Optimal Redistribution with a Shadow Economy

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Abstract

We examine the constrained efficient allocations in the Mirrlees (1971) model with a shadow economy. There are two labor markets: formal and informal. The income from the formal market is observed by the planner, while the income from the informal market is not. There is a distribution of workers that differ with respect to the formal and the informal productivity. We show that when the planner does not observe individual productivities some workers may optimally work in the shadow economy. Moreover, the social welfare of the model with the shadow economy can be higher than the welfare of the model without the informal sector. These results hold even when each agent is more productive formally than in the shadow economy. In order to apply our theory, we propose a novel way of estimating workers' productivities in the two sectors from micro data. Calibrating the model to Colombia, where 58% of workers are employed informally, we find that the optimal shadow economy is less than half that size. The optimal income tax schedule is very different then the one implied by the Mirrlees (1971) model without the informal sector.

1 Introduction

Informal activity, defined broadly as any endeavor which is not necessarily illegal but evades taxation, accounts for a large fraction of economic activity in both developing and developed economies. According to Jutting, Laiglesia, et al. (2009) more then half of the jobs in the non-agricultural sector worldwide can be considered informal. Schneider, Buehn, and Montenegro (2011) estimate the share of informal production in the GDP of high income OECD countries in the years 1999-2007 as 13.5%. Given this evidence, the informal sector should be considered in the design of fiscal

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policy. This paper extends the theory of the optimal redistributive taxation by Mirrlees (1971) to the economies with an informal labor market.

The ability of the state to redistribute income depends on how responsive to taxes individuals are. When incomes are very elastic, differential taxation of different individuals is hard, because workers adjust their earnings to minimize the tax burden.[1] The shadow economy allows workers to earn additional income which is unobserved by the government. Without shadow economy, workers respond to taxes only by changing their total labor supply. With the shadow economy, they can additionally shift labor between the formal and the informal sector, which increases the elasticity of their formal income. As incomes in the formal economy become more elastic, redistribution becomes more difficult.

We show that the government can exploit differences in informal productivity between workers to improve redistribution. Suppose there are two types of workers: skilled and unskilled. The responsiveness of the skilled workers determines the taxes they pay and the transfers the unskilled receive. In the world without the shadow economy, this responsiveness depends on how easy it is for the skilled to reduce income to the level of the unskilled worker. If that happens, the government cannot tax differentially the two types of individuals. In the world with the shadow economy, the government can improve redistribution in the following way. By increasing taxes at low levels of formal income, the unskilled workers are pushed to informality. If the unskilled workers can easily find a good informal job, this transition will not hurt them much. Now the skilled workers can avoid taxes only if they too move to the shadow economy. Hence, the responsiveness of the skilled workers depends on their informal productivity. If the skilled workers suffer a large productivity loss by moving to the other sector, the government can tax them more in the formal sector and provide higher transfers to the unskilled informal workers. In the opposite case, however, when the skilled can easily move between sectors while the unskilled cannot, the government cannot use the shadow economy to discourage the skilled workers from reducing formal income. In such a case, redistribution will be reduced.

The shadow economy also affects the efficiency of labor allocation by sheltering workers from tax distortions.[2] The labor supply of formal workers is determined jointly by their formal productivity and a marginal tax rate they face. In contrast, the labor supply of informal workers depends only on their informal production opportunity and is unaffected by tax distortions. When their informal productivity is not much lower than the formal one, informal workers will produce more than if they stayed in the formal sector. In this way the shadow economy improves the allocation of labor and raises efficiency.

Whether the shadow economy is harmful or beneficial from the social welfare perspective depends on its joint impact on redistribution and efficiency. The informal sector improves redistribution if the workers that pay high taxes cannot easily move to the shadow economy. It benefits efficiency if informal workers have similar productivities in formal and informal sector. As a rule of a thumb,

[1] Diamond (1998) and Saez (2001) expressed the optimal tax rates in the Mirrlees model with elasticities. The higher is the elasticity of labor supply, the lower is the optimal marginal tax rate at this level of income.

[2] This effect corresponds to what La Porta and Shleifer (2008) call the romantic view on the shadow economy. In this view, associated with the works of Hernando de Soto (de Soto (1990, 2000)), the informal sector protects productive firms from harmful regulation and taxes.

we can say that the shadow economy raises welfare if it allows poor workers who collect transfers to earn some additional money, but does not tempt the rich taxpayers to reduce their formal income.

