ is denoted by $L^{p,q}(X, \mu)$ and is called the *Lorentz space*.

Lorentz spaces were named after G. G. Lorentz. One may also note that this is $L^q(\mathbb{R}_{>0}, d\nu)$ norm of $t^{1/p}f^*(t)$, where $d\nu(t) = dt/t$ is the Haar measure associated to the group $\mathbb{R}_{>0}$ under multiplication. Thus, $\nu(rS) = \nu(S)$ for any $r \in \mathbb{R}_{>0}$. From this definition, we see that $L^{\infty,\infty} = L^\infty$. We may also note that by Proposition 3.3 and some properties from Proposition 3.6, one has

$$\|f\|_{L^p}^p = p\int_0^\infty \alpha^{p-1} d_f(\alpha)d\alpha = p\int_0^\infty \alpha^{p-1} d_{f^*}(\alpha)d\alpha = \|f^*\|_{L^p}^p ,$$

which then proves that $L^{p,p} = L^p$. It is also easy to see that

$$\left\||f|^r\right\|_{L^{p,q}} = \|f\|_{L^{pr,qr}}^r \tag{3.4}$$

for $0 < p, r < \infty$ and $0 < q \leq \infty$.

Example 3.8. Assume we have the same notation and function as in Example 1.5. Let $0 < q < \infty$, then

$$\|f\|_{L^{p,q}} = \begin{cases} \left(\dfrac{p}{q}\right)^{1/q} \left[\displaystyle\sum_{i=1}^N a_i^q \left(B_i^{q/p} - B_{i-1}^{q/p}\right)\right]^{1/q} & \text{if } q < \infty, \\ \displaystyle\sup_{1\leq j\leq N} a_j B_j^{1/p} & \text{if } q = \infty. \end{cases}$$

Remark 3.9. Example 3.8 then says that the only simple function with finite $L^{\infty,q}$ is 0 μ−a.e. Hence, $L^{\infty,q} = \{0\}$ for all $0 < q < \infty$.

Proposition 3.10. *Suppose $0 < p \leq \infty$ and $0 < q < r \leq \infty$. Then there exist some constant $c_{p,q,r}$ such that*

$$\|f\|_{L^{p,r}} \leq c_{p,q,r} \|f\|_{L^{p,q}} . \tag{3.5}$$

That is, $L^{p,q} \subset L^{p,r}$.

Proof. As seen in Remark 3.9, the case $p = \infty$ is trivial. Assume $p < \infty$. We will calculate $\|f\|_{p,\infty}$.

$$\begin{aligned} t^{1/p}f^*(t) &= \left(\frac{q}{p}\int_0^t s^{q/p}\frac{ds}{s}\right)^{1/q} f^*(t) \\ &= \left(\frac{q}{p}\int_0^t [f^*(t)s^{1/p}]^q \frac{ds}{s}\right)^{1/q} \\ &\leq \left(\frac{q}{p}\int_0^t [f^*(s)s^{1/p}]^q \frac{ds}{s}\right)^{1/q} \\ &\leq \left(\frac{q}{p}\int_0^\infty [f^*(s)s^{1/p}]^q \frac{ds}{s}\right)^{1/q} = \left(\frac{q}{p}\right)^{1/q} \|f\|_{L^{p,q}} . \end{aligned}$$

Taking the sup over all $t > 0$, we obtain

$$\|f\|_{p,\infty} \leq \left(\frac{q}{p}\right)^{1/q} \|f\|_{L^{p,q}} . \tag{3.6}$$

Now, for $r < \infty$, we have

$$\begin{aligned} \|f\|_{L^{p,r}} &= \left\{ \int_0^\infty [t^{1/p} f^*(t)]^{r-q+q} \frac{dt}{t} \right\}^{1/r} \\ &= \left\{ \int_0^\infty [t^{1/p} f^*(t)]^{r-q} [t^{1/p} f^*(t)]^{q} \frac{dt}{t} \right\}^{1/r} \\ &\leq \|f\|_{L^{p,\infty}}^{(r-q)/r} \|f\|_{L^{p,q}}^{q/r} . \end{aligned} \tag{3.7}$$

Another way of proving this inequality is by using Hölder's inequality and (3.4):

$$\begin{aligned} \|f\|_{L^{p,r}}^r = \||f|^r\|_{L^{\frac{p}{r},1}} &\leq \left\||f|^{r-q}\right\|_{L^{\frac{p}{r-q},\infty}} \||f|^q\|_{L^{\frac{p}{q},1}} \\ &= \|f\|_{L^{p,\infty}} \|f\|_{L^{p,q}} . \end{aligned}$$

Now, using (3.5), and substituting into (3.6) we then see that

$$\|f\|_{L^{p,r}} \leq \|f\|_{p,\infty}^{(r-q)/r} \|f\|_{p,q}^{q/r} \leq \left(\frac{q}{p}\right)^{\frac{r-q}{qr}} \|f\|_{L^{p,q}}^{\frac{r-q}{r}} \|f\|_{L^{p,q}}^{q/r} = \left(\frac{q}{p}\right)^{\frac{r-q}{qr}} \|f\|_{p,q} .$$

□

The notation for the functional $\|\cdot\|_{L^{p,q}}$ suggests that this is a norm. However, we can prove it is *not* a norm, as it fails to satisfy the triangular inequality (Minkowski's inequality). That is, taking $f(x) = x$ and $g(x) = 1 - x$ for $x \in [0,1]$, we have $f^*(t) = g^*(t) = (1-t)\chi_{[0,1]}(t)$. Then,

$$\begin{aligned} \|f\|_{L^{p,q}} + \|g\|_{L^{p,q}} &= \left(\int_0^\infty \left(t^{1/p} f^*(t)\right)^q \frac{dt}{t} \right)^{1/q} + \left(\int_0^\infty \left(t^{1/p} g^*(t)\right)^q \frac{dt}{t} \right)^{1/q} \\ &= 2 \left(\int_0^1 t^{q/p} (1-t)^q \frac{dt}{t} \right)^{1/q} \\ &= 2 \left(\int_0^1 t^{q/p-1} \left[1-t\right]^q dt \right)^{1/q} \\ &= 2 \left(\frac{\Gamma(q/p)\Gamma(q+1)}{\Gamma(q+1+q/p)} \right)^{1/q} . \end{aligned}$$

On the other hand,

$$
\begin{aligned}
\|f+g\|_{L^{p,q}}^q &= \int_0^\infty \left(t^{1/p}(f+g)^*(t)\right)^q \frac{dt}{t} \\
&= \int_0^\infty \left(t^{1/p}(1)^*(t)\right)^q \frac{dt}{t} \\
&= \int_0^1 t^{q/p} \frac{dt}{t} \\
&= \frac{p}{q}.
\end{aligned}
$$

Thus, if we were to assume that the triangle inequality holds, we would have

$$
\frac{p}{q} \leq 2^q \frac{\Gamma(q/p)\Gamma(q+1)}{\Gamma(q+1+q/p)}.
$$

for all $0 < p, q < \infty$. However, this does not hold in general. For example, when $p = q = 1/2$, we have

$$
1 > \frac{2\sqrt{2}}{3} = 2^{1/2} \frac{\Gamma(1/2+1)}{\Gamma(1/2+1+1)}.
$$

Fortunately, it is a quasi-norm, as $\|f\|_{L^{p,q}} = 0$ if and only if $f = 0$ $\mu-$a.e., and the estimate

$$
\|f+g\|_{L^{p,q}} \leq c_{p,q}(\|f\|_{L^{p,q}} + \|g\|_{L^{p,q}})
$$

holds for $c_{p,q} = 2^{1/p}\max(1, 2^{(1-q)/q})$. Moreover, it can be proven that this is a Banach space for $1 < p, q < \infty$ (see [3]).

Chapter 4

Interpolation theorems

4.1 Marcinkiewicz interpolation theorem

As seen in chapter 3, L^p and $L^{p,q}$ spaces have many similarities. In L^p spaces, the classical Marcinkiewicz interpolation theorem was first proved by Józef Marcinkiewicz in 1939, which bounds the norms of non-linear operators acting on L^p spaces. Informally, it says that if T is a bounded quasi-linear operator from L^p to $L^{p,\infty}$ and at the same time from L^q to $L^{q,\infty}$, then T is also a bounded operator from L^r to L^r for any $r \in (p,q)$. In the setting of Lorentz spaces, one can prove a similar result.

4.2 The off-diagonal Marcinkiewicz interpolation theorem

Definition 4.1. An operator T is of weak type (r,p) if

$$\|Tf\|_{L^{p,\infty}} \leq k \|f\|_{L^r}.$$

for all f belonging to the domain of T.

Remark 4.2. Since $\|f\|_{L^{r,r}} = \|f\|_{L^r}$, we have that this operator satisfies the inequality

$$\|Tf\|_{L^{p,\infty}} \leq k \|f\|_{L^{r,1}}.$$

for all f belonging to the domain of T. If this estimate is satisfied for $f = \chi_A$ for any set A of finite measure, we say that T is of restricted weak type.

We say that an operator T is quasi-linear if it satisfies

$$|T(\lambda f)| = |\lambda| \, |T(f)|, \qquad \text{and} \qquad |T(f+g)| \leq K(|T(f)| + |T(g)|)$$

for some $K > 0$, $\lambda \in \mathbb{C}$, and all f, g in the domain of T. We will assume $K \geq 1$ to avoid trivial cases. Now we state a classical estimate known as *Hardy's inequality*.

Lemma 4.3 (Hardy's inequality)**.** *If $q \geq 1$, $r > 0$ and g is a nonnegative function defined on $(0, \infty)$ then*

$$\text{(i)} \quad \left(\int_0^\infty \left[\int_0^t g(u) du \right]^q t^{-r-1} dt \right)^{1/q} \leq \frac{q}{r} \left(\int_0^\infty [ug(u)]^q u^{-r-1} du \right)^{1/q},$$

$$\text{(ii)} \quad \left(\int_0^\infty \left[\int_t^\infty g(u) du \right]^q t^{r-1} dt \right)^{1/q} \leq \frac{q}{r} \left(\int_0^\infty [ug(u)]^q u^{r-1} du \right)^{1/q}.$$

A proof can be found in [10]. We now state and prove the central theorem to this chapter. Throughout the proof, we will assume that T is an operator defined on a linear space D of measurable functions on (X, μ) having values that are measurable functions defined on (Y, ν).

Theorem 4.4. *Suppose T is a quasi-linear operator of restricted weak types (r_0, p_0) and (r_1, p_1), with $r_0 < r_1$ and $p_0 \neq p_1$, then there exists a constant $B = B_\theta$ such that*

$$\|Tf\|_{L^{p,q}} \leq B \, \|f\|_{L^{r,q}}$$

for all f belonging to the domain of T and to $L^{r,q}$, where $1 \leq q \leq \infty$,

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \qquad \frac{1}{r} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1} \qquad \text{and} \qquad 0 < \theta < 1.$$

Proof. Fix $f \in D \cap L^{r,q}$ and split $f(x) = f_t(x) + f^t(x)$ for

$$f^t(x) = \begin{cases} f(x) & \text{if } |f(x)| > f^*(t^\gamma) \\ 0 & \text{if } |f(x)| \leq f^*(t^\gamma) \end{cases}$$

and

$$f_t(x) = \begin{cases} 0 & \text{if } |f(x)| > f^*(t^\gamma) \\ f(x) & \text{if } |f(x)| \leq f^*(t^\gamma), \end{cases}$$

where

$$\gamma = \frac{(1/p_0) - (1/p)}{(1/r_0) - (1/r)} = \frac{(1/p) - (1/p_1)}{(1/r) - (1/r_1)}.$$

Note that

$$d_{f^t}(\alpha) = \begin{cases} d_f(\alpha) & \text{when } \alpha > f^*(t^\gamma) \\ d_f(f^*(t^\gamma)) & \text{when } \alpha \leq f^*(t^\gamma) \end{cases}$$

$$d_{f_t}(\alpha) = \begin{cases} 0 & \text{when } \alpha \geq f^*(t^\gamma) \\ d_f(\alpha) - d_f(f^*(t^\gamma)) & \text{when } \alpha < f^*(t^\gamma) \end{cases}$$

To find the non-increasing rearrangement, we make the following calculations:

$$\begin{aligned} s \geq t^\gamma \implies (f^t)^*(s) &= \inf\{\alpha > 0 : d_{f^t}(\alpha) \leq s\} \\ &\leq \inf\{s \in (0, f^*(t^\gamma)] : d_{f^t}(\alpha) \leq s\} \\ &= \inf\{s \in (0, f^*(t^\gamma)] : d_f(f^*(t^\gamma)) \leq s\} \\ &= \inf(0, f^*(t^\gamma)] \\ &= 0, \end{aligned}$$

$$
\begin{aligned}
s < t^\gamma \implies (f^t)^*(s) &\le \inf\{\alpha > f^*(t^\gamma) : d_{f^t}(\alpha) \le s\} \\
&= \inf\{\alpha > f^*(t^\gamma) : d_f(f^*(t^\gamma)) \le s\} \\
&= \inf\{\{\alpha > 0 : d_f(\alpha) \le s\} \cap (f^*(t^\gamma), \infty)\} \\
&= f^*(s),
\end{aligned}
$$

$$
\begin{aligned}
s \ge t^\gamma \implies (f_t)^*(s) &= \inf\{\alpha > 0 : d_{f_t}(\alpha) \le s\} \\
&\le \inf\{\alpha > 0 : d_f(\alpha) \le s\} \\
&= f^*(s),
\end{aligned}
$$

where the second equality follows from $d_{f_t} \le d_f$.

$$
\begin{aligned}
s < t^\gamma \implies (f_t)^*(s) &= \inf\{\alpha > 0 : d_{f_t}(\alpha) \le s\} \\
&\le f^*(t^\gamma).
\end{aligned}
$$

where this last ineqality follows from $d_{f_t}(f^*(t^\gamma)) = 0$. We may also note that $f_t^*(s) \le \min\{f^*(t^\gamma), f^*(s)\}$.

Thus we have

$$
(f^t)^*(s) \le \begin{cases} f^*(s) & \text{if } 0 < s < t^\gamma \\ 0 & \text{if } t^\gamma \le s \end{cases} \tag{4.1}
$$

and

$$
f_t^*(s) \le \begin{cases} f^*(t^\gamma) & \text{if } 0 < s < t^\gamma \\ f^*(s) & \text{if } t^\gamma \le s \end{cases} \tag{4.2}
$$

Since T is quasi-linear, it follows that

$$
|Tf(y)| = \left|T(f^t + f_t)(y)\right| \le \left|Tf^t(y)\right| + |Tf_t(y)|
$$

for almost every $y \in Y$. Thus, if $s > 0$,

$$
\begin{aligned}
&\{y \in Y : |Tf(y)| > (Tf^t)^*(s) + (Tf_t)^*(s)\} \\
&\subset \{y \in Y : \left|Tf^t(y)\right| > (Tf^t)^*(s)\} \cup \{y \in Y : |Tf_t(y)| > (Tf_t)^*(s)\}.
\end{aligned}
$$

Let d_{Tf}, d_{Tf^t} and d_{Tf_t} denote the distribution functions of Tf, Tf^t and Tf_t, respectively. Then,

$$
d_{Tf}((Tf)^*(s) + (Tf_t)^*(s)) \le d_{Tf^t}((Tf^t)^*(s)) + d_{Tf_t}((Tf_t)^*(s)) \le 2s.
$$

By definition of $(Tf)^*$, we thus have

$$
(Tf)^*(2s) \le (Tf^t)^*(s) + (Tf_t)^*(s) \quad \text{for all } s > 0. \tag{4.3}
$$

Now suppose $r_1 < \infty$ and $q < \infty$. Setting $s = t$ in (4.3), we may now see that

$$\begin{aligned}
\|Tf\|_{L^{p,q}} &= \left(\int_0^\infty (t^{1/p}(Tf)^*(t))^q \frac{dt}{t}\right)^{1/q} \\
&= 2^{1/p}\left(\int_0^\infty (t^{1/p}(Tf)^*(2t))^q \frac{dt}{t}\right)^{1/q} \\
&\leq 2^{1/p}\left(\int_0^\infty (t^{1/p}[(Tf^t)^*(t) + (Tf_t)^*(t)])^q \frac{dt}{t}\right)^{1/q} \\
&\leq 2^{1/p}\left\{\left(\int_0^\infty (t^{1/p}(Tf^t)^*(t))^q \frac{dt}{t}\right)^{1/q}\right. \\
&\qquad \left. + \left(\int_0^\infty (t^{1/p}(Tf_t)^*(t))^q \frac{dt}{t}\right)^{1/q}\right\},
\end{aligned}$$

where the last inequality follows from Minkowski's inequality. Since T is of restricted weak types (r_0, p_0) and (r_1, p_1), we have

$$t^{1/p_0}(Tf^t)^*(t) \leq k_0 \left\|f^t\right\|_{L^{r_0,1}} \quad \text{and} \quad t^{1/p_1}(Tf_t)^*(t) \leq k_1 \left\|f_t\right\|_{L^{r_1,1}}.$$

Thus, the sum of integrals in between the brackets is at most

$$\begin{aligned}
&k_0\left(\int_0^\infty \left(t^{1/p-1/p_0}\left\|f^t\right\|_{L^{r_0,1}}\right)^q \frac{dt}{t}\right)^{1/q} \\
&\qquad + k_1\left(\int_0^\infty \left(t^{1/p-1/p_1}\left\|f_t\right\|_{L^{r_1,1}}\right)^q \frac{dt}{t}\right)^{1/q}.
\end{aligned}$$

Now, by Hardy's inequality and Equation 4.1,

$$\begin{aligned}
&\left(\int_0^\infty \left[t^{1/p-1/p_0}\left\|f^t\right\|_{L^{r_0,1}}\right]^q \frac{dt}{t}\right)^{1/q} \\
&\quad = \left(\int_0^\infty \left[t^{1/p-1/p_0}\left(\int_0^\infty s^{1/r_0-1}f^t(s)ds\right)\right]^q \frac{dt}{t}\right)^{1/q} \\
&\quad \leq \left(\int_0^\infty \left[t^{1/p-1/p_0}\left(\int_0^{t^\gamma} s^{1/r_0-1}f^*(s)ds\right)\right]^q \frac{dt}{t}\right)^{1/q} \\
&\quad = |\gamma|^{-1/q}\left(\int_0^\infty \left[u^{1/r-1/r_0}\left(\int_0^u s^{1/r_0-1}f^*(s)ds\right)\right]^q \frac{du}{u}\right)^{1/q} \\
&\quad \leq \frac{|\gamma|^{-1/q}}{1/r-1/r_0}\left(\int_0^\infty [u^{1/r_0}f^*(u)]^q u^{q/r-q/r_0}\frac{du}{u}\right)^{1/q} \\
&\quad = \frac{|\gamma|^{-1/q}}{1/r-1/r_0}\left\|f\right\|_{L^{r,q}}.
\end{aligned}$$

