

Optimal Redistribution with a Shadow Economy

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Abstract

We extend the theory of the optimal redistributive taxation to economies with an informal labor market. The optimal tax formula contains two new terms capturing reported income responses of informal workers on an intensive and an extensive margin. Both terms decrease the optimal tax rates. We quantitatively show that this reduction can be substantial, exceeding 30 percentage points, and we document a large welfare gain of up to 6.4% of consumption from following our tax formula rather than the standard formula. We also provide a novel decomposition of the welfare impact of the shadow economy into labor efficiency and redistribution components. In the quantitative model estimated with Colombian data the shadow economy benefits efficiency at the expense of redistribution. As a result, conditional on the optimal tax policy, the presence of the informal sector does not substantially affect social welfare unless social preferences for redistribution are strong.

JEL Codes: H21, H26, J46.

1. Introduction.

Informal activity, broadly defined as any economic endeavor which evades taxation, accounts for a large fraction of economic activity in both developing and developed economies. The share of informal production in GDP is consistently estimated to be on average above 10% in high income OECD countries and above 30% in developing and

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transition countries, in extreme cases reaching up to 70% (Schneider and Enste 2000; Schneider, Buehn, and Montenegro 2011). The shadow economy allows workers to earn additional income which is unobserved by the government. Intuitively, this additional margin of response to taxation makes income redistribution more difficult. On the other hand, the informal jobs seem to be less productive and attract mostly the poor.¹ If the informal sector benefits those in need, perhaps it can be useful from a social justice perspective. Our aim is to evaluate these claims within an optimal taxation framework. We pose the following questions:

1. What is the optimal income tax schedule in the presence of a shadow economy?
2. How does a shadow economy affect social welfare?

Concerning the first question, we find that the shadow economy substantially reduces optimal tax rates. The tax rates are lower due to shadow workers' income responses on intensive and extensive margins, which are not fully accounted for in the standard optimal tax formulas. In the model estimated to match the Colombian informal sector we find that the tax rate reduction can exceed 30 percentage points, reducing the share of the shadow workers by 15 percentage points and lifting welfare by 6.4% of consumption.

To answer the second question, we decompose the social welfare impact of the informal sector into efficiency and redistribution components. We analytically show that, conditional on the optimal policy, the shadow economy can harm or enhance welfare on either of the two dimensions. Using the calibrated model, we find that the shadow economy in Colombia benefits labor efficiency at the expense of possible redistribution. As a result, the presence of the Colombian informal sector does not substantially affect social welfare unless social preferences for redistribution are strong.

Building on the seminal work of Mirrlees (1971), we consider a framework with heterogeneous agents equipped with distinct formal and shadow productivities. Workers face an idiosyncratic fixed cost of working in the shadow economy, which may reflect either ethical or technological constraints. The government observes only formal incomes and introduces taxation to maximize its redistributive welfare criterion. Importantly, we allow workers to supply labor to the formal sector and the shadow sector simultaneously.

Our first contribution is a novel sufficient statistics optimal tax formula for economies with an informal sector. The tax formula contains two new terms which capture a deadweight loss of taxation due to shadow workers' responses on an extensive margin (getting an informal job) and an intensive margin (shifting hours between a formal and an informal job). Importantly, these terms are not always accounted for in the standard sufficient statistic tax formulas from the models with the intensive margin of labor supply only (Diamond 1998, Saez 2001) or both the intensive and the extensive margins of labor supply (Jacquet, Lehmann, and Van der Linden 2013). To see it concretely, note

¹For instance, focusing on the main jobs, we find that the shadow economy in Colombia accounts for 58% of jobs but for only 31.4% of earnings.

that according to the standard formulas and absent wealth effects, workers respond on the intensive margin only when the marginal tax rate at their formal income level is changed. We show that shadow workers can respond on the intensive margin to a tax rate perturbation which happens at a strictly lower formal income level.² Such responses are not captured by the standard formulas. Furthermore, [Jacquet, Lehmann, and Van der Linden \(2013\)](#) consider a binary participation decision: working or not working. In our setting it would correspond to allowing agents to work only formally or only informally. We generalize this notion of extensive margin responses by allowing workers to retain some formal earnings when getting an informal job.

We compare our formula to the benchmark [Diamond \(1998\)](#) sufficient statistics formula. It is a meaningful comparison, since the two formulas coincide when all workers of a given productivity are formal. However, when some workers of a given productivity are informal, the terms corresponding to the shadow workers' responses on the intensive and the extensive margins tend to reduce the optimal tax rate in comparison to the benchmark, conditional on the distribution of formal income. Moreover, we analytically show that the shadow economy reduces the optimal top tax rate both via the new terms in the tax formula and via the endogenous adjustment of income distribution. The informality among top productivity workers shrinks the thickness of the upper tail of the formal income distribution, which is an additional factor towards a lower tax rate at the top.

Our second contribution is a decomposition of the welfare impact of the shadow economy. We compare the optimal allocations when a shadow economy is present and when it is costlessly shut down. The difference between these two allocations can be expressed as a sum of an *efficiency gain* and a *redistribution gain*. In a simplified framework we analytically derive the comparative statics of both gains and show that, depending on the joint distribution of formal and shadow productivities, the informal sector can harm or enhance welfare on either of the two dimensions. Our result sheds light on non-trivial welfare implications of informality. [Kopczuk \(2001\)](#) provides an example of welfare-improving tax evasion in which, according to our decomposition, the tax evasion allows for more redistribution at the cost of efficiency. It may suggest that welfare gains from the informal sector arise only due to a more equitable division of a smaller pie. We show that this is not the case. Specifically, when the shadow economy augments efficiency but restricts redistribution, a costless shutdown of the informal sector could reduce utility of all agents. In this case the presence of the shadow economy is Pareto improving.

To gain intuition on the welfare impact of the shadow economy, consider the efficiency gain first. If the productivity loss from moving to a shadow economy is low/high for

²The intuition is as follows. Shadow workers choose their formal and shadow labor supply such that the net returns to both are equal. When the marginal tax rates are non-monotone, there may be multiple formal income levels which satisfy this condition and some of them will constitute local optima. Increasing the marginal tax rate between the two locally optimal formal income levels affects utility in the higher one, but not in the lower one. As a result, it may trigger a jump to the lower local optimum.

agents that face high marginal tax rates, the shadow economy will raise/reduce labor efficiency. When agents do not lose much of their productivity by working informally, the shadow sector effectively shelters them from tax distortions of the formal economy.³ Conversely, when the productivity loss is large, the distortions implied by the lower shadow productivity may well dominate tax distortions. Now consider the redistribution gain. If agents who pay high total taxes suffer a low/high productivity loss for moving to the informal sector, then the shadow economy is likely to limit/expand the scope of a possible redistribution. When the productivity loss of these agents is small, they can reduce their formal earnings and adjust their shadow earnings at low cost, which limits redistribution. Conversely, when their shadow productivity is low, they are less tempted by low formal incomes, since their final consumption would be much lower. As a result, they are willing to maintain high formal earnings even when taxes are high.

We explore the importance of our results quantitatively. We extract the information on formal and shadow incomes from the Colombian household survey and estimate the model by maximum likelihood. The model replicates well the empirical sorting of workers between the formal and the informal sectors. We find that the optimal formula leads to a large reduction of tax rates in comparison to the [Diamond \(1998\)](#) formula when social preferences for redistribution are strong. In this case the tax rate reduction covers the bottom 95% of workers and at some formal income levels exceeds 30 percentage points. Lower tax rates reduce the share of shadow workers by 15 percentage points in comparison to the allocation implied by the Diamond formula and lead to a large welfare gain, equivalent to a 6.4% increase in consumption. In contrast, when the social welfare function places little weight on equality, the optimal formula, while leading to a slightly lower tax rates and a smaller shadow economy, does not bring noticeable welfare gains.

We find that the informal sector in Colombia strengthens labor efficiency by providing less productive workers with relatively high shadow productivity and by reducing marginal tax rates in the formal sector. On the other hand, the shadow economy restricts redistribution. The efficiency and redistribution gains are of the same order of magnitude and tend to cancel each other out unless the social preferences for redistribution are strong. For highly redistributive social preferences the redistribution component dominates and the presence of the shadow economy leads to a welfare loss of 1.27% of consumption. Our results point out that even if the informal sector could be shut down at no cost, such policy would not yield substantial welfare gains unless the government had a strong preference for redistribution.

Related literature. Following [Allingham and Sandmo \(1972\)](#), tax evasion has been studied in a framework with probabilistic audits and penalties, taking a tax rate as given. [Andreoni, Erard, and Feinstein \(1998\)](#) and [Slemrod and Yitzhaki \(2002\)](#) review

³This effect is related to what [Porta and Shleifer \(2008\)](#) call the romantic view on the shadow economy. In this view, associated with works of Hernando de Soto ([de Soto \(1990, 2000\)](#)) and modeled formally by [Choi and Thum \(2005\)](#), the informal sector protects productive firms from harmful regulation.

this strand of literature. We take a complementary approach and study the optimal non-linear tax schedule conditional on fixed tax evasion abilities of workers. Although we do not model tax audits and penalties explicitly, they are one of the possible justifications for different productivities in the formal and the shadow sector. Under this interpretation, our results on the welfare-improving informal sector can provide insights into the optimal design of tax audits. Some early results from merging both optimal taxation and optimal tax compliance policies were derived by [Cremer and Gahvari \(1996\)](#), [Kopczuk \(2001\)](#) and [Slemrod and Kopczuk \(2002\)](#). [Kopczuk \(2001\)](#) also shows that the standard formula for the optimal linear tax is still valid with tax evasion. In contrast, we show that the standard formula for the optimal *non-linear* tax no longer holds in the presence of a shadow economy.⁴

This paper is closely related to the literature on the optimal taxation with multiple sectors. [Rothschild and Scheuer \(2014\)](#) consider uniform taxation of multiple sectors when agents can work in many sectors simultaneously. [Kleven, Kreiner, and Saez \(2009\)](#), [Scheuer \(2014\)](#) and [Gomes, Lozachmeur, and Pavan \(2017\)](#) study differential taxation of broadly understood sectors (e.g. individual tax filers and couples, employees and entrepreneurs), when agents can belong to one sector only. [Jacobs \(2015\)](#) studies a complementary problem when all agents work in all sectors at the same time. Our analysis differs in that we consider a particular case of differential taxation (only one sector is taxed) when agents face an idiosyncratic fixed cost of participation in one of the sectors. This structure implies that some agents can effectively work in one sector only, while others are unconstrained in supplying labor to two sectors simultaneously. We show that a typical result on the sufficiency of local incentive constraints is no longer valid.⁵ We find an alternative set of incentive constraints which ensures global incentive compatibility.

[Emran and Stiglitz \(2005\)](#) and [Boadway and Sato \(2009\)](#) study commodity taxation in the presence of informality. Both papers assume that the commodity tax affects only the formal sector.⁶ Hence, it is equivalent to a proportional tax on formal income, provided that formal and shadow goods are perfect substitutes. Under these assumptions our focus on non-linear income tax is without loss of generality. A related literature on optimal taxation with home production ([Kleven, Richter, and Sørensen 2000](#); [Olovsson 2015](#)) studies the case of non-perfect substitutability between market and home produced

⁴Our settings is not identical to Kopczuk's, since we consider a fixed cost of shadow employment. In a previous working paper version ([Doligalski and Rojas 2016](#)), we show that the standard formula for the optimal non-linear tax is not valid even if we abstract from the fixed cost of shadow employment.

⁵The planner's problem in our setting is an example of multidimensional screening, as agents are heterogeneous with respect to the productivity and the fixed cost of shadow employment. [Carroll \(2012\)](#) shows that the local incentive constraints are sufficient in the multidimensional setting when the appropriately defined space of agents' types is convex. This condition is not satisfied in our setting. The local incentive constraints are insufficient to prevent deviations in both dimensions simultaneously.

⁶In principle, VAT taxation covers informal firms indirectly if they purchase intermediate goods from the formal firms. [De Paula and Scheinkman \(2010\)](#) show that exactly for this reason informal firms tend to make transactions with other informal firms.

goods.

Structure of the paper. In Section 2, using a simplified framework, we analytically characterize the efficiency and the redistribution impacts of the shadow economy. In Section 3 we derive the optimal tax formula and compare it to the standard formulas. Section 4 is devoted to the quantitative exploration of our theoretical results. The last section provides conclusions.

2. Welfare impact of the shadow economy.

In this section we decompose the welfare impact of the presence of the shadow economy into redistribution and efficiency gains. We consider a simplified version of the full model which allows us to characterize analytically comparative statics of both welfare components. Specifically, we consider an economy with two types of workers, no fixed cost of shadow employment and no possibility of working simultaneously in the formal and the informal sectors.

There are two types of individuals, indexed by letters L and H , with population shares μ_L and $\mu_H = 1 - \mu_L$. They care about consumption c and labor supply n according to a quasilinear utility function $U(c, n) \equiv c - v(n)$. We assume that v is increasing, strictly convex, twice differentiable and satisfies $v'(0) = 0$.

There are two labor markets and, correspondingly, each agent is equipped with two linear production technologies. An agent of type $i \in \{L, H\}$ produces with productivity w_i^f in a formal labor market and with productivity w_i^s in an informal labor market. Income in each sector is given by $y_i^x = w_i^x n_i^x$, where n_i^x denotes labor supply in sector $x \in \{f, s\}$. We identify type H as the one with higher formal productivity: $w_H^f > w_L^f$. Moreover, in this section we assume that each type is more productive formally: $\forall_i w_i^f > w_i^s$. It implies that the shadow economy is inefficient and is never used in the first-best, when the planner can observe individual types. We relax this assumption when we consider the full model.

2.1. The planner's problem.

The social planner observes only the formal income of each individual. Furthermore, the planner can transfer resources between agents with taxes T_i . We can think about y_i^f and $y_i^f - T_i$ as a pre-tax and an after-tax reported income. It is convenient to express agents' choices of shadow income as a function of their formal income:

$$y_i^s(y^f) \equiv w_i^s v'^{-1}(w_i^s) \times \mathbf{1}(y^f = 0). \quad (1)$$

If agents have any formal earnings, their shadow earnings are zero. If instead they have no formal earnings, they are unconstrained in choosing their shadow income. Given this function, we can specify agents' consumption $c_i = y_i^f + y_i^s(y_i^f) - T_i$ and labor supply $n_i = y_i^f/w_i^f + y_i^s(y_i^f)/w_i^s$, conditional on a truthful revelation of types.

The social planner maximizes the sum of utilities weighted with Pareto weights λ_i

$$W = \max_{\{(y_i^f, T_i) \in \mathbb{R}_+ \times \mathbb{R}\}_{i \in \{L, H\}}} \lambda_L \mu_L U(c_L, n_L) + \lambda_H \mu_H U(c_H, n_H) \quad (2)$$

subject to a resource constraint

$$\mu_L T_L + \mu_H T_H \geq 0, \quad (3)$$

and incentive-compatibility constraints

$$U(c_i, n_i) \geq U\left(y_{-i}^f + y_i^s(y_{-i}^f) - T_{-i}, \frac{y_{-i}^f}{w_i^f} + \frac{y_i^s(y_{-i}^f)}{w_i^s}\right) \quad i \in \{H, L\}. \quad (4)$$

The incentive compatibility constraints capture the limited information available to the planner. They imply that no agent can be better off by choosing formal income of the other type and, if this income level is zero, freely adjusting shadow earnings.

Lemma 1. *Suppose that $\lambda_i > \lambda_{-i}$. In the optimum,*

- *type $-i$ faces no labor distortions and does not work in the shadow economy.*
- *type i faces labor distortions and may work in the shadow economy.*

Proof. See Appendix A. □

Lemma 1 is a generalization of the classic *no distortion at the top* result. When $\lambda_i > \lambda_{-i}$, the planner wants to redistribute from type $-i$ to type i . The incentive constraint of type $-i$ will bind, and hence the planner cannot improve the allocation by distorting labor of type $-i$. Since an agent works in the shadow economy only if his formal labor is sufficiently distorted downwards (and equal to zero), the agent $-i$ will never work in the shadow economy in the optimum. On the other hand, distorting the labor choice of type i relaxes the binding incentive constraint and allows for more redistribution. Hence, type i can potentially work in the shadow economy in the optimum.

2.2. Welfare decomposition

Suppose that $\lambda_i > \lambda_{-i}$, such that the planner wants to redistribute resources from type $-i$ to type i . There are two candidate allocations for the optimum: a *Mirrleesian*

allocation in which type i works formally (denoted with superscript M) and a *shadow economy allocation* in which type i works informally (denoted with superscript SE). Note that the *Mirrleesian allocation* is also the optimum in the setting without the shadow economy. We examine the welfare impact of the shadow economy by comparing these two allocations.

Proposition 1. *Suppose that $\lambda_i > \lambda_{-i}$. The welfare difference between the shadow economy allocation and the Mirrleesian allocation can be decomposed in the following way*

$$\underbrace{W^{SE} - W^M}_{\text{welfare impact}} = \underbrace{\lambda_i \mu_i \left(U(w_i^s n_i^{SE}, n_i^{SE}) - U(w_i^f n_i^M, n_i^M) \right)}_{\text{efficiency gain}} + \underbrace{(\lambda_i - \lambda_{-i}) \mu_i (T_i^M - T_i^{SE})}_{\text{redistribution gain}},$$

where

- the efficiency gain is increasing with w_i^s and is positive when $w_i^s > \bar{w}_i^s$,
- the redistribution gain is decreasing with w_{-i}^s and is positive when $w_{-i}^s < \bar{w}_{-i}^s$,
- the productivity thresholds satisfy $\bar{w}_i^s < w_i^f$ and $\bar{w}_{-i}^s < w_{-i}^f$.

Proof. See Appendix A. □

Proposition 1 decomposes the welfare impact of the shadow economy into an *efficiency gain*, measuring the difference in distortions imposed on type i , and a *redistribution gain*, capturing the change in the level of transfers received by type i .

Efficiency gain. In the shadow economy allocation, type i supplies the efficient level of labor to the inefficient shadow sector. In the Mirrleesian allocation, due to the distortions imposed by the planner, type i supplies an inefficient amount of labor to the efficient formal sector. The relative inefficiency of the shadow sector depends on the productivity difference $w_i^f - w_i^s$. When this difference is sufficiently small ($w_i^s > \bar{w}_i^s$), distortions in the shadow sector are smaller than distortions in the formal sector and the shadow economy improves the efficiency of labor allocation. Intuitively, in this case the shadow economy provides a shelter against tax distortions. If instead the shadow economy distortions are large ($w_i^s < \bar{w}_i^s$), the efficiency impact of the informal sector will be negative.