We derive the formula for the optimal tax with a shadow economy. The informal sector imposes an upper bound on the marginal tax rate, which depends on the distribution of formal and informal productivities. The optimal tax rate at each formal income level is given by either the usual Diamond (1998) formula or the upper bound, if the Diamond formula prescribes rates that are too high. In contrast to the standard Mirrlees (1971) model, in the model with shadow economy different types of workers are likely to be bunched at the single level of formal income. Specifically, all agents that supply shadow labor are subject to bunching. We develop the optimal bunching condition which complements the Diamond formula.[3]

The model is calibrated to Colombia, where 58% of workers are employed informally. We derive the joint distribution of formal and shadow productivity from a household survey. The main difficulty is that most individuals work only in one sector at a time. We infer their productivity in the other sector by estimating a factor: a linear combination of workers' and jobs' characteristics that explains most of the variability of shadow and formal productivities. The factor allows us to match similar individuals and infer their missing productivities. When we apply the actual tax schedule to the calibrated economy, the model replicates well the actual size of the informal sector.

We find that the optimal share of shadow workers in the total workforce is close to 25% under the Rawlsian planner and no more than 6.5% under the Utilitarian planner. This means that the optimal shadow economy is much smaller than than 58%, the actual share of shadow workers in Colombia. In comparison the Colombian income tax at the time, the optimal tax schedule has lower marginal rates at the bottom and higher rates elsewhere. Lower tax rates at the bottom displace less workers to the shadow economy, while higher tax rates above raise more revenue from high earners, yielding high welfare gains. The optimal tax rates are generally lower then the ones implied by the Mirrlees (1971) model without the informal sector. The application of the Mirrlees (1971) income tax would displace an excessive number of workers to the shadow economy.

Related literature. Tax evasion has been studied at least since Allingham and Sandmo (1972). For us, the most relevant paper from this literature is Kopczuk (2001). He derives the optimal linear income tax in a general environment with two-dimensional heterogeneity in productivity and tax avoidance skills. Moreover, he points at the possibility of tax evasion in the optimum. We, in contrast to Kopczuk, focus on the optimal non-linear income tax and provide a sharp characterization of the optimal shadow economy. The impact of income taxes on informal activity has been studied empirically as well. Frías, Kumler, and Verhoogen (2013) show that underreporting of wages decreases, once reported income is linked to pension benefits. Waseem (2013) documents that an increase of taxes of partnerships in Pakistan led to a massive shift to other business forms as well as a large spike in income underreporting.

[3] In the Mirrlees (1971) model without wealth effects the optimal allocation is described by the Diamond formula if and only if the resulting income schedule is non-decreasing, which is usually verified ex post. If the Diamond formula implies the income schedule that is decreasing at some type, our optimal bunching condition recovers the optimum.

Our model is focused on the workers' heterogeneity with respect to formal and informal productivities. A similar approach was taken by Albrecht, Navarro, and Vroman (2009), who study the impact of labor market institutions in a model with the formal and informal labor markets and a search friction. There is a complementary approach to modeling the shadow economy, which focuses on firms' rather than workers' heterogeneity. In Rauch (1991) managers with varying skills decide in which sector to open a business. He finds that less productive managers choose informal sector in order to avoid costly regulation. Meghir, Narita, and Robin (2015) consider heterogeneous firms that decide in which sector to operate and who are randomly matched with homogeneous workers. They find that policies aimed at reduction of the shadow economy increase competition for workers in the formal labor market and improve welfare. Amaral and Quintin (2006) to the best of our knowledge provide the only framework with the shadow economy where heterogeneity of both firms and workers is present. They extend the Rauch (1991) model by allowing for physical and human capital accumulation. Due to complementarity between the two types of capital, educated workers tend to stay in the more capital intensive formal sector.

The following two papers derive the optimal policy in related environments. Gomes, Lozachmeur, and Pavan (2014) study the optimal sector-specific income taxation when individuals can work in one of the two sectors of the economy. In our setting there are also two sectors, but the government can impose tax only on one of them. Moreover, we allow agents to work in the two sectors simultaneously. Alvarez-Parra and Sánchez (2009) study the optimal unemployment insurance with the moral hazard in search effort and an informal labor market. It is another environment with information frictions in which the informal employment is utilized in the optimal allocation.

Structure of the paper. In the next section we illustrate the main ideas of the paper with a simple model. In Section 3 we solve a model with a large number of types and general social preferences. In Section 4 we introduce our methodology of extracting shadow productivities from the micro data and apply it to Colombia. We derive the optimal Colombian tax schedule in Section 5. The last section concludes.

2 Simple model

Imagine an economy inhabited by people that share preferences but differ in productivity. There are two types of individuals, indexed by letters l and h, with strictly positive population shares μ_l and μ_h. They all care about consumption c and labor supply n according to the utility function

$$U(c,n) = c - v(n). \tag{1}$$

We assume that v is increasing, strictly convex and twice differentiable. We also impose that $v'(0) = 0$ and $\lim_{n\to\infty} v'(n) = \infty$. We denote the inverse function of v' by g.