In a similar fashion,

$$
\begin{aligned}
&\left(\int_0^\infty \left[t^{1/p-1/p_1}\|f_t\|_{L^{r_1,1}}\right]^q \frac{dt}{t}\right)^{1/q} \\
&\quad = \left(\int_0^\infty \left[t^{1/p-1/p_1}\left(\int_0^\infty s^{1/r_1-1} f_t(s)ds\right)\right]^q \frac{dt}{t}\right)^{1/q} \\
&\quad \le \left(\int_0^\infty \left[t^{1/p-1/p_1}\left(\int_0^{t^\gamma} s^{1/r_1-1} f^*(t^\gamma)ds + \int_{t^\gamma}^\infty s^{1/r_1-1} f^*(s)ds\right)\right]^q \frac{dt}{t}\right)^{1/q} \\
&\quad \le \left(\int_0^\infty \left[t^{1/p-1/p_1} f^*(t^\gamma)t^{\gamma/r_1}\right]^q \frac{dt}{t}\right)^{1/q} \\
&\qquad + \left(\int_0^\infty \left[t^{1/p-1/p_1}\int_{t^\gamma}^\infty s^{1/r_1-1} f^*(s)ds\right]^q \frac{dt}{t}\right)^{1/q} \\
&\quad = |\gamma|^{-1/q}\left\{\left(\int_0^\infty \left[u^{1/r}f^*(u)\right]^q \frac{du}{u}\right)^{1/q}\right. \\
&\qquad \left. + \left(\int_0^\infty \left[u^{1/r-1/r_1}\int_u^\infty s^{1/r_1-1} f^*(s)ds\right]^q \frac{du}{u}\right)^{1/q}\right\} \\
&\quad \le |\gamma|^{-1/q}\left\{\|f\|_{L^{r,q}} + \frac{1}{1/r-1/r_0}\left(\int_0^\infty u^{q/r-q/r_1}\left[f^*(u)u^{1/r^1}\right]^q \frac{du}{u}\right)^{1/q}\right\} \\
&\quad \le \left[|\gamma|^{-1/q} + \frac{|\gamma|^{-1/q}}{1/r-1/r_1}\right]\|f\|_{L^{r,q}}.
\end{aligned}
$$

Bringing everything together, we have proven that

$$\|Tf\|_{L^{p,q}} \lesssim \|f\|_{L^{r,q}}.$$

Now, assume $r_1 < \infty$ and $q = \infty$. For $t > 0$, we estimate $t^{1/p}(Tf)^*(t)$ using (4.1) and (4.2).

$$
\begin{aligned}
t^{1/p}(Tf)^*(t) &\le c_1 t^{1/p-1/p_0}\int_0^{t^\gamma} f^*(s)s^{1/r_0-1}ds+ \\
&\quad + c_2 t^{1/p-1/p_1}\int_0^{t^\gamma} f^*(t^\gamma)s^{1/r_1-1}ds + c_3 t^{1/p-1/p_1}\int_{t^\gamma}^\infty f^*(t^\gamma)s^{1/r_1-1}ds.
\end{aligned}
$$

Using that $f^*s^{1/r} \le \|f\|_{L^{r,\infty}}$, we have that

$$t^{1/p}(Tf)^*(t) \le \left\{\frac{c_1}{1/r_0-1/r} + c_2 r_1 + \frac{c_2}{1/r-1/r_1}\right\}\|f\|_{L^{r,\infty}},$$

so we have $\|Tf\|_{L^{p,\infty}} \lesssim \|f\|_{L^{r,\infty}}$. The case $r_1 = \infty$ and $q = \infty$ follows from the estimate $\|f_t\|_{L^{\infty,\infty}} \le f^*(t^\gamma)$ and some modifications on the above estimates. $\square$

Corollary 4.5. *For T as in the statement of Theorem 4.4, with $0 < p_0 \neq p_1 \leq \infty$, and $0 < r_0 \neq r_1 \leq \infty$. If T is such that $T(L^{r_0}) \subset L^{p_0,\infty}$ and $T(L^{r_1}) \subset L^{p_0,\infty}$, and for some $0 < \theta < 1$ we have*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \qquad \frac{1}{r} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1}, \quad \textit{and} \quad r \leq p$$

then $\|Tf\|_{L^r} \leq C \|f\|_{L^p}$ for all f in the domain of T.

Proof. Taking $q = p$ in Theorem 4.4 gives

$$\|Tf\|_{L^p} = \|Tf\|_{L^{p,p}} \leq B \|f\|_{L^{r,p}} \leq B \|f\|_{L^{r,r}} = B \|f\|_{L^r}.$$

□

This corollary in particular will be used in the next section. It is a powerful tool for reducing strong L^p estimates to Lorentz space estimates.

4.3 An "equivalent" norm

As we mentioned in the previous sections, the functional $\|\cdot\|_{L^{p,q}}$ is not a norm in general. However, making a slight modification on the definition of this functional will result in a norm.

Given $0 < p, q < \infty$, fix $r = r(p,q) > 0$ such that $r \leq 1$, $r \leq q$ and $r < p$. For $t \leq \mu(X)$ define

$$f^{**}(t) = \sup_{\mu(E) \geq t} \left(\frac{1}{\mu(E)} \int_E |f|^r \, d\mu \right)^{1/r},$$

while for $t > \mu(X)$ (if $\mu(X) < \infty$) let

$$f^{**}(t) = \left(\frac{1}{t} \int_X |f|^r \, d\mu \right)^{1/r}.$$

Also define

$$|||f|||_{L^{p,q}} = \left(\int_0^\infty (t^{1/p} f^{**}(t))^q \frac{dt}{t} \right)^{1/q}.$$

It can be shown that this is a norm for $r = 1$ by noting that

$$[(f+g)^{**}(t)]^r \leq [f^{**}(t)]^r + [g^{**}(t)]^r$$

for all $t \geq 0$. It is clear that since f^* is non-increasing, we must have $f^{**}(t) \geq f^*(t)$. It then follows that

$$\|f\|_{L^{p,q}} \leq |||f|||_{L^{p,q}}.$$

We may also show that for $1 < p < \infty$ and $f \in L^{p,q}$,

$$\left\lvert\!\left\lvert\!\left\lvert f \right\rvert\!\right\rvert\!\right\rvert_{L^{p,q}} \leq \frac{p}{p-1} \left\lVert f \right\rVert_{L^{p,q}} .$$

For $1 < p \leq \infty$ and $1 \leq q < \infty$, we note that

$$\begin{aligned} \left\lvert\!\left\lvert\!\left\lvert f \right\rvert\!\right\rvert\!\right\rvert_{L^{p,q}} &= \left(\int_0^\infty t^{-q(q-q/p)-1} \left[\int_0^t f^*(u)\, du \right]^q dt \right)^{1/q} \\ &\leq \frac{p}{p-1} \left(\int_0^\infty u^{-q(1-1/p)-1} [u f^*(u)]^q du \right)^{1/q} = \frac{p}{p-1} \left\lVert f \right\rVert_{L^{p,q}} . \end{aligned}$$

For the cases $1 < p < \infty$ and $q = \infty$, we have

$$\begin{aligned} t^{1/p} f^{**}(t) &= t^{1/p-1} \int_0^t f^*(u)\, du = t^{1/p-1} \int_0^t u^{-1/p} u^{1/p} f^*(u)\, du \\ &\leq \left\lVert f \right\rVert_{L^{p,q}} t^{1/p-1} \int_0^t u^{-1/p}\, du = \frac{p}{p-1} \left\lVert f \right\rVert_{L^{p,\infty}} . \end{aligned}$$

Hence, $L^{p,q}$ is normable for $1 < p < \infty$ and $1 \leq q \leq \infty$.

Part III

Restriction Conjecture

Chapter 5

Tao's ε-removal theorem

This result was first stated and proven by Terence Tao in [12]. Briefly, in the language of extensions, says that if

$$\|E_{\mathbb{S}^{n-1}} f\|_{L^p(B_R)} \lesssim R^{\varepsilon} \|f\|_{L^p(\mathbb{S}^{n-1})}, \tag{5.1}$$

for some $2 < p < \infty$ and all $0 < \varepsilon \ll 1$, we have

$$\|E_{\mathbb{S}^{n-1}} f\|_{L^q(\mathbb{R}^n)} \lesssim \|f\|_{L^q(\mathbb{S}^{n-1})}$$

for all $p < q < \infty$. That is, if we are able to control the norm of the extension operator of f supported on N large balls of radius R whose centers are at a large enough distance, then one can obtain an estimate which does not depend on the number of balls in which f is supported on. This last remark is important, as one can obtain *very* trivial estimates such as

$$\|E_{\mathbb{S}^{n-1}} f\|_{L^q(\mathbb{R}^n)} \lesssim N R^{\varepsilon} \|f\|_{L^q(\mathbb{S}^{n-1})},$$

which would be a quite useless estimate, for we want to cover all of $\mathbb{R}^n$ with a large amount of balls. Another not so trivial estimate follows from Hölder's inequality:

$$\left(\left[\int_{B_1} |f|^p \right]^{1/p}, \ldots, \left[\int_{B_N} |f|^p \right]^{1/p} \right) \cdot (1, \ldots, 1) \leq N^{1/p'} R^{\alpha} \left[\int_{\mathbb{R}^n} |f|^p \right]^{1/p}.$$

Nevertheless, these estimates will become trivial when attempting to obtain an estimate for a general f.

5.1 A covering lemma

In this section, we will state and prove a lemma that involves covering sets which are the union of cubes of constant side length comparable to 1.

Lemma 5.1. *Suppose E is the union of c-cubes, and $N \geq 1$. Then there exist $O(N\,|E|^{1/N})$ sparse collections of balls which cover E, such that the balls in each collection have radius $O(|E|^{C^N})$.*

It is important to notice that the collections need not have the same radius (this will become apparent in the proof), but they are bounded by some number depending on the size of E, C and N. Also note that the union over all collections (of different radii) will cover E, not necessarily a single collection of balls of the same radius.

Proof. Define the radii R_k for $0 \leq k \leq N$ by $R_0 = 1$ and $R_{k+1} = |E|^C R_k^C$, so that

$$R_k = |E|^C R_k^C = |E|^C \cdot |E|^{C^2} \cdots |E|^{C^k} \leq |E|^{kC^k},$$

so if we replace C by $k^{1/k}C$, we have that $R_k = O(|E|^{C^k})$.

Let E_1 be the set of all $x \in E$ such that

$$|E \cap B_{R_1}(x)| \leq |E|^{1/N}.$$

Recursively, we define E_k to be the set of $x \in E \backslash (\bigcup_{j<k} E_j)$ and are such that

$$|E \cap B_{R_k}(x)| \leq |E|^{k/N}.$$

For every $1 \leq k \leq N$ and $x \in E_k$, we have by construction and hypothesis that

$$\left|E \cap B_{R_{k-1}}(x)\right| \gtrsim |E|^{(k-1)/N}.$$

Furthermore, for every $x \in E_k$, once again by these previous conditions, we need $O(|E|^{1/N})$ balls of radius R_{k-1} to cover the set $E_k \cap B_{R_k}(x)$. This implies that the entire set E_k can be covered by $O(|E|^{1/N})$ collections of R_{k-1}-balls which are R_k-separated. That is, if we consider a countable subset of $x_i \in E_k$ such that $\bigcup_{i\geq 1} B_{R_k}(x_i)$, then by the previous remark each can be covered by $O(|E|^{1/N})$ balls of radius R_{k-1}. Call this collection of balls R_{k-1}^i. Then,for $i \neq j$, the every R_{k-1}-ball from R_{k-1}^i is at a distance at least R_{k-1}^C from any other R_{k-1}-ball from R_{k-1}^j. Since there are at most $|E|^C$ R_{k-1}-balls, these are $R_{k-1}^C\,|E|^C = R_k$ separated. Since the cardinality of these collections can be at most $O(|E|)$, the collections must be sparse. The lemma now follows from $E = \bigcup_{k=1}^N E_k$. $\square$

5.2 Statement and proof of the theorem

In the introduction section, we introduced the notation of $R_S(p \to q)$. Here, we make a slight modification in the form of an additional parameter. We will write $R_S(p \to q; \alpha)$ for the statement

$$\|R_S f\|_{L^q(S)} \lesssim_{p,q,S} R^\alpha \|f\|_{L^p(B_R)}.$$

That is, an estimate which has a dependence on the radius of the ball in which we integrate. We define $E_S(p \to q; \alpha)$ similarly. Also, we write $R_S(p; \alpha) := R_S(p \to p; \alpha)$ to reduce notation.

Theorem 5.2. *If $1 < p < 2$ and $0 < \alpha \ll 1$, then $R_{\mathbb{S}^{n-1}}(p; \alpha) \implies R_{\mathbb{S}^{n-1}}(q; 0)$ whenever*

$$\frac{1}{q} > \frac{1}{p} + \frac{C}{\log \frac{1}{\alpha}}.$$

By duality, one can conclude a theorem in the language of extensions.

Theorem 5.3. *If $2 < p < \infty$ and $0 < \alpha \ll 1$, then $E_{\mathbb{S}^{n-1}}(p; \alpha) \implies E_{\mathbb{S}^{n-1}}(q; 0)$ whenever*

$$\frac{1}{q} < \frac{1}{p} - \frac{C}{\log \frac{1}{\alpha}}.$$

Here are the key ideas to proving Theorem 5.2:

1. We first want to prove the local estimate $R(p, \alpha)$ for functions which are supported on a union of N balls of radius R which are at distance at least a fixed number depending on both R and N. The main idea to prove this ingredient is to use that since the functions are supported on balls which apart from each other, their Fourier transforms will be essentially pairwise orthogonal. To use this, we start by analyzing the restriction operator to an R-neighborhood of $\mathbb{S}^{n-1}$, then use interpolation.

2. Once we have proved 1, we want to show that it implies that a global estimate $R(q, 0)$ holds for any f, given a slightly worse exponent q, as illustrated in the statement of the Theorem. We then reduce the problem to an estimate in Lorentz spaces, and use results from chapter 1 to conclude the Theorem whenever the aforementioned estimate holds.

3. To prove that the Lorentz space estimate holds in general, we will actually only need prove an estimate on the restriction operator of characteristic functions on cubes.

4. This last estimate will in turn be implied by our control on both the number of sets of sparse balls and their radius which cover a set which is a union of cubes.

Before going into the arguments of the proof, we will state and prove a standard result of great use.

Lemma 5.4 (Schur's test)**.** *Let X, Y be measurable spaces. Let T be an integral operator with non-negative Schwartz kernel $K(x, y)$, $x \in X$ and $y \in Y$:*

$$Tf(x) = \int_Y K(x, y) f(y) dy.$$

If there exist real functions $p(x) > 0$ and $q(y) > 0$ and numbers $\alpha, \beta > 0$ such that

$$\int_Y K(x, y) q(y) dy \leq \alpha p(x)$$

for almost all x and

$$\int_X p(x)K(x,y)dx \leq \beta q(y)$$

for almost all y, then T extends to a continuous operator $T : L^2 \to L^2$ with operator norm

$$\|T\|_{L^2 \to L^2} \leq \sqrt{\alpha\beta}.$$

Proof. The proof follows from a use of Cauchy-Schwartz and Fubini:

$$\begin{aligned}
\|Tf\|_{L^2}^2 &= \int_X \left| \int_Y K(x,y)f(y)dy \right|^2 dx \\
&\leq \int_X \left(\int_Y K(x,y)q(y)dy \right) \left(\int_Y \frac{K(x,y)f(y)^2}{q(y)} dy \right) dx \\
&\leq \alpha \int_Y \frac{f(y)^2}{q(y)} \left(\int_X p(x)K(x,y)dx \right) dy \\
&\leq \alpha\beta \int_Y f(y)^2 dy = \alpha\beta \, \|f\|_{L^2}^2 ,
\end{aligned}$$

for all $f \in L^2(Y)$, and the result follows. □

Definition 5.5. A collection $\{B(x_i, R)\}_{i=1}^N$ of $R-$balls is *sparse* if the centers x_i are $R^C N^C$ separated. (C denotes a positive constant that will change from line to line).

We may see that $R \geq 2^{-1}$, $N \geq 1$ and $C \geq 1$ are necessary to make sense of the word sparse. We will make a reduction on notation to make the section easily readable. We will write $\mathfrak{R} = R_{\mathbb{S}^{n-1}}$, the restriction operator to the $n-1$ sphere.

Lemma 5.6. *Suppose $R_{\mathbb{S}^{n-1}}(p; \alpha)$ holds for some $\alpha > 0$ and $1 < p < 2$. Then*

$$\|\mathfrak{R}f\|_{L^p(\mathbb{S}^{n-1})} \lesssim R^\alpha \|f\|_{L^p(\mathbb{R}^n)}$$

whenever f is supported on $\bigcup_{i=1}^N B(x_i, R)$ and $\{B(x_i, R)\}_{i=1}^N$ is a sparse collection of balls.

Proof. We start by modifying the localized restriction hypothesis slightly. Let $\tilde{\mathfrak{R}}f$ denote the restriction operator of the Fourier transform of f to the annulus A_R around $\mathbb{S}^{n-1}$ of thickness $\sim 1/R$. A simple calculation shows this set has measure $O(1/R)$. Note that if we assume $R(p; \alpha)$ and then express the integral in polar coordinates, we have

$$\begin{aligned}
\left\|\tilde{\mathfrak{R}}f\right\|_{L^p(A_R)}^p &= \int_{A_R} \left|\tilde{\mathfrak{R}}f(\xi)\right|^p d\sigma(\xi) \\
&= \int_{1-\frac{1}{R}}^{1+\frac{1}{R}} \int_{\mathbb{S}^{n-1}} \left|\tilde{\mathfrak{R}}f(\xi)\right|^p r^{n-1}\, d\sigma(\xi)dr \\
&= \int_{1-\frac{1}{R}}^{1+\frac{1}{R}} \int_{\mathbb{S}^{n-1}} \left|r^{-n}\mathfrak{R}f(\xi)\right|^p r^{n-1}\, d\sigma(\xi)dr \\
&\lesssim R^{\alpha p} \left\|f\right\|_{L^p(B_R)}^p \int_{1-\frac{1}{R}}^{1+\frac{1}{R}} r^{n-1-np}\, dr \\
&\lesssim R^{-1} R^{\alpha p} \left\|f\right\|_{L^p(\mathbb{R}^n)}^p .
\end{aligned}$$

Hence,

$$\left\|\tilde{\mathfrak{R}}f\right\|_{L^p(A_R)} \lesssim R^{-1/p} R^{\alpha} \left\|f\right\|_{L^p(\mathbb{R}^n)} . \tag{5.2}$$

whenever f is supported on B_R. By transnational symmetry, this estimate also holds whenever f is supported on $B_R(x)$ for any $x \in \mathbb{R}^n$.