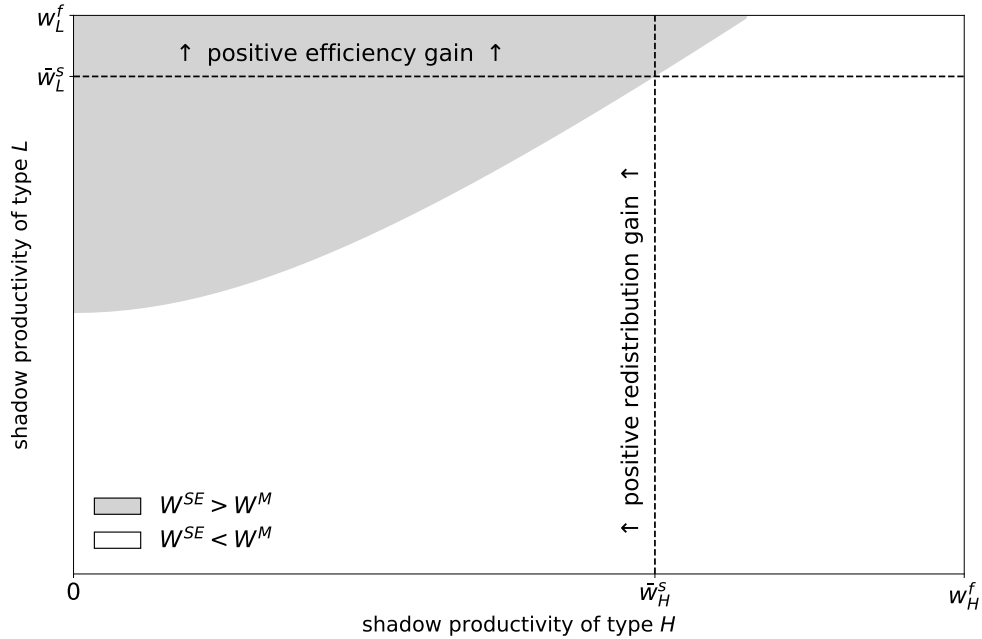
Redistribution gain. The shadow economy improves redistribution if the planner is able to provide type i with a higher transfer (or equivalently raise a higher tax from type $-i$). The scale of redistribution is determined by the payoff of type $-i$ from misreporting. In the Mirrleesian allocation the deviating worker works formally and can earn only as much as type i . In the shadow economy allocation the deviating worker cannot supply any formal labor, but is unconstrained in supplying shadow labor. As the shadow productivity of type $-i$ increases, the payoff from misreporting in the shadow economy allocation rises and the redistribution is reduced. On the other hand, when

w_{-i}^s is sufficiently low ($w_{-i}^s < \bar{w}_{-i}^s$), the shadow economy deters the deviation of type $-i$, helping the planner to tell the two types of agents apart. In this case the informal sector is used as a screening device.

Proposition 1 is illustrated in Figure 1, where we assume that the planner maximizes the utility of type L : $\lambda_H = 0$. Intuitively, the shadow economy does not have to strengthen both redistribution and efficiency simultaneously to be welfare improving. Particularly interesting is the region where the redistribution gain is negative, but the efficiency gain is sufficiently high such that welfare is higher with the shadow economy. In this case the shadow economy allocation Pareto dominates the Mirrleesian allocation. Type L gains, since the welfare is higher with the shadow economy. Type H benefits as well, as the negative redistribution gain implies a lower tax burden on this type.

Kopczuk (2001) provides an example in which, starting from the allocation without tax evasion, a marginal increase in evasion yields welfare gains.⁷ According to our decomposition, in his example welfare improves due to greater redistribution, but at the cost of efficiency. It may suggest that the shadow economy improves welfare by allowing for more even division of a smaller aggregate output. We show that such a scenario is only one of many possibilities. The shadow economy can reduce redistribution, while still being welfare-improving, in which case all agents benefit from the presence of the shadow economy.

Figure 1: Welfare impact of the shadow economy



⁷Kopczuk (2001) also presents a second example of welfare-improving tax evasion in which some agents have a distaste for paying taxes. We abstract from agents having preferences directly over tax payments.

3. Model with a continuum of types

In this section we derive and characterize the optimal tax schedule in the model with a continuum of productivity types and an idiosyncratic fixed cost of shadow employment. The fixed cost can be interpreted either as a technological constraint on tax evasion or a utility cost of violating social norms. The idiosyncratic fixed cost allows two agents of the same formal productivity to have different shadow employment opportunities, which is an important feature of the data.⁸ For the model with a continuum of productivity types, but without the fixed cost of shadow employment, see the earlier working paper version (Doligalski and Rojas 2016).

We follow Diamond (1998) by assuming that agents' preferences do not exhibit wealth effects - the assumption we make for tractability. We maintain the quasilinear preferences from the simple model.⁹ The assumption of linear utility from consumption is not restrictive, since we are characterizing the entire Pareto frontier which is invariant to any increasing transformation of the utility function. Hence, our results are applicable also with GHH preferences $G(c - v(n))$, where G is an increasing and strictly concave function. In fact, this is the formulation we use in the quantitative exercise.

Individuals have two privately observed characteristics: the productivity parameter θ and the cost parameter κ . $\theta \in [0, 1]$ determines the productivity in the formal economy $w^f(\theta)$ and in the shadow economy $w^s(\theta)$. We assume that both productivity functions are non-negative and continuously differentiable with respect to θ and that the formal productivity is strictly increasing. θ has a cumulative distribution function $F(\theta)$ and density $f(\theta)$. The parameter $\kappa \in [0, \infty)$ is a fixed cost of engaging in shadow employment. Conditional on θ , it has cumulative distribution function $G_\theta(\kappa)$ and density $g_\theta(\kappa)$.¹⁰

To solve the model with a continuum of types, it is useful to recover the *Spence-Mirrlees* single crossing property. This property ensures that formal income is increasing in productivity type θ .

Lemma 2. *Agents' preferences satisfy a strict Spence-Mirrlees single crossing condition if and only if $w^s(\theta)/w^f(\theta)$ is decreasing with θ or $w^s(\theta) = 0$ for all θ .*

Proof. See Appendix B. □

⁸In the quantitative section we show that observable individual characteristics alone are not sufficient to explain informality patterns (see the second panel of Figure 6).

⁹The agents' utility function over consumption and labor, net of the fixed cost of shadow employment, is $U(c, n) \equiv c - v(n)$, where v is increasing, strictly convex, twice differentiable and satisfies $v'(0) = 0$.

¹⁰In principle, we could introduce a fixed cost of formal employment as well. This would correspond to what Magnac (1991) calls a segmentation approach to informal labor markets, according to which shadow workers are restricted from formal employment by various labor regulations. An alternative, competitive approach is that individuals sort between the two sectors according to their individual advantage, which corresponds more closely to our framework. Magnac (1991) shows that the data on married women in Colombia favor the latter, competitive approach.

The single crossing requires that the comparative advantage in shadow labor is decreasing with formal productivity. This natural assumption is maintained throughout this section. In Section 4 we verify that it holds for Colombia.

3.1. Implementability

We characterize the income choices of all agents by focusing on two classes of agents: *low-cost workers* with no fixed cost of shadow employment ($\kappa = 0$) and *high-cost workers* with a prohibitively high fixed cost (denoted by $\kappa = \infty$). We will describe the implementable income schedules of these agents shortly. For now, take as given the formal income schedule of the low-cost workers $y^f(\cdot, 0)$ and of the high-cost workers $y^f(\cdot, \infty)$. Suppose they face an income tax schedule $T : \mathbb{R}_+ \rightarrow \mathbb{R}$. Their indirect utility function is

$$V(\theta, \kappa) \equiv \max_{y^s \geq 0} U \left(y^f(\theta, \kappa) + y^s - T(y^f(\theta, \kappa)), \frac{y^f(\theta, \kappa)}{w^f(\theta)} + \frac{y^s}{w^s(\theta)} \right) - \kappa \mathbb{1}_{y^s > 0}, \quad (5)$$

where $\kappa \in \{0, \infty\}$. Denote the informal earnings of low-cost workers, implicit in the above definition, by $y^s(\cdot, 0)$.

Define a *formality threshold* $\tilde{\kappa}(\theta) \equiv V(\theta, 0) - V(\theta, \infty)$. This threshold is positive when the low-cost workers earn some shadow income and obtain a strictly higher utility than the high-cost workers of the same productivity type. Take a worker of an arbitrary type $(\theta, \kappa) \in [0, 1] \times [0, \infty)$. Depending on whether the cost parameter κ is above or below the formality threshold, the agent behaves as either a high-cost or a low-cost worker:

$$(y^f(\theta, \kappa), y^s(\theta, \kappa)) = \begin{cases} (y^f(\theta, \infty), 0) & \text{if } \kappa \geq \tilde{\kappa}(\theta) \\ (y^f(\theta, 0), y^s(\theta, 0)) & \text{otherwise,} \end{cases} \quad (6)$$

and consequently the indirect utility function is:

$$V(\theta, \kappa) = \begin{cases} V(\theta, \infty) & \text{if } \kappa \geq \tilde{\kappa}(\theta) \\ V(\theta, 0) - \kappa & \text{otherwise.} \end{cases} \quad (7)$$

The agents with a fixed cost κ above the formality threshold $\tilde{\kappa}(\theta)$ work only formally and are called *formal workers*. The agents with a cost below the threshold supply some shadow labor (and possibly some formal labor as well) and are called *shadow workers*.

We have described the income choices of all agents conditional on the formal income schedules of low and high-cost workers. Now we will characterize the income choices of these two classes of agents. Without loss of generality we focus on right-continuous formal income schedules.

Definition 1. *Formal income schedules $y^f(\cdot, \infty)$ and $y^f(\cdot, 0)$ are implementable if there exists a tax schedule $T(\cdot)$ such that, for any $\theta, \theta' \in [0, 1]$ and $\kappa, \kappa' \in \{0, \infty\}$:*

1. The incentive compatibility constraint of the high-cost worker holds:

$$V(\theta, \infty) \geq U \left(y^f(\theta', \kappa') - T(y^f(\theta', \kappa')), \frac{y^f(\theta', \kappa')}{w^f(\theta)} \right). \quad (8)$$

2. The incentive compatibility constraint of the low-cost worker holds:

$$V(\theta, 0) \geq \max_{y^s \geq 0} U \left(y^f(\theta', \kappa') + y^s - T(y^f(\theta', \kappa')), \frac{y^f(\theta', \kappa')}{w^f(\theta)} + \frac{y^s}{w^s(\theta)} \right). \quad (9)$$

The incentive constraints of the high-cost workers prevent these agents from choosing the formal earnings assigned to any other agent. The incentive constraints of the low-cost workers additionally allow the deviating agent to adjust shadow earnings.¹¹

In the typical optimal taxation or screening model it is enough to restrict attention to *local* incentive constraints, making sure that no agent has incentives to misreport their productivity type *marginally* (see e.g. Fudenberg and Tirole 1991).¹² The local incentive constraints of a formal worker of type (θ, κ) can be expressed as the standard first-order condition with respect to formal income.¹³ It requires that the net marginal return to formal income is equal to the marginal disutility from higher earnings:

$$\left(1 - T' \left(y^f(\theta, \kappa) \right)\right) w^f(\theta) = v' \left(\frac{y^f(\theta, \kappa)}{w^f(\theta)} \right). \quad (10)$$

The local incentive constraint of a shadow worker (θ, κ) additionally equalizes the net return to formal and shadow labor:

$$\left(1 - T' \left(y^f(\theta, \kappa) \right)\right) w^f(\theta) = v' \left(\frac{y^f(\theta, \kappa)}{w^f(\theta)} + \frac{y^s(\theta, \kappa)}{w^s(\theta)} \right) = w^s(\theta). \quad (11)$$

Finally, when the formal income schedule of shadow workers is discontinuous at some type (θ_d, κ) then the local incentive constraint makes sure that the utility schedule remains continuous. It implies that the *average* net returns to formal and shadow labor

¹¹Denote the image of $y^f(\cdot, \cdot)$ by Y . Thus, Y is the set that contains all assigned formal income levels. The definition of implementability requires that no agent has incentives to deviate from the assigned formal income to any other income from the set Y . It does not, on the other hand, prevent deviations to formal income levels from $\mathbb{R}_+ \setminus Y$. However, these deviations can always be ruled out by modifying the tax schedule at $\mathbb{R}_+ \setminus Y$, which has no utility or resource cost. Hence, in further analysis we ignore these kinds of deviations.

¹²In our model, the local incentive constraint of type (θ, κ) can be represented as

$$\frac{d}{d\theta'} \left[\max_{y^s \geq 0} U \left(y^f(\theta', \kappa) + y^s - T \left(y^f(\theta', \kappa) \right), \frac{y^f(\theta', \kappa)}{w^f(\theta)} + \frac{y^s}{w^s(\theta)} \right) - \kappa \mathbb{1}_{y^s > 0} \right] \Big|_{\theta' = \theta} = 0.$$

By Proposition 2, the implementable formal income schedules are non-decreasing and hence continuous almost everywhere. When the formal income schedule is discontinuous, we require the continuity of the utility schedule: $V(\theta^-, \kappa) = V(\theta, \kappa)$.

¹³Whenever the formal income schedule is locally flat with respect to θ , the local incentive constraint is satisfied automatically. Then (10) and the left equality in (11) hold as \leq inequalities. It can happen e.g. at the tax kink.

coincide:

$$\left(1 - \frac{T(y^f(\theta_d, \kappa)) - T(y^f(\theta_d^-, \kappa))}{y^f(\theta_d, \kappa) - y^f(\theta_d^-, \kappa)}\right) w^f(\theta_d) = w^s(\theta_d). \quad (12)$$

The local incentive constraints are not sufficient in our setting. Specifically, there exist formal income schedules of the low-cost workers that are non-decreasing, satisfy local incentive constraints and yet violate some of the *global* incentive constraints.¹⁴ There are two reasons for this. First, the planner cannot adjust the tax schedule for high and low-cost workers independently, as they face the same tax schedule. Second, when tax rates are such that the net returns to formal and shadow labor are equal, the low-cost workers can shift labor between sectors at no cost. They can make large formal income adjustments which cannot be prevented by local constraints alone. In order to ensure implementability, we need to verify that, apart from the local incentive constraints, some non-local incentive constraints hold as well.

Proposition 2. *Formal income schedules $y^f(\cdot, \infty)$ and $y^f(\cdot, 0)$ are implementable if and only if there exists a tax schedule $T(\cdot)$ which satisfies the following conditions:*

1. $y^f(\cdot, \infty)$ and $y^f(\cdot, 0)$ are non-decreasing and satisfy local incentive constraints.
2. The bottom high-cost agent cannot gain by deviating to any lower formal income.
3. The top low-cost agent cannot gain by deviating to any higher formal income.
4. Suppose that $y^f(\cdot, 0)$ is discontinuous at θ_d . The agent $(\theta_d, 0)$ cannot gain by deviating to any formal income from the interval $(y^f(\theta_d^-, 0), y^f(\theta_d, 0))$.

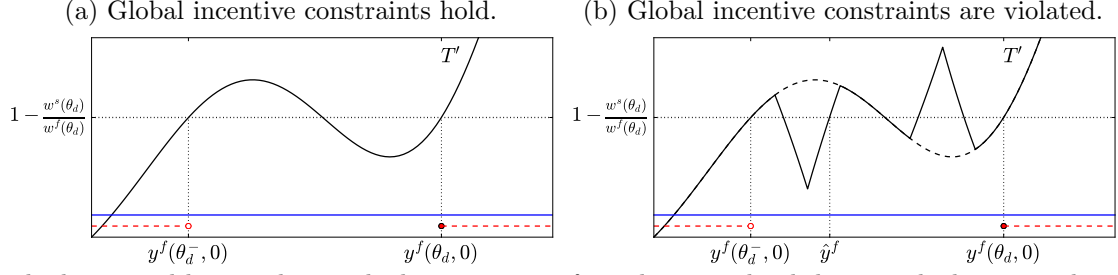
Proof. See Appendix B. □

The intuition behind the proof is following. The local incentive constraints are sufficient to prevent deviations of agents *within* their cost class (the class of either high or low-cost workers). Additional constraints are required to prevent deviations *between* the cost classes. Many such deviations are already covered by the local incentive constraints, since the images of the two formal income schedules are partially overlapping. Therefore, we need to focus only on deviations to formal income levels which are earned by some high-cost (low-cost) workers but by no low-cost (high-cost) worker. We capture these deviations by imposing non-local incentive constraints for the bottom high-cost agent, top low-cost agent and at each discontinuity point of $y^f(\cdot, 0)$.

Figure 2 shows graphically the insufficiency of the local incentive constraints. Consider an interval of low-cost workers that supply shadow labor. By the assumed decreasing comparative advantage $w^s(\theta)/w^f(\theta)$ and the local incentive constraint (9), the marginal

¹⁴Kleven, Kreiner, and Saez (2009), Scheuer (2014) and Gomes, Lozachmeur, and Pavan (2017) recover sufficiency of local incentive constraints in two-dimensional settings with two sectors, under the assumption that agents can work in one sector at a time. Rothschild and Scheuer (2014) allow workers to supply labor in multiple sectors simultaneously, but the government observes and taxes the sum of all incomes, which implies that the local incentive constraints are sufficient.

Figure 2: Insufficiency of the local incentive constraints.