There are two labor markets and, correspondingly, each agent is equipped with two linear production technologies. An agent $i \in \{l,h\}$ produces with productivity w_i^f in a formal labor market, and with productivity w_i^s in an informal labor market. Type h is more productive in the formal market than type l: $w_h^f > w_l^f$. Moreover, in this section we assume that each type's informal productivity is lower than formal productivity. We relax this assumption when we consider the full model.

$$\forall_i \; w_i^f > w_i^s. \tag{2}$$

Any agent may work formally, informally, or in both markets simultaneously. An agent of type i works n_i hours in total, which is the sum of n_i^f hours at the formal job and n_i^s hours in the shadow economy. The formal and the informal income, denoted by y_i^f and y_i^s respectively, is a product of the relevant productivity and the relevant labor supply. The allocation of resources may involve transfers across types, so one's consumption may be different than the sum of formal and informal income. In order to capture these flows of resources, we introduce a tax T_i, equal to the gap between total income and total consumption

$$T_i \equiv y_i^f + y_i^s - c_i. \tag{3}$$

A negative tax is called a transfer, and we are going to use these terms interchangeably.

The social planner follows John Rawls' theory of justice and wants to improve the well-being of the least well-off agents,[4] but is limited by imperfect knowledge. The planner knows the structure and parameters of the economy, but, as in the standard Mirrlees model, does not observe the type of any individual. In addition, shadow income and labor are unobserved by the planner as well. The only variables at the individual level the planner sees and can directly verify are the formal income y_i^f and the tax T_i. We can think about y_i^f and $y_i^f - T_i$ as a pre-tax and an after-tax reported income. Although shadow labor cannot be controlled directly, it is influenced by the choice of formal labor. Formal labor affects the marginal disutility from labor and hence changes the agent's optimal choice of shadow hours. Two types of labor are related according to the following function, implied by the agent's first order condition

$$n_i^s\left(n^f\right) = \max\left\{g\left(w_i^s\right) - n^f, 0\right\}.$$

When the agent works a sufficient number of hours in the formal sector, the marginal disutility from labor is too high to work additionally in the shadows. However, if the formal hours fall short of $g\left(w_i^s\right)$, the resulting gap is filled with shadow labor.

The planner maximizes the Rawlsian social welfare function, given by a utility level of the worst-off agent

$$\max_{\left\{\left(n_i^f, T_i\right) \in \mathbb{R}_+ \times \mathbb{R}\right\}_{i \in \{l,h\}}} \min\left\{U\left(c_l, n_l\right), U\left(c_h, n_h\right)\right\}, \tag{4}$$

subject to the relation between formal and shadow labor

$$n_i^s\left(n^f\right) = \max\left\{g\left(w_i^s\right) - n^f, 0\right\}, \tag{5}$$

[4] We pick this particular point of the Pareto frontier because it allows us to show the interesting features of the model with relatively easy derivations. At the end of this section we discuss how other constrained efficient allocations look like.

the accounting equations

$$\forall_{i\in\{l,h\}}\; c_i = w_i^f n_i^f + w_i^s n_i^s\left(n_i^f\right) - T_i, \tag{6}$$

$$\forall_{i\in\{l,h\}}\; n_i = n_i^f + n_i^s\left(n_i^f\right), \tag{7}$$

a resource constraint

$$\sum_{i\in\{l,h\}} \mu_i T_i \geq 0, \tag{8}$$

and incentive-compatibility constraints

$$\forall_{i\in\{l,h\}}\; U\left(c_i, n_i\right) \geq U\left(w_{-i}^f n_{-i}^f + w_i^s n_i^s\left(\frac{w_{-i}^f}{w_i^f} n_{-i}^f\right) - T_{-i}, \frac{w_{-i}^f}{w_i^f} n_{-i}^f + n_i^s\left(\frac{w_{-i}^f}{w_i^f} n_{-i}^f\right)\right). \tag{9}$$

We denote the generic incentive constraint by $IC_{i,-i}$. It means that an agent i cannot be better off by earning the formal income of the other type and simultaneously adjusting informal labor.

2.1 First-best

What if the planner is omniscient and directly observes all variables? The planner knows types and can choose the shadow labor supply directly. The optimal allocation is a solution to the welfare maximization problem (4) where planner chooses both formal and shadow labor and a tax of each type subject only to the accounting equations (6) and (7) and the resource constraint (8). All types are more productive in the formal sector than in the shadow economy, so no agent will work informally. Each agent will supply the formal labor efficiently, equalizing the marginal social cost and benefit of working. Moreover, the planner redistributes income from h to l in order to achieve the equality of well-being.

Proposition 1. *In the first-best both types work only formally and supply an efficient amount of labor:* $\forall_i v'\left(n_i\right) = w_i^f$. *Utility levels of the two types are equal:* $U\left(c_l, n_l\right) = U\left(c_h, n_h\right)$.