Write $f = \sum_{i=1}^N f_i\varphi_i$, where each f_i is supported on $B_R(x_i)$, $\varphi_i(x) = \varphi(\frac{x-x_i}{R})$, and φ such that $\varphi \in \mathcal{S}$, $\varphi(x) > 0\ \forall x \in \mathbb{B}^n$ and $\operatorname{supp}\hat{\varphi} \subset \mathbb{B}^n$. We have

$$\left\|f\right\|_{L^p(\mathbb{R}^n)}^p = \left\|\sum_{i=1}^N f_i\varphi_i\right\|_{L^p(\mathbb{R}^n)}^p = \sum_{i=1}^N \left\|f_i\varphi_i\right\|_{L^p(\mathbb{R}^n)}^p \sim \sum_{i=1}^N \left\|f_i\right\|_{L^p(\mathbb{R}^n)}^p , \tag{5.3}$$

where the second equality is possible due to the disjointedness of their supports. The last estimate can be found by taking either the minimum of maximum of φ. Moreover,

$$\mathfrak{R}f = \mathfrak{R}\left(\sum_{i=1}^N f_i\varphi_i\right) = \sum_{i=1}^N \mathfrak{R}(f_i\varphi_i) = \sum_{i=1}^N \tilde{\mathfrak{R}}f_i * \hat{\varphi}_i|_{S^{n-1}}. \tag{5.4}$$

Note that we can prove the lemma if the following estimate holds for all $F_i \in L^p(\mathbb{R}^n)$:

$$\left\|\sum_{i=1}^N F_i * \hat{\varphi}_i|_{S^{n-1}}\right\|_{L^p(\mathbb{R}^n)} \lesssim R^{1/p}\left(\sum_{i=1}^N \left\|F_i\right\|_{L^p(\mathbb{R}^n)}^p\right)^{1/p} . \tag{5.5}$$

Indeed, using (5.2), (5.3), (5.4) and the previous estimate,

$$
\begin{aligned}
\|\mathfrak{R}f\|_{L^p(S^{n-1})} &= \left\| \sum_{i=1}^{N} \tilde{\mathfrak{R}} f_i * \hat{\varphi}_i|_{S^{n-1}} \right\|_{L^p(S^{n-1})} \\
&\lesssim R^{1/p} \left(\sum_{i=1}^{N} \left\| \tilde{\mathfrak{R}} f_i \right\|_{L^p(A_R)}^p \right)^{1/p} \\
&\lesssim R^{1/p} \left(\sum_{i=1}^{N} R^{-1} R^{p\alpha} \|f_i\|_{L^p(\mathbb{R}^n)}^p \right)^{1/p} \\
&\lesssim R^{\alpha} \|f\|_{L^p(\mathbb{R}^n)} .
\end{aligned}
$$

Thus, we only need to prove (5.5). We will prove it for $p = 1$ and $p = 2$, then use the Marcinkiewicz' interpolation theorem to prove it for every $1 < p < 2$. For $p = 1$, we do a direct computation:

$$
\begin{aligned}
\left\| \sum_{i=1}^{N} F_i * \hat{\varphi}_i|_{S^{n-1}} \right\|_{L^1(\mathbb{S}^{n-1})} &= \int_{\mathbb{S}^{n-1}} \left| \sum_{i=1}^{N} F_i * \hat{\varphi}_i|_{S^{n-1}}(\xi) \right| d\sigma(\xi) \\
&\leq \sum_{i=1}^{N} \int_{\mathbb{S}^{n-1}} \left| \int_{\mathbb{R}^n} F_i(\eta) \hat{\varphi}_i(\xi - \eta) d\eta \right| d\sigma(\xi) \\
&\leq \sum_{i=1}^{N} \int_{\mathbb{S}^{n-1}} \int_{\mathbb{R}^n} |F_i(\eta) \hat{\varphi}_i(\xi - \eta)| \, d\eta d\sigma(\xi) \\
&= \sum_{i=1}^{N} \int_{\mathbb{R}^n} |F_i(\eta)| \left(\int_{\mathbb{S}^{n-1}} |\hat{\varphi}_i(\xi - \eta)| \, d\sigma(\xi) \right) d\eta \\
&\lesssim R \sum_{i=1}^{N} \|F_i\|_{L^1(\mathbb{R}^n)} .
\end{aligned}
$$

For $p = 2$, we use Plancharel's theorem to see that the estimate is equivalent to

$$
\left\| \mathfrak{R} \left(\sum_{i=1}^{N} f_i \varphi_i \right) \right\|_{L^2(S^{n-1})} \lesssim R^{1/2} \left(\sum_{i=1}^{N} \|f_i\|_{L^2(\mathbb{R}^n)}^2 \right)^{1/2} \tag{5.6}
$$

for all $f_i \in L^2(\mathbb{R}^n)$. Recall that

$$
\langle Tf, Tf \rangle_{L^2(S)} = \langle T^* T f, f \rangle_{L^2(\mathbb{R}^n)} ,
$$

for some L^2 operator T. The dual operator of $f \mapsto \mathfrak{R}(f \cdot \varphi_i)$ is then $f \mapsto \varphi_i \cdot \mathfrak{R}^*(f)$. Indeed,

$$
\langle \mathfrak{R}(f\varphi), g \rangle = \int \mathfrak{R}(f\varphi) g = \int (f\varphi) \cdot \mathfrak{R}^* g = \int f \cdot (\varphi \mathfrak{R}^*(g)) = \langle f, \varphi \mathfrak{R} g \rangle .
$$

The case in which $f = \sum f_i\varphi_i$ follows from bi-linearity of the inner product. We will now show that proving (5.6) is equivalent to proving

$$\left(\sum_{j=1}^{N}\left\|\varphi_j\mathfrak{R}^*\mathfrak{R}\left(\sum_{i=1}^{N} f_i\varphi_i\right)\right\|^2_{L^2(\mathbb{S}^{n-1})}\right)^{1/2} \lesssim R\left(\sum_{i=1}^{N}\|f_i\|^2_{L^2(\mathbb{R}^n)}\right)^{1/2}. \tag{5.7}$$

First assume (5.6), then

$$\begin{aligned}
\left\|\varphi_j\mathfrak{R}^*\mathfrak{R}\left(\sum_{i=1}^{N} f_i\varphi_i\right)\right\|_{L^2} &= \sup_{\|g\|_{L^2}=1}\left|\left\langle\varphi_j\mathfrak{R}^*\mathfrak{R}\left(\sum_{i=1}^{N} f_i\varphi_i\right), g\right\rangle\right| \\
&= \sup_{\|g\|_{L^2}=1}\left|\left\langle\mathfrak{R}\left(\sum_{i=1}^{N} f_i\varphi_i\right), \mathfrak{R}(g\varphi_j)\right\rangle\right| \\
&\le \left\|\mathfrak{R}\left(\sum_{i=1}^{N} f_i\varphi_i\right)\right\|_{L^2}\|\mathfrak{R}(g\varphi_j)\|_2 \\
&\lesssim R^{1/2}\left(\sum_{i=1}^{N}\|f_i\|_2^2\right)^{1/2}\|\mathfrak{R}(g\varphi_j)\|_2\,,
\end{aligned}$$

and thus

$$\begin{aligned}
\sum_{j=1}^{N}\left\|\varphi_j\mathfrak{R}^*\mathfrak{R}\left(\sum_{i=1}^{N} f_i\varphi_i\right)\right\|^2_{L^2(\mathbb{S}^{n-1})} &\lesssim R\sum_{i=1}^{N}\|f_i\|_2^2\sum_{j=1}^{N}\|\mathfrak{R}(g\varphi_j)\|_2^2 \\
&\sim R\sum_{i=1}^{N}\|f_i\|_2^2\left\|\mathfrak{R}\left(\sum_{j=1}^{N} g\varphi_j\right)\right\|_2^2 \\
&\lesssim R^2\sum_{i=1}^{N}\|f_i\|_2^2\,.
\end{aligned}$$

Conversely, assume (5.7), then

$$\begin{aligned}
\left|\left\langle\mathfrak{R}\left(\sum_{i=1}^{N} f_i\varphi_i\right), \mathfrak{R}\left(\sum_{j=1}^{N} f_j\varphi_j\right)\right\rangle\right| &= \left|\left\langle\mathfrak{R}^*\mathfrak{R}\left(\sum_{i=1}^{N} f_i\varphi_i\right), \left(\sum_{i=j}^{N} f_j\varphi_j\right)\right\rangle\right| \\
&= \sum_{j=1}^{N}\left|\left\langle\varphi_j\mathfrak{R}^*\mathfrak{R}\left(\sum_{i=1}^{N} f_i\varphi_i\right), f_j\right\rangle\right| \\
&= \sum_{j=1}^{N}\left|\int\varphi_j\mathfrak{R}^*\mathfrak{R}\left(\sum_{i=1}^{N} f_i\varphi_i\right)\cdot f_j\right| \\
&\le \sum_{j=1}^{N}\left\|\varphi_j\mathfrak{R}^*\mathfrak{R}\left(\sum_{i=1}^{N} f_i\varphi_i\right)\right\|_{L^2}\|f_j\|_{L^2}\,,
\end{aligned}$$

where the last inequality follow from Hölder's inequality. To prove the bound for (5.7), we will first prove that

$$\sup_j \sum_i \|\varphi_j \mathfrak{R}^* \mathfrak{R} \varphi_i\|_{L^2 \to L^2} \lesssim R, \tag{5.8}$$

and then use Schur's test. This estimate will in turn follow from proving

$$\|\varphi_i \mathfrak{R}^* \mathfrak{R} \varphi_i\|_{L^2 \to L^2} \lesssim R \tag{5.9}$$

and

$$\|\varphi_j \mathfrak{R}^* \mathfrak{R} \varphi_i\|_{L^2 \to L^2} \lesssim R^{-C} N^{-C}, \qquad j \neq i. \tag{5.10}$$

To prove (5.9), we use Plancharel's theorem to obtain the equivalent

$$\|\hat{\varphi}_i * (d\sigma(\hat{\varphi} * g))\|_{L^2} \lesssim R \|g\|_{L^2},$$

To use Schur's test, we first find the kernel of the operator.

$$\begin{aligned}
\hat{\varphi}_i * (d\sigma(\hat{\varphi}_i * g))(x) &= \int_{\mathbb{S}^{n-1}} \hat{\varphi}_i(x-y) \int_{\mathbb{R}^n} \hat{\varphi}_i(y-z) g(z) dz \, d\sigma(y) \\
&= \int_{\mathbb{R}^n} \underbrace{\left[\int_{\mathbb{S}^{n-1}} \hat{\varphi}_i(x-y) \hat{\varphi}_i(y-z) d\sigma(y) \right]}_{K(x,z)} g(z) dz.
\end{aligned}$$

Now, we note that

$$\begin{aligned}
\int_{\mathbb{R}^n} |K(x,z)| \, dz &= \int_{\mathbb{R}^n} \int_{\mathbb{S}^{n-1}} |\hat{\varphi}_i(x-y) \hat{\varphi}_i(y-z)| \, d\sigma(y) dz \\
&= \int_{\mathbb{S}^{n-1}} \int_{\mathbb{R}^n} |\hat{\varphi}_i(x-y) \hat{\varphi}_i(y-z)| \, dz d\sigma(y) \\
&\leq \int_{\mathbb{S}^{n-1}} R^n |\hat{\varphi}(Ry)| \, d\sigma(y) \int_{\mathbb{R}^n} R^n |\hat{\varphi}(Rz)| \, dz \\
&= R \int_{\mathbb{S}^{n-1}} R^{n-1} |\hat{\varphi}(Ry)| \, d\sigma(y) \int_{\mathbb{R}^n} |\hat{\varphi}(z)| \, dz \\
&\lesssim R.
\end{aligned}$$

A verbatim argument works for integration with respect to x. Thus, by Schur's test, (5.9) follows. For (5.10), we first note that

$$\begin{aligned}
\varphi_j \cdot (\hat{d\sigma} * (\varphi_i f))(y) &= \varphi_j(y) \int_{\mathbb{R}^n} \int_{\mathbb{S}^{n-1}} e^{-2\pi i (y-x)\cdot \xi} d\sigma(\xi) f(x) \varphi_i(x) dx \\
&= \varphi_j(y) \int_{\mathbb{S}^{n-1}} e^{-2\pi i y \cdot \xi} \int_{\mathbb{R}^n} e^{2\pi i x \cdot \xi} f(x) \varphi_i(x) dx d\sigma(\xi) \\
&= \varphi_j \mathfrak{R}^* \mathfrak{R} \varphi_i f(y)
\end{aligned}$$

where now the kernel is given by

$$K(x,y) = \varphi_j(y)\widehat{d\sigma}(x-y)\varphi_i(x).$$

We now estimate the integral of the kernel to use Schur's test.

$$\begin{aligned}\int_{\mathbb{R}^n} |K(x,y)|\, dx &= \int_{\mathbb{R}^n} |\varphi_j(y)|\, |\widehat{d\sigma}(x-y)\varphi_i(x)| dx \\ &\lesssim \int_{\mathbb{R}^n} \left|(x-y)^{-\frac{n-1}{2}}\varphi_i(x)\right| dx \\ &\lesssim \int_{\mathbb{R}^n} (R^C N^C)^{-\frac{n-1}{2}} R^n\, |\varphi(x)|\, dx \\ &\lesssim R^{-C(\frac{n-1}{2})+n} N^{-C},\end{aligned}$$

so we know that $C > \frac{2n}{n-1}$ is needed to have a negative exponent. □

Recall that $L^p \subset L^{p,\infty}$, and thus

$$\begin{aligned}\|\mathfrak{R}f\|_{L^{p,\infty}} \lesssim \|\mathfrak{R}f\|_{L^p} \lesssim \|f\|_{L^{q_0,1}}\,, \\ \|\mathfrak{R}f\|_{L^{\infty,\infty}} \lesssim \|\mathfrak{R}f\|_{L^\infty} \lesssim \|f\|_{L^{1,1}}\,.\end{aligned}$$

That is, $\mathfrak{R}$ is of weak restricted types $(1,\infty)$ and (q_0,p). Hence, for some $0<\theta<1$ with

$$\frac{1}{q} = \frac{1-\theta}{p}, \qquad \frac{1}{r} = \frac{1-\theta}{q_0} + \frac{\theta}{1}.$$

Note that for each $0<\theta<1$, we have $r \le q$. In particular, for $0 < \theta \ll 1$, $\frac{1}{q} \approx \frac{1}{p}$ and $\frac{1}{r} \approx \frac{1}{q_0}$, so that one has, by Corollary 4.2,

$$\|\mathfrak{R}f\|_{L^p} \lesssim \|f\|_{L^{q_0}}\,.$$

Now, by Hölder's inequality, we may see that since $q_0 < p$,

$$\begin{aligned}\|\mathfrak{R}f\|_{L^{q_0}}^{q_0} = \int_{\mathbb{S}^{n-1}} |\mathfrak{R}f|^{q_0} \cdot 1 &\le \left(\int_{\mathbb{S}^{n-1}} |\mathfrak{R}f|^p\right)^{q_0/p} \left(\int_{\mathbb{S}^{n-1}} 1\right)^{1-\frac{q_0}{p}} \\ &\lesssim \|\mathfrak{R}f\|_{L^p}^{q_0} \\ &\lesssim \|f\|_{L^{q_0}}^{q_0}\,,\end{aligned}$$

which proves our globalized restriction theorem. To prove the estimate (5.11), we make some other reductions on the problem. The following theorem basically reduces the global behavior of an operator to its behavior on finite measure sets. Its proof can be found in [10].

Theorem 5.7. *Suppose T is a linear operator which maps the finite linear combinations of characteristic functions χ_E of sets $E \subset M$ of finite measure into a vector space B that is endowed with an order preserving norm $\|\cdot\|_B$. If*

$$\|T\chi_E\|_B \le C\,\|\chi_E\|_{L^{r,1}} = C\{\mu(E)\}^{1/r},$$

where C is independent of E, then there exists a constant A such that

$$\|Tf\|_B \leq A \|f\|_{L^{r,1}}$$

for all f in the domain of T.

In our case, T will be the restriction operator, B will be an L^p space with its usual norm, $M = \mathbb{R}^n$ and μ is the Lebesgue measure. Thus, if we can prove that

$$\|\mathfrak{R}\chi_E\|_{L^p(S^{n-1})} \leq C_\alpha |E|^{1/p+C/\log(1/\alpha)}, \tag{5.11}$$

Theorem 5.7 will imply (5.11). However, we do not need to prove this for every finite measure set. It is enough to consider functions which are constant on cubes of side lengths a proper constant c. Indeed, we first note that

$$\int_{[-1/2,1/2]^n} |f|^p = \int_{[-1/2,1/2]^n} |\tilde{f}|^p,$$

where $\tilde{f}$ is defined as the average of $|f|$ on cube $Q = [-1/2, 1/2]^n$. Thus, setting $F(k) = \tilde{f}_k$, where $\tilde{f}_k$ is the average of $|f|$ on $\prod_{i=1}^n [k_i - 1/2, k_i + 1/2]$ for $k_i \in \mathbb{Z}$, then $F(k)$ is supported on $\mathbb{Z}^n$, and

$$\|f\|_{L^p}^p \int |f|^p = \sum_{k\in\mathbb{Z}^n} \int_Q |f(k-y)|^p \, dy = \sum_{k\in\mathbb{Z}^n} \int_Q |\tilde{f}(k-y)|^p dy = \|F\|_{\ell^p}^p.$$

That is, it suffices to prove it for measure functions supported on $\mathbb{Z}^n$, and the $L^{q_0,1}$ norm is now $L^{q_0,1}(\mathbb{Z}^n) = \ell^{q_0,1}$ (see also [2]). The next step is to revert to the continuous $L^{q_0,1}$ norm. Consider χ, a characteristic function on a cube of side lengths c, with $c \sim 1$ such that $\hat{\chi} > 0$ on the unit ball. We may now suppose $f = f\chi$, then the $\ell^{q_0,1}$ turns into the continuous $L^{q_0,1}$ norm.