The horizontal lines indicate whether at a given formal income level there are high-cost workers (solid, blue) or low-cost workers (dashed, red). In both panels agent $(\theta_d, 0)$ is indifferent between $y^f(\theta_d^-, 0)$ and $y^f(\theta_d, 0)$. Hence, the local incentive constraint (12) holds. However, in the right panel the worker strictly prefers formal income level \hat{y}^f , since the average tax rate between $y^f(\theta_d^-, 0)$ and \hat{y}^f is below $1 - w^s(\theta_d)/w^f(\theta_d)$.

tax rate they face is increasing in θ . When the marginal tax rates are not monotone increasing in formal income, the formal income schedule of the low-cost workers must be discontinuous, as illustrated in the figure. The local incentive constraint of the agent at the discontinuity $(\theta_d, 0)$, given by equation (12), requires this worker to be indifferent between the two formal income levels across the discontinuity: $y^f(\theta_d^-, 0)$ and $y^f(\theta_d, 0)$. In the right panel we modify the marginal tax rates in a way that total tax levels at $y^f(\theta_d^-, 0)$ and $y^f(\theta_d, 0)$ do not change. Thus, the local incentive constraint of agent $(\theta_d, 0)$ still holds. However, this agent has a profitable deviation to \hat{y}^f . The average tax rate between $y^f(\theta_d^-, 0)$ and \hat{y}^f is below $1 - w^s(\theta)/w^f(\theta)$, which implies that the utility from deviation to \hat{y}^f is higher than the utility at $y^f(\theta_d, 0)$. Therefore, the formal income schedule of the low-cost types is not globally incentive compatible.

3.2. The planner's problem.

The social planner maximizes the average of individual utilities, weighted with Pareto weights $\lambda(\theta, \kappa)$. We normalize the weights such that $\mathbb{E}\{\lambda(\theta, \kappa)\} = 1$, which implies that the Pareto weights and the marginal social welfare weights coincide.¹⁵ The planner solves

$$\max_{\substack{y^f(\cdot, \infty) : [0, 1] \rightarrow \mathbb{R}_+ \\ y^f(\cdot, 0) : [0, 1] \rightarrow \mathbb{R}_+ \\ T : \mathbb{R}_+ \rightarrow \mathbb{R}}} \int_0^1 \int_0^\infty \lambda(\theta, \kappa) V(\theta, \kappa) dG_\theta(\kappa) dF(\theta) \quad (13)$$

subject to the implementability conditions from Proposition 2 and the budget constraint

$$\int_0^1 \int_0^\infty T(y^f(\theta, \kappa)) dG_\theta(\kappa) dF(\theta) \geq E, \quad (14)$$

¹⁵The marginal social welfare weights describe the welfare impact of marginally increasing consumption of a given type, expressed in the unit of tax revenue (see e.g. Piketty and Saez (2013)). In our environment they are simply $\lambda(\theta, \kappa)/\eta$, where η is the multiplier of the budget constraint. It is easy to show that at the optimum $\eta = \mathbb{E}\{\lambda(\theta, \kappa)\}$.

where E stands for exogenous government expenditures. By solving the planner's problem for arbitrary Pareto weights, we recover the entire Pareto frontier of the model without wealth effects.¹⁶

We proceed with the theoretical analysis under the standard assumptions that the monotonicity constraints on formal income schedules are not binding (i.e. there is no bunching) and additionally that the implementability condition 2 is not binding. Moreover, since we derive the optimal allocation by perturbing the implementable formal income schedules, we do not need to account for implementability conditions 3 and 4 explicitly in our tax formulas. In the computational algorithm we verify the monotonicity constraints and condition 2 ex post and we numerically optimize with respect to formal income choices of the low-cost workers, which effectively replaces conditions 3 and 4 with a stronger incentive compatibility condition (9).

3.3. Derivation of the optimal tax formulas.

So far we have stated the problem of finding the optimal tax schedule using the mechanism design approach. It is instructive, however, to think about it in terms of tax perturbations as in Saez (2001). In this section we will derive the optimal tax formulas with the tax perturbation approach by considering a small variation of the marginal tax rate at some formal income level. In Appendix C we derive the tax formulas with the mechanism design approach and we provide the exact correspondence between the two perspectives.

From now on we will focus on the endogenous distribution of formal income. Denote the density of formal income by $h(\cdot)$. We can decompose it into the density of formal workers $h^f(\cdot)$ and the density of shadow workers $h^s(\cdot)$, such that at each income level y we have $h(y) = h^f(y) + h^s(y)$.¹⁷

Suppose that agents face a twice differentiable tax schedule $T(\cdot)$. Consider perturbing the marginal tax rate in the formal income interval $[y, y + dy]$ by a small $d\tau > 0$. This perturbation influences tax revenue via: (i) intensive margin responses of formal and shadow workers, (ii) extensive margin responses due to workers changing their informality status, (iii) mechanical and welfare effects.

¹⁶Suppose that the planner follows the social welfare function $\int_0^1 \int_0^\infty \Gamma(V(\theta, \kappa)) dG_\theta(\kappa) dF(\theta)$, where Γ is an increasing and differentiable function. Γ is typically assumed to be strictly concave and it can represent either decreasing marginal utility of consumption or the planner's taste for equality. We find the optimal allocation in this case by setting the Pareto weights in the planner's problem according to $\lambda(\theta, \kappa) = \Gamma'(V(\theta, \kappa))$, where V is the indirect utility function at the optimum. In this case the Pareto weights are endogenous, since they explicitly depend on the optimal allocation, and the model needs to be solved iteratively: in each iteration the Pareto weights are updated to reflect the indirect utility function implied by the previous solution of the model.

¹⁷Formally, $h^f(\cdot)$ and $h^s(\cdot)$ are not densities, as they do not integrate to 1 but to a share of formal and shadow workers in total employment, respectively. Keeping this slight abuse of terminology in mind, we will continue calling them densities.

Intensive margin responses of formal and shadow workers. In response to the increase in the marginal tax rate, the agents with income y or slightly higher will reduce their formal earnings. The income reduction of formal workers is standard and equal approximately to

$$h^f(y)\tilde{\varepsilon}^f(y)y\frac{d\tau dy}{1-T'(y)}, \quad \text{where } \tilde{\varepsilon}^f(y) \equiv \left(\frac{1}{\varepsilon(y)} + \frac{T''(y)y}{1-T'(y)}\right)^{-1}. \quad (15)$$

$\tilde{\varepsilon}^f(y)$ is the elasticity of formal income of formal workers with respect to the marginal tax rate along the non-linear tax schedule. It depends both on $\varepsilon(y)$, the elasticity along the *linear* tax schedule, and the local tax curvature. For instance, the typical isoelastic disutility of labor $v(n) = n^{1+1/\epsilon}$ implies $\varepsilon(y) = \epsilon$. When the tax schedule is non-linear, the income responses depend not only on this structural elasticity $\varepsilon(y)$, but also on the curvature of the tax schedule. For instance, when the tax is locally strictly progressive ($T''(y) > 0$), an increase of income in response to a tax rate cut is reduced, as an increase in income leads to higher tax rates due to progressivity. Hence, local progressivity (regressivity) of the tax schedule reduces (increases) the elasticity of income.

What are the formal income responses of shadow workers? Suppose that there are some shadow workers with formal income y . In Appendix C we show that the reduction of formal income of shadow workers is equal to

$$h^s(y)\tilde{\varepsilon}^s(y)y\frac{d\tau dy}{1-T'(y)}, \quad \text{where } \tilde{\varepsilon}^s(y) \equiv \frac{1-T'(y)}{T''(y)y} > \tilde{\varepsilon}^f(y). \quad (16)$$

The elasticity of the formal income of *shadow* workers is strictly greater than that of *formal* workers: $\tilde{\varepsilon}^s(y) > \tilde{\varepsilon}^f(y)$. This is because the elasticity of shadow workers along the *linear* tax schedule is infinite. Suppose that a shadow worker of type θ faces a linear tax with tax rate $1 - w^s(\theta)/w^f(\theta)$. This agent is indifferent between supplying formal and shadow labor and the optimality condition (11) pins down only the total labor supply, but not its sectoral split. Suppose that the tax rate is increased marginally. Now the return to shadow labor is strictly greater than the return to formal labor. Thus, the agent shifts the entire labor supply to the shadow economy and reduces his formal income all the way to zero. This dramatic reduction of formal income means that the elasticity of formal income of shadow workers along the *linear* tax schedule is infinite. In contrast, if the tax schedule were nonlinear and locally strictly progressive ($T''(y) > 0$), the shadow worker would reduce his formal income only until the marginal tax rate is again equal $1 - w^s(\theta)/w^f(\theta)$, which implies a finite elasticity.¹⁸

So far we considered the situation in which there are some shadow workers with formal income y at which we perturb the marginal tax rate. Since the income schedule of low-

¹⁸Since the second-order optimality condition of shadow workers is $T''(y) \geq 0$, we do not need to consider a locally strictly regressive tax. No shadow worker would choose such a formal income level.

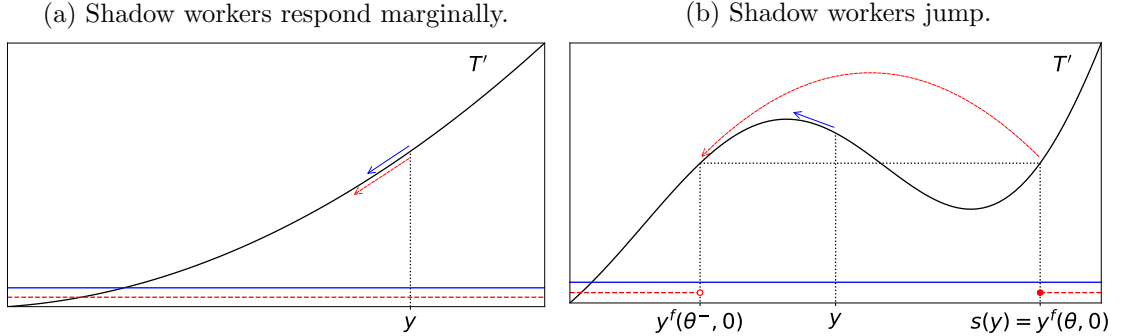
cost workers can easily become discontinuous, we also need to discuss the case in which there are no low-cost workers at y , but there are some with strictly higher formal income. In general, the perturbation of $T'(y)$ triggers an income response of the low-cost workers at formal income $s(y)$, where $s(y)$ is defined as

$$s(y) \equiv \min_{\theta} \{y^f(\theta, 0) \text{ s.t. } y^f(\theta, 0) \geq y\}. \quad (17)$$

When there are some low-cost workers with formal income y , then $s(y) = y$, as in the case discussed above. To consider the other case, suppose that the income schedule of low-cost workers $y^f(\cdot, 0)$ has discontinuity at θ . It means that for any $y \in (y^f(\theta^-, 0), y^f(\theta, 0))$ we have $s(y) = y^f(\theta, 0)$. Furthermore, by (12) the shadow worker with income $s(y)$ is exactly indifferent between earning $s(y)$ and $y^f(\theta^-, 0)$. Consider an increase in $T'(y)$. As the tax burden at $s(y)$ increases, the agent strictly prefers $y^f(\theta^-, 0)$ to $s(y)$ and jumps to the lower formal income level.

Figure 4 illustrates the two types of formal income responses of shadow workers. On the left panel, the formal income schedule of the shadow workers is locally continuous at y and the tax schedule is locally strictly progressive. Consequently, the shadow workers respond to an increase of $T'(y)$ by marginally reducing their formal income. On the right panel, the formal income schedule of the shadow workers is discontinuous. In the response to an increase in $T'(y)$ the shadow worker discretely jumps to a lower formal income level.

Figure 4: Formal income responses to the increase of $T'(y)$.



The horizontal lines indicate whether there are high-cost workers (solid, blue) or low-cost workers (dashed, red) at a given formal income level. The arrows represent formal income responses of high-cost workers (solid, blue) and low-cost, shadow workers (dashed, red).

When the formal income schedule of shadow workers is discontinuous, formal income responses of shadow workers to a marginal change in tax rates are non-marginal, discrete. Surprisingly and very conveniently, they can be still described with the intensive margin elasticity $\tilde{\varepsilon}^s(\cdot)$. The perturbation increases the tax burden at $s(y)$ by $d\tau dy$ and makes some shadow workers discretely decrease their formal income from $s(y)$ to $s(y) - \Delta y$. The measure of shadow workers at income level $s(y)$ that decides to jump is given by

$h^s(s(y))ds(y)$. By differentiating (12), we obtain $ds(y) = [T''(s(y))\Delta y]^{-1}d\tau dy$. Therefore, the overall income reduction is exactly as in the case when shadow workers adjust income marginally:

$$\Delta y h^s(s(y))ds(y) = h^s(s(y))\varepsilon^s(s(y))s(y)\frac{d\tau dy}{1 - T'(s(y))}, \quad (18)$$

The intuition is that, although the formal income elasticity of each individual worker who jumps is infinite, the measure of jumping individuals is small enough such that the overall elasticity at $s(y)$ is finite.

Suppose that $y \leq y^f(1, 0)$. We can express the tax revenue impact of the intensive margin responses of formal and shadow workers, no matter whether they are responding marginally or jumping, as

$$- \left(\frac{T'(y)}{1 - T'(y)} h^f(y) \tilde{\varepsilon}^f(y) y + \frac{T'(s(y))}{1 - T'(s(y))} h^s(s(y)) \tilde{\varepsilon}^s(s(y)) s(y) \right) d\tau dy. \quad (19)$$

In the remaining case of $y > y^f(1, 0)$, no shadow workers are distorted and the second term in the bracket is equal to zero.

Extensive margin responses. The perturbation of $T'(y)$ increases the tax burden for workers with incomes above y . Consequently, it increases incentives for informality for agents who, conditional on working informally, would earn less than y in the formal sector.

Define the formal income gap between high and low-cost agents as $\Delta_\kappa(y) \equiv y^f(\theta, \infty) - y^f(\theta, 0)$, where θ is such that $y^f(\theta, \kappa) = y$. $\Delta_\infty(y)$ tells us by how much the formal worker with income y would decrease his formal income if he had a lower realization of the fixed cost and worked informally. Conversely, $\Delta_0(y)$ tells us by how much the shadow worker with formal income y would increase his formal earnings if he did not work in the shadow economy. Note that if all θ -workers are formal, then $\Delta_\infty(y^f(\theta, \infty)) = \Delta_0(y^f(\theta, 0)) = 0$.

Suppose that $y \leq y^f(1, 0)$. The perturbation of $T'(y)$ increases incentives for informality for formal workers in the income interval $(y, s(y + dy) + \Delta_0(s(y + dy)))$. Workers with income below y are unaffected, since their tax schedule is unchanged. Workers with income above $s(y + dy) + \Delta_0(s(y + dy))$ pay taxes higher by $d\tau dy$ no matter whether they stay formal or move to the shadow economy, so their incentives for informality are unchanged as well. In the following derivations we focus on a subinterval $[y + dy, s(y) + \Delta_0(s(y))]$, since the terms corresponding to the remaining parts of the original interval are of second order (i.e. they are proportional to dy^2) and vanish as we consider an arbitrarily small dy .

Denote the impact of the perturbation on the density of formal workers at income y' by $dh^f(y')$. Also denote the *tax burden of staying formal* at formal income by $\Delta T(y') \equiv T(y') - T(y' - \Delta_\infty(y'))$. It captures the tax revenue loss when a formal worker with

income y starts supplying informal labor. The tax revenue impact of the perturbation via the adjustment of the distribution of formal workers is

$$\int_{y+dy}^{s(y)+\Delta_0(s(y))} dh^f(y') \Delta T(y') dy' d\tau dy = - \int_{y+dy}^{s(y)+\Delta_0(s(y))} \pi(y') h^f(y') dy' d\tau dy, \quad (20)$$

where $\pi(y^f(\theta, \infty)) \equiv \frac{g_\theta(\tilde{\kappa}(\theta))}{1-G_\theta(\tilde{\kappa}(\theta))} \Delta T(y^f(\theta, \infty))$ is the elasticity of the density of formal workers at y' with respect to the tax burden of staying formal. Intuitively, the more elastic is the density of formal workers, the higher is the tax revenue loss due to increasing participation in the shadow economy.

In the case of $y > y^f(1, 0)$, all shadow workers have formal incomes below y . Consequently, the tax perturbation increases incentives for informality at all formal income levels above $y + dy$, rather than only up to $s(y) + \Delta_0(s(y))$.

Mechanical and welfare impact. Consider the tax schedule at incomes above $y + dy$. The perturbation keeps the tax rate fixed, while increasing the tax level by $d\tau dy$. On the one hand, an increase in the tax level mechanically raises the tax revenue. On the other hand, it reduces utility of agents with higher incomes, resulting in a welfare loss. Denote the average Pareto weight at a given formal income level y by $\bar{\lambda}(y)$. Ignoring the second-order terms, the combined mechanical and welfare impact of the perturbation is

$$\int_{y+dy}^{\infty} (1 - \bar{\lambda}(y')) h(y') dy' d\tau dy. \quad (21)$$

Optimal tax formulas. Optimality requires that no small tax perturbation can increase welfare-adjusted tax revenue. Hence, the sum of all the impacts of the tax perturbation: (19), (20) and (21), needs to be zero for any $d\tau$ and an arbitrary small dy .

Theorem 1. *Suppose that bunching does not occur. For $y \leq y^f(1, 0)$, the optimal tax rate satisfies*

$$\begin{aligned} & \frac{T'(y)}{1 - T'(y)} h^f(y) \varepsilon^f(y) y + \frac{T'(s(y))}{1 - T'(s(y))} h^s(s(y)) \varepsilon^s(s(y)) s(y) \\ &= \int_y^{\infty} [1 - \bar{\lambda}(y)] h(y) dy - \int_y^{s(y)+\Delta_0(s(y))} \pi(y') h^f(y') dy'. \end{aligned} \quad (22)$$

For $y > y^f(1, 0)$, the optimal tax rate satisfies

$$\frac{T'(y)}{1 - T'(y)} h^f(y) \varepsilon^f(y) y = \int_y^{\infty} [1 - \bar{\lambda}(y') - \pi(y')] h(y') dy'. \quad (23)$$

Tax formula (22) equates the *deadweight loss* from distorting the formal workers and the shadow workers on the left-hand side, with the *tax revenue gain* from higher tax on

formal incomes above y net of the tax loss from increased participation in the shadow economy on the right-hand side. The second tax formula (23) captures the case when no low-cost workers are distorted by a perturbation of the tax rate at the given formal income level.

The deadweight loss of both formal and shadow workers increases in (i) the marginal tax rate, as the reduction in formal income implies a higher tax loss if it is taxed at the higher rate, (ii) the density of formal income and (iii) the formal income reduction per worker in response to a higher tax rate, i.e. the product of formal income and the elasticity at this income level. There are two important differences between the deadweight losses of formal and shadow workers. First, the distorted shadow workers may have formal income that is strictly higher than y , while the distorted formal workers have always exactly y . Second, conditional on the local progressivity of the tax schedule, the formal income of shadow workers is more elastic than the income of formal workers.

