

Denotational Semantics of Untyped λ -Calculus in Agda

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Abstract

In synthetic domain theory, all sets are predomains, domains are pointed sets, and functions are implicitly continuous. The denotational semantics of the untyped lambda-calculus presented here illustrates how it might look if synthetic domain theory can be implemented in Agda. As a work-around, the code presented here uses unsatisfiable postulates to allow Agda to type-check the definitions.

The (currently illiterate) Agda source code used to generate this document can be downloaded from <https://github.com/pdmosses/xds-agda>, and browsed with hyperlinks and highlighting at <https://pdmosses.github.io/xds-agda/>.

```
{-# OPTIONS --rewriting --confluence-check #-}
```

```
module ULC.All where
```

```
import ULC.Variables
```

```
import ULC.Terms
```

```
import ULC.Domains
```

```
import ULC.Environments
```

```
import ULC.Semantics
```

```
import ULC.Checks
```

```
module ULC.Variables where
```

```
open import Data.Bool using (Bool)
```

```
open import Data.Nat using ( $\mathbb{N}$ ;  $\equiv^b$  _)
```

```
data Var : Set where
```

```
  x :  $\mathbb{N}$  → Var -- variables
```

```
variable v : Var
```

```
 $\_ == \_$  : Var → Var → Bool
```

```
x n == x n' = (n  $\equiv^b$  n')
```

```
module ULC.Terms where
```

```
open import ULC.Variables
```

```
data Exp : Set where
```

```
  var _ : Var → Exp -- variable value
```

```
  lam : Var → Exp → Exp -- lambda abstraction
```

```
  app : Exp → Exp → Exp -- application
```

```
variable e : Exp
```

```

module ULC.Domains where

open import Relation.Binary.PropositionalEquality.Core using (_≡_; refl) public

Domain = Set

postulate ⊥      : {D : Domain} → D
postulate fix    : {D : Domain} → (D → D) → D
postulate fix-fix : ∀ {D} → (f : D → D) → fix f ≡ f (fix f)
postulate fix-app : ∀ {P D} → (f : (P → D) → (P → D)) (p : P) → fix f p ≡ f (fix f) p

open import Function using (Inverse; _↔_) public

postulate D∞ : Domain
postulate instance iso : D∞ ↔ (D∞ → D∞)
open Inverse [{ ... }] using (to; from) public

variable d : D∞

```

```

module ULC.Environments where

open import ULC.Variables
open import ULC.Domains
open import Data.Bool using (if _then_else _)

Env : Domain
Env = Var → D∞
-- the initial environment for a closed term is λ v → ⊥

variable ρ : Env

_[_/ _] : Env → D∞ → Var → Env
ρ [ d / v ] = λ v' → if v == v' then d else ρ v'

```

```

module ULC.Semantics where

open import ULC.Variables
open import ULC.Terms
open import ULC.Domains
open import ULC.Environments

[[_]] : Exp → Env → D∞
-- [[ e ]] ρ is the value of e with ρ giving the values of free variables

[[ var v ]] ρ      = ρ v
[[ lam v e ]] ρ    = from ( λ d → [[ e ]] (ρ [ d / v ]) )
[[ app e1 e2 ]] ρ = to ( [[ e1 ]] ρ ) ( [[ e2 ]] ρ )

```

```

{-# OPTIONS --rewriting --confluence-check #-}

open import Agda.Builtin.Equality
open import Agda.Builtin.Equality.Rewrite

module ULC.Checks where

open import ULC.Domains
open import ULC.Variables
open import ULC.Terms
open import ULC.Environments
open import ULC.Semantics

open Inverse using (inversel; inverser)

to-from : (f : D∞ → D∞) → to (from f) ≡ f
from-to : (d : D∞)       → from (to d) ≡ d

to-from f = inversel iso refl
from-to f = inverser iso refl

{-# REWRITE to-from from-to #-}

-- The following proofs are potentially unsound, due to unsafe postulates.

-- (λx1.x1)x42 = x42
check-id :
  [ app (lam (x 1) (var x 1))
    (var x 42) ] ≡ [ var x 42 ]
check-id = refl

-- (λx1.x42)x0 = x42
check-const :
  [ app (lam (x 1) (var x 42))
    (var x 0) ] ≡ [ var x 42 ]
check-const = refl

-- (λx0.x0 x0)(λx0.x0 x0) = ...
-- check-divergence :
--   [ app (lam (x 0) (app (var x 0) (var x 0)))
--     (lam (x 0) (app (var x 0) (var x 0))) ]
--   ≡ [ var x 42 ]
-- check-divergence = refl

-- (λx1.x42)((λx0.x0 x0)(λx0.x0 x0)) = x42
check-convergence :
  [ app (lam (x 1) (var x 42))
    (app (lam (x 0) (app (var x 0) (var x 0)))
      (lam (x 0) (app (var x 0) (var x 0)))) ]
  ≡ [ var x 42 ]
check-convergence = refl

```

```

-- (λx1.x1)(λx1.x42) = λx2.x42
check-abs :
  [ app (lam (x 1) (var x 1))
    (lam (x 1) (var x 42)) ]
  ≡ [ lam (x 2) (var x 42) ]
check-abs = refl

-- (λx1.(λx42.x1)x2)x42 = x42
check-free :
  [ app (lam (x 1)
    (app (lam (x 42) (var x 1))
      (var x 2)))
    (var x 42) ] ≡ [ var x 42 ]
check-free = refl

```