We now make the final arguments to prove Theorem 5.2. If E is the union of c-cubes, by Lemma 5.1 we may cover E with $O(N|E|^{1/N})$ sets E_j which are the union of sparse collections of balls of radius $O(|E|^{C^N})$. Now, we consider $E_j' = E_j \cap E$, then applying Lemma 5.6 to each E_j' we obtain

$$\left\|\mathfrak{R}\chi_{E_j'}\right\|_{L^p} \lesssim \left(|E|^{C^N}\right)^\alpha |E|^{1/p}.$$

Hence, by Minkowski's inequality, and the bound for the number of sets E_j,

$$\|\mathfrak{R}\chi_E\|_{L^p} \leq \left\|\sum_j \mathfrak{R}\chi_{E_j'}\right\|_{L^p} \lesssim N|E|^{1/N}\left(|E|^{C^N}\right)^\alpha |E|^{1/p}.$$

Choosing $N = C^{-1}\log\frac{1}{\alpha}$, we have

$$\begin{aligned} N|E|^{1/N}\left(|E|^{C^N}\right)^\alpha |E|^{1/p} &= C^{-1}\log\frac{1}{\alpha}\,|E|^{1/p+C/\log\frac{1}{\alpha}+\alpha C^{\log\frac{1}{\alpha}C^{-1}}} \\ &= C^{-1}\log\frac{1}{\alpha}\,|E|^{1/p+C/\log\frac{1}{\alpha}}\,|E|^{\alpha C^{\log\frac{1}{\alpha}C^{-1}}}, \end{aligned}$$

and note that $\alpha C^{\log\frac{1}{\alpha}C^{-1}} = e^{\log\alpha(1-\frac{\log C}{C})}$, so the end term is $|E|^{\alpha^{(1-\frac{\log C}{C})}}$, which is a constant depending on α since $\max\limits_{C\geq 1} C^{-1}\log C = e^{-1}$. Then, choosing a big enough C,

$$\|\mathfrak{R}\chi_E\|_{L^p} \leq C_\alpha |E|^{1/p + C/\log\frac{1}{\alpha}},$$

and the proof is complete.

5.3 ε-removal for parabolic surfaces

Although Theorem 5.2 was proven for the special case of $S = \mathbb{S}^{n-1}$, we can make a limit argument to also obtain the restriction estimate on a paraboloid. This will result useful later on, as [5] deals with surfaces which resemble parabolas. These arguments follow a problem presented in [13].

Let $S \subset \mathbb{R}^n$ be any subset with some measure $d\sigma$, and let $T : \mathbb{R}^n \to \mathbb{R}^n$ be any invertible affine transformation on $\mathbb{R}^n$ (that is, $Tx = Lx + y$ for some fixed $y \in \mathbb{R}^n$ and some invertible linear transformation $L : \mathbb{R}^n \to \mathbb{R}^n$). The image $T(S)$ of S under the transformation of T is thus endowed with the *push-forward measure* $T_*(d\sigma)$, defined by

$$\int_{T(S)} f(\xi) T_*(d\sigma)(\xi) \coloneqq \int_S f(T\xi) d\sigma(\xi).$$

It is not difficult to prove that for any $1 \leq p, q \leq \infty$, the estimate $R_S(p \to q)$ holds if and only if $R_{T(S)}(p \to q)$ holds.

Now suppose $1 \leq p, q \leq \infty$ obey the scaling relationship $\frac{n+1}{p'} = \frac{n-1}{q}$. This relation was encountered in iv. Suppose that the restriction estimate $R_{\mathbb{S}^{n-1}}(p \to q)$ holds. We will prove that the restriction estimate $R_{S_{parab}}(p \to q)$ must hold for the paraboloid S. We start by re-scaling $\mathbb{S}^{n-1}$ to a paraboloid. Denote by T the translation

$$T(x_1, \ldots, x_n) = (x_1, \ldots, x_n - 1),$$

and a scaling S given by

$$S(x_1, \ldots, x_n) = (\lambda x_1, \ldots, \lambda x_{n-1}, \lambda^2 x_n),$$

then $A \coloneqq TS$ is an invertible affine transformation which sends the sphere to an ellipsoid. Taking $\lambda \to 0$, the image $A(\mathbb{S}^{n-1})$ approaches the paraboloid $2x_n = \sum_{i=1}^{n-1} x_i^2$. Indeed,

$$(\lambda x_1)^2 + (\lambda x_2)^2 + \cdots + (\lambda x_{n-1})^2 + (\lambda^2 x_n - 1)^2 = 1$$

is equivalent to

$$x_1^2 + x_2^2 + \cdots + x_{n-1}^2 + \lambda^2 x_n^2 - 2x_n = 0.$$

The Jacobian of this transformation is λ^{n+1}. By Fatou's Lemma,

$$\lim_{\lambda\to 0} \int_{A(\mathbb{S}^{n-1})} f(\lambda\underline{\xi}, \lambda^2\xi_n - 1) A(d\sigma(\xi)) = \int_{S_{parab}} f(\underline{\xi}, \tfrac{1}{2}|\underline{\xi}|^2) d\underline{\xi},$$

Hence, if

$$\|R_{\mathbb{S}^{n-1}} f\|_{L^q(\mathbb{S}^{n-1})} \lesssim \|f\|_{L^p(\mathbb{R}^n)}\,, \tag{5.12}$$

holds, then setting $\underline{\xi} = A(\xi)$ on the right hand side of (5.12), we have

$$\begin{aligned}&\left(\lambda^{n-1} \int_{A(\mathbb{S}^{n-1})} \left|R_S f(\lambda\underline{\xi}, \lambda^2 \xi_n - 1)\right|^q d\sigma(\xi)\right)^{1/q} \\ &= \lambda^{\frac{n-1}{q}} \lambda^{-n-1} \left(\int_{A(\mathbb{S}^{n-1})} |R_S f_\lambda(\xi)|^q \, d\sigma(\xi)\right)^{1/q},\end{aligned}$$

where $f_\lambda(x) = f(\lambda\underline{x}, \lambda^2 x_n)$. Then, using that the restriction property is invariant under invertible affine transformation, this last expression becomes

$$\begin{aligned}\lambda^{\frac{n-1}{q}} \lambda^{-n-1} \left(\int_{A(\mathbb{S}^{n-1})} |R_S f_\lambda(\xi)|^q \, d\sigma(\xi)\right)^{1/q} &\lesssim \lambda^{\frac{n-1}{q}} \lambda^{-n-1} \|f_\lambda\|_{L^p} \\ &= \lambda^{\frac{n-1}{q}} \lambda^{-(n+1)} \lambda^{\frac{n+1}{p}} \|f\|_{L^p}\,.\end{aligned}$$

Then, using $1/p + 1/p' = 1$ and the scaling condition $\frac{n-1}{q} - \frac{n+1}{p'} = 0$, the exponent of λ becomes 0. Now, taking the limit as $\lambda \to 0$, we have

$$\left\|R_{S_{parab}} f\right\|_{L^q} \lesssim \|f\|_{L^p}\,.$$

This now allows us to also use Theorem 5.3 on parabolic surfaces.

Chapter 6

Parabolic scaling and broad points

This chapter is mainly focused on reducing the proof of Theorem 0.1 to an estimate which deals with a certain set of points on the surface. The proof of such estimate will be presented in the next chapter. Here we will only prove how such estimates are enough to conclude Theorem 0.1.

Henceforth, we will work on $\mathbb{R}^3$, as opposed to the general $\mathbb{R}^n$ managed in the previous chapters.

6.1 Nearly parabolic surfaces

Suppose S is a compact hypersurface given by the graph of a function $f : \mathbb{B}^2 \to \mathbb{R}$ satisfying, for some large $L \in \mathbb{Z}_+$,

Conditions 6.1.

$$\frac{1}{2} \leq \mathbf{H}_f \leq 2, \tag{6.1}$$

$$0 = f(0) = \nabla f(0), \tag{6.2}$$

$$f \in C^L \text{ and, for } 3 \leq l \leq L, \quad \left\|\partial^l f\right\|_{C^0} \leq 10^{-9}. \tag{6.3}$$

Note that (6.1) refers to the eigenvalues of the Hessian matrix of f

$$\mathbf{H}_f = \begin{pmatrix} \dfrac{\partial^2 f}{\partial x_1^2} & \dfrac{\partial^2 f}{\partial x_1 \partial x_2} \\ \dfrac{\partial^2 f}{\partial x_2 \partial x_1} & \dfrac{\partial^2 f}{\partial x_2^2} \end{pmatrix}$$

being between the values of $1/2$ and 2.

Definition 6.2. We say a smooth surface S is *parabolic up to order* L if it satisfies Conditions 6.1 with L derivatives.

We will simply say that S is *nearly parabolic* if the order in Definition 6.2 is irrelevant.

A trivial example of a surface satisfying the conditions is the truncated paraboloid $z = x^2 + y^2$, $z \leq 1$.

6.2 Parabolic scaling

The reason why we consider these types of surfaces is because one can reduce the behavior of a compact smooth surface with positive second fundamental form to various local estimates which can be transformed to be viewed as parabolic surfaces.

Henceforth, we write $\omega_3 = h(\omega_1, \omega_2) = h(\vec{\omega})$, where $\vec{\omega} \in \mathbb{R}^2$ denotes the first two coordinates of $\omega \in \mathbb{R}^3$. The change of variables goes as follows. Define $\tilde{h}$ to be h minus its first order Taylor expansion at $\vec{\omega}_0$:

$$\tilde{h}(\vec{\omega}) = h(\vec{\omega}) - h(\vec{\omega}_0) - (\vec{\omega} - \vec{\omega}_0) \cdot \nabla h(\vec{\omega}_0)$$

Let $\vec{\eta} \in B^2(1)$, and define a parametric equation for $B^2_r(\vec{\omega}_0)$ as

$$\vec{\omega} = \vec{\omega}_0 + r\vec{\eta}.$$

Now define

$$h_1(\vec{\eta}) = r^{-2}\tilde{h}(\vec{\omega}) = r^{-2}\tilde{h}(\vec{\omega}_0 + r\vec{\eta}).$$

Let S_1 denote the graph of h_1. It is crucial to note that whenever the graph of h is parabolic up to order L, the graph of h_1 will also be nearly parabolic. It is straightforward to verify that

$$0 = \tilde{h}(\vec{\omega}_0) = \nabla\tilde{h}(\vec{\omega}_0).$$

It is also to check that $0 = h_1(0) = \nabla h_1(0)$ by definition. Moreover, we have

$$\partial^2_{ij} h_1(\vec{\eta}) = \partial^2_{ij} h(\vec{\omega}_0 + r\vec{\eta}).$$

In particular, for all $\vec{\eta} \in B^2_1$,

$$1/2 \leq \mathbf{H}_{h_1} \leq 2.$$

It is also clear that h_1 is C^∞. Moreover, for $l \geq 2$,

$$\left\|\partial^l h_1\right\|_{C^0} = r^{l-2} \left\|\partial^l h\right\|_{C^0},$$

so $\left\|\partial^l h_1\right\|_{C^0} \leq \left\|\partial^l h\right\|_{C^0}$ for all $l \geq 3$.

We now prove a lemma which relates the behavior of the extension operator E_{S_0} on B_R to E_{S_1} on a smaller ball.

Lemma 6.3. *Suppose the graph of h is parabolic up to order L. Let S_0 be as above: the restriction of the graph of h to the ball of radius r. If E_{S_1} satisfies the inequality*

$$\|E_{S_1} g\|_{L^p(B_{10rR})} \le M \|g\|_{L^\infty(S_1)},$$

then E_{S_0} satisfies the inequality

$$\|E_{S_0} f\|_{L^p(B_R)} \le C r^{2-\frac{4}{p}} M \|f\|_{L^\infty(S_0)}.$$

Proof. Let $f \in L^p(S_0)$. We will express $E_{S_0} f$ using E_{S_1}:

$$|E_{S_0} f(x)| = \left| \int_{S_0} e^{i\omega x} f(\omega) d\sigma_{S_0}(\omega) \right| = \left| \int_{B_r^2(\vec{\omega}_0)} e^{i\vec{\omega}\cdot\vec{x}} e^{ih(\vec{\omega})x_3} f \, |Jh| \, d\vec{\omega} \right|,$$

where $|Jh_0|$ is the Jacobian $(1+|\nabla h|^2)^{1/2}$. We now write the integral in terms of $\tilde{h}$.

$$= \left| \int_{B_r^2(\vec{\omega}_0)} e^{i\vec{\omega}\cdot\vec{x}} e^{ix_3(\tilde{h}(\vec{\omega}) + (\vec{\omega}-\vec{\omega}_0)\partial h(\vec{\omega}_0) + h(\vec{\omega}_0))} f \, |Jh| \, d\vec{\omega} \right|$$

$$= \left| \int_{B_r^2(\vec{\omega}_0)} e^{i\vec{\omega}\cdot(\vec{x}+\partial h(\vec{\omega}_0)x_3)} e^{i\tilde{h}(\vec{\omega})x_3} f \, |Jh| \, d\vec{\omega} \right|.$$

Now, we rewrite it using h_1.

$$= \left| \int_{B_1^2} e^{i(\vec{\omega}_0 + r\vec{\eta})\cdot(\vec{x}+\partial h(\vec{\omega}_0)x_3)} e^{ir^2 h_1(\vec{\eta})x_3} f \, |Jh| \, r^2 d\vec{\eta} \right|$$

$$= \left| \int_{B_1^2} e^{ir\vec{\eta}\cdot(\vec{x}+\partial h(\vec{\omega}_0)x_3)} e^{ir^2 h_1(\vec{\eta})x_3} f \, |Jh| \, r^2 d\vec{\eta} \right|.$$

Define

$$g(\vec{\eta}) = f(\vec{\omega}_0 + r\vec{\eta}) r^2 \, |Jh| \, |Jh_1|^{-1} \tag{6.4}$$

and

$$\bar{x} = (rx_1 + r\partial_1 h(\vec{\omega}_0)x_3, rx_2 + r\partial_2 h(\vec{\omega}_0)x_3, r^2 x_3), \tag{6.5}$$

then

$$|E_{S_1} g(\bar{x})| = \left| \int_{B_1^2} e^{ir\vec{\eta}\cdot(\vec{x}+\partial h(\vec{\omega}_0)x_3)} e^{ir^2 h_1(\vec{\eta})x_3} f \, |Jh| \, r^2 d\vec{\eta} \right|.$$

Now, since h, h_1 are both nearly parabolic, we must have $\nabla h(x), \nabla h_1(x) \to 0$ as $x \to 0$. Since $\left|\nabla^2 h\right|, \left|\nabla^2 h_1\right| \le 4$, then $|\nabla h|, |\nabla h_1| \le 4$. Hence, $|Jh| = (1+|\nabla h|^2)^{1/2} \lesssim 1$ and $|Jh_1| = (1+|\nabla h_1|^2)^{1/2} \lesssim 1$. Thus using (6.4) we see that

$$\|g\|_{L^\infty(S_1)} \lesssim r^2 \|f\|_{L^\infty(S_0)}.$$

Since $|\partial h(\omega_0)| \leq 2$, we may use (6.5) to see that is $x \in B_R$, then

$$|\bar{x}| \leq \sqrt{(rR + 2rR)^2 + (rR + 2rR)^2 + (r^2R)^2} \leq 10rR,$$

so $\bar{x} \in B_{10rR}$. Let Φ denote the linear change of variables $x \mapsto \bar{x}$. That is,

$$\Phi(x_1, x_2, x_3) = (rx_1 + r\partial_1 h(\vec{\omega}_0)x_3, rx_2 + r\partial_2 h(\vec{\omega}_0)x_3, r^2x_3),$$

then

$$\Phi = \begin{pmatrix} r & 0 & r\partial_1 h(\vec{\omega}_0) \\ 0 & r & r\partial_2 h(\vec{\omega}_0) \\ 0 & 0 & r^2 \end{pmatrix},$$

so $\det \Phi = r^4$. Recall we proved earlier that $|E_{S_0} f(x)| = |E_{S_1} g(\bar{x})|$. Therefore,

$$\begin{aligned} \left(\int_{B_R} |E_{S_0} f(x)|^p \, dx \right)^{1/p} &= \left(\int_{B_R} |E_{S_1} g(\bar{x})|^p \, dx \right)^{1/p} \\ &= \left(|\Phi|^{-1} \int_{B_{10rR}} |E_{S_1} g(\bar{x})|^p \, d\bar{x} \right)^{1/p} \\ &= r^{-4/p} \left\| E_{S_1} g \right\|_{L^p(B_{10rR})} \\ &\leq Mr^{-4/p} \left\| g \right\|_{L^\infty(S_1)} \\ &\lesssim Mr^{2-4/p} \left\| f \right\|_{L^\infty(S_0)} . \end{aligned}$$

□

Remark 6.4. We emphasize that the constant M remains on the concluded estimate. This will give a certain control on future estimates.

6.3 Broad points

Let S be nearly parabolic. We divide S into $\sim K^2$ caps τ of diameter $\sim K^{-1}$. Let f_τ denote the restriction of f to τ, so that $f = \sum_\tau f_\tau$.

For $\alpha \in (0, 1)$, we say that $x \in \mathbb{R}^3$ is $\alpha-$broad for Ef if

$$\max_\tau |Ef_\tau(x)| \leq \alpha \, |Ef(x)| \, .$$

Define

$$\mathrm{Br}_\alpha \, Ef(x) = \begin{cases} |Ef(x)| & \text{if } x \text{ is } \alpha\text{-broad for } Ef, \\ 0 & \text{otherwise.} \end{cases}$$

If a point x is not α-broad, we roughly have $|Ef(x)| \sim |Ef_\tau(x)|$ for some cap τ. It is also important to note that broad points depend on our choice of K and τ. The following theorem deals with the broad points of Ef, which are the hardest ones to control. The proof of it will be presented in the next chapter.

Theorem 6.5. *For any $\varepsilon > 0$, there exists $K = K(\varepsilon)$ and $L = L(\varepsilon)$ so that if S is parabolic up to order L, then for any radius R,*

$$\|\mathrm{Br}_{K^{-\varepsilon}} Ef\|_{L^{3.25}(B_R)} \leq C_\varepsilon R^\varepsilon \|f\|_2^{12/13} \|f\|_\infty^{1/13},$$

with $\lim_{\varepsilon\to 0} K(\varepsilon) = +\infty$.

Assuming this estimate on broad points is actually enough to obtain an estimate for all of Ef. For the proof, we need to introduce the concept of *induction on scales*. As the name suggests, it has an essence of the usual induction on the natural numbers. Here, of course, we will be dealing with an uncountable set. Namely, the positive real numbers. Then, suppose we have an estimate $A \leq P_R$ for some R. We then want to conclude an estimate $A \leq P_\rho$ for some $\rho > 2R$. This will come in the form of utilizing Lemma 6.3.