The tax revenue gain in formulas (22) and (23) consists of two terms. The first one summarizes the mechanical and welfare impact from increased taxation of all workers with higher formal income. A perturbation in the marginal tax rate increases their taxes and reduces welfare proportionally to their Pareto weight. The second term captures the tax revenue cost of increased participation in the shadow economy. Note that in formula (22) the participation in the shadow economy of workers with formal income above $s(y) + \Delta_0(s(y))$ is unchanged. These workers have no incentives to start informal employment. Even if they decided to work in the shadow economy, their formal income would be high enough such that they would still pay higher taxes. Thus, they have no additional incentives for informality. In contrast, when formula (23) applies, the perturbation of the marginal tax rate increases incentives for informality for all workers with higher formal income. That is because all affected agents would have formal income below the income at which we perturb the rate if they worked informally.

3.4. How does a shadow economy affect optimal tax rates?

First consider an income level y where $\Delta_\infty(y) = 0$, which means that all workers of productivity type θ , where $y^f(\theta, \infty) = y$, are formal. In this case the optimal tax formula (22) collapses into the Diamond (1998) formula from the standard Mirrlees model expressed with sufficient statistics:

$$\frac{T'(y)}{1 - T'(y)} h(y) \bar{\varepsilon}(y) y = \int_y^\infty [1 - \bar{\lambda}(y')] h(y') dy', \quad (24)$$

where $\bar{\varepsilon}(y)$ stands for the mean elasticity of formal income at y .¹⁹ Importantly, in this particular case not only do the sufficient statistics formulas coincide, but so do the values

¹⁹Diamond (1998) derived the optimal tax formula in the Mirrlees model without wealth effects. Saez (2001) expressed the optimal tax formula with sufficient statistics. Acknowledging that, we will refer to (24) as the Diamond formula.

of the sufficient statistics.²⁰ The optimal tax rate at this income level is thus *exactly* as in the standard model without the shadow economy. This result holds regardless of whether there are shadow workers at higher or lower income levels.

To see why all the terms related to the shadow economy vanish, first note that no shadow worker is distorted. More specifically, the distorted low-cost worker works only formally, which implies that $h^s(s(y)) = 0$. Second, $\Delta_\infty(y) = 0$ implies both that $s(y) = y$ and $\Delta_0(y) = 0$. As a result, the integral capturing the impact of the tax perturbation on the distribution of formal income disappears. Since there is no formal worker with income above y who after starting informal employment would choose formal income below y , varying $T'(y)$ does not affect the extensive margin incentives for shadow employment of any agent.

Now suppose that $\Delta_\infty(y) > 0$, which means that some workers of productivity type θ , where $y^f(\theta, \infty) = y$, are working informally. First, consider the case of $s(y) = y$, so that there are some shadow workers with formal income y . Then the formula (22) can be written as

$$\frac{T'(y)}{1 - T'(y)} h(y) \bar{\varepsilon}(y) y = \int_y^\infty [1 - \bar{\lambda}(y')] h(y') dy' - \int_y^{y+\Delta_0(y)} \pi(y') h^f(y') dy'. \quad (25)$$

Since the distorted formal and shadow workers have exactly the same formal earnings, we can combine their deadweight losses into one term: the average deadweight loss at formal income y , as in the Diamond formula (24). The optimal formula differs from Diamond's due to the additional term on the right-hand side, which stands for the tax revenue loss from increased participation in the shadow economy. Whenever the marginal tax rates at the interval $[y - \Delta_\infty(y), y + \Delta_0(y)]$ are positive, the additional term is positive as well. Then the increased participation in the shadow economy is costly for tax revenue and the tax formula leads to lower marginal tax rates than the Diamond formula.²¹

We can also compare formula (25) to the tax formula by [Jacquet, Lehmann, and Van der Linden \(2013\)](#), derived in the model with both intensive and extensive margins of labor supply. They do not consider the shadow economy and their extensive margin responses are binary: each agent can either work or withdraw entirely from the labor market. Translating this approach to our setting would mean that agents can work either only formally or only informally. We generalize these extensive margin responses by allowing workers who start shadow employment to retain some formal earnings. This limits the size of the term related to extensive responses for two reasons. First, if the total tax is increasing with formal income, joining the shadow economy and keeping some formal income implies a lower tax revenue loss than leaving the formal economy entirely. Second, the tax perturbation at y does not increase incentives for shadow employment for all

²⁰This statement ceases to be true once the Pareto weights are allowed to depend on the allocation.

²¹Suppose on the contrary that the marginal tax rates at this interval are negative. Then there is a fiscal benefit of pushing workers to the shadow economy, as they end up paying higher taxes. In that case the optimal tax rates will be higher than the ones implied by the Diamond formula.

workers with higher incomes, but only for those who would earn less than y formally if they started shadow employment. Given that, the tax formula (25) implies marginal tax rates that are weakly higher than the ones of [Jacquet, Lehmann, and Van der Linden \(2013\)](#).

Note that when $y > y(1, 0)$ holds, the optimal tax formula is given by (23), which can be expressed similarly to (25), the only difference being that the extensive margin term includes all the formal income levels above y , rather than being limited by $y + \Delta_0(y)$. The conclusions based on formula (25) extend to this case as well.

Finally, suppose that $\Delta_\infty(y) > 0$ and $s(y) > y$. It implies that the distorted shadow workers have strictly higher formal income than the distorted formal workers. We can express the formula (22) as

$$\begin{aligned} \frac{T'(y)}{1 - T'(y)} h(y) \bar{\varepsilon}(y) y = \int_y^\infty [1 - \bar{\lambda}(y')] h(y') dy' - \int_y^{s(y) + \Delta_0(s(y))} \pi(y') h^f(y') dy' \\ - \frac{T'(s(y))}{1 - T'(s(y))} h^s(s(y)) \tilde{\varepsilon}^s(s(y)) s(y). \end{aligned} \quad (26)$$

Since the distorted shadow workers have formal income that is higher than y , their deadweight loss is not included in the average deadweight loss at y . As a result, the Diamond formula (as well as the formula of [Jacquet, Lehmann, and Van der Linden \(2013\)](#)) misses the shadow deadweight loss of taxation. As long as $T'(s(y))$ is positive, the deadweight loss of shadow workers contribute to lower tax rates than according to the Diamond formula.

We conclude this section by considering the optimal taxation of top incomes.

Proposition 3 (Tax rate at the top). *Suppose that (i) formal income distribution has a Pareto tail: $\lim_{y \rightarrow \infty} \frac{y h(y)}{1 - H(y)} = \alpha$, (ii) the fixed cost of shadow employment has a Pareto tail: $\forall \theta \lim_{\kappa \rightarrow \infty} \frac{\kappa g_\theta(\kappa)}{1 - G_\theta(\kappa)} = \gamma$, (iii) the mean labor elasticity at the top converges to $\bar{\varepsilon}(\infty)$, (iv) the mean Pareto weight at the top converges to $\bar{\lambda}(\infty)$.*

The top net-of-tax rate according to the Diamond formula is $1 - T'_D(\infty) \equiv \frac{\alpha \bar{\varepsilon}(\infty)}{1 - \bar{\lambda}(\infty) + \alpha \bar{\varepsilon}(\infty)}$. The optimal top net-of-tax rate $1 - T'(\infty)$ satisfies:

$$1 - T'(\infty) = \begin{cases} 1 - T'_D(\infty) & \text{if } 1 - T'_D(\infty) \geq w^s(1)/w^f(1), \\ w^s(1)/w^f(1) & \text{if } 1 - T'_D(\infty) < w^s(1)/w^f(1) \text{ and } \gamma \geq \bar{\gamma}, \\ \frac{\alpha \bar{\varepsilon}(\infty)}{1 - \bar{\lambda}(\infty) + \alpha \bar{\varepsilon}(\infty) - \pi(\infty)} & \text{if } 1 - T'_D(\infty) < w^s(1)/w^f(1) \text{ and } \gamma \leq \bar{\gamma}, \end{cases} \quad (27)$$

for some threshold $\bar{\gamma} > 0$.

Proof. See Appendix [B](#). □

Suppose that the Pareto weight at the top is below the average Pareto weight: $1 > \bar{\lambda}(\infty)$. It implies that the top tax rate and the elasticity $\pi(\infty)$, capturing the fiscal cost of pushing workers to the shadow economy, are both positive.

When the top tax rate implied by the Diamond formula is low enough, the Diamond top tax rate is optimal with a shadow economy as well. In this case the top *net-of-tax* rate is increasing in the labor elasticity at the top and the welfare weight at the top, and it is decreasing in the thickness of the upper tail of the formal income distribution $1/\alpha$.

In the opposite case, when the Diamond top tax rate would push some of the top workers to the shadow economy, the planner has incentives to decrease the top net-of-tax rate at least to $w^s(1)/w^f(1)$. At this level, an additional marginal decrease in the top net-of-tax rate has a discrete cost due to a displacement of a fraction of top workers to the shadow economy. This fraction increases with γ , the inverse of the tail parameter of the fixed cost distribution. High γ means that there are many top productivity workers with a relatively low fixed cost of shadow employment. Hence, for sufficiently high γ the optimal net-of-tax rate will be equal to $w^s(1)/w^f(1)$. When γ is sufficiently low, there are few top workers with a low value of the fixed cost of shadow employment and the planner is willing to decrease the top net-of-tax rate below $w^s(1)/w^f(1)$.

The analysis so far took the distribution of formal income as given. In the case of the top tax rate we can also determine how the shadow economy affects the thickness of the upper tail of the formal income distribution.

Lemma 3. *Suppose that the formal productivity distribution has a Pareto tail:*

$$\lim_{\theta \rightarrow 1} \frac{f(\theta)w^f(\theta)}{1 - F(\theta)} \left(\frac{dw^f(\theta)}{d\theta} \right)^{-1} = \alpha_w. \quad (28)$$

Under the assumptions of Proposition 3, the distribution of formal income has a Pareto tail with parameter

$$\alpha = \begin{cases} \frac{\alpha_w}{1 + \bar{\varepsilon}(\infty)} & \text{if } 1 - T'(\infty) \geq w^s(1)/w^f(1), \\ \frac{\alpha_w}{1 + \bar{\varepsilon}(\infty)} + \gamma & \text{otherwise.} \end{cases} \quad (29)$$

Proof. See Appendix B. □

When there are shadow workers among the most productive types, the thickness of the upper tail of the formal income distribution is reduced: the tail parameter of formal income increases by γ . Therefore, the difference in sufficient statistics reinforces the difference in tax formulas. In this case the optimal top tax rate with a shadow economy is lower than in the standard model both because of the extensive margin of shadow labor at the top and because the upper tail of the income distribution is thinner.

4. Quantitative exploration

In this section we quantitatively explore the importance of our theoretical results. First, we estimate the continuum of types model using the data from Colombia, an economy with a large informal sector. Second, we compare the allocations implied by the optimal tax formula and the standard Diamond formula. We document large differences in terms of tax rates, the size of the shadow economy and welfare. Finally, we decompose the welfare impact of the shadow economy in Colombia and show that both efficiency and redistribution components are quantitatively important. Additional quantitative results are available in Appendix E, where we examine the social preferences implicit in the actual Colombian income tax schedule.

4.1. Estimation

We estimate the continuum of types model using survey data from Colombia. The Colombian household survey allows us to reliably extract information on individual informal incomes from the main job, which we describe in detail below. The International Labour Organization ranked Colombia as 9th among 14 Latin American countries in terms of share of informal employment in total workforce (ILO 2014), which shows that Colombia is not an outlier with respect to the size of the informal sector.

Although we have expressed the optimal tax rates in terms of sufficient statistics, some of these statistics are very local in nature. We know from the previous section that shadow workers are very responsive to the shape of the tax schedule. As a result, the density of shadow workers at some formal income level, even if reliably estimated, is of limited use unless we know exactly how it changes as we vary the income tax. To overcome this obstacle, we follow the suggestion of Chetty (2009) and estimate the structural model to extrapolate the values of sufficient statistics out of sample.

Below we explain how we identify informality in the data and introduce our estimation strategy. The detailed descriptions of the data and of the estimation procedure are provided in Appendix D.

Identifying informality. We identify the main job of a given worker as informal if the worker reports not contributing to the mandatory social security programs. Since social security contributions are paid jointly with payroll taxes and the withheld part of the personal income tax, a worker who contributes to the social security is automatically subject to income taxation. Thus, this approach is particularly well suited for our exercise.²² We find that 58% of all workers in Colombia in 2013 were employed informally

²²Detecting informality via social security contributions is broadly consistent with the methodology of the International Labour Organization (ILO 2013) and is used by the Ministry of Labor of Colombia (ILO 2014), as well as by Mora and Muro (2017) and Guataquí, García, and Rodríguez (2010) in the studies of Colombia.

at the main job, a result consistent with other indicators of informality in Colombia.²³

Sample selection. We restrict attention to individuals aged 24-50 years without children (26,000 individuals). We choose this sample, since these workers face a tax and transfer schedule which is not means-tested and does not depend on choices absent from our modeling framework, such as number of children or college attainment.

Estimation strategy. The main challenge in estimating the model is identifying the joint distribution of formal and shadow productivities. For each worker we observe the hourly wage at the main job, which we interpret as productivity, and a sector of the main job, which can be either formal or informal. Crucially, we do not observe the counterfactual productivity in the sector in which the worker *is not employed* at the main job. Heckman and Honore (1990) and French and Taber (2011) show that the data on wages and the sector in which workers participate is in general not sufficient to identify the sectoral productivity profiles, since workers self-select to a sector in which they are more productive. Heckman and Honore (1990) prove that the model can be identified with additional regressors which influence wages. We follow this approach. Denote the vector of regressors, which includes worker's and job's characteristics, by X .²⁴ We assume that X is informative about the worker's productivity type: $\theta \sim N(X\beta, \sigma_\theta^2)$, where a vector β and a scalar σ_θ are parameters to be estimated. This assumption allows us to match similar individuals who, due to different realizations of the fixed cost of shadow employment, ended up having the main job in different sectors. Given that, we can infer a counterfactual shadow productivity of each formal worker from the observed shadow productivity of the matched shadow workers, and *vice versa*.

Additionally, we assume that (i) the sectoral log-productivity schedules $w^f(\cdot)$ and $w^s(\cdot)$ are affine with respect to the productivity type, (ii) the fixed cost of shadow employment κ is drawn from a generalized Pareto distribution, the parameters of which are allowed to vary with the productivity type θ , (iii) disutility from labor is given by $v(n) = \Gamma \frac{n^{1+1/\varepsilon}}{1+1/\varepsilon}$, implying a constant intensive margin labor elasticity ε , which is standard in the literature. We obtain the density of the productivity type $F(\theta)$ with kernel density estimation and we fit a Pareto tail to the distribution of top wages. Given these assumptions, we formulate the likelihood function and estimate the model using maximum likelihood. The likelihood function and parameter estimates are available in Appendix D.

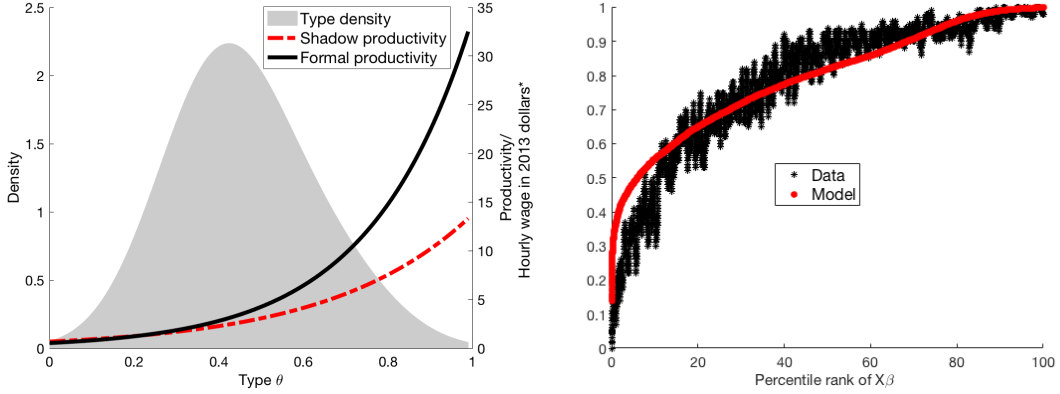
²³The official statistical agency of Colombia (DANE) follows an alternative measure of informality based on size of the establishment, status in employment and educational level of workers. They find that 57.3% and 56.7% of workers were informal in the first two quarters of 2013 (ILO 2014), which is very close to 58% we find for the entire 2013.

²⁴In our estimation the vector X contains typical regressors from Mincerian wage equations such as age, gender, education level and experience. We also include job and firm characteristics, such as the task performed by the worker and the size of the firm.

Estimation results. The left panel of Figure 6 presents the estimated productivity profiles and the density of productivity types. The bottom 8% of workers are more productive in the shadow sector while the median worker is 32% more productive formally. We find that the comparative advantage in the shadow economy decreases with the productivity type.²⁵ Thus, as assumed in the theoretical analysis, the single crossing condition holds. The density of productivity types in the main part of the distribution is approximately normal, which means that sectoral wages are distributed approximately log-normally with a Pareto tail. The intensive margin labor elasticity is equal to 0.358, close to the value of 0.33 suggested by Chetty (2012).

