Market Risk Management - Assignment 1

Peter Millington

October 23, 2023

Question 1

$$\mathbf{r} = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix}, \mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}, \mathbf{e} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \mathbf{V} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 9 \end{pmatrix} \tag{1}$$

The Lagrangian is

$$L(\mathbf{w}, \lambda_1, \lambda_2) = \frac{1}{2}\mathbf{w^T V w} + \lambda_1(1 - \mathbf{w^T 1}) + \lambda_2(\mu_p - \mathbf{w^T e}) \tag{2}$$

Expanding each part of the equation

$$\frac{1}{2}\mathbf{w^T V w} = \frac{1}{2}\begin{pmatrix} w_1 & w_2 & w_3 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ 1 & 4 & 0 \\ 0 & 0 & 9 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \tag{3}$$

$$= \frac{1}{2}(w_1^2 + 4w_2^2 + 9w_3^2 + 2w_1w_2) \tag{4}$$

$$\tag{5}$$

$$\lambda_1(1 - \mathbf{w^T 1}) = \lambda_1(1 - w_1 - w_2 - w_3) \tag{6}$$

$$\lambda_2(\mu_p - \mathbf{w^T e}) = \lambda_2(\mu_p - \begin{pmatrix} w_1 & w_2 & w_3 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}) \tag{7}$$

$$= \lambda_2(1 - w_1 - 2w_2 - 3w_3) \tag{8}$$

So the Lagrangian may be written:

$$L = \frac{1}{2}(w_1^2 + 4w_2^2 + 9w_3^2 + 2w_1w_2) + \lambda_1(1 - w_1 - w_2 - w_3) + \lambda_2(\mu_p - w_1 - 2w_2 - 3w_3) \tag{9}$$

The first order conditions give us the following simultaneous equations to solve:

$$\frac{\partial L}{\partial w_1} = w_1 + w_2 - \lambda_1 - \lambda_2 = 0 \tag{10}$$

$$\frac{\partial L}{\partial w_2} = w_1 + 4w_2 - \lambda_1 - 2\lambda_2 = 0 \tag{11}$$

$$\frac{\partial L}{\partial w_3} = 9w_3 - \lambda_1 - 3\lambda_2 = 0 \tag{12}$$

$$\frac{\partial L}{\partial \lambda_1} = 1 - w_1 - w_2 - w_3 = 0 \tag{13}$$

$$\frac{\partial L}{\partial \lambda_2} = \mu_p - w_1 - 2w_2 - 3w_3 = 0 \tag{14}$$

Using the first three equations, we express the weights in terms of λ_1 and λ_2:

$$w_1 = \lambda_1 + \frac{2}{3}\lambda_2 \tag{15}$$

$$w_2 = \frac{1}{3}\lambda_2 \tag{16}$$

$$w_3 = \frac{1}{9}(\lambda_1 + 3\lambda_2) \tag{17}$$

Replacing the values for w in the constraints with the above we obtain:

$$10\lambda_1 + 12\lambda_2 = 9 \tag{18}$$

$$4\lambda_1 + 7\lambda_2 = 3\mu_p \tag{19}$$

$$\begin{pmatrix} 10 & 12 \\ 4 & 7 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 9 \\ 3\mu_p \end{pmatrix} \tag{20}$$

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \frac{1}{22} \begin{pmatrix} 7 & -12 \\ -4 & 10 \end{pmatrix} \begin{pmatrix} 9 \\ 3\mu_p \end{pmatrix} \tag{21}$$

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \frac{1}{22} \begin{pmatrix} 63 - 36\mu_p \\ -36 + 30\mu_p \end{pmatrix} \tag{22}$$

$$\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} \lambda_1 + \frac{2}{3}\lambda_2 \\ \frac{1}{3}\lambda_2 \\ \frac{1}{9}(\lambda_1 + 3\lambda_2) \end{pmatrix} \tag{23}$$

$$= \frac{1}{22} \begin{pmatrix} 39 - 16\mu_p \\ -12 + 10\mu_p \\ -5 + 6\mu_p \end{pmatrix} \tag{24}$$

The variance is given by $\sigma^2 = \mathbf{w^T V w}$:

$$\sigma_p^2 = (w_1^2 + 4w_2^2 + 9w_3^2 + 2w_1w_2) \tag{25}$$

$$= (\frac{1}{22})^2((39 - 16\mu_p)^2 + 4(-12 + 10\mu_p)^2 + 9(-5 + 6\mu_p)^2 + 2(39 - 16\mu_p)(-12 + 10\mu_p)) \tag{26}$$

$$= \frac{1}{22}(30\mu_p^2 - 72\mu_p + 63) \tag{27}$$

From the above, a, b and c are given as:

$$a = \frac{15}{11} \tag{28}$$

$$b = -\frac{36}{11} \tag{29}$$

$$c = \frac{63}{22} \tag{30}$$

The above equation can be rearranged into a standard hyperbola form with a lot of arithmetic:

$$\frac{\sigma_p^2}{(\frac{9}{10})} - \frac{(\mu_p - \frac{6}{5})^2}{(\frac{33}{50})} = 1 \tag{31}$$

The minimum variance portfolio is situated at the vertex of this curve, at $(\frac{3}{\sqrt{10}}, \frac{6}{5})$
The corresponding weights w are obtained by substituting in for μ_p into equation (24) and are given by:

$$\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \frac{1}{110} \begin{pmatrix} 99 \\ 0 \\ 11 \end{pmatrix} \tag{32}$$

$$\tag{33}$$

Question 2.