Theorem 6.6. *For any $\varepsilon > 0$, there is some L so that if S is parabolic up to order L, then for any radius R, then*

$$\|E_S f\|_{L^{3.25}(B_R)} \leq C_\varepsilon R^\varepsilon \|f\|_\infty .$$

Proof. Assuming Theorem 6.5, we will use the aforementioned induction on the size of the radius R. For the base case, we note that

$$\|E_S f\|_{L^{3.25}(B_1)} \leq |B_1| \, \|E_S f\|_\infty \lesssim \|f\|_1 \leq \|f\|_\infty$$

Thus, we assume the We want to prove that $\|Ef\|_{L^{3.25}(B_R)} \leq \bar{C}_\varepsilon R^\varepsilon \|f\|_\infty$ for some constant $\bar{C}_\varepsilon$ independent of R. By Theorem 6.5, we have

$$\|\mathrm{Br}_{K^{-\varepsilon}} Ef\|_{L^{3.25}(B_R)} \leq C_\varepsilon R^\varepsilon \|f\|_\infty .$$

We will now give a bound for $\|Ef\|^{3.25}_{L^{3.25}(B_R)}$. If x is $K^{-\varepsilon}-$broad, then $|Ef(x)| = \mathrm{Br}\, Ef$. If not, then there is some $K^{-\varepsilon}-$cap τ so that $|Ef(x)| \leq K^\varepsilon |Ef_\tau(x)|$. Therefore,

$$\int_{B_R} |Ef|^{3.25} \leq \int_{B_R} (\mathrm{Br}_{K^{-\varepsilon}} Ef)^{3.25} + K^{O(\varepsilon)} \sum_\tau \int_{B_R} |Ef_\tau|^{3.25} . \tag{6.6}$$

The broad terms are bounded by $(C_\varepsilon R^\varepsilon \|f\|_\infty^{1/13} \|f\|_2^{12/13})^{3.25} \leq (C_\varepsilon R^\varepsilon \|f\|_\infty)^{3.25}$. To bound the Ef_τ terms, we will use Lemma 6.3. Let τ be the graph of h over $B^2_{K^{-1}(\omega_0)}$, and let S_1 be the corresponding surface. We know that S_1 is nearly parabolic. Let K be large enough so that

$$10K^{-1}R < R/2 \iff K > 20.$$

Using induction on R (that is, we assume that the estimate on Theorem 6.6 holds for some R) and applying Lemma 6.3 with $r = K^{-1}$ yields

$$\int_{B_R} |Ef_\tau|^{3.25} \leq CK^{-2.5} (\bar{C}_\varepsilon R^\varepsilon \|f_\tau\|_{L^\infty})^{3.25}.$$

Since there are $\sim K^2$ caps τ, the non-broad terms in (6.6) end up having a contribution

$$\leq CK^{O(\varepsilon)-1/2}(\bar{C}_\varepsilon R^\varepsilon \|f\|_{L^\infty})^{3.25}.$$

Since $\lim_{\varepsilon\to 0} K(\varepsilon) = \infty$, choose ε small enough so that $CK^{O(\varepsilon)-1/2} \leq 1/100$. Choosing $\bar{C}_\varepsilon = 10C_\varepsilon$ closes the induction. $\square$

One may then use Tao's ε-removal theorem on Theorem 6.6 (see Theorem 5.3) to obtain a slightly weaker exponent. That is, we have the following corollary.

Corollary 6.7. *If S is parabolic up to order L, then for $p > 3.25$ we have*

$$\|E_S f\|_{L^p(\mathbb{R}^n)} \lesssim_{p,S} \|f\|_\infty.$$

This corollary can be used to deduce Theorem 0.1 via a partitioning of the surface and using parabolic re-scaling on each of them. The argument is as follows.

Proof. If S is a compact C^∞ surface with strictly positive second fundamental form $Ldx^2 + 2Mdxdy + Ndy^2$. That is, the symmetric matrix

$$\begin{pmatrix} L & M \\ M & N \end{pmatrix}$$

is positive-definite. Then, we can divide S into $C(S)$ pieces so that each piece is contained in the graph of a smooth function. For each chart, we choose an orthonormal system of coordinates. Then, each graph has the form $\omega_3 = h(\vec{\omega})$ for $\vec{\omega}$ contained in a ball of radius $\sim_S 1$. We can assume that $0 = h(0) = \nabla h(0)$. Given that S has positive second fundamental form, the matrix

$$\begin{pmatrix} L' & M' \\ M' & N' \end{pmatrix}$$

associated to h has positive real eigenvalues. That is, for $\lambda, \Lambda \in \mathbb{R}_+$, we have that $0 < \lambda \leq \mathbf{H}_h \leq \Lambda$. Moreover, h is C^∞. For any positive integer L, we do parabolic re-scaling on h for some radius $r = r(\lambda, \Lambda, \|h\|_{C^L})$ as discussed at the start of this section. Recall that we defined

$$\tilde{h}(\vec{\omega}) = h(\vec{\omega}) - (\vec{\omega} - \vec{\omega}_0)\cdot\nabla h(\vec{\omega}_0) - h(\vec{\omega})$$

and, parametrizing $B_r(\vec{\omega}_0)$ by a coordinate $\vec{\eta} \in B_1(0)$ with

$$\vec{\omega} = \vec{\omega}_0 + r\vec{\eta}.$$

Then, we defined

$$h_1(\vec{\eta}) = r^{-2}\tilde{h}(\vec{\omega}_0 + r\vec{\eta}).$$

Recall that we also had

$$\left\|\partial^l h_1\right\|_{C^0} = r^{l-2}\left\|\partial^l h\right\|_{C^0}$$

Thus, we choose r so that

$$\left|\partial^l h_1\right| \leq 10^{-9}$$

for all $3 \leq l \leq L$. We make another change of coordinates so that the Hessian matrix at zero $\mathbf{H}_{h_1}(0)$ is the identity matrix. Thus, $1/2 \leq \mathbf{H}_{h_1} \leq 2$. This change of coordinates may alter the estimate $\left|\partial^l h_1\right| \leq 10^{-9}$ for the larger values of l. To fix this, we once again do parabolic re-scaling on h_1. This function is now nearly parabolic, so we may apply Corollary 6.7 to it. Thus, summing over all $C(S)$ pieces S' of S, we have

$$\|E_S f\|_{L^p(\mathbb{R}^3)} \leq \sum_{S' \subset S} \|E_{S'} f\|_{L^p(\mathbb{R}^3)} \leq C_{S,p} \|f\|_{L^\infty}$$

for all $p > 3.25$. □

Remark 6.8. The choice of p in Theorem 6.5 will become apparent in a later section. This p is sharp given the right hand side of the inequality (see section 7.6 for a detailed argument). However, one can find refinements in [8] and [6], where $E_S(p \to q)$ is proven for $q > 3.25$ and $q \geq 2p'$, which is the full possible range of exponents on p.

Chapter 7

Broad points estimate

7.1 Wave packet decomposition

In this section, we will discuss a common technique used in harmonic analysis, which (roughly) consists in analyzing a small burst sample of a wave function. The results about wave packet decomposition will not be proven, as the proofs do not provide further insight on how the polynomial partitioning works.

7.1.1 Basics of wave packets

Suppose we have a rectangle $\tau \subset \mathbb{R}^2$ centered at ω_τ, given by $M^{-1} \times M^{-2}$. Suppose $\hat{f}_\tau$ is supported on τ. Let $\tau^* \subset \mathbb{R}^2$ denote the dual rectangle given by $M^1 \times M^2$. Now, let $\mathbb{T}_\tau$ denote the tiling of $\mathbb{R}^2$ generated by τ^*. The special structure given to f_τ is what is called a *wave packed decomposition*. With these conditions, one expects to have, for each $T \in \mathbb{T}_\tau$,

$$f_\tau(x) \approx a_T e^{2\pi i \omega_T x},$$

where $a_T \in \mathbb{C}$ is a constant. In particular, $|f_\tau(x)|$ is expected to be constant on every rectangle $T \in \mathbb{T}_\tau$. According to these heuristics, we can express f_τ on all of $\mathbb{R}^2$ as a sum of its components in each rectangle:

$$f_\tau(x) \approx \sum_{T \in \mathbb{T}_\tau} a_T e^{2\pi i \omega_\tau x} \chi_T,$$

where χ_T is the characteristic function of T (or a test function with support in T). Each term in the right hand side is called a wave packet, and the whole sum is called a wave packet decomposition.

The above heuristics can be formally proved with some more technical details. But to give a reason as to why these should hold true, we remember that the Fourier transform behaves nicely under linear transformations and translations. It now becomes apparent that trying to understand the wave packet decomposition on $\tau = [-1,1]^2$ is sufficient. Then, suppose $\hat{f}$ is supported on $\tau = [-1,1]^2$, so that by the Fourier inversion formula,

$$f(x) = \int_{[-1,1]^2} \hat{f}(\omega) e^{2\pi i \omega \cdot x}\, d\omega.$$

For $|x| \leq 1$, the functions $e^{2\pi i \omega \cdot x}$ will vary slowly as ω runs through $[-1,1]^2$, and will look roughly constant in any scale smaller than a unit square. Since f is a linear combination of these functions with a slight decay/increase in each unit square, one expects f to be nearly constant in scales smaller than a unit square.

7.1.2 Wave packets on paraboloids

One can extend these ideas to surfaces such as paraboloids in a rather simple way. Suppose we want to decompose the extension of f to the paraboloid S. As before, we want to create a tiling of $\mathbb{R}^2$, so taking *caps* of the surface is a natural way doing so. We now make these ideas precise. Decompose S into

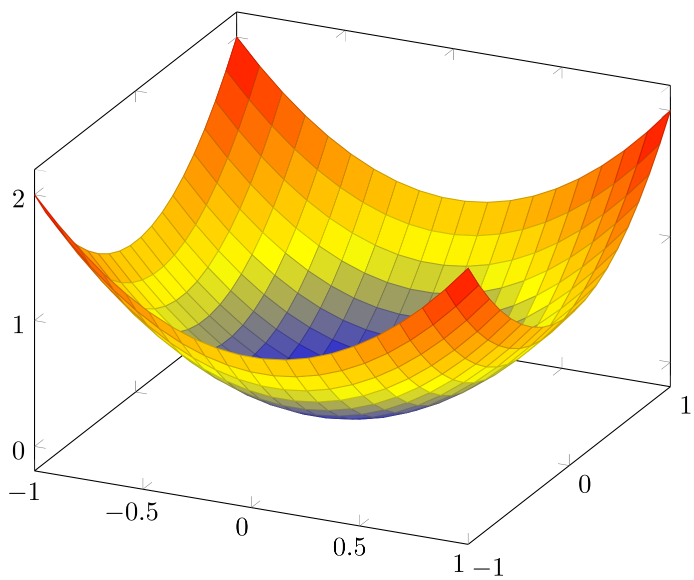

Figure 7.1: A section of a paraboloid partitioned into caps.

$R^{-1/2}-$caps θ. Let ω_θ denote a vector near the center of $S \cap \theta$. The reason why we make explicit that it need to be the center of the intersection is due to the possibility of some caps being truncated when truncating S. Let v_θ denote the unit normal vector to S at ω_θ (which points outwards).

Let $\delta > 0$ be a small parameter. For each cap θ, define $\mathbb{T}_\theta$ to be the set of cylindrical tubes parallel to v_θ, with radius $R^{(1/2)+\delta}$ and length $\sim R$. This set now covers B_R. The choice to choose a radius bigger than the dual of the radius of the cap, which would be $R^{1/2}$, is because we want a very sharp decay in the wave packets outside of these tubes. As mentioned before, $\mathbb{T}_\theta$ covers B_R, but we can also note that for any $x \in B_R$, x lies in $O(1)$ of tubes $T \in \mathbb{T}_\theta$. We let $\mathbb{T} = \bigcup_\theta \mathbb{T}_\theta$. One can refer to Figure 7.1 to better the direction of the tubes in $\mathbb{T}$.

Let θ be a cap, and let $h : \mathbb{R}^2 \to \mathbb{R}$ be a function whose graph is precisely θ when taking $B_r^2(\vec{\omega}_\theta)$ as its domain. Let 3θ denote a larger cap containing θ, given by the same $h : \mathbb{R}^2 \to \mathbb{R}$ when taking $B_{3r}^2(\vec{\omega}_\theta)$ as its domain.

If T is a tube in T_θ, we let $v(T) = v_\theta$ be the direction of the tube. With this decomposition and notation, we now state several estimates on the wave packet decomposition of Ef.

Proposition 7.1. *Suppose S is nearly parabolic. Let $\mathbb{T}$ be as described above, with $\delta > 0$. Suppose R is sufficiently large, depending on δ. If f is a function in $L^2(S)$, then for each $T \in \mathbb{T}$, we can choose a function f_T so that the following holds.*

(1) *If $T \in T_\theta$, then* $\operatorname{supp} f_T \subset 3\theta$.

(2) *If $x \in B_R \backslash T$, then $|Ef_T(x)| \leq R^{-1000} \|f\|_{L^2}$.*

(3) *For any $x \in B_R$, $\left|Ef(x) - \sum_{T\in\mathbb{T}} Ef_T(x)\right| \leq R^{-1000} \|f\|_{L^2}$.*

(4) *(essential orthogonality) If $T_1, T_2 \in \mathbb{T}_\theta$ and T_1, T_2 are disjoint, then we have $\int f_{T_1} \bar{f}_{T_2} \leq R^{-1000} \int_\theta |f|^2$.*

(5) $\sum_{T\in T_\theta} \int_S |f_T|^2 \lesssim \int_\theta |f|^2$.

Remark 7.2. We will recurrently use Proposition 7.1 on the functions f_τ. The first property then says that if $\operatorname{supp} f_\tau \subset \tau$, then for every $T \in \mathbb{T}$, the support of $f_{\tau,T}$ is in an $O(R^{-1/2})$ neighborhood of τ.

Let $\mathbb{T}_i \subset \mathbb{T}$, for i in an indexing set $\mathbb{I}$. For each τ and i, we define

$$f_{\tau,i} \coloneqq \sum_{T\in\mathbb{T}_i} f_{\tau,T}.$$

Lemma 7.3. *Consider some subsets $\mathbb{T}_i \subset \mathbb{T}$ indexed by $i \in I$. If each tube T belongs to at most μ of the subsets $\{\mathbb{T}_i\}_{i\in I}$, then for every θ,*

$$\sum_{i\in I} \int_{3\theta} |f_{\tau,i}|^2 \lesssim \mu \int_{10\theta} |f_\tau|^2 .$$

Also,

$$\sum_{i\in I} \int_S |f_{\tau,i}|^2 \lesssim \mu \int_S |f_\tau|^2 .$$

Lemma 7.4. *If $\mathbb{T}_i \subset \mathbb{T}$, then for any cap θ, and any τ,*

$$\int_{3\theta} |f_{\tau,i}|^2 \lesssim \int_{10\theta} |f_\tau|^2 .$$

Remark 7.5. Note that Lemma 7.4 is simply the case where the indexing set has cardinality 1.

7.2 Reductions on the broad estimate

This section will contain most of the proof of Theorem 6.5. It will use polynomial partitioning (see section 2.2 and section 2.3), wave packet decomposition, presented in the previous section, and induction on scales (see page 44).

7.2.1 The inductive setup

We will first assume that the caps which split our surface are not necessarily disjoint. Suppose that each cap τ is the graph of h over a ball $B_r(\vec{\omega}_\tau)$, and that the union of the τ is S. Consider a decomposition $f = \sum_\tau f_\tau$, where $\operatorname{supp} f_\tau \subset \tau$.

Assume that the centers $\{\vec{\omega}_\tau\} \subset B_1(0)$ are K^{-1}−separated. We define the *multiplicity* μ of the covering by saying that the radius r of each cap τ lies in the range $[K^{-1}, \mu^{1/2}K^{-1}]$. Since the area of each cap grows by a square factor of the radius, and the caps are K^{-1} separated, any point in S belongs to at most μ caps τ.

Let $⨍_\theta |f|^2$ denote the average of $|f|^2$ over a cap θ.

Theorem 7.6. *For any $\varepsilon > 0$, there exists K, L and a small $\delta_{trans} \in (0, \varepsilon)$, depending only on ε, so that the following holds.*

Suppose that S is the graph of a function h which is parabolic up to order L. Suppose that the caps τ cover S with multiplicity at most μ, and suppose that $\alpha \geq K^{-\varepsilon}$.

If for any τ and $\omega \in S$,

$$⨍_{B_{R^{-1/2}}(\omega)\cap S} |f_\tau|^2 \leq 1,$$

then

$$\int_{B_R} (\mathrm{Br}_\alpha\, Ef)^{3.25} \leq C_\varepsilon R^\varepsilon \left(\sum_\tau \int_S |f_\tau|^2 \right)^{(3/2)+\varepsilon} R^{\delta_{trans}\log(K^\varepsilon \alpha\mu)}.$$

Moreover, $\lim_{\varepsilon\to 0} K(\varepsilon) = +\infty$.

We can deduce Theorem 6.5 from Theorem 7.6 as follows. Fix $\varepsilon > 0$. We may assume that f is such that $\|f\|_{L^\infty} = 1$ (by normalizing). We then divide the surface S into a disjoint union of K^{-1}-caps τ. By definition of the multiplicity of the cover μ, we must have $\mu \lesssim 1$, as the caps are essentially disjoint. Take $f_\tau = f\chi_\tau$. Then we have $\sum_\tau \int_S |f_\tau|^2 = \int_S |f|^2$. We know that $\|f\|_{L^\infty} = 1$, so $\sup|f| \leq 1$ for a.e. point on the surface. Therefore, the average value of $|f_\tau|^2$ on any region of the surface cannot exceed 1. Take $\alpha = K^{-\varepsilon}$. Moreover, one has $R^{\delta_{trans}\log(K^\varepsilon \alpha\mu)} \leq R^{C\delta_{trans}} \leq R^{O(\varepsilon)}$. Thus, applying Theorem 6.5, and using that $\|f\|_{L^\infty} = 1$ we have

$$\begin{aligned}
\int_{B_R} (\mathrm{Br}_\alpha Ef)^{3.25} &\lesssim C_\varepsilon R^{O(\varepsilon)} \left(\int_S |f|^2\right)^{(3/2)+\varepsilon} \\
&= C_\varepsilon R^{O(\varepsilon)} \|f\|_{L^2}^3 \|f\|_{L^2}^{2\varepsilon} \\
&\lesssim C_\varepsilon R^{O(\varepsilon)} \|f\|_{L^2}^3 \|f\|_{L^\infty}^{2\varepsilon} \\
&= C_\varepsilon R^{O(\varepsilon)} \|f\|_{L^2}^3 \|f\|_{L^\infty}^{1/4}.
\end{aligned}$$

Equivalently, taking the $4/13 = 3.25$ power from both sides, we obtain

$$\|\mathrm{Br}_\alpha Ef\|_{L^{3.25}(B_R)} \leq C_\varepsilon R^{O(\varepsilon)} \|f\|_{L^2}^{12/13} \|f\|_{L^\infty}^{1/13}.$$

Since $\varepsilon > 0$ was arbitrary, this proves Theorem 6.5.