Figure 6: Estimation results

(a) Productivity profiles and type distribution (b) Probability of having a formal main job



The right panel of Figure 6 shows the estimated probability of having a main job in a formal sector for each percentile of $X\beta$. The probability of having a formal main job is strictly increasing and ranges from 14% to 99%. To illustrate the fit of the model we also plot the share of shadow workers in a rolling window of 200 workers centered around each observed $X\beta$ in the sample. The model tracks the data well, showing that our parametric specification is compatible with the observed sorting of workers across sectors.²⁶

4.2. Optimal tax schedules.

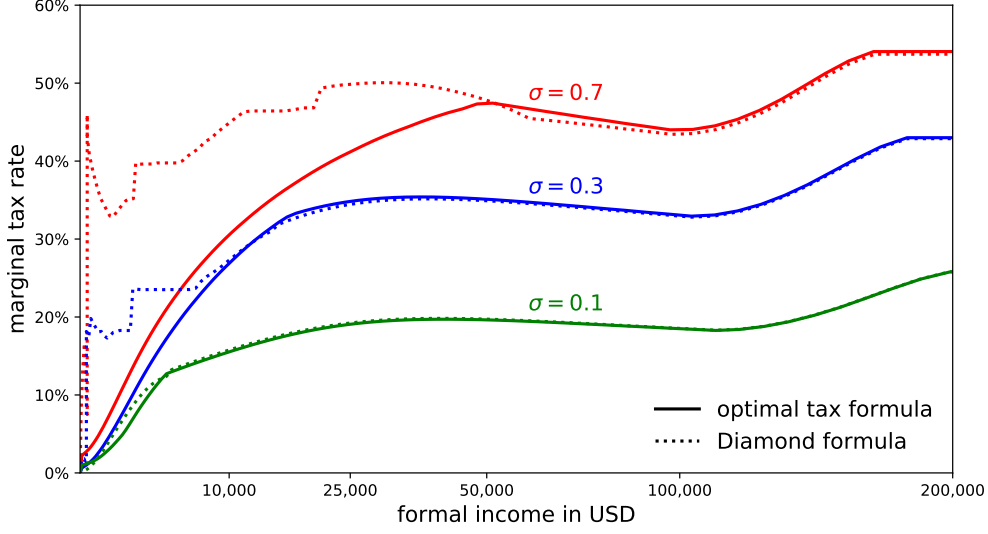
Consider the following social welfare function

$$\int_0^{\theta^{top}} \int_0^\infty \frac{\tilde{V}(\theta, \kappa)^{1-\sigma}}{1-\sigma} dG_\theta(\kappa) dF(\theta) + \lambda^{top} \int_{\theta^{top}}^1 \int_0^\infty \tilde{V}(\theta, \kappa) dG_\theta(\kappa) dF(\theta). \quad (30)$$

²⁵Under our parametric assumptions, the comparative advantage in the shadow economy follows $w^s(\theta)/w^f(\theta) = w^s(0)/w^f(0) \exp\{(\rho^s - \rho^f)\theta\}$ and is decreasing when $\rho^s - \rho^f < 0$. The point estimate of $\rho^s - \rho^f$ is -1.16 with a standard error of 0.002.

²⁶ The model matches the wage distribution as well: see Figure 13 in Appendix D.

Figure 8: Comparison of tax schedules



For σ equal to 0.1 the median formal income is approx. \$9,600, while the 95th percentile of formal income is approx. \$60,000. For higher values of σ the median formal income is lower.

where $\tilde{V}(\theta, \kappa)$ is utility shifted by a constant to ensure positive values. This social welfare function implies a preference for redistribution which increases in parameter σ . We find the optimal income tax for various values of σ . In each case we require the same net tax revenue as the one generated by the actual Colombian tax. To facilitate computations, we assume that the Pareto weights are constant in the upper tail.²⁷

Since the distribution of income is endogenous to tax policy, we find the tax schedules implied by each tax formula iteratively: a tax schedule implies an income distribution which, together with a tax formula, results in a new tax schedule. The tax schedules we present are the fixed points of this mapping.²⁸ Note that, in principle, each tax formula can have multiple fixed points. In practice, we find multiple solutions only for the Diamond formula. All of these solutions share a common feature: an excessive size of the shadow economy. We report the solution which yields the highest welfare, which coincides with the smallest size of the informal sector.

Figure 8 compares the tax schedules implied by the optimal tax formula and by the Diamond formula. We have shown in Section 3 that the Diamond formula leads to excessively high tax rates at income levels where some formal workers are tempted to join the shadow economy. On Figure 8 we see that the Diamond tax rates are excessively high in the main part of the formal income distribution, while leaving the top tax rate mostly unaffected.²⁹ The difference in tax schedules is much more pronounced for higher

²⁷The Pareto weight at the upper tail, λ^{top} , is set equal to $\tilde{V}(\theta^{top}, \infty)^{-\sigma}$, the implicit Pareto weight of θ^{top} agent. As a result, the schedule of implicit Pareto weights is continuous.

²⁸Rothschild and Scheuer (2016) call these fixed points ‘Self-confirming Policy Equilibria’.

²⁹Note that at high incomes the Diamond tax rates are slightly below the optimal ones, particularly for high σ . The Diamond formula leads to an excessive redistribution from the top to the bottom which reduces utility levels of top earners and raises their implicit welfare weights. As a result, the

Table 1: Comparison of allocations implied by the optimal and the Diamond formula

	$\sigma = 0.1$		$\sigma = 0.3$		$\sigma = 0.7$	
	optimal	Diamond	optimal	Diamond	optimal	Diamond
share of shadow workers	1.2%	1.4%	2.1%	6.7%	4.5%	25.9%
share of shadow income	0.1%	0.1%	0.2%	1.1%	0.6%	11.1%
welfare gain rel. to Diamond	0.0%		0.1%		6.4%	

values of σ , i.e. when the planner has a stronger preference for redistribution. When σ is low, the optimal tax formula reduces the tax rates only at low incomes and by at most 2 percentage points. For the intermediate value of σ , the tax rates below the median formal income are reduced by up to 17 percentage points. When σ is high, the optimal tax formula reduces tax rates below the 95th percentile of formal income and the maximal reduction exceeds 30 percentage points.

Table 1 compares the size of the shadow economy implied by the optimal tax formula and by the Diamond formula. The optimal fraction of shadow workers increases with σ from 1.2% up to 4.5%. The share of shadow income is proportionally smaller, as only the least productive agents work informally. The Diamond formula, which prescribes higher tax rates at low income levels, leads to an excessive informal employment: the share of shadow workers exceeds the optimal level from 0.2 p.p. for the low value of σ to close to 15 p.p. for high σ . The welfare gain from using the optimal tax formula, expressed in terms of consumption, increases steeply with σ from close to 0% up to 6.4% for the most redistributive planner.

4.3. Welfare impact of the shadow economy

How does the Colombian shadow economy affect social welfare? We generalize the notions of efficiency and redistribution gains from Section 2 to a model with a continuum of types and a fixed cost of shadow employment. Suppose that all variables with superscript M correspond to the optimum of the Mirrlees model without the shadow economy:³⁰

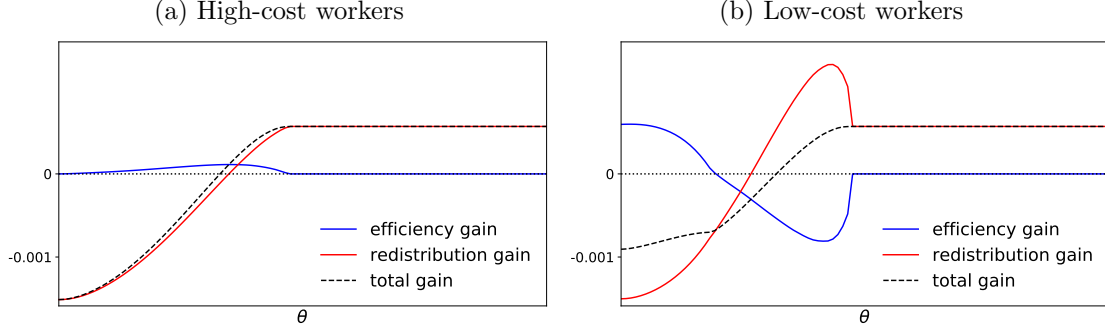
$$\underbrace{V(\theta, \kappa) - V^M(\theta)}_{\text{total gain of agent } (\theta, \kappa)} = \underbrace{V(\theta, \kappa) + T(y^f(\theta, \kappa)) - V^M(\theta) - T^M(y^M(\theta))}_{\text{efficiency gain of agent } (\theta, \kappa)} + \underbrace{T^M(y^M(\theta)) - T(y^f(\theta, \kappa))}_{\text{redistribution gain of agent } (\theta, \kappa)}. \quad (31)$$

As before, the redistribution gain captures the change in the level of transfers, while the efficiency gain measures the reduction of distortions in the shadow economy allocation

Diamond tax rates at high incomes are adjusted downwards.

³⁰Specifically, we take the implied welfare weights from the optimum with the shadow economy and use them to derive the optimal Mirrleesian allocation.

Figure 9: Welfare decomposition ($\sigma = 0.3$)



relative to the Mirrleesian allocation. Note that the fixed cost of shadow employment becomes another source of distortion in the shadow economy allocation.

Figure 9 depicts the utility decomposition for the high-cost and the low-cost workers when $\sigma = 0.3$. The high-cost workers with low to medium productivity type benefit modestly from lower marginal tax rates, but suffer from lower transfers with the shadow economy. For sufficiently productive high-cost workers the redistribution gain becomes positive, as lower marginal tax rates at low incomes imply lower tax burdens at higher incomes. In the case of the low-cost workers, low productivity types work only in the shadow economy and gain on the efficiency side due to higher shadow productivity, but lose due to lower transfers. For the middle types the picture is reversed. These agents work simultaneously in both sectors and choose formal income levels with relatively high marginal tax rates, which implies high distortions, but low total taxes. Finally, the most productive low-cost workers stay fully formal and gain from lower total taxes. Since the shadow economy does not affect the tax rates at the top, the efficiency gain is zero for agents with high θ . The picture is analogous for other values of σ , though the absolute magnitude of both gains increases with the strength of redistributive preferences.

In Table 2 we aggregate the efficiency and the redistribution gains of different agents using the social welfare function and expressed them in consumption-equivalent terms relative to the Mirrleesian allocation. The sum of the two aggregate gains is the total welfare gain from the presence of the shadow economy. We find that the shadow economy improves aggregate efficiency, but restricts redistribution. There are two effects at play. First, the least productive agents are actually more productive in the shadow economy, which improves efficiency. Second, the marginal tax rates with shadow economy are lower. On the one hand it alleviates distortions, on the other it reduces the tax burden at higher incomes. The sum of the two effects depends on the redistributive preferences of the planner. With weak redistributive preferences the first effect dominates and the shadow economy very modestly improves welfare. As σ increases, the redistribution gain falls faster than the efficiency gain increases, eventually leading to a substantial welfare loss for planners with strong redistributive preferences.

Table 2: Welfare impact of the shadow economy

	$\sigma = 0.1$	$\sigma = 0.3$	$\sigma = 0.7$
efficiency gain	0.02%	0.23%	2.68%
redistribution gain	-0.0%	-0.25%	-3.95%
welfare gain	0.01%	-0.02%	-1.27%

5. Conclusions

In this paper we study optimal income taxation when agents have access to the informal labor market, the income from which is unobserved by the government. We show that the optimal tax formulas with a shadow economy contain additional terms, capturing income responses of shadow workers, which lead to lower tax rates. We quantitatively demonstrate that, when social preferences for redistribution are strong, the tax rate reduction is substantial, at some income levels exceeding 30 percentage points, and leads to large welfare gains. We also show that, depending on the distribution of formal and shadow productivities, the shadow economy can improve or deteriorate social welfare through two channels: labor efficiency and redistribution. We find that the shadow economy in Colombia strengthens efficiency of labor supply at the expense of possible redistribution. For low to medium strength of redistributive preferences the two forces roughly cancel out and the shadow economy is neutral for welfare. For strongly redistributive governments the redistribution channel dominates and the shadow economy reduces social welfare.

Our analysis could be extended in several directions. First, suppose that the government can use audits and penalties to differentially affect tax evasion opportunities of different agents.³¹ The optimal design of tax audits could, rather than minimizing overall tax evasion, tailor individual evasion opportunities to maximize the welfare improving potential of the shadow economy. Second, informal activity is closely related to home production. In some developed economies home production may be quantitatively more important than tax evasion. When a home produced good is a perfect substitute for market income, our framework can be directly applied to study home production.

Acknowledgements

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³¹For instance, conducting tax audits at medium levels of formal income restricts tax evasion of highly productive agents, but not of low productivity workers who would never choose such income level. See [Cremer and Gahvari \(1996\)](#) for the analysis of tax audits in the optimal taxation model with two types.

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A. Proofs from Section 2

Proof of Lemma 1. The planner can increase social welfare by transferring consumption from type $-i$ to type i , so at the optimum the incentive constraint of $-i$ will bind and

the incentive constraint of i will be slack. Denote the undistorted level of formal income of type $-i$ by $y_{-i}^{f*} \equiv w_{-i}^f v'^{-1}(w_{-i}^f)$. If $y_{-i}^f \neq y_{-i}^{f*}$, the planner can extract more resources without violating the incentive constraint by setting $y_{-i}^f = y_{-i}^{f*}$ and increasing T_{-i} to keep the utility level of type $-i$ constant. Since $y_{-i}^{f*} > 0$, type $-i$ will not work in the shadow economy.

To see that the planner optimally distorts the labor supply of type i , notice that a marginal adjustment of y_i^f , starting from the undistorted level y_i^{f*} , has no direct impact on welfare of type i by the Envelope Theorem. However, the distortion in a correct direction will reduce the payoff of $-i$ from misreporting, relax the incentive constraint and, hence, allow for more redistribution. In particular, if $w_i^f < w_{-i}^f$ ($w_i^f > w_{-i}^f$), a marginal decrease (increase) of y_i^f will relax the incentive constraint. \square

Proof of Proposition 1. The difference in the utility level of type i between the two allocations is

$$U(c_i^{SE}, n_i^{SE}) - U(c_i^M, n_i^M) = U(w_i^s n_i^{SE}, n_i^{SE}) - U(w_i^f n_i^M, n_i^M) + T_i^M - T_i^{SE}. \quad (32)$$

The difference in utility level of type $-i$ is

$$U(c_{-i}^{SE}, n_{-i}^{SE}) - U(c_{-i}^M, n_{-i}^M) = T_{-i}^M - T_{-i}^{SE} = -\frac{\mu_i}{\mu_{-i}} (T_i^M - T_i^{SE}), \quad (33)$$

where the first equality follows from Lemma 1, since in the two allocations the labor supply of $-i$ is undistorted, and the second equality follows from the resource constraint. Using both utility differences, we can decompose $W^{SE} - W^M$ as stated in the proposition.

Define an function $\Psi(w) \equiv U(wv'^{-1}(w), v'^{-1}(w))$, equal to the utility level of an individual with productivity w who supplies labor efficiently and receives no transfers. The efficiency term can be restated as $\lambda_i \mu_i (\Psi(w_i^s) - U(w_i^f n_i^M, n_i^M))$. Since Ψ is an increasing function, the efficiency gain is increasing in w_i^s and changes sign at $\bar{w}_i^s \equiv \Psi^{-1}(U(w_i^f n_i^M, n_i^M))$. To see that $\bar{w}_i^s < w_i^f$, note that since n_i^M is distorted, $\Psi(w_i^f) > U(w_i^f n_i^M, n_i^M)$.

To characterize the redistribution term, note that, due to the binding incentive constraints, we have

$$U(c_{-i}^{SE}, n_{-i}^{SE}) - U(c_{-i}^M, n_{-i}^M) = \Psi(w_{-i}^s) - T_{-i}^{SE} - U(w_i^f n_i^M, w_i^f n_i^M / w_{-i}^f) + T_i^M. \quad (34)$$

Combining it with (33), we find that $T_i^M - T_i^{SE} = \mu_{-i} (U(w_i^f n_i^M, w_i^f n_i^M / w_{-i}^f) - \Psi(w_{-i}^s))$. It implies that the redistribution term is decreasing in w_{-i}^s and changes sign at $\bar{w}_{-i}^s \equiv \Psi^{-1}(U(w_i^f n_i^M, w_i^f n_i^M / w_{-i}^f))$. $\bar{w}_{-i}^s < w_{-i}^f$ holds, since $U(w_i^f n_i^M, w_i^f n_i^M / w_{-i}^f) < \Psi(w_{-i}^f)$ due to the optimal distortion of n_i^M . \square

B. Proofs of Section 3

Proof of Lemma 2. The strict *Spence-Mirrlees* single crossing condition holds if, keeping the formal income level fixed, the marginal rate of substitution $v' \left(\frac{y^f}{w^f(\theta)} + \frac{y^s}{w^s(\theta)} \right) / w^f(\theta)$ is decreasing with θ . For formal workers it follows from the convexity of v . For shadow workers we have $v'(n) = w^s(\theta)$ and the single-crossing follows from $w^s(\theta)/w^f(\theta)$ being decreasing. \square

Proof of Proposition 2. Given the single crossing condition, the necessity of non-decreasing formal income schedules and local incentive constraints follows from Theorem 7.2 in [Fudenberg and Tirole \(1991\)](#). By Theorem 7.3 in [Fudenberg and Tirole \(1991\)](#), non-decreasing formal income schedule and local incentive constraints are sufficient to prevent deviations *within* the cost class, i.e. deviations of some high-cost (low-cost) worker to formal income level earned by another high-cost (low-cost) worker. We will show that conditions 2. - 4. are sufficient to prevent deviations *between* the cost classes.

Denote the image of formal income schedule of types with fixed cost $\kappa \in \{0, \infty\}$ by $Y(\kappa) \equiv \{y \in \mathbb{R}_+ : \exists \theta \in [0, 1] y^f(\theta, \kappa) = y\}$. Deviations between the cost classes may arise if the formal income schedules of the two classes do not have identical images: $Y(0) \neq Y(\infty)$. The difference in images may occur when suprema or infima of the two sets do not coincide: either $y^f(0, 0) < y^f(0, \infty)$ or $y^f(1, 0) < y^f(1, \infty)$. Conditions 2. and 3. take care of these possibilities. Alternatively, one of the income schedules can exhibit a discontinuous jump between formal income values where the other schedule remains continuous. Condition 4. prevents potential deviations when $y^f(\cdot, 0)$ is discontinuous, while $y^f(\cdot, \infty)$ remains continuous, i.e. when there is $y \in (y^f(0, \infty), y^f(1, 0))$ such that $y \notin Y(0)$ and $y \in Y(\infty)$.³² Below we show that the reverse situation never happens: when there is $y \in (y^f(0, \infty), y^f(1, 0))$ such that $y \in Y(0)$, then always $y \in Y(\infty)$.

We will show that for any θ we can find $\tilde{\theta}$ such that $y^f(\tilde{\theta}, \infty) = y^f(\theta, 0)$. Take some implementable income schedules $y^f(\cdot, 0)$ and $y^f(\cdot, \infty)$, the corresponding tax schedule $T(\cdot)$ and any θ such that $y^f(\theta, 0) > y^f(0, \infty)$ and $y^s(\theta, 0) > 0$. Consider a productivity type $\tilde{\theta}$ such that

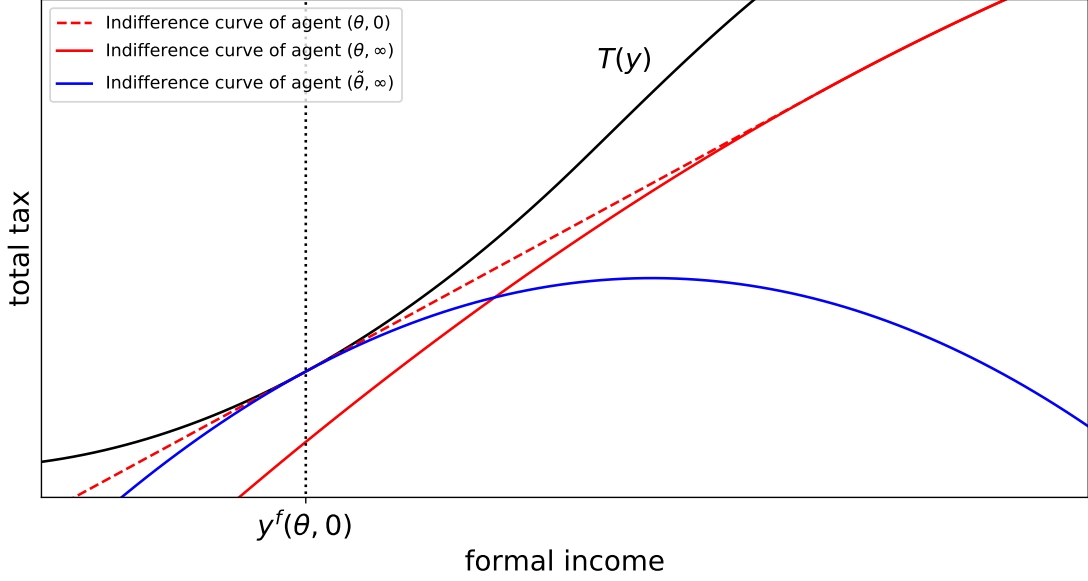
$$\frac{v'(y^f(\theta, 0)/w^f(\tilde{\theta}))}{w^f(\tilde{\theta})} = \frac{w^s(\theta)}{w^f(\theta)}. \quad (35)$$

We will show that $y^f(\tilde{\theta}, \infty) = y^f(\theta, 0)$. It means that at any formal income level above $y^f(0, \infty)$ which is chosen by some low-cost worker there is also some high-cost worker.³³

³²Note that, by the single crossing, it is sufficient to impose additional constraints only on particular types: top, bottom, or type at the discontinuity. If these constraints hold, other types are not tempted by a deviations, since incentive-compatible income schedules are non-decreasing.

³³If $w^f(0) > 0$, we need to make sure that $\tilde{\theta}$ always exists. Suppose on the contrary that $v'(y^f(\theta, 0)/w^f(0))/w^f(0) < w^s(\theta)/w^f(\theta)$, so that there is no $\tilde{\theta} \geq 0$ which satisfies (35). It means that if agent $(\theta, 0)$ prefers $y^f(\theta, 0)$ to $y^f(0, \infty)$, so does agent $(0, \infty)$. It is a contradiction, since income schedule $y^f(\cdot, \infty)$ is implementable.

Figure 11: Indifference curves.



Consider indifference curves of agents $(\theta, 0)$ and $(\tilde{\theta}, \infty)$ in the (y^f, T) space, depicted in Figure 11. The indifference curve of the low-cost θ -worker and the high-cost $\tilde{\theta}$ -worker are tangential at formal income $y^f(\theta, 0)$. Furthermore, the indifference curve of the low-cost worker is a straight line whenever this agent supplies shadow labor, while the indifference curve of the high-cost worker is strictly concave. Finally, the two indifference curves never cross. Otherwise, the indifference curves of agents $(\tilde{\theta}, \infty)$ and (θ, ∞) would cross more than once and the single crossing condition would be violated. Altogether, it means that $y^f(\theta, 0)$ is the incentive-compatible formal income choice of the high-cost $\tilde{\theta}$ -worker. Suppose on the contrary that agent $(\tilde{\theta}, \infty)$ prefers some $\tilde{y}^f \neq y^f(\theta, 0)$. This is a profitable deviation for agent $(\theta, 0)$ as well, since his indifference curve is weakly higher. It contradicts the original assumption of implementability of $y^f(\cdot, 0)$. \square

Proof of Proposition 3. Define a *net deadweight loss* as a the difference between the total deadweight loss (the left-hand side of the tax formula) and the tax revenue gain (the right-hand side of the tax formula), divided by a share of income above a given income level. Below we write the net deadweight loss at the top according to the Diamond formula as a function of the top tax rate $T'(\infty)$:

$$\begin{aligned} \lim_{y \rightarrow \infty} \frac{T'(y)}{1 - T'(y)} \frac{h(y)y}{1 - H(y)} \bar{\varepsilon}(y) - \mathbb{E}\{(1 - \bar{\lambda}(y)) \mid y' > y\} \\ = \frac{T'(\infty)}{1 - T'(\infty)} \alpha \bar{\varepsilon}(\infty) - (1 - \bar{\lambda}(\infty)) \equiv NDWL_D(T'(\infty)). \end{aligned} \quad (36)$$

The optimal top tax rate according to the Diamond formula, which we denote by $T'_D(\infty)$, is given implicitly by $NDWL_D(T'_D(\infty)) = 0$.

We will derive the net deadweight loss at the top in the model with a shadow economy: $NDWL_{SE}(T'(\infty))$. When $T'(\infty) < 1 - w^s(1)/w^f(1)$ we have $NDWL_{SE}(T'(\infty)) = NDWL_D(T'(\infty))$. In the opposite case $T'(\infty) > 1 - w^s(1)/w^f(1)$ the highest formal income level of the low-cost workers $y^f(1, 0)$ is finite and the relevant tax formula is (23). We can show that

$$\lim_{\theta \rightarrow 1} \frac{\Delta T(y^f(\theta, \infty))}{\tilde{\kappa}(\theta)} = (1 + \bar{\varepsilon}(\infty)) \frac{T'(\infty)}{1 - T'(\infty)} \left(\left(\frac{w^s(1)/w^f(1)}{1 - T'(\infty)} \right)^{1+\bar{\varepsilon}} - 1 \right)^{-1} \equiv \delta(T'(\infty)), \quad (37)$$

which implies that the elasticity of the density of formal workers at the top converges to

$$\pi(y^f(\theta, \infty)) = \frac{g_\theta(\tilde{\kappa}(\theta)) \tilde{\kappa}(\theta)}{1 - G_\theta(\tilde{\kappa}(\theta))} \frac{\Delta T(y^f(\theta, \infty))}{\tilde{\kappa}(\theta)} \rightarrow \gamma \delta(T'(\infty)) \equiv \pi(\infty). \quad (38)$$

Now, $NDWL_{SE}(T'(\infty))$ can be written as

$$\lim_{y \rightarrow \infty} \left\{ \frac{T'(y)}{1 - T'(y)} \bar{\varepsilon}(y) \frac{yh(y)}{1 - H(y)} - \mathbb{E}\{1 - \bar{\lambda}(y') - \pi(y') \mid y' \geq y\} \right\} \quad (39)$$

$$= \frac{T'(\infty)}{1 - T'(\infty)} \alpha \bar{\varepsilon}(\infty) - (1 - \bar{\lambda}(\infty) - \pi(\infty)) \equiv NDWL_{SE}(T'(\infty)) \quad (40)$$

Thus, for $T'(\infty) > 1 - w^s(1)/w^f(1)$ we have

$$NDWL_{SE}(T'(\infty)) = NDWL_M(T'(\infty)) + \gamma \delta(T'(\infty)). \quad (41)$$

Note that (i) $\delta(T'(\infty)) > 0$ for $T'(\infty) \in (1 - w^s(1)/w^f(1), 1)$ and (ii) $\delta(T'(\infty))$ diverges to infinity as $T'(\infty)$ approaches $1 - w^s(1)/w^f(1)$ from the right (see Figure 12). As γ converges to 0, $NDWL_{SE}(T'(\infty))$ is arbitrarily close to $NDWL_D(T'(\infty))$ for any $T'(\infty) > 1 - w^s(1)/w^f(1)$. Hence, for sufficiently small γ the optimal top tax rate is given by \tilde{T}' , defined implicitly by $NDWL_{SE}(\tilde{T}') = 0$.³⁴ On the other hand, when γ is sufficiently large, the optimal top tax rate is $1 - w^s(1)/w^f(1)$.

More formally, suppose that $T'_D(\infty) > 1 - w^s(1)/w^f(1)$. The optimal top tax rate solves

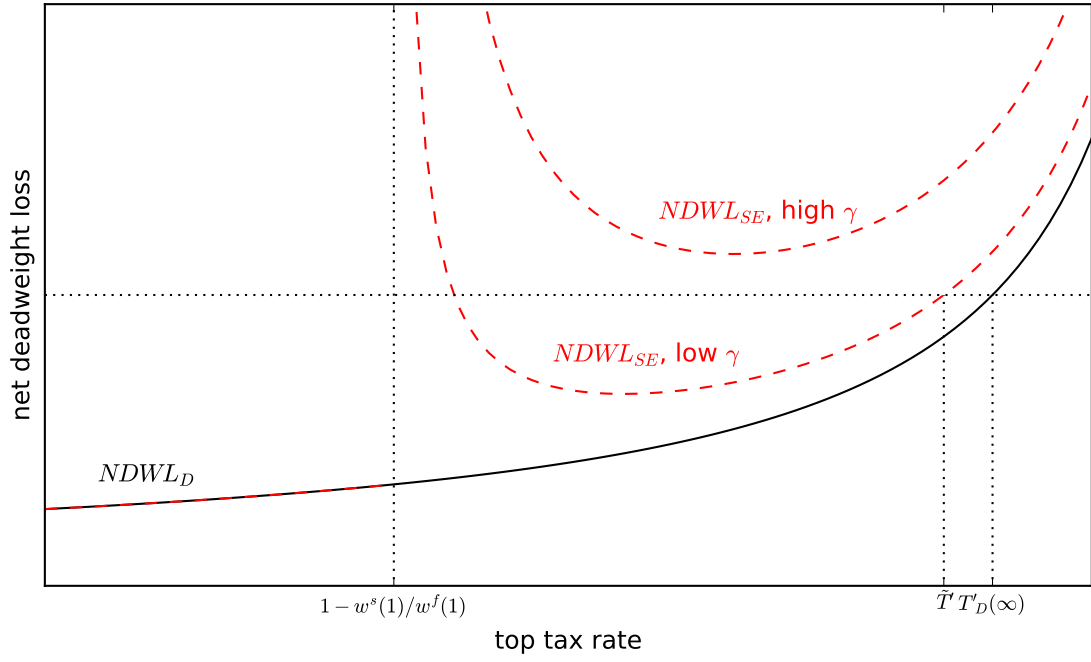
$$\min_{T'(\infty) \geq 1 - \frac{w^s(1)}{w^f(1)}} \int_{1 - \frac{w^s(1)}{w^f(1)}}^{T'(\infty)} NDWL_D(\tau) d\tau + \gamma \int_{1 - \frac{w^s(1)}{w^f(1)}}^{T'(\infty)} \delta(\tau) d\tau. \quad (42)$$

Suppose that at some γ the solution is $T'(\infty) = 1 - w^s(\theta)/w^f(\theta)$. Since $\delta(T'(\infty)) > 0$, this tax rate is optimal also for higher values of γ . It proves the existence of threshold $\bar{\gamma}$. \square

Proof of Lemma 3. Since the labor elasticity converges to $\bar{\varepsilon}(\infty)$, the disutility from labor converges to $\tilde{v}(n) = c_0 + \frac{c_1}{1 + 1/\bar{\varepsilon}(\infty)} n^{1 + 1/\bar{\varepsilon}(\infty)}$ for some constants c_0, c_1 . If all top workers

³⁴In this case there are at least two solutions to $NDWL_{SE}(T'(\infty)) = 0$, a local maximum and a local minimum. If there are only two solutions, the higher one is the maximum.

Figure 12: Finding the optimal top tax rate.



Net deadweight loss is the difference between the deadweight loss and the tax revenue gain - negative net deadweight loss indicates gains from increasing the tax rate. $NDWL_D$ is the net deadweight loss according to the Diamond formula, $NDWL_{SE}$ is the net deadweight loss with the shadow economy (see the proof of Proposition 3 for formal definitions). When γ is high, the optimal top tax rate is $1 - w^s(1)/w^f(1)$. When γ is low, the optimal top tax rate is \tilde{T}' .

are formal, the distribution of formal income satisfies

$$\lim_{y \rightarrow \infty} \frac{1 - H(y)}{h(y)y} = \lim_{\theta \rightarrow 1} \frac{1 - F(\theta)}{f(\theta)w^f(\theta)} \frac{dw^f(\theta)}{d\theta} \frac{dy^f(\theta, \infty)}{dw^f(\theta)} \frac{w^f(\theta)}{y^f(\theta, \infty)} = \frac{1 + \bar{\varepsilon}(\infty)}{\alpha_w}. \quad (43)$$

When there are some shadow workers among the top productivity types, we have

$$\lim_{y \rightarrow \infty} \frac{1 - H(y)}{h(y)y} = \lim_{\theta \rightarrow 1} \frac{1 - \int_0^\theta (1 - G_{\theta'}(\tilde{\kappa}(\theta'))) dF(\theta')}{(1 - G_\theta(\tilde{\kappa}(\theta))) f(\theta)w^f(\theta)} \frac{dw^f(\theta)}{d\theta} \frac{dy^f(\theta, \infty)}{dw^f(\theta)} \frac{w^f(\theta)}{y^f(\theta, \infty)}. \quad (44)$$

One can show that threshold $\tilde{\kappa}(\theta)$ is asymptotically proportional to $w^f(\theta)^{1+\bar{\varepsilon}(\infty)}$:

$$\lim_{\theta \rightarrow 1} \frac{\tilde{\kappa}(\theta)}{w^f(\theta)^{1+\bar{\varepsilon}(y^f(\theta, \infty))}} = \frac{1}{c_1^{\bar{\varepsilon}(\infty)}(1 + \bar{\varepsilon}(\infty))} \left(\left(\frac{w^s(1)}{w^f(1)} \right)^{1+\bar{\varepsilon}(\infty)} - (1 - T'(\infty))^{1+\bar{\varepsilon}(\infty)} \right) \quad (45)$$

Consequently, $1 - G_\theta(\tilde{\kappa}(\theta))$ is asymptotically proportional to $w^f(\theta)^{-\gamma(1+\bar{\varepsilon}(\infty))}$ and

$$\begin{aligned} \lim_{\theta \rightarrow 1} \frac{1 - \int_0^\theta (1 - G_{\theta'}(\tilde{\kappa}(\theta'))) dF(\theta')}{(1 - G_\theta(\tilde{\kappa}(\theta))) f(\theta)w^f(\theta)} \frac{dw^f(\theta)}{d\theta} &= \lim_{w^f \rightarrow \infty} \frac{\int_{w^f}^\infty 1/(w)^{1+\alpha_w+\gamma(1+\bar{\varepsilon}(\infty))} dw}{1/(w^f)^{1+\alpha_w+\gamma(1+\bar{\varepsilon}(\infty))} w^f} \\ &= \frac{1}{\alpha_w + \gamma(1 + \bar{\varepsilon}(\infty))}. \end{aligned} \quad (46)$$

Plugging it into (44), we get $\lim_{y \rightarrow \infty} \frac{1-H(y)}{h(y)y} = \frac{1+\bar{\varepsilon}(\infty)}{\alpha_w + \gamma(1+\bar{\varepsilon}(\infty))}$. \square

C. Derivation of the optimal tax rates.

In Lemma C.1 we derive the optimal tax rates in terms of model primitives, using the mechanism design approach. Then we define the sufficient statistics used to derive the optimal tax rates in the main text and show the equivalence between the sufficient statistics approach and the mechanism design approach.

Lemma C.1 (Optimal tax formulas). *Suppose that bunching does not occur. Denote the sectoral productivity growth rates by $\rho^x(\theta) \equiv w_\theta^x(\theta)/w^x(\theta)$, $x \in \{f, s\}$; the elasticity of labor supply by $\varepsilon(\theta, \kappa) \equiv \frac{v'(n(\theta, \kappa))}{n(\theta, \kappa)v''(n(\theta, \kappa))}$, where $n(\theta, \kappa)$ is the total labor supply of agent (θ, κ) ; the tax loss from worker of productivity type θ joining the shadow economy by $\tilde{\Delta}T(\theta) \equiv T(\theta', \infty) - T(\theta', 0)$. Define $\tilde{s}(\theta) \equiv \min\{\theta' \in [0, 1] \text{ s.t. } y^f(\theta', 0) \geq y^f(\theta, \infty)\}$.*

When $y^f(\theta, \infty) \leq y^f(1, 0)$, the optimal tax rate satisfies

$$\begin{aligned} & \frac{T'(y^f(\theta, \infty))}{1 - T'(y^f(\theta, \infty))} \frac{(1 - G_\theta(\tilde{\kappa}(\theta)))f(\theta)}{\rho^f(\theta)(1 + \varepsilon^{-1}(\theta, \infty))} + \frac{w^f(\tilde{s}(\theta) - w^s(\tilde{s}(\theta)))}{w^s(\tilde{s}(\theta))} \frac{G_{\tilde{s}(\theta)}(\tilde{\kappa}(\tilde{s}(\theta)))f(\tilde{s}(\theta))}{\rho^f(\tilde{s}(\theta)) - \rho^s(\tilde{s}(\theta))} \\ &= \int_\theta^{\tilde{s}(\theta)} \left[\int_{\tilde{\kappa}(\theta')}^\infty (1 - \lambda(\theta', \kappa)) dG_{\theta'}(\kappa) - g_{\theta'}(\tilde{\kappa}(\theta')) \tilde{\Delta} T(\theta') \right] dF(\theta') \\ & \quad + \int_{\tilde{s}(\theta)}^1 \int_0^\infty (1 - \lambda(\theta', \kappa)) dG_{\theta'}(\kappa) dF(\theta'). \quad (47) \end{aligned}$$

When $y^f(\theta, \infty) > y^f(1, 0)$, the optimal tax rate satisfies

$$\begin{aligned} & \frac{T'(y^f(\theta, \infty))}{1 - T'(y^f(\theta, \infty))} \frac{(1 - G_\theta(\tilde{\kappa}(\theta)))f(\theta)}{\rho^f(\theta)(1 + \varepsilon^{-1}(\theta, \infty))} \\ &= \int_\theta^1 \left[\int_{\tilde{\kappa}(\theta')}^\infty (1 - \lambda(\theta', \kappa)) dG_{\theta'}(\kappa) - g_{\theta'}(\tilde{\kappa}(\theta')) \tilde{\Delta} T(\theta') \right] dF(\theta'). \quad (48) \end{aligned}$$

Proof of Lemma C.1.