The parameter in the previous theorem has a special name because there will be various parameters we will keep track of: $\delta, \delta_{trans}, \delta_{\deg}, \varepsilon$ and K. We will actually choose $\delta_{trans} = \varepsilon^6$, $K = e^{\varepsilon^{-10}}$, $\delta = \varepsilon^2$ and $\delta_{\deg} = \varepsilon^4$. Thus, we have

$$\delta_{trans} \ll \delta_{\deg} \ll \delta \ll \varepsilon.$$

We will also need $K \gg \delta_{trans}$ so that $R^{\delta_{trans}\log(10^{-6}K^\varepsilon)} \geq R^{1000}$.

7.3 Using polynomial partitioning

During the following sections, we will use the notation $A \lesssim B$ for $A \leq C(\varepsilon)B$.

The polynomial partitioning method presented in chapter 3 will now be utilized to prove Theorem 7.6. We first pick a degree $D = R^{\delta_{\deg}}$ where $\delta_{\deg} = \varepsilon^4$. Then we apply polynomial partitioning with this degree to the function $\chi_{B_R} \mathrm{Br}_\alpha Ef^{3.25}$. By Corollary 2.9, there is a polynomial P of degree at most D so that $\mathbb{R}^n \backslash Z(P)$ is the disjoint union of $\sim D^3$ cells O_i, and so that for each i one has

$$\int_{O_i \cap B_R} (\mathrm{Br}\, Ef)^{3.25} \sim D^{-3} \int_{B_R} (\mathrm{Br}_\alpha Ef)^{3.25}.$$

Moreover, the polynomial P is a product of non-singular polynomials.

Let $A \subset \mathbb{R}^n$. Define

$$N_r A = \bigcup_{x \in A} B_r(x).$$

That is, an r-neighborhood around A. Also define $\mathbb{T} = \bigcup_\theta \mathbb{T}(\theta)$. Then, define

$$W := N_{R^{(1/2)+\delta}} Z(P), \qquad O_i' := (O_i \cap B_R) \backslash W$$

That is, W is the zero set of the polynomial with the girth of a tube (and a small bit more). Then, one can perfectly fit a portion of some tube $T \in \mathbb{T}(\theta)$ (for a proper cap θ).We also define

$$\mathbb{T}_i := \{T \in \mathbb{T} \text{ so that } T \cap O_i' \neq \varnothing\}.$$

We let

$$f_{\tau,i} = \sum_{T \in \mathbb{T}_i} f_{\tau,T} \qquad \text{and} \qquad f_i = \sum_{\tau} f_{\tau,i}.$$

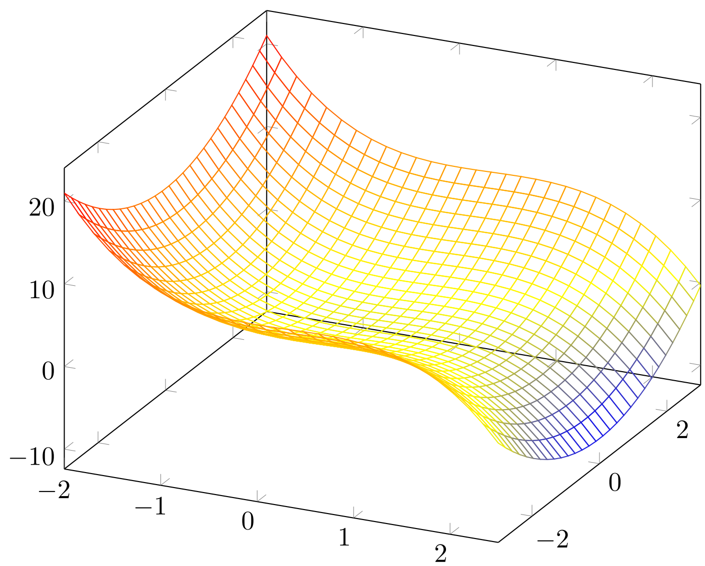

Figure 7.2: The cubic curve $z = y^2 - x(x-2)^2$ in $\mathbb{R}^3$.

Moreover, note that if $T \in \mathbb{T}_i$, then $T \cap O_i'$ is non-empty, which means that the central line of the tube T must intersect O_i. In 2 dimensions, this just means that the center of the interval (cylinder which is of width exactly the girth of W), cannot be the zero of the polynomial, otherwise the intersection with O_i' would be empty. Since a line can cross $Z(P)$ at most D times, a tube $T \in \mathbb{T}$ with this central line will cross at most $D+1$ regions O_i'. We state this remark as a lemma.

Lemma 7.7. *Each tube $T \in \mathbb{T}$ lies in at most $D+1$ of the sets $\mathbb{T}_i$.*

The integral of $\mathrm{Br}_\alpha Ef^{3.25}$ on a cell O_i' will be controlled by induction. We will also have to control it on W.

We cover B_R with $\sim R^{3\delta}$ balls B_j of radius $R^{1-\delta}$. This radius is rather close in size to R, which means that the number of balls will be relatively low. If $B_j \cap W \neq \varnothing$, then we must identify which tubes of $\mathbb{T}$ are tangent to $Z(P)$ in B_j and which are transverse by identifying the angles they make with the tangent

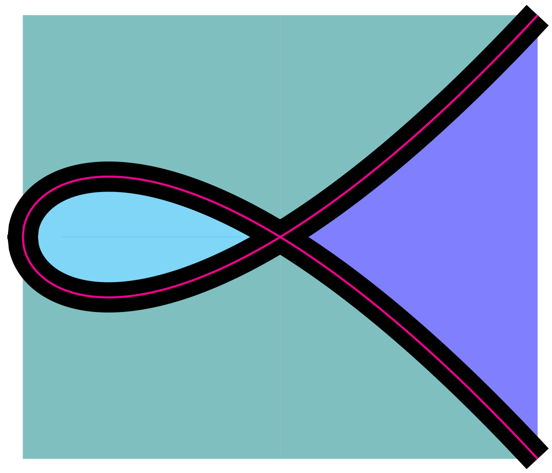

Figure 7.3: The set W projected into the x, y-plane in black, the zero set of $z = y^2 - x(x-2)$ in magenta, and the three regions partitioned by the curve in different shades of blue.

planes at points of the zero set. We now define these specific sets of tubes in $\mathbb{T}$ to make a distinction in their contributions to the integral of $\mathrm{Br}_\alpha Ef^{3.25}$.

Definition 7.8. We denote by $\mathbb{T}_{j,tang}$ the set of all $T \in \mathbb{T}$ obeying the two following conditions.

- $T \cap W \cap B_j \neq \varnothing$.
- If z is any non-singular point of $Z(P)$ lying in $2B_j \cap 10T$, then

$$\mathrm{Angle}(v(T), T_z Z) \leq R^{-(1/2)+2\delta},$$

  where $v(T)$ denotes the unit vector with direction of the tube T and $T_z Z$ is the tangent plane at z of $Z(P)$.

Definition 7.9. We denote by $\mathbb{T}_{j,trans}$ the set of all $T \in \mathbb{T}$ obeying the two following conditions.

- $T \cap W \cap B_j \neq \varnothing$.
- There exists a non-singular point z of $Z(P)$ lying in $2B_j \cap 10T$, so that

$$\mathrm{Angle}(v(T), T_z Z) > R^{-(1/2)+2\delta}.$$

The reason behind the choice of angle is directly related to the choice of radius for our tubes and the decomposition of B_R. We do not give explicit details, as it is essentially a purely geometrical problem, but we remark that the radius for $10T$ is $10R^{1/2+\delta}$, the angle given is $R^{-1/2+2\delta}$, and *it can be proven*[1] that we can decompose $10T$ into tubes of length $10R^{1/2+\delta} \cdot R^{1/2-2\delta} = 10R^{1-\delta}$, which is 10 times the radius of the balls which cover B_R.

We claim that any tube $T \in \mathbb{T}$ that intersects $W \cap B_j$ lies in exactly one of $T_{j,tang}$ and $T_{j,trans}$. Given the definitions, if $T \cap W \cap B_j \neq \varnothing$, all we need to do is prove that there is a non-singular point of $Z(P)$ in $10T \cap 2B_j$, given that the inequalities are mutually disjoint. Now, using the fact that W in an $R^{(1/2)+\delta}$ neighborhood of $Z(P)$ and the $T's$ are tubes of radius also $R^{(1/2)+\delta}$, then if $x \in T \cap W \cap B_j$, then there is a point $z \in Z(P)$ with $d(x,z) \leq R^{(1/2)+\delta}$. This point z lies in $10T \cap 2B_j$. Since P is a product of non-singular polynomials, the set of non-singular values is dense in $Z(P)$, so we can assume z is non-singular.

We now state two lemmas about the tangent and transversal tubes. The proofs will be omitted, as they fall beyond the scope of this text. They can also be found in the last section of [5].

Lemma 7.10. *Each tube $T \in \mathbb{T}$ belongs to at most $\mathrm{Poly}(D) = R^{O(\delta_{\deg})}$ different sets $\mathbb{T}_{j,trans}$.*

Remark 7.11. A tube T intersects R^{δ} distinct balls B_j. We choose $\delta_{\deg} = \varepsilon^4 \ll \varepsilon^2 = \delta$. So T belongs to $\mathbb{T}_{j,trans}$ for only a small fraction of these balls.

Lemma 7.12. *For each j, the number of different θ so that $\mathbb{T}_{j,tang} \cap \mathbb{T}(\theta) \neq \varnothing$ is at most $R^{(1/2)+O(\delta)}$.*

Remark 7.13. There are $\sim R$ different caps $\theta \subset S$. Lemma 7.12 says that only $R^{1/2}$ of the caps will contribute to $\mathbb{T}_{j,trans}$. For instance, if $Z(P)$ is a plane, then only the directions tangent to the plane can appear in $\mathbb{T}_{j,tang}$.

We define $f_{\tau,j,tang} \coloneqq \sum_{T \in \mathbb{T}_{j,tang}} f_{\tau,T}$ and $f_{j,tang} \coloneqq \sum_{\tau} f_{\tau,T,tang}$ and similarly for $f_{\tau,j,trans}$ and $f_{j,trans}$.

7.4 The inductive step

We will now split the integral $\int_{B_R} \mathrm{Br}_\alpha Ef^{3.25}$ into pieces coming from f_i, $f_{j,trans}$ and $f_{j,tang}$. We call these the cellular, transverse and tangential pieces, respectively. The tangential pieces can be bounded by a direct calculation, while the other two pieces will require induction. In this section, we will only prove bounds for transverse and cellular pieces.

If $x \in O'_i$, then as seen in Proposition 7.1, $Ef_\tau(x)$ is almost equal to $Ef_{\tau,i}(x)$ for each τ. We would also want to relate the $\alpha-$broad parts of $Ef(x)$ and $Ef_i(x)$.

Lemma 7.14. *If $x \in O'_i$ and R is large enough, then*

$$\mathrm{Br}_\alpha Ef(x) \leq 2\,\mathrm{Br}_{2\alpha} Ef_i(x) + R^{-900} \sum_{\tau} \|f_\tau\|_2 \,.$$

[1] See the last section of [5].

Proof. By Proposition 7.1, we have

$$Ef_\tau(x) = \sum_{T\in\mathbb{T}} Ef_{\tau,T}(x) + O(R^{-1000}\left\|f_\tau\right\|_2).$$

If $x \in T$, then $x \in O_i' \cap T$, so $T \in \mathbb{T}_i$. If $x \notin T$, then Proposition 7.1 gives $|Ef_{\tau,T}| \le R^{-1000}\left\|f_\tau\right\|_2$. The total contribution of these $T \notin \mathbb{T}_i$ is small, leaving

$$Ef_\tau(x) = Ef_{\tau,i}(x) + O(R^{-990}\left\|f_\tau\right\|_2). \tag{7.1}$$

Summing over τ, we have

$$Ef(x) = Ef_i(x) + O(R^{-990}\sum_\tau \left\|f_\tau\right\|_2). \tag{7.2}$$

We now deal with the $\alpha-$broad points. First assume that we have the inequality $|Ef(x)| \ge R^{-900}\sum_\tau \left\|f_\tau\right\|_2$, and thus $|Ef_i| \ge (1/2)R^{-900}\sum_\tau \left\|f_\tau\right\|_2$. We can also assume x is $\alpha-$broad for Ef. We must now show that under these assumptions, x is $2\alpha-$broad for Ef_i. using (5.1) and (5.2), we have

$$\begin{aligned}|Ef_{\tau,i}(x)| &\le |Ef_\tau(x)| + O(R^{-990}\left\|f_\tau\right\|_2) \le \alpha\,|Ef(x)| + O(R^{-990}\left\|f_\tau\right\|_2)\\ &\le \alpha\,|Ef_i(x)| + O(R^{-990}\sum_\tau \left\|f_\tau\right\|_2) \le 2\alpha\,|Ef_i(x)|\,.\end{aligned}$$

□

On the counterpart, if $x \in W \cap B_j$, its components are harder to estimate. In this case, $Ef(x)$ is almost equal to $Ef_{j,trans}(x) + Ef_{j,tang}(x)$. But in order for the $\alpha-$broad terms to behave well, we will need some other related functions.

Recall S is divided into $\sim K^2$ caps τ of diameter K^{-1}. If I is any subset of these caps, we let $f_{I,j,trans} \coloneqq \sum_{\tau\in I} f_{\tau,j,trans}$. The function $f_{I,j,trans}$ comes with a natural decomposition: if $\tau \in I$, we let $f_{\tau,I,trans} = f_{\tau,j,trans}$, and if $\tau \notin I$, then $f_{\tau,I,j,trans} = 0$.

For the computation of the terms involving $f_{\tau,tang}$, we define a bilinear version of it. We say two caps τ_1, τ_2 are non-adjacent if the distance between them is $\ge K^{-1}$:

$$\operatorname{Bil}(Ef_{j,tang}) \coloneqq \sum_{\substack{\tau_1,\tau_2\\ \text{non-adjacent}}} |Ef_{\tau_1,j,tang}|^{1/2}\,|Ef_{\tau_2,j,tang}|^{1/2}\,.$$

With these definitions in hand, we can now state our lemma connecting $\operatorname{Br}_\alpha Ef$ with f_i, $f_{j,trans}$ and $f_{j,tang}$.

Lemma 7.15. *If $x \in B_j \cap W$ and $\alpha\mu \le 10^{-5}$, then $\operatorname{Br}_\alpha |Ef(x)|$ is bounded by*

$$2\left(\sum_I \operatorname{Br}_{2\alpha}|Ef_{I,j,trans}(x)| + K^{100}\operatorname{Bil}(Ef_{j,tang})(x) + R^{-900}\sum_\tau \left\|f_\tau\right\|_2\right).$$

Remark 7.16. Note that there are roughly 2^{K^2} sets over which we are summing. Since K is a constant depending on ε, the coefficient 2^{K^2} will not pose a problem.

Proof. Suppose $x \in B_j \cap W$. We can assume that x is an $\alpha-$broad point and that $|Ef(x)| \geq R^{-900} \sum_\tau \|f_\tau\|_2$.

Let I be the set of K^{-1}-caps τ so that $|Ef_{\tau,j,tang}|(x) \leq K^{-100} |Ef(x)|$. In other words, I^c is the set of caps τ so that $|Ef_{\tau,j,tang}(x)| \geq K^{-100} |Ef(x)|$. If I^c contains two non-adajcent caps, then $|Ef(x)| \leq K^{100} \operatorname{Bil}(Ef_{j,tang}(x))$, and the conclusion holds.

If I^C does not contain two non-adjacent caps, then I^C consists of at most $10^4\mu$ caps, because the centers of the caps are K^{-1} separated, and the radius of each cap is at most $\mu^{1/2}K^{-1}$. Since x is $\alpha-$broad for Ef, and $\alpha\mu \leq 10^{-5}$, we have

$$\sum_{\tau \in I^C} |Ef_\tau(x)| \leq 10^4 \mu\alpha \, |Ef(x)| \leq (1/10) \, |Ef(x)| \, .$$

Therefore, $|Ef_I(x)| \geq (9/10) \, |Ef(x)|$. Next, we split Ef_I into tangential and transverse contributions.

If $T \in \mathbb{T}$ and $T \cap B_j \cap W \neq \varnothing$, then T belongs to $\mathbb{T}_{j,trans}$ or $\mathbb{T}_{j,tang}$. If T does not intersect $B_j \cap W$, then $|f_{\tau,T}(x)| = O(R^{-1000} \|f_\tau\|_2)$. Therefore, for any cap τ, we have

$$|Ef_\tau(x)| \leq |Ef_{\tau,j,trans}(x)| + |Ef_{\tau,j,trans}(x)| + O(R^{-990} \|f_\tau\|_2). \tag{7.3}$$

Summing over all $\tau \in I$, we have

$$|Ef_I(x)| \leq |Ef_{I,j,trans}(x)| + \left(\sum_{\tau \in I} |Ef_{\tau,j,trans}(x)| \right) + O(R^{-990} \sum_\tau \|f_\tau\|_2). \tag{7.4}$$

But for each cap $\tau \in I$, one has $|Ef_{\tau,j,trans}(x)| \leq K^{-100} |Ef(x)|$, and so $\sum_{\tau \in I} |Ef_{\tau,j,i}| \leq K^{-98} |Ef(x)|$. Plugging this in and using that $|Ef_I(x)| \geq (9/10)$, we have

$$(9/10) \, |Ef(x)| \leq |Ef_{I,j,trans}(x)| + K^{-98} \, |Ef(x)| + O(R^{-980} \sum_\tau \|f_\tau\|_2).$$

Since $|Ef(x)| \geq R^{-900} \sum_\tau \|f_\tau\|_2$, we see that

$$|Ef(x)| \leq (3/2) \, |Ef_{I,j,trans}(x)| \, . \tag{7.5}$$

In this case, it remains to prove that x is $2\alpha-$broad for $Ef_{I,j,trans}$. Given (7.5), it suffices to prove that for each $\tau \in I$,

$$|Ef_{\tau,j,trans}(x)| \leq (1.1)\alpha \, |Ef(x)| \, .$$

Rewriting (7.4) we may obtain

$$|Ef_{\tau,j,trans}(x)| \leq |Ef_\tau(x)| + |Ef_{\tau,j,tang}(x)| + O(R^{-990} \|f_\tau\|_2).$$

Since $\tau \in I$, $|Ef_{\tau,j,trans}(x)| \leq K^{-100} |Ef(x)|$. Therefore, we have

$$|Ef_{\tau,j,trans}(x)| \leq \alpha |Ef(x)| + K^{-100} |Ef(x)| + O(R^{-990} \|f_\tau\|_2).$$

Because $|Ef(x)| \geq R^{-900} \sum_\tau \|f_\tau\|_2$ and $\alpha \geq K^{-\varepsilon}$, we have

$$|Ef_{\tau,j,trans}(x)| \leq (1.1)\alpha |Ef(x)|.$$

Hence, the point x is $2\alpha-$broad for $Ef_{I,j,trans}$. □

We may now state an estimate for the tangential terms.

Proposition 7.17.

$$\int_{B_j} \operatorname{Bil}(Ef_{j,tang})^{3.25} \lesssim R^{O(\delta)} \left(\sum_\tau \int |f_\tau|^2 \right)^{3/2}.$$

The proof will be given in the next section. For now, we can use this estimate to prove Theorem 7.6 using induction.

Proof of Theorem 7.6. We do induction on the radius R. For each radius R, we also induct on $\sum_\tau \int |f_\tau|^2$. As a base case, the theorem holds when $R = 1$ or when $\sum_\tau \int |f_\tau|^2 \leq R^{-1000}$. For $R = 1$, the estimate follows from the trivial

$$\int_{\mathbb{B}} (\operatorname{Br}_\alpha Ef)^{3.25} \leq \int_{\mathbb{B}} |Ef|^{3.25} \leq \|f\|_{L^2(S)}^{3+2\varepsilon} \sim \left(\sum_\tau \int_S |f|^2 \right)^{3/2+\varepsilon}.$$

If instead we have $\sum_\tau \int |f_\tau|^2 \leq R^{-1000}$, the theorem follows from noting that

$$\sup |\operatorname{Br}_\alpha Ef| \leq \left(\sum_\tau \int_S |f_\tau| \right) \leq C R^{O(\varepsilon)} \left(\sum_\tau \int_S |f|^2 \right)^{1/2}.$$

Therefore,

$$\begin{aligned}
\int_{B_R} |\operatorname{Br}_\alpha Ef|^{3.25} &\leq C R^3 \left(\sum_\tau \int_S |f_\tau| \right)^{3.25} \\
&\leq C R^4 \left(\sum_\tau \int_S |f_\tau|^2 \right)^{3/2+1/8} \\
&\leq C R^{-100} \left(\sum_\tau \int_S |f_\tau|^2 \right)^{3/2+\varepsilon}.
\end{aligned}$$

Thus, we assume that Theorem 7.6 holds for all radii $\leq R/2$ or for functions g with

$$\sum_\tau \int_S |g|^2 \leq (1/2) \sum_\tau \int |f_\tau|^2.$$

If $\mu\alpha \geq 10^{-6}$, then the estimate becomes trivial, as the term $R^{\delta_{trans}\log(K^\varepsilon \mu\alpha)}$ is very large: we choose $K(\varepsilon) = e^{\varepsilon^{-10}}$ and so the exponent $\varepsilon^6 \log(K^\varepsilon 10^{-6}) \gtrsim \varepsilon^{-4}$. For ε small enough ($\varepsilon \lesssim 0.1$) so that $R^{\delta_{trans}\log(K^\varepsilon \mu\alpha)}$ is at least R^{1000}, so the bound is trivially true. Therefore, we assume that $\alpha\mu \leq 10^{-6}$.

We decompose the integral into pieces in the cells and a piece from the walls between cells. That is, the boundary of W.

$$\int_{B_R} (\mathrm{Br}_\alpha Ef)^{3.25} = \sum_i \int_{B_R \cap O_i'} (\mathrm{Br}_\alpha Ef)^{3.25} + \int_{B_R \cap W} (\mathrm{Br}_\alpha Ef)^{3.25}.$$

If the cellular terms ($B_R \cap O_i'$) dominate, then we first note that $\int_{B_R \cap O_i'} (\mathrm{Br}_\alpha Ef)^{3.25}$ is essentially independent of i, so there must be $\sim D^3$ different cells O_i' so that

$$\int_{B_R \cap O_i'} (\mathrm{Br}_\alpha Ef)^{3.25} \sim D^{-3} \int_{B_R} (\mathrm{Br}_\alpha Ef)^{3.25}. \tag{7.6}$$

For each such i, we apply Lemma 7.14 and thus

$$\begin{aligned}
\int_{B_R} (\mathrm{Br}_\alpha Ef)^{3.25} &\lesssim D^3 \int_{B_R \cap O_i'} (\mathrm{Br}_\alpha Ef)^{3.25} \\
&\leq D^3 \int_{B_R} (2\,\mathrm{Br}_{2\alpha} Ef_i + R^{-900} \sum_\tau \|f_\tau\|_2)^{3.25} \\
&\lesssim D^3 \int_{B_R} \left[(\mathrm{Br}_{2\alpha} Ef_i)^{3.25} + (R^{-900} \sum_\tau \|f_\tau\|_2)^{3.25} \right] \\
&\lesssim D^3 \int_{B_R} \left[(\mathrm{Br}_{2\alpha} Ef_i)^{3.25} + R^{-1000} \sum_\tau \|f_\tau\|_2^{3.25} \right]
\end{aligned}$$

If the error $R^{-1000} \sum_\tau \|f_\tau\|_2^{3.25}$ dominates the broad term, then

$$\begin{aligned}
\int_{B_R} (\mathrm{Br}_\alpha Ef)^{3.25} \lesssim D^3 R^{-1000} \sum_\tau \|f_\tau\|_2^{3.25} &= D^3 R^{-1000} \sum_\tau \left(\int_S |f_\tau|^2 \right)^{3/2+1/8} \\
&\leq D^3 R^{-1000} \left(\sum_\tau \int_S |f_\tau|^2 \right)^{3/2+1/8}.
\end{aligned}$$

Now, we consider $\sum_\tau \int |f_{\tau,i}|^2$. By Lemma 7.7, each tube $T \in \mathbb{T}$ lies in at most $D+1$ of the sets $\mathbb{T}_i$. By **??**, we know that

$$\sum_i \int |f_{\tau,i}|^2 \lesssim D \int |f_\tau|^2 .$$

Choosing an index i such that (7.6) is satisfied, we have

$$\sum_\tau \int |f_{\tau,i}|^2 \lesssim D^{-2} \sum_\tau \int |f_\tau|^2 . \tag{7.7}$$

We claim that we can apply Theorem 7.6 to $f_i = \sum_\tau f_{\tau,i}$. By Proposition 7.1, we know that $\operatorname{supp} f_{\tau,i}$ is in a small neighborhood of τ. Therefore, the new multiplicity μ is barely larger than μ (at most 2μ). By Lemma 7.4, we have that for any $\omega \in S$,

$$\fint_{B_{R^{-1/2}}(\omega)\cap S} |f_{\tau,i}|^2 \lesssim \fint_{B_{10R^{-1/2}}(\omega)\cap S} |f_\tau|^2 \lesssim 1.$$

Equivalently,

$$C\fint_{B_{R^{-1/2}}(\omega)\cap S} |f_{\tau,i}|^2 \le 1$$

for a suitable constant C. Hence, multiplying f_i by a constant gives the necessary conditions to apply Theorem 7.6. Moreover, $\sum_\tau \int |f_{\tau,i}|^2 \le (1/2)\sum_\tau \int |f_\tau|^2$. By induction on $\sum_\tau \int |f_\tau|^2$, we can apply Theorem 7.6 to f_i. This leads to the estimate

$$\begin{aligned}\int_{B_R} (\operatorname{Br}_\alpha Ef)^{3.25} &\lesssim D^3 \int_{B_R} (\operatorname{Br}_{2\alpha} Ef_i)^{3.25} \\ &\lesssim D^3 C_\varepsilon R^\varepsilon R^{\delta_{trans}\log(4\alpha\mu K^\varepsilon)} \left(\sum_\tau \int |f_{\tau,i}|^2\right)^{3/2+\varepsilon}.\end{aligned}$$

Note that we have obtained a 4 inside of the log because of the new multiplicity (2μ) and the 2α-broad terms. Using (7.7) on the last estimate, we arrive to

$$\int_{B_R} \operatorname{Br}_\alpha Ef^{3.25} \le (CD^{-2\varepsilon}R^{4\delta_{trans}})C_\varepsilon R^\varepsilon R^{\delta_{trans}\log(\mu\alpha K^\varepsilon)} \left(\sum_\tau \int |f_\tau|^2\right)^{3/2+\varepsilon}.$$

To close the induction, we only need to prove that the term $CD^{-2\varepsilon}R^{4\delta_{trans}}$ is at most 1. First note that the term is at most $R^{-\delta_{\deg}\varepsilon+4\delta_{trans}}$. Since $\delta_{\deg} = \varepsilon^4$ and $\delta_{trans} = \varepsilon^6$, the exponent of R is negative, so the term is at most 1 and the induction closes. This covers one case.

If, instead, the decomposition

$$\int_{B_R} (\operatorname{Br}_\alpha Ef)^{3.25} = \sum_i \int_{B_R\cap O_i'} (\operatorname{Br}_\alpha Ef)^{3.25} + \int_{B_R\cap W} (\operatorname{Br}_\alpha Ef)^{3.25}$$

is dominated by the contribution of the cell walls, we use Lemma 7.15 and recall that we considered a cover for B_R with $\sim R^{3\delta}$ balls B_j of radius $R^{1-\delta}$, so we first obtain an estimate on each $B_j \cap W$:

$$\begin{aligned}\int_{B_j\cap W} (\operatorname{Br}_\alpha Ef)^{3.25} \lesssim \sum_I \int_{B_j\cap W} &(\operatorname{Br}_{2\alpha} Ef_{I,j,trans})^{3.25} \\ &+ K^{100}\int_{B_j\cap W} (\operatorname{Bil}(Ef_{j,tang}))^{3.25} \\ &+ R^{-1000}\int_{B_j\cap W} \sum_\tau \|f_\tau\|_2^{3.25}.\end{aligned}$$

Thus, summing over all j, we have

$$\int_{B_R} (\mathrm{Br}_\alpha\, Ef)^{3.25} \lesssim \sum_{j,I} \int_{B_j \cap W} (\mathrm{Br}_{2\alpha}\, Ef_{I,j,trans})^{3.25} + K^{100} \sum_j \int_{B_j \cap W} (\mathrm{Bil}(Ef_{j,tang}))^{3.25} + O\Big(R^{-1000} \sum_\tau \|f_\tau\|_2^{3.25}\Big).$$

If the last O-term dominates, we simply use the fact that $\sum_\tau \|f_\tau\|_2^2 \lesssim 1$ to conclude the proof. If not, we note that by Proposition 7.17, we have

$$\int_{B_j} (\mathrm{Bil}(Ef_{j,tang}))^{3.25} \le R^{O(\delta)} \left(\sum_\tau \int |f_\tau|^2 \right)^{3/2} \le R^\varepsilon \left(\sum_\tau |f_\tau|^2 \right)^{3/2}.$$

Hence, if the tangential terms dominate, we are also done. We are then left with the case where the 2α-broad terms dominate. Explicitly, when

$$\int_{B_R} (\mathrm{Br}_\alpha\, Ef)^{3.25} \lesssim \sum_{j,I} \int_{B_j} (\mathrm{Br}_{2\alpha}\, Ef_{I,j,trans})^{3.25}. \tag{7.8}$$

We claim that we can apply Theorem 7.6 to each integral on the right side of (7.8). Given that the radius of each B_j is $R^{1-\delta} < R/2$,the induction applies. We now verify that $f_{I,j,trans}$ satisfies the hypothesis of the theorem. By Proposition 7.1, supp $f_{\tau,I,j,trans}$ lies in a small neighborhood of τ. Similar to the case treated above, the multiplicity is now at most 2μ. By Lemma 7.4, for any $\omega \in S$ we have

$$\fint_{B_{R^{-1/2}}(\omega)\cap S} |f_{\tau,I,j,trans}|^2 \lesssim \fint_{B_{10R^{-1/2}}(\omega)\cap S} |f_\tau|^2 \lesssim 1.$$

Hence, applying the theorem to each term yields the upper bound

$$\sum_{B_j} (\mathrm{Br}_{2\alpha}\, Ef_{I,j,trans})^{3.25} \lesssim C_\varepsilon R^{(1-\delta)\varepsilon} R^{\delta_{trans} \log(4\alpha\mu K^\varepsilon)} \left(\sum_\tau \int_S |f_{\tau,j,trans}|^2 \right)^{3/2+\varepsilon}.$$

Now, we need to find a bound for the sum over all j, I. We use Lemma 7.10, which says that a tube $T \in \mathbb{T}$ lies in $\mathbb{T}_{j,trans}$ for at most $\mathrm{Poly}(D)$ values of j. For the index I, Remark 7.16 assures that there is a constant (depending on ε) number of terms. Therefore, by **??**, we have

$$\sum_j \int |f_{\tau,j,trans}|^2 \lesssim \mathrm{Poly}(D) \sum_\tau \int |f_\tau|^2,$$

and hence

$$\sum_{j,I} \left(\sum_{\tau\in I} \int |f_{\tau,j,trans}|^2 \right)^{3/2+\varepsilon} \lesssim \mathrm{Poly}(D) \left(\sum_\tau \int |f_\tau|^2 \right)^{3/2+\varepsilon}.$$

Hence, (7.8) becomes

$$\int_{B_R} (\mathrm{Br}_\alpha\, Ef)^{3.25} \le \mathrm{Poly}(D) C_\varepsilon R^{(1-\delta)\varepsilon} R^{\delta_{trans}\log(4\alpha\mu K^\varepsilon)} \left(\sum_\tau \int |f_\tau|^2\right)^{3/2+\varepsilon}$$
$$= (C\,\mathrm{Poly}(D) R^{-\delta\varepsilon} R^{C\delta_{trans}}) C_\varepsilon R^\varepsilon R^{\delta_{trans}\log(\alpha\mu K^\varepsilon)} \left(\sum_\tau \int |f_\tau|^2\right)^{3/2+\varepsilon}.$$

To close the induction, it only remains to show that the term in parenthesis, $C\,\mathrm{Poly}(D)R^{-\delta\varepsilon}R^{C\delta_{trans}}$, is at most 1. For some sufficiently large R, the term is bounded by $R^{C\delta_{\deg}-\delta\varepsilon+C\delta_{trans}}$. Since $\delta = \varepsilon^2$, $\delta_{\deg} = \varepsilon^4$ and $\delta_{trans} = \varepsilon^6$, the exponent of R is negative and thus the induction closes. □

7.5 Tangential term estimate

In this section, we will prove Proposition 7.17. That is,

$$\int_{B_j\cap W} \mathrm{Bil}(Ef_{j,tang})^{3.25} \lesssim R^{O(\delta)} \left(\sum_\tau \int |f_\tau|^2\right)^{3/2}.$$

We first cover $B_j \cap W$ with cubes Q of side length $R^{1/2}$. For each cube Q, we let $\mathbb{T}_{j,tang,Q}$ be the set of tubes in $\mathbb{T}_{j,tang}$ that intersect Q. On Q, Proposition 7.1 assert we have

$$Ef_{\tau,j,tang} = \sum_{T\in\mathbb{T}_{j,tang,Q}} Ef_{\tau,T} + O(R^{-990}\|f_\tau\|_2). \tag{7.9}$$

By Lemma 7.10, the terms of the form $O(R^{-990}\|f_\tau\|_2)$ will always be negligible for big R in the calculation of the tangential terms. Thus, in this section, we write

$$Ef_{\tau,j,tang} = \sum_{T\in\mathbb{T}_{j,tang,Q}} Ef_{\tau,T} + \iota. \tag{7.10}$$

instead of (7.9). That is, ι represents a negligible term.

We will now use Córdoba's L^4 argument (see Córdoba) to obtain a bilinear estimate on Q. In an extremely short way of explaining the method, it exploits the fact that $4 = 2^2 = 2 + 2$ to reduce an L^4 estimate to an L^2 estimate.

Lemma 7.18. *If τ_1 and τ_2 are non-adjacent caps, then*

$$\int_Q |Ef_{\tau_1,j,tang}|^2\,|Ef_{\tau_2,j,tang}|^2$$

is bounded by

$$CR^{O(\delta)}R^{-1/2}\left(\sum_{T_1\in\mathbb{T}_{j,tang,Q}} \|f_{\tau_1,T_1}\|_2^2\right)\left(\sum_{T_2\in\mathbb{T}_{j,tang,Q}} \|f_{\tau_2,T_2}\|_2^2\right) + \iota.$$

Proof. Let $S = h(\vec{\omega})$ denote a parabolic surface up to degree L. On Q, we have (7.10). Let η_Q be a bump function equal to 1 on Q, with support on $10Q$. We then have

$$\int_Q |Ef_{\tau_1,j,tang}|^2 |Ef_{\tau_2,j,tang}|^2$$
$$\leq \sum_{T_1,\overline{T}_1,T_2,\overline{T}_2\in\mathbb{T}_{j,tang,Q}} \int_Q \eta_Q Ef_{\tau_1,T_1}\overline{Ef_{\tau_1,\overline{T}_1}}Ef_{\tau_2,T_2}\overline{Ef_{\tau_2,\overline{T}_2}} + \iota.$$

Using Plancharel's theorem, the sum equals

$$\sum_{T_1,\overline{T}_1,T_2,\overline{T}_2\in\mathbb{T}_{j,tang,Q}} \int_{\mathbb{R}^3} (\hat{\eta}_Q * f_{\tau_1,T_1}d\sigma_S * f_{\tau_2,T_2}d\sigma_S)\overline{(f_{\tau_1,\overline{T}_1}d\sigma_S * f_{\tau_2,\overline{T}_2}d\sigma_S)}.$$

For each tube T, let $\theta(T)$ denote the cap θ from which T was constructed (see 48), and let $\omega(T)$ be the center of $\theta(T)$. By Proposition 7.1, $f_{\tau,T}d\sigma_S$ is supported on $3\theta(T)$, so the support lies in an $O(R^{1/2})$-neighborhood of $\omega(T)$. Since η_Q is of rapid decay, so is $\hat{\eta}_Q$. Hence, a term in the sum above will be negligible unless

$$\omega(T_1) + \omega(T_2) = \omega(\overline{T}_1) + \omega(\overline{T}_2) + O(R^{-1/2+\delta}). \tag{7.11}$$

We will now prove that (7.11) ensures the estimates

$$\left|\omega(T_1) - \omega(\overline{T}_1)\right| \leq CR^{-1/2+\delta}$$

and

$$\left|\omega(T_2) - \omega(\overline{T}_2)\right| \leq CR^{-1/2+\delta}.$$

That is, the centers of the caps of same index are very close. Since $v(T_i)$ and $v(\overline{T}_i)$ all lie in a common plane T and $v(T_i)$ is essentially a normal vector to S at $\omega(T_i)$, we have that each point $\vec{\omega}(T_i)$ and $\vec{\omega}(\overline{T}_i) \in \mathbb{B}^2$, we have

$$m \cdot \nabla h(\vec{\omega}) + b = 0$$

for some $m \in \mathbb{R}^2$ with $|m| \leq 1$ and $b \in \mathbb{R}$ with $|b| \lesssim 1$. Furthermore, this is a curve in $\mathbb{B}^2$. If h is a quadratic surface, then it defines a line. Since S is nearly parabolic, we have $1/2 \leq \mathbf{H}_h \leq 2$ and $\left|\partial^3 h\right| \leq 10^{-9}$ pointwise, so that the curve is almost a straight line. After a proper rotation in the ω_1, ω_2 plane, the curve can be expressed by a function $g(\omega_1) = \omega_2$, where $|\nabla g|, \left|\nabla^2 g\right| \leq 10^{-6}$.