By Corollary 1 from [Milgrom and Segal \(2002\)](#) the value function $V(\theta, \kappa)$ is differentiable with respect to θ almost everywhere. The derivative is given by

$$\frac{dV(\theta, \kappa)}{d\theta} = \left(\rho^f(\theta) \frac{y^f(\theta, \kappa)}{w^f(\theta)} + \rho^s(\theta) \frac{y^s(\theta, \kappa)}{w^s(\theta)} \right) v' \left(\frac{y^f(\theta, \kappa)}{w^f(\theta)} + \frac{y^s(\theta, \kappa)}{w^s(\theta)} \right) \equiv V_\theta(\theta, \kappa), \quad (49)$$

where $\rho^x(\theta) \equiv w_\theta^x(\theta)/w^x(\theta)$ stands for the productivity growth rate in sector $x \in \{f, s\}$. Hence, we can represent the value function in the integral form

$$V(\theta, \kappa) = V(0, \kappa) + \int_0^\theta V_\theta(\theta', \kappa) d\theta'. \quad (50)$$

Take some high-cost worker (θ, ∞) . We will derive the optimality condition by perturbing formal income of this worker by small dy^f and adjusting the tax paid such that the utility level $V(\theta, \infty)$ is unchanged. This perturbation affects the slope $V_\theta(\theta, \infty)$, which in turn implies via equation (50) a uniform shift of utility levels of all high-cost types above.

Moreover, since all agents face the same tax schedule, we need to adjust the allocation of the low-cost workers as well. We can distinguish three cases. First, when $y^f(\tilde{s}(\theta), \infty) = y^f(\theta, \infty)$, the distorted shadow workers respond by marginally decreasing formal income. Second, when $y^f(\tilde{s}(\theta), \infty) > y^f(\theta, \infty)$, the distorted shadow workers respond by jumping to a discretely lower formal income level. These two cases have identical fiscal impact and lead to the same optimal tax formula. Finally, when $y^f(\theta, \infty) > y^f(1, 0)$, all the shadow workers have lower formal income and hence are unaffected by the perturbation.

Distortion of formal workers. A formal income perturbation dy^f affects the utility of type (θ, ∞) by $\left(1 - \frac{v'(n(\theta, \infty))}{w^f(\theta)}\right) dy^f$, or equivalently by $T'(y^f(\theta, \infty)) dy^f$. We need to adjust the total tax paid by the same amount such that the utility level stays constant. The fiscal impact of doing so is

$$T'(y^f(\theta, \infty))(1 - G_\theta(\tilde{\kappa}(\theta)))f(\theta)dy^f. \quad (51)$$

The impact of this perturbation on the slope of the utility schedule is

$$dV_\theta(\theta, \infty) = \rho^f(\theta) \left(1 - T'(y^f(\theta, \infty))\right) \left(1 + \frac{1}{\varepsilon(\theta, \infty)}\right) dy^f. \quad (52)$$

Hence, a perturbation that leads to a change of slope $dV_\theta(\theta, \infty)$ implies a change in tax revenue from the formal workers by

$$\frac{T'(y^f(\theta, \infty))}{1 - T'(y^f(\theta, \infty))} \left(1 + \frac{1}{\varepsilon(\theta, \infty)}\right)^{-1} \frac{1}{\rho^f(\theta)} (1 - G_\theta(\tilde{\kappa}(\theta)))f(\theta)dV_\theta(\theta, \infty). \quad (53)$$

Distortion of shadow workers. Let's consider the case of $y^f(\theta, \infty) \leq y^f(1, 0)$, otherwise there is no tax loss from the shadow workers. First, suppose that $y^f(\tilde{s}(\theta), 0) = y^f(\theta, \infty)$. A perturbation of formal income dy_2^f affects the utility level of $(\tilde{s}(\theta), 0)$ -type worker by $\left(1 - \frac{w^s(\tilde{s}(\theta))}{w^f(\tilde{s}(\theta))}\right) dy_2^f = T'(y^f(\tilde{s}(\theta), 0)) dy_2^f$. We need to adjust the tax paid by the same amount, which affects the resource constraint by

$$T'(y^f(\tilde{s}(\theta), 0))G_{\tilde{s}(\theta)}(\tilde{\kappa}(\tilde{s}(\theta)))f(\tilde{s}(\theta))dy_2^f. \quad (54)$$

The slope of the utility schedule of low-cost workers changes by

$$dV_\theta(\tilde{s}(\theta), 0) = \left(\rho^f(\tilde{s}(\theta)) - \rho^s(\tilde{s}(\theta))\right) \frac{w^s(\tilde{s}(\theta))}{w^f(\tilde{s}(\theta))} dy_2^f. \quad (55)$$

The perturbation needs to respect the common tax schedule at higher formal incomes - the slopes of $V(\theta, \infty)$ and $V(\tilde{s}(\theta), 0)$ have to change by the same amount, which can be achieved by appropriately adjusting dy_2^f . Then, by using the first-order condition of workers $(\tilde{s}(\theta), 0)$, we can express the tax loss as

$$\frac{w^f(\tilde{s}(\theta)) - w^s(\tilde{s}(\theta))}{w^s(\tilde{s}(\theta))} \frac{G_{\tilde{s}(\theta)}(\tilde{\kappa}(\theta))f(\tilde{s}(\theta))}{\rho^f(\tilde{s}(\theta)) - \rho^s(\tilde{s}(\theta))} dV(\theta, \infty). \quad (56)$$

Second, suppose that $y^f(\tilde{s}(\theta), 0) > y^f(\theta, \infty)$. In this case there is a discontinuity at $\tilde{s}(\theta)$ in the formal income schedule of the low-cost workers. Denote by superscripts $\{-, +\}$ the directional limit of a given variable, e.g. $y^f(\tilde{s}(\theta)^-, 0)$ stands for the left limit of formal income of the low-cost workers at $\tilde{s}(\theta)$. From the definition of the mapping s we know that $y^f(\tilde{s}(\theta)^-, 0) < y^f(\theta^+, 0)$.

The perturbation in the formal income of (θ, ∞) decreased the utility of all workers with formal income above $y^f(\theta, \infty)$, including $\tilde{s}(\theta)$, by $dV_\theta(\theta, \infty)$. It means that the perturbation, absent behavioral responses, leads to a discontinuity at $\tilde{s}(\theta)$ in the utility schedule of the low-cost workers, which is not incentive compatible. The behavioral responses will restore the continuity of $V(\theta, 0)$ by adjusting the mapping $\tilde{s}(\theta)$. Denote this adjustment by $d\tilde{s}(\theta)$.

Continuity of $V(\theta, 0)$ at $\tilde{s}(\theta)$ means that $V(\tilde{s}(\theta)^-, 0) = V(\tilde{s}(\theta)^+, 0)$. Suppose that the utility of worker $(\tilde{s}(\theta), 0)$ is decreased by dT . Continuity of the utility schedule requires that

$$\begin{aligned} V_\theta(\tilde{s}(\theta)^-, 0)d\tilde{s}(\theta) &= V_\theta(\tilde{s}(\theta)^+, 0)d\tilde{s}(\theta) - dT \\ \implies d\tilde{s}(\theta) &= \frac{w^f(\tilde{s}(\theta))/w^f(\tilde{s}(\theta))}{\rho^f(\tilde{s}(\theta)) - \rho^s(\tilde{s}(\theta))} \frac{dT}{y^f(\tilde{s}(\theta)^+, 0) - y^f(\tilde{s}(\theta)^-, 0)}. \end{aligned}$$

This adjustment of $\tilde{s}(\theta)$ is associated with a tax loss

$$\left(T(y^f(\tilde{s}(\theta)^+, 0)) - T(y^f(\tilde{s}(\theta)^-, 0)) \right) f(\tilde{s}(\theta)) G_{\tilde{s}(\theta)}(\tilde{\kappa}(\theta)) d\tilde{s}(\theta). \quad (57)$$

Note that $V(\tilde{s}(\theta)^-, 0) = V(\tilde{s}(\theta)^+, 0)$ implies that

$$\frac{T(\tilde{s}(\theta)^+, 0) - T(\tilde{s}(\theta)^-, 0)}{y^f(\tilde{s}(\theta)^+, 0) - y^f(\tilde{s}(\theta)^-, 0)} = 1 - \frac{w^s(\tilde{s}(\theta))}{w^f(\tilde{s}(\theta))}. \quad (58)$$

Using this result, we can express the tax loss as

$$\frac{w^f(\tilde{s}(\theta)) - w^s(\tilde{s}(\theta))}{w^s(\tilde{s}(\theta))} \frac{G_{\tilde{s}(\theta)}(\tilde{\kappa}(\tilde{s}(\theta))) f(\tilde{s}(\theta))}{\rho^f(\tilde{s}(\theta)) - \rho^s(\tilde{s}(\theta))} dT. \quad (59)$$

Notice the dT is equal to $dV_\theta(\theta, \infty)$. Hence, the tax loss is the same as in the previous case, when $y^f(\tilde{s}(\theta), 0) = y^f(\theta, \infty)$.

Impact on workers with higher formal income. First, suppose that $y^f(\theta, \infty) \leq y^f(1, 0)$. The perturbation implies a shift $dV_\theta(\theta, \kappa)$ in utility levels of formal workers above type θ and shadow workers above $\tilde{s}(\theta)$. Recall that the marginal social welfare weights are equal to the Pareto weights. The fiscal and welfare impact of such change is

$$\begin{aligned} \int_\theta^{\tilde{s}(\theta)} \int_{\tilde{\kappa}(\theta')}^\infty (\lambda(\theta', \kappa) - 1) dG(\kappa) dF(\theta') dV_\theta(\theta, \infty) \\ + \int_{\tilde{s}(\theta)}^1 \int_0^\infty (\lambda(\theta', \kappa) - 1) dG(\kappa) dF(\theta') dV_\theta(\theta, \infty). \end{aligned} \quad (60)$$

Note that among the productivity types in the segment $(\theta, \tilde{s}(\theta))$ the high-cost workers are affected by the perturbation, but the low-cost worker are not. Hence, the perturbation changes the threshold $\tilde{\kappa}$ at this segment. Denote by $\tilde{\Delta}T(\theta) \equiv T(y^f(\theta, \infty)) - T(y^f(\theta, 0))$

the tax loss from worker of type θ moving to the shadow economy. The fiscal impact of the change in participation is

$$\int_{\theta}^{\tilde{s}(\theta)} \tilde{\Delta}T(\theta')g_{\theta'}(\tilde{\kappa}(\theta'))dF(\theta')dV_{\theta}(\theta, \infty). \quad (61)$$

In the case of $y^f(\theta, \infty) > y^f(1, 0)$ only the formal workers are affected by a tax reform. The total fiscal and welfare impact on agents with higher formal income is

$$\int_{\theta}^1 \left[\int_{\tilde{\kappa}(\theta')}^{\infty} (\lambda(\theta', \kappa) - 1) dG(\kappa) + \tilde{\Delta}T(\theta')g_{\theta'}(\tilde{\kappa}(\theta')) \right] dF(\theta')dV_{\theta}(\theta, \infty). \quad (62)$$

Collecting the terms. At the optimum, the total impact of a small perturbation is zero. First, consider the case of $y^f(\theta, \infty) \leq y^f(1, 0)$. The sum of the distortion cost of a high-cost worker (53), the distortion cost of the low-cost worker (56) as well as of impacts on the workers with higher formal income (60) and (61) needs to be zero, which results in tax formula (47). If the perturbation affects no shadow workers ($y^f(\theta, \infty) > y^f(1, 0)$), the terms (53) and (62) should sum up to zero, which yields tax formula (48). This concludes the proof of Lemma C.1.

Definitions of sufficient statistics.

$\varepsilon^x(\theta)$ and $\tilde{\varepsilon}^x(\theta)$ stand for the formal income elasticity of workers in sector $x \in \{f, s\}$ with respect to the marginal tax rate along the linear and nonlinear tax schedule, respectively. $\varepsilon_{w^f}^x(\theta)$ and $\tilde{\varepsilon}_{w^f}^x(\theta)$ stand for the formal income elasticity of workers in sector $x \in \{f, s\}$ with respect to the gross formal wage along the linear and nonlinear tax schedule, respectively. The elasticities of formal workers are derived from the optimality condition $y^f(\theta, \infty) = w^f(\theta)(v')^{-1}((1 - T'(y^f(\theta, \infty)))w^f(\theta))$, while the elasticities of shadow workers are derived from the optimality condition $(1 - T'(y^f(\theta, 0)))w^f(\theta) = w^s(\theta)$.

The elasticities of formal workers are

$$\varepsilon^f(y^f(\theta, \infty)) \equiv \frac{v'(n(\theta, \infty))}{n(\theta, \infty)v''(n(\theta, \infty))}, \quad (63)$$

$$\tilde{\varepsilon}^f(y) \equiv \left[\frac{1}{\varepsilon^f(y)} + \frac{T''(y)y}{1 - T'(y)} \right]^{-1}, \quad (64)$$

$$\varepsilon_{w^f}^f(y) \equiv 1 + \varepsilon^f(y), \quad (65)$$

$$\tilde{\varepsilon}_{w^f}^f(y) \equiv \frac{\tilde{\varepsilon}^f(y)}{\varepsilon^f(y)} \varepsilon_{w^f}^f(y). \quad (66)$$

The elasticities of shadow workers are

$$\tilde{\varepsilon}^s(y) \equiv \frac{1 - T'(y)}{T''(y)y}, \quad (67)$$

$$\tilde{\varepsilon}_{wf}^s(y^f(\theta, 0)) \equiv \left(1 - \frac{\rho^s(\theta)}{\rho^f(\theta)}\right) \tilde{\varepsilon}^s(y^f(\theta, 0)). \quad (68)$$

Note that shadow workers have infinite elasticities of formal income along the linear tax schedule: as soon as the net formal wage departs from the shadow wage, the shadow worker either stops supplying formal labor entirely or becomes a formal worker. Nevertheless, elasticities along the non-linear tax schedule are well defined, as long the tax schedule is not locally linear.

Denote the derivative of formal income w.r.t. the productivity type along the non-linear tax schedule as

$$\tilde{y}_\theta^f(\theta, \kappa) \equiv \begin{cases} \tilde{\varepsilon}_{wf}^f(y^f(\theta, \kappa))\rho^f(\theta)y^f(\theta, \kappa) & \text{if } \kappa \geq \tilde{\kappa}(\theta), \\ \tilde{\varepsilon}_{wf}^s(y^f(\theta, \kappa))\rho^f(\theta)y^f(\theta, \kappa) & \text{otherwise.} \end{cases} \quad (69)$$

The density of formal workers at formal income $y^f(\theta, \infty)$, scaled by the share of formal workers, is defined as $h^f(y^f(\theta, \infty)) \equiv (1 - G_\theta(\tilde{\kappa}(\theta)))f(\theta)/\tilde{y}_\theta^f(\theta, \infty)$. The density of shadow workers at formal income $y^f(\theta, 0)$, scaled by the share of shadow workers, is $h^s(y^f(\theta, 0)) \equiv G_\theta(\tilde{\kappa}(\theta))f(\theta)/\tilde{y}_\theta^f(\theta, 0)$ and $h^s(y^f) \equiv 0$ for $y^f \notin y^f([0, 1], 0)$. The density of formal income is simply $h(y) \equiv h^f(y) + h^s(y)$. The mean elasticity at income level y is $\bar{\varepsilon}(y) \equiv h^f(y)\tilde{\varepsilon}^f(y) + h^s(y)\tilde{\varepsilon}^s(y)$.

Define the elasticity of the density of formal workers with respect to the tax burden of staying formal $\tilde{\Delta}T(\theta)$ by

$$\pi(y^f(\theta, \infty)) \equiv \frac{g_\theta(\tilde{\kappa}(\theta))\tilde{\Delta}T(\theta)}{1 - G_\theta(\tilde{\kappa}(\theta))}. \quad (70)$$

Define the average welfare weights of formal and shadow workers at a given formal income as

$$\bar{\lambda}^f(y^f(\theta, \infty)) \equiv \int_{\tilde{\kappa}(\theta)}^{\infty} \lambda(\theta, \kappa) \frac{dG_\theta(\kappa)}{1 - G_\theta(\tilde{\kappa}(\theta))}, \quad \bar{\lambda}^s(y^f(\theta, 0)) \equiv \int_0^{\tilde{\kappa}(\theta)} \lambda(\theta, \kappa) \frac{dG_\theta(\kappa)}{G_\theta(\tilde{\kappa}(\theta))}. \quad (71)$$

Then the average welfare weight at formal income y is $\bar{\lambda}(y) \equiv (h^f(y)\bar{\lambda}^f(y) + h^s(y)\bar{\lambda}^s(y)) / h(y)$.

The equivalence of the mechanism design approach and the sufficient statistics approach.

Note that $s(y^f(\theta, \infty)) = y^f(\tilde{s}(\theta, 0))$. By substituting the terms defined above, we can represent the left-hand sides of (47) and (48) as in the sufficient statistics formulas (22)

and (23), respectively.

Let's focus on the tax revenue gain (the right-hand side of the tax formulas). We can represent the tax revenue gain (the right-hand side of (47)) as

$$\begin{aligned} \int_{\theta}^1 \left[1 - \bar{\lambda}^f(y^f(\theta', \infty)) \right] (1 - G_{\theta'}(\tilde{\kappa}(\theta'))) dF(\theta') + \int_{\tilde{s}(\theta)}^1 \left[1 - \lambda^s(y^f(\theta', 0)) \right] G(\tilde{\kappa}(\theta')) dF(\theta') \\ - \int_{\theta}^{\tilde{s}(\theta)} \frac{g_{\theta'}(\tilde{\kappa}(\theta')) \tilde{\Delta} T(\theta')}{1 - G_{\theta'}(\tilde{\kappa}(\theta'))} (1 - G_{\theta'}(\tilde{\kappa}(\theta'))) dF(\theta'). \end{aligned} \quad (72)$$

By changing variables we obtain the right-hand side of (22):

$$\int_{y^f(\theta, \infty)}^{\infty} \left[1 - \bar{\lambda}^f(y) \right] h^f(y) dy + \int_{y^f(\tilde{s}(\theta), 0)}^{\infty} \left[1 - \bar{\lambda}^s(y) \right] h^s(y) dy - \int_{y^f(\theta, \infty)}^{y^f(\tilde{s}(\theta), \infty)} \pi(y) h^f(y) dy \quad (73)$$

$$= \int_{y^f(\theta, \infty)}^{\infty} \left[1 - \bar{\lambda}(y) \right] h(y) dy - \int_{y^f(\theta, \infty)}^{y^f(\tilde{s}(\theta), \infty)} \pi(y) h^f(y) dy. \quad (74)$$

Finally, note that $y^f(\tilde{s}(\theta), \infty) = s(y^f(\theta, \infty)) + \Delta_0(s(y^f(\theta, \infty)))$. We can express the right-hand side of (48) as the right-hand side of (23) analogously.

D. Estimation details.

First we describe the data and explain how we recover wages and sectoral participation. Second, we list the identifying assumptions and formulate the likelihood function. Last we present the parameter estimates.

Data. We use the 2013 wave of the household survey by the official statistical agency of Colombia (DANE). We restrict attention to individuals aged 24-50 years without children (26,000 individuals). We choose this sample, since these workers face the tax and transfer schedule which is not means-tested and does not depend on choices absent from our modeling framework, such as a number of children or college attainment.

The information we use in the estimation is given by $\{w_i, f_i, X_i, s_i\}_{i=1}^N$ where w_i is the hourly wage of worker i before taxes; f_i an indicator variable for having a main job in the formal sector; X_i a vector of worker characteristics; s_i the sampling weights and N the total number of observations in our sample. The indicator variable f_i is set equal to one if the worker reports to be affiliated to all three components of social security: pension system, health insurance and labor accidents insurance. A fraction (about 3%) of workers also have a second job. If the first job is formal we cannot identify if the worker's second job is shadow or formal. Therefore $f_i = 1$ indicates formality of the

Table 3: Variables included in X_i

Variable	Description	Values
<i>Individual characteristics</i>		
Gender	Dummy variable equal to 1 for women	0-1
Age	Age of the worker	16-90
Age ²	Age squared	
Educ	Number of education years	0-26
Degree	Highest degree achieved (No degree to Doctorate)	1-5
Work	Number of months worked in the last year	1-12
Exper	Number of months worked in the last job	0-720
1stJob	Dummy for the first job (1 if it is the first job)	0-1
<i>Job characteristics</i>		
S-Man	Dummy for the manufacturing sector	0-1
S-Fin	Dummy for financial intermediation	0-1
S-Ret	Dummy for the sales and retailers sector	0-1
B-city	Dummy for a firm in one of the two largest cities	0-1
Size	Categories for the number of workers	1-9
Lib	Dummy for a liberal occupation	0-1
Admin	Dummy for an administrative task	0-1
Seller	Dummy for sellers and related	0-1
Services	Dummy for a service task	0-1
<i>Worker-firm relationship</i>		
Union	Dummy for labor union affiliation (1 if yes)	0-1
Agency	Dummy for agency hiring (1 if yes)	0-1
Senior	Number of months of the worker in the firm	0-720

main job and does not imply that the worker is exclusively formal. We compute the hourly wage before taxes w_i as the ratio of the income and worked hours at the main job the month prior to the survey.³⁵ If the worker is identified to be formal at the main job we include the statutory payroll taxes that are paid by the employer in the computation of the pre-tax income at the main job. Variables included in vector X_i are listed in Table 3.

Modeling assumptions. We assume that productivity in each sector $j \in \{s, f\}$ features a constant, sector specific growth rate ρ^j with respect to the productivity type θ :

$$\log(w^j(\theta)) = \log(w^j(0)) + \rho^j\theta, \quad j \in \{s, f\}. \quad (75)$$

³⁵We further assume that survey respondents correctly reveal their gross income from the main job, regardless of whether the main job is formal or informal. Other papers making this assumption include Meghir, Narita, and Robin (2015) for Brazil and López García (2015) for Chile.

The above assumption is not restrictive for the unconditional distribution of formal wages, as long as we are free to choose any distribution of the productivity types $F(\theta)$. This assumption, however, restricts the joint distribution of formal and shadow wages. The comparative advantage in the shadow economy becomes

$$\frac{w^s(\theta)}{w^f(\theta)} = \frac{w^s(0)}{w^f(0)} \exp \left\{ \left(\rho^s - \rho^f \right) \theta \right\}. \quad (76)$$

Heckman and Honore (1990) and French and Taber (2011) show that the data on wages and the sector the worker participates is in general not sufficient to identify the productivity profiles. Heckman and Honore (1990) also prove that the model can be identified with additional regressors that affect the location parameters of the skill distribution. We follow this approach. Denote the vector of regressors, which includes worker's and job's characteristics, by X .³⁶ We assume that the regressors are useful in predicting the worker's productivity type:

$$\theta \sim N(X\beta, \sigma_\theta^2), \quad (77)$$

where β is a vector of parameters. We obtain $F(\theta)$ using (77) and a kernel density estimation of the $X\beta$ distribution. To capture the right tail of the wage distribution, we fit a Pareto distribution with parameter α_w to the top 1% of formal wages.

The fixed cost of shadow employment κ follows a generalized Pareto distribution with density

$$g_\theta(\kappa) = \frac{1}{\sigma_\kappa (w^f(\theta) - w_\kappa)^{\alpha_\kappa}} \left(1 + \frac{\kappa}{\sigma_\kappa (w^f(\theta) - w_\kappa)^{\alpha_\kappa}} \right)^{-2}, \quad (78)$$

where parameters σ_κ , α_κ and w_κ determine how the distribution of fixed cost is affected by the productivity type θ . Finally, we assume that agents' preferences over labor supply follow

$$v(n) = \Gamma \frac{n^{1+1/\varepsilon}}{1+1/\varepsilon}, \quad (79)$$

where ε is the common elasticity of labor supply. Together, assumptions (75), (77), (78) and (79) identify the model. We estimate the model by Maximum Likelihood.

Likelihood function. We can decompose the probability of the realization $\{w_i, f_i, X_i\}$ into three elements:

$$P(w = w_i, f = f_i, X = X_i; B) = P(X_i) \times P(w = w_i | X_i; B) \times P(f = f_i | w_i; B)$$

where

$$B = \left(\beta, \varepsilon, \Gamma, w^s(0), \rho^s, w^f(0), \rho^f, \sigma_\theta, \sigma_\kappa, w_\kappa \right)$$

and the elements correspond to:

³⁶In our estimation the vector X contains typical regressors from Mincerian wage equations such as age, gender, education level and experience. We also include job and firm characteristics, such as the task performed by the worker and the size of the firm. See the detailed description in the Appendix XXX.

- $P(X_i) = s_i$ is the sampling weight assigned in the survey. Measures how representative is the observation at the population level.
- $P(w = w_i | X_i; B)$ is the probability that a worker with characteristics X_i has the wage w_i at the sector j where she is participating. Let $\epsilon_i = \theta_i - X_i\beta$, then $\epsilon_i \sim N(0, \sigma_\theta^2)$ and we have

$$\begin{aligned}
P(w = w_i | X_i; B) &= P(w_i = w^j(0) \exp\{\rho^j (X_i\beta + \epsilon_i)\}) \\
&= P(\epsilon_i = \log(w_i) - \log(w^j(0)) - \rho^j X_i\beta) \\
&= \frac{1}{\sqrt{2\pi\sigma_\theta^2}} \exp\left\{-\frac{(\log(w_i) - \log(w^j(0)) - \rho^j X_i\beta)^2}{2\sigma_\theta^2}\right\}
\end{aligned}$$

If the worker belongs to the top 1% then $P(w = w_i)$ is given by the Pareto distribution with parameter α_w .

- $P(f = f_i | w_i; B)$ is the probability that a worker with observed wage w_i at sector j has a participation decision in the formal sector given by f_i . Note that the observed wage at the given sector reveals the type of the worker as follows:

$$\theta_i = \frac{1}{\rho^j} \log\left(\frac{w_i}{w^j(0)}\right)$$

Given the type of the worker there are two possibilities: i) The participation cost is above the threshold $\tilde{\kappa}(\theta_i)$ and formal income is given by $y^f(\theta_i, \infty)$, or ii) the participation cost is below $\tilde{\kappa}(\theta_i)$ and formal income $y^f(\theta_i, 0)$. Then, considering these two possibilities we can write the participation probability as

$$P(f = f_i | w_i; B) = \begin{cases} G_{\theta_i}(\tilde{\kappa}(\theta_i)) I_{(y^f(\theta_i, 0) > 0)} + [1 - G_{\theta}(\tilde{\kappa}(\theta_i))] I_{(y^f(\theta_i, \infty) > 0)} & \text{if } f_i = 1 \\ G_{\theta_i}(\tilde{\kappa}(\theta_i)) I_{(y^f(\theta_i, 0) = 0)} + [1 - G_{\theta}(\tilde{\kappa}(\theta_i))] I_{(y^f(\theta_i, \infty) = 0)} & \text{if } f_i = 0 \end{cases}$$

where G_{θ_i} is the cumulative distribution of κ for worker i and I is an indicator function.

Parameter estimates. The parameter estimates are reported in Table 4. The estimated density of types as well as the fit of the model along the shadow economy participation margin are demonstrated in Figure 6 in the main text. Figure 13 compares the estimated and the empirical distribution of wages. Specifically, we plot the average shadow and formal productivities implied by the model along the percentile rank of $X\beta$. The data counterpart is constructed using a rolling window of 100 workers centered around each $X\beta$ observed in the sample and averaging the observed sectoral wages.³⁷ Figure 13 shows that the model generates wage distribution that matches the data well. We take this as a further indication that our parametric specification is compatible with the data.

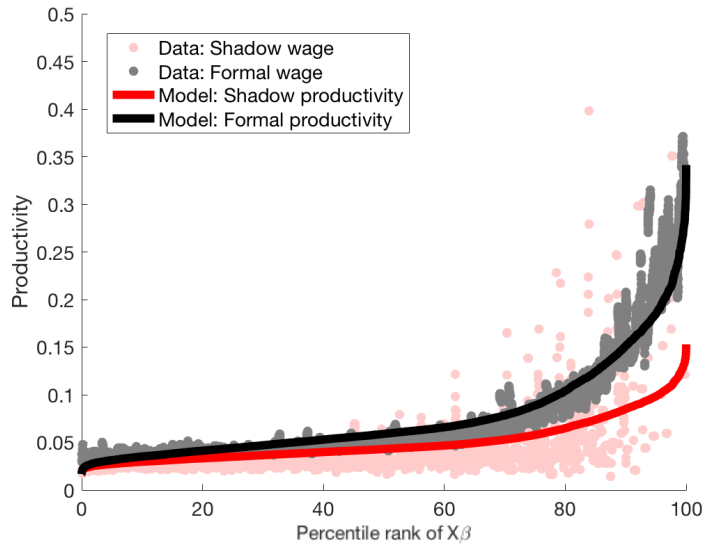
³⁷Note that the empirical average shadow wage tends to be more volatile at the top levels of $X\beta$ as shadow participation is low and there are few observations inside the rolling window.

Table 4: Parameter estimates

preferences		productivity schedules					distributions of θ and κ			
ε	Γ	$w^s(0)$	ρ^s	$w^f(0)$	ρ^f	α_w	σ_θ	σ_κ	α_κ	w_κ
0.358	0.497	0.011	2.99	0.0085	4.15	2.4	0.51	0.59	0.05	0.008
(.01)	(.002)	(.003)	(.04)	(.003)	(.04)	(.03)	(.03)	(.02)	(.03)	(.0001)
β individual characteristics								β worker-firm		
Gender	Age	Age ²	Educ	Degree	Work	Exper	1stJob	Union	Agency	
-0.10	0.02	-.0002	0.02	0.13	0.03	.001	.007	0.07	-0.15	
(.001)	(.001)	(.0001)	(.002)	(.006)	(.002)	(.0001)	(.001)	(.02)	(.01)	
β job characteristics										
S-Man	S-Fin	S-Ret	B-city	Size	Lib	Admin	Seller	Services	Senior	
-0.07	0.21	-0.14	0.15	0.03	0.42	-0.03	-0.08	-0.19	0.001	
(.01)	(.02)	(.01)	(.001)	(.008)	(.001)	(.012)	(.011)	(.013)	(.0001)	

Standard errors are reported in brackets. Standard errors are obtained from the information matrix.

Figure 13: Fit of wages



E. Social preferences implicit in the Colombian tax schedule.

The optimal tax formula can be used to extract social preferences, i.e. Pareto weights at each income level, implicit in a given tax schedule. This methodology is typically used to study the Pareto efficiency of the actual tax, as a negative Pareto weight would mean that the tax system is inefficient and the government can increase tax revenue without reducing utility of any agent.³⁸

To extract the welfare weights, differentiate the optimal tax formula (22) evaluated at some formal income level y to get

$$\bar{\lambda}(y) = \mathbb{E}(\bar{\lambda}) + \underbrace{\frac{\partial DWL(y)}{\partial y} \frac{1}{h(y)}}_{\text{intensive margin}} - \underbrace{\pi(y) \frac{h^f(y)}{h(y)}}_{\text{extensive margin}}, \quad (80)$$

where $DWL(\cdot)$ stands for the total deadweight loss, i.e. the left-hand side of (22), at formal income level y . The mean Pareto weight at a given income level can be explained by three components. The first one is the average Pareto weight across all income levels, equal to 1. The second is the contribution of the intensive margin. The total deadweight loss, including both formal and shadow workers, increases faster at income levels associated with higher Pareto weights. That is because higher $\bar{\lambda}(y)$ reduces the deadweight loss below y and does not affect the deadweight loss above y (see equation (22)). The third component captures the extensive margin: a decision to participate in the shadow economy. Recall that $\pi(y)$ is the elasticity of the density of formal workers with respect to the tax burden of staying formal. The impact of extensive margin is similar to that of the Pareto weight: it implies a higher derivative of the deadweight loss. Hence, higher $\pi(y)$ means that a smaller part of the increase of deadweight loss remains to be explained by social preferences.

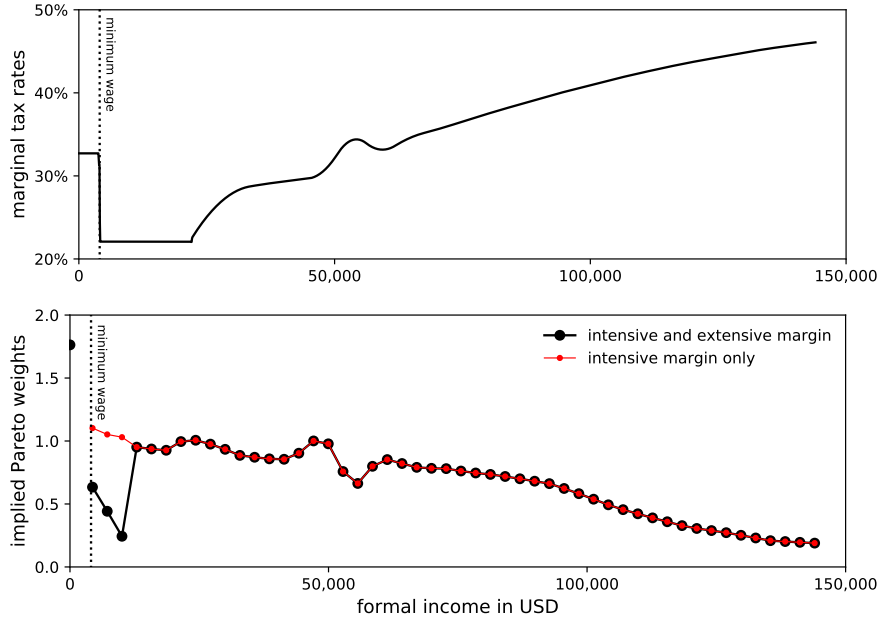
Figure 14 shows the actual tax schedule in Colombia in 2013 as well as the implied Pareto weights. The marginal tax rates are high at the bottom due to the phase-out of transfers, then drop to 22% - the rate of payroll taxation - and then increase as the personal income tax starts at around \$22,000. The personal income tax is roughly progressive, but highly irregular: it is a step function of more than 70 steps of varying length. To abstract from inefficiencies associated with multiple small notches which are not the focus of the paper, we approximate this part of the Colombian tax schedule with cubic splines.³⁹

We find no evidence of negative Pareto weights - the smoothed Colombian tax schedule is Pareto efficient. However, the implicit weights exhibit a peculiar pattern: they are

³⁸The original test of Pareto efficiency was proposed by [Werning \(2007\)](#). The methodology was further developed and applied by [Bourguignon and Spadaro \(2012\)](#); [Brendon \(2013\)](#); [Lorenz and Sachs \(2016\)](#); [Jacobs, Jongen, and Zoutman \(2017\)](#).

³⁹The minimum wage is another source of inefficiency. Since our framework is not designed to address this issue, we do not optimize with respect to the the minimum wage level.

Figure 14: Income tax in Colombia



much lower for workers with earnings close to the minimum wage than for workers with slightly higher earnings. For instance, formal workers earning \$10,000 annually have an implied Pareto weight which is *four times smaller* than the weight of workers earning \$13,000. Although the income interval of unusually low Pareto weights is relatively small, it contains 34% of all formal workers. None of the workers with formal earnings in this interval has shadow earnings.

Unusually low Pareto weights at low incomes are driven by the incentives to participate in the shadow economy, i.e. the extensive margin term in formula (80). If instead we consider only the intensive margin term (see the red line in the bottom panel of Figure 14), the Pareto weights look much more regular. In particular, they are decreasing with income at low income levels.⁴⁰ A plausible interpretation of this result is that the Colombian tax schedule was set without taking into account the extensive margin incentives for informality. The marginal rate of the actual tax schedule is constant where the Pareto weights are unusually low. The tax schedule which accounts for informality and follows the more intuitive, decreasing schedule of Pareto weights would instead have increasing, rather than flat, marginal rates.

⁴⁰Pareto weights are also locally increasing when marginal rates of the personal income tax are increasing rapidly. It has nothing to do with the informal sector.