Let $j(\omega_1) = h(\omega_1, g(\omega_1))$. We then have by the chain rule and $\partial_1^2 h \geq 1/2$ that

$$j'' \geq 1/4.$$

Let $\omega_1(T_i)$ be the ω_1-coordinate of $\omega(T_i)$. Then, (7.11) is equivalent to the following system of equations

$$\omega_1(T_1) + \omega_1(T_2) = \omega_1(\overline{T}_1) + \omega_1(\overline{T}_2) + O(R^{-1/2+\delta}) \tag{7.12}$$

$$j(\omega_1(T_1)) + j(\omega_1(T_2)) = j(\omega_1(\overline{T}_1)) + j(\omega_1(\overline{T}_2)) + O(R^{-1/2+\delta}). \tag{7.13}$$

(7.12) tells us that the midpoint of $\omega_1(T_1)$ and $\omega_1(T_2)$ is essentially equal to the midpoint of $\omega_1(\overline{T}_1)$ and $\omega_1(\overline{T}_2)$. We now assume, without loss of generality, that $\omega_1(\overline{T}_1) < \omega_1(T_1) < \omega_1(T_2) < \omega_1(\overline{T}_2)$. Since $\omega(T_1)$ and $\omega(T_2)$ lie in (or very near) τ_1 and τ_2, respectively, and these caps are $K^{-1}-$separated, we have $|\omega_1(T_1) - \omega_1(T_2)| \gtrsim K^{-1}$. Let $I_1 = [\omega_1(\overline{T}_1), \omega(T_1)]$ and $I_2 = [\omega_1(T_2), \omega(\overline{T}_2)]$. By (7.12), these intervals are of the same length up to an error of $O(R^{-1/2+\delta})$. We then have that for any $s_1 \in I_1$ and $s_2 \in I_2$, $j'(s_2) - j'(s_1) = K^{-1}j''(c) \geq (4K)^{-1}$. Using this bound and the fundamental theorem of calculus, we obtain

$$\begin{aligned}
|I_1| + |I_2| &\lesssim \int_{I_2} j'(s)ds - \int_{I_1} j'(s)ds + O(R^{-1/2+\delta}) \\
&= j(\omega_1(\overline{T}_2)) + j(\omega_1(\overline{T}_1)) - j(\omega_1(T_1)) - j(\omega_1(T_2)) + O(R^{-1/2+\delta}) \\
&= O(R^{-1/2+\delta}).
\end{aligned}$$

This proves that $\left|\omega(T_i) - \omega(\overline{T}_i)\right| \lesssim R^{-1/2+\delta}$ for $i = 1, 2$.

For each θ, only $O(1)$ tubes in $\mathbb{T}(\theta)$ intersect Q. That is, there are $O(1)$ tubes of $\mathbb{T}(\theta)$ in $\mathbb{T}_{j,tang,Q}$. Therefore, the expression

$$\sum_{T_1,\overline{T}_1,T_2,\overline{T}_2 \in \mathbb{T}_{j,tang,Q}} \int_{\mathbb{R}^3} (\hat{\eta}_Q * f_{\tau_1,T_1}d\sigma_S * f_{\tau_2,T_2}d\sigma_S)\overline{(f_{\tau_1,\overline{T}_1}d\sigma_S * f_{\tau_2,\overline{T}_2}d\sigma_S)}.$$

is bounded by

$$R^{O(\delta)} \sum_{T_1,T_2 \in \mathbb{T}_{j,tang,Q}} \int_{\mathbb{R}^3} |f_{\tau_1,T_1}d\sigma_S * f_{\tau_2,T_2}d\sigma_S|^2 . \tag{7.14}$$

Once again using that the caps τ_1, τ_2 are K^{-1} separated, the angle between the tangent spaces of S at $\theta(T_1)$ and $\theta(T_2)$ is $\gtrsim K^{-1}$.
Let $f_i d\sigma_{S_i} := f_{\tau_i,T_i}d\sigma_S$ for $i = 1, 2$, where S_i is a cap containing $\operatorname{supp} f_i$ with radius $\sim R^{-1/2}$. Because of the angle condition between S_1 and S_2, we can *foliate* S_1 by curves γ_s, $s \in [0, R^{-1/2}]$ so that the tangent direction of γ_s is significantly transversal to the tangent plane of S_2. The foliation gives $d\sigma_{S_1} = J \cdot d\sigma_{\gamma_s} ds$ for some Jacobian factor $J \sim 1$. One then has

$$f_1 d\sigma_{S_1} * f_2 d\sigma_{S_2} = \int_0^{R^{-1/2}} (J f_1 d\sigma_{\gamma_s} * f_2 d\sigma_{S_2}) ds.$$

It follows from Minkowski's inequality and Cauchy-Schwartz that

$$\begin{aligned}
\|f_1 d\sigma_{S_1} * f_2 d\sigma_{S_2}\|_2^2 &\leq \left(\int_0^{R^{-1/2}} \|J f_1 d\sigma_{\gamma_s} * f_2 d\sigma_{S_2}\|_2 \, ds \right)^2 \\
&\leq R^{-1/2} \int_0^{R^{-1/2}} \|J f_1 d\sigma_{\gamma_s} * f_2 d\sigma_{S_2}\|_2^2 \, ds.
\end{aligned}$$

We also have, after a change of coordinates,

$$\|Jf_1 d\sigma_{\gamma_s} * f_2 d\sigma_{S_2}\|_2^2 \sim \int_{\gamma_S} |f_1|^2 \int_{S_2} |f_2|^2 . \tag{7.15}$$

Therefore, by (7.15), we have

$$\begin{aligned}\|f_1 d\sigma_{S_1} * f_2 d\sigma_{S_2}\|_2^2 &\le R^{-1/2} \int_0^{R^{-1/2}} \left(\int_{\gamma_s} |f_1|^2 \right) ds \int_{S_2} |f_2|^2 \\ &\lesssim R^{-1/2} \int_{S_1} |f_1|^2 \int_{S_2} |f_2|^2 .\end{aligned}$$

Writing the explicit definitions for this inequality gives

$$\int_{\mathbb{R}^3} |f_{\tau_1,T_1} d\sigma_S * f_{\tau_2,T_2} d\sigma_S|^2 \lesssim R^{-1/2} \|f_{\tau_1,T_1}\|_2^2 \|f_{\tau_2,T_2}\|_2^2 . \tag{7.16}$$

Thus, by (7.16) and the bound given by (7.14), we arrive to the expression in Lemma 7.18. □

As expressed in section 7.1, wave packets have the intention of approximating f_τ by $\sum a_T e^{2\pi i \omega_\tau x} \chi_T$. In this same train of thought, one would like to approximate $|Ef_{\tau,T}|$ by $\chi_T \|f_{\tau,T}\| \lesssim \chi_T R^{-1/2} \|f_{\tau,T}\|$. This motivates the function

$$S_{\tau,j,tang} := \left(\sum_{T \in \mathbb{T}_{j,tang}} \left(\chi_T R^{-1/2} \|f_{\tau,T}\|_2 \right)^2 \right)^{1/2} .$$

This definition is quite similar to the sums appearing in Lemma 7.18. In fact, using that the volume of Q is $R^{3/2}$ and the previous definition, we have

$$\begin{aligned}&\int_Q |Ef_{\tau_1,j,tang}|^2 |Ef_{\tau_2,j,tang}|^2 \\ &\quad\lesssim R^{O(\delta)} R^{-1/2} \left(\sum_{T_1 \in \mathbb{T}_{j,tang,Q}} \|f_{\tau_1,T_1}\|_2^2 \right) \left(\sum_{T_2 \in \mathbb{T}_{j,tang,Q}} \|f_{\tau_2,T_2}\|_2^2 \right) + \iota \\ &\quad\lesssim R^{O(\delta)} \int_Q S^2_{\tau_1,j,tang} S^2_{\tau_1,j,tang} + \iota.\end{aligned}$$

Summing over all $Q \subset B_j \cap W$, we have

$$\int_{B_j \cap W} |Ef_{\tau_1,j,tang}|^2 |Ef_{\tau_2,j,tang}|^2 \lesssim R^{O(\delta)} \int_{B_j \cap W} S^2_{\tau_1,j,tang} S^2_{\tau_1,j,tang} + \iota.$$

Using the definition of the S functions, we have the following bound:

$$\int_{B_j \cap W} S^2_{\tau_1,j,tang} S^2_{\tau_1,j,tang} \le \sum_{T_1,T_2 \in \mathbb{T}_{j,tang}} R^{-2} \|f_{\tau_1,T_1}\|_2^2 \|f_{\tau_2,T_2}\|_2^2 \int \chi_{T_1} \chi_{T_2} .$$

This last integral is the volume of the intersection of the tubes, which is $\sim R^{3/2}$. Since these come from K^{-1}-separated caps, the integral is $\lesssim KR^{3/2}$. This last sum is then

$$\lesssim R^{-1/2}\left(\sum_{T_1\in\mathbb{T}_{j,tang}} \|f_{\tau_1,T_1}\|_2^2\right)\left(\sum_{T_2\in\mathbb{T}_{j,tang}} \|f_{\tau_2,T_2}\|_2^2\right).$$

By Proposition 7.1, the functions $\{f_{\tau,T}\}_{T\in\mathbb{T}}$ are essentially orthogonal, so

$$\sum_{T\in\mathbb{T}_{j,tang}} \|f_{\tau_2,T_2}\|_2^2 \lesssim \|f_{\tau,j,tang}\|_2^2 + \iota.$$

Altogether, we have

$$\int_{B_j\cap W} |Ef_{\tau_1,j,tang}|^2\,|Ef_{\tau_2,j,tang}|^2 \lesssim R^{O(\delta)}R^{-1/2}\,\|f_{\tau_1,j,tang}\|_2^2\,\|f_{\tau_2,j,tang}\|_2^2 + \iota.$$

Summing over all non-adjacent τ_1,τ_2, and then taking the $1/4$ power from both sides, we have

$$\|\mathrm{Bil}(Ef_{j,tang})\|_{L^4(B_j\cap W)} \lesssim R^{O(\delta)}R^{-1/8}\Big(\sum_{\tau}\|f_{\tau,j,tang}\|_2^2\Big)^{1/2} + \iota.$$

Now we need to use interpolation to conclude the estimate for all $2\le p\le 4$. That is, we need to prove a bound for $\|\mathrm{Bil}(Ef_{j,tang})\|_{L^2(B_j\cap W)}$. For this, we use that $E_S(2\to 2;1/2)$ holds[2]. That is,

$$\|Ef\|_{L^2} \lesssim R^{1/2}\,\|f\|_2\,.$$

Hence, we have

$$\|\mathrm{Bil}(Ef_{j,tang})\|_{L^2(B_j\cap W)} \lesssim R^{1/2}\left(\sum_{\tau}\|f_{\tau,j,tang}\|_2^2\right)^{1/2}.$$

Now we can use interpolation between L^2 and L^4. That is, for $\theta\in(0,1)$,

$$\frac{1}{p}=\frac{\theta}{2}+\frac{1-\theta}{4} \iff \theta=\frac{4}{p}-1,$$

we have

$$\begin{aligned}\|\mathrm{Bil}(Ef_{j,tang})\|_{L^p(B_j\cap W)} &\lesssim R^{O(\delta)}R^{-\frac{1-\theta}{8}}R^{\frac{\theta}{2}}\left(\sum_{\tau}\|f_{\tau,j,tang}\|_2^2\right)^{1/2}\\ &= R^{O(\delta)}R^{-\frac{3}{4}+\frac{5}{2p}}\left(\sum_{\tau}\|f_{\tau,j,tang}\|_2^2\right)^{1/2}.\end{aligned}$$

[2]The proof of this follows from interpolation of L^1 and L^∞ estimates, which boil down to calculating an integral of a Schwartz function.

Equivalently,

$$\int_{B_j \cap W} |\mathrm{Bil}(Ef_{j,tang})|^p \lesssim R^{O(\delta)} R^{-\frac{3p}{4}+\frac{5}{2}} \left(\sum_\tau \|f_{\tau,j,tang}\|_2^2 \right)^{p/2}. \tag{7.17}$$

We now turn to the terms $\|f_{\tau,j,tang}\|_2$. We have by Lemma 7.4 $\|f_{\tau,j,tang}\|_2 \lesssim \|f_\tau\|_2$, on any cap θ, but we can use the tangential property to obtain a better estimate on some of these caps. More precisely, we will use Lemma 7.12 to optimize the estimate on the tangential terms. The lemma says that $f_{\tau,j,tang}$ is supported on $R^{O(\delta)}R^{1/2}$ caps θ. On each of these caps, we now use Lemma 7.4 to get the bound

$$\fint_\theta |f_{\tau,j,tang}|^2 \lesssim \fint_{10\theta} |f_\tau|^2 \lesssim 1.$$

Adding the contribution of $R^{1/2+O(\delta)}$ caps, we have

$$\int |f_{\tau,j,tang}|^2 \lesssim R^{O(\delta)} R^{-1/2}.$$

Combining these two estimates for $\|f_{\tau,j,tang}\|_2$, we have, for $p \geq 3$,

$$\left(\sum_\tau \|f_{\tau,j,tang}\|_2^2 \right)^{p/2} \lesssim R^{O(\delta)} R^{\frac{3}{4}-\frac{p}{4}} \left(\sum_\tau \|f_\tau\|_2^2 \right)^{3/2}.$$

Substituting this into (7.17), we obtain

$$\int_{B_j \cap W} |\mathrm{Bil}(Ef_{j,tang})|^p \lesssim R^{O(\delta)} R^{\frac{13}{4}-p} \left(\sum_\tau \|f_\tau\|_2^2 \right)^{3/2}.$$

Choosing $p = 13/4 = 3.25$ gives the bound for Proposition 7.17.

7.6 Sharpness of p

As mentioned in , $p = 3.25$ is sharp. Indeed, suppose S is the truncated paraboloid $\omega_3 = \omega_1^2 + \omega_2^2$, $\omega_3 \leq 1$, then $E_S f$ is essentially supported in a planar slab of dimensions $R^{1/2} \times R \times R$. There are $R^{1/2}$ caps $\theta \subset S$ for which the normal vector lies within an angle $\sim R^{-1/2}$ of the plane $\omega_3 = 0$. For each of the caps, there are $\sim R^{1/2}$ tubes $T \in \mathbb{T}(\theta)$ that lie in the planar slab. Fix a number N between 1 and $R^{1/2}$, and for each of the $R^{1/2}$ caps θ we pick N tubes of $\mathbb{T}(\theta)$ that lie in the planar slab, so that we have $\sim NR^{1/2}$ tubes T. On average, a point in the planar slab lies in $\sim N$ of the tubes, as they were picked randomly.

For each of the chose tubes T, we choose f_T so that $|E_S f_T| \gtrsim \chi_T$, $\|f_T\|_2 \sim R^{1/2}$ and $\|f_T\|_\infty \sim R$. Let f be a sum of random signs of f_T:

$$f = \sum_T \varepsilon_T f_T,$$

where ε_T, for each tube T, has $1/2$ probability of being -1 and $1/2$ of being $+1$. Because of the random signs, on most points of the planar slab we have

$$|E_S f(x)| = \left|\sum_T \varepsilon_T E_S f_T(x)\right| \gtrsim \left|\sum \varepsilon_T \left|E_S f_T(x)\right|\right| \gtrsim \left|\sum \varepsilon_T \chi_T\right| \geq N^{1/2}.$$

Since the slab has volume $R^{5/2}$, $\|E_S f\|_{L^p(B_R)} \gtrsim N^{1/2} R^{\frac{5}{2p}}$. If $N \geq K^{10\varepsilon}$, then almost every point will be $K^{-\varepsilon}$-broad. Therefore,

$$\|\mathrm{Br}_{K^{-\varepsilon}} Ef\|_{L^p(B_R)} \gtrsim N^{1/2} R^{\frac{5}{2p}}. \tag{7.18}$$

Now, since f_T are essentially orthogonal, we have

$$\|f\|_2 \sim N^{1/2} R^{3/4}.$$

We also have

$$\|f\|_\infty \leq N \max_T \|f_T\|_\infty \sim NR.$$

Consider the case $N \sim K^{10\varepsilon}$, N independent of R. If $\|\mathrm{Br}_{K^{-\varepsilon}} Ef\|_{L^p(B_R)} \leq C_\varepsilon R^\varepsilon \|f\|^{12/13} \|f\|_\infty^{1/13}$, then

$$\begin{aligned}
\|\mathrm{Br}_{K^{-\varepsilon}} Ef\|_{L^p(B_R)} &\leq C_\varepsilon R^\varepsilon \|f\|^{12/13} \|f\|_\infty^{1/13} \\
&\sim N^{\frac{12}{13}\cdot\frac{1}{2}} R^{\frac{12}{13}\cdot\frac{3}{4}} N^{\frac{1}{13}} R^{\frac{1}{13}} \\
&= N^{\frac{7}{13}} R^{\frac{10}{13}}.
\end{aligned}$$

Using the condition in (7.18), we obtain $p \geq 3.25$.

Conclusions

In conclusion, the restriction conjecture on $\mathbb{R}^3$ for smooth compact manifolds of positive second fundamental form has gained a range of exponents due to the use of polynomial partitioning. The range $p > 3.25$ is getting closer to the desired $p > 3$, and the mere fact that this method was able to extend the range of p tells us that we should not limit our tools to only those which seem directly related to harmonic analysis.

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