We can slightly restrict the amount of information available to the planner without affecting the optimal allocation. Suppose that the planner still observes the formal productivity, but shadow labor and income are hidden. The optimal allocation is a solution to (4) subject to the relation between shadow and formal labor (5), the accounting equations (6) and (7) and the resource constraint (8).

Proposition 2. *If the planner knows types, but does not observe shadow labor and income, the planner can achieve the first-best.*

When the types are known, the planner can use the lump-sum taxation and implement the first-best. Without additional frictions, the hidden shadow economy does not constrain the social planner.

2.2 Second-best

Let's consider the problem in which neither type nor informal activity is observed. The planner solves (4) subject to all the constraints (5) - (9). We call the solution to this problem the second-best or simply the optimum.

In the first-best, both types work only on the formal market and their utilities are equal. If h could mimic the other type, higher formal productivity would allow h to increase utility. Hence, the first-best does not satisfy $IC_{h,l}$ and this constraint limits the welfare at the optimum. On the other hand, $IC_{l,h}$ never binds at the optimum. It would require the redistribution of resources from type l to h, which is clearly suboptimal.

Proposition 3. *The optimum exists and is not the first-best. $IC_{h,l}$ is binding, while $IC_{l,h}$ is slack.*

2.2.1 Optimal shadow economy

The standard Mirrlees model typically involves labor distortions, since they can relax the binding incentive constraints. If type i is tempted to pretend to be of the type $-i$, distorting number of hours of $-i$ will discourage the deviation. Agents differ in labor productivity, so if i is more (less) productive than the other type, decreasing (increasing) number of hours worked by $-i$ will make the deviation less attractive. Proposition 3 tells us that no agent wants to mimic type h, hence the planner has no reason to distort the labor choice of these agents. Moreover, according to (5) shadow labor is supplied only if formal labor is sufficiently distorted. Hence, the classic result of no distortions at the top implies here that h will work only formally.

Corollary 1. *Type h faces no distortions and never works in the shadow economy.*

On the other hand, the planner can improve social welfare by distorting the formal labor supply of type l. Stronger distortions relax the binding incentive constraint and allow the planner to redistribute more. If distortions are strong enough, type l will end up supplying shadow labor. Optimality of doing so depends on whether and by how much increasing shadow labor of type l relaxes the binding incentive constraint. As Proposition 4 demonstrates, a comparative advantage of type l in shadow labor plays a crucial role. In the proof we use the optimality condition derived in the Appendix (see Lemma 3 in the Appendix 2). In order to make sure that this condition is well behaved, we require that v'' is nondecreasing.[5]

Proposition 4. *Suppose that v'' is nondecreasing. Type l may optimally work in the shadow economy only if*

$$\left(\frac{w_l^s}{w_l^f} - \frac{w_h^s}{w_h^f}\right)\mu_h \geq \frac{w_l^f - w_l^s}{w_l^f}\mu_l. \tag{10}$$

Condition (10) is also a sufficient condition for type l to optimally work in the shadow economy if $\frac{w_h^f}{w_l^f} g\left(w_h^s\right) \geq g\left(w_l^s\right)$. Otherwise, the sufficient (but not necessary) condition is

$$\left(\frac{w_l^s}{w_l^f} - \frac{v'\left(\frac{w_l^f}{w_h^f} g\left(w_l^s\right)\right)}{w_h^f}\right)\mu_h \geq \frac{w_l^f - w_l^s}{w_l^f}\mu_l. \tag{11}$$

[5] In the canonical case of isoelastic utility, it means that the elasticity of the labor supply is not greater than 1.

Inequality (10) provides a necessary condition for the optimal shadow economy by comparing the marginal benefit and cost of increasing shadow labor of type l. The left hand side is the comparative advantage of type l over type h in the shadow labor, multiplied by the share of type h. This advantage has to be positive for type l to optimally work in the shadow economy. Otherwise, increasing shadow labor of this type does not relax the binding incentive constraint. Since the shadow economy does not facilitate screening of types, there are no benefits from the productivity-inferior shadow sector. The welfare gains from the relaxed incentive constraint are proportional to the share of type h, as the planner obtains more resources for redistribution by imposing a higher tax on this type. On the right hand side, the cost of increasing shadow labor is given by the productivity loss from using the inferior shadow production, multiplied by the share of types that supply shadow labor.

Condition (10) is also a sufficient condition for type l to work in the shadow economy if the shadow productivity of type h is not much lower than the shadow productivity of type l. If that is not the case, the optimality condition derived in Lemma 3 (see Appendix 2) is not sufficient and we have to impose a stronger sufficiency condition (11).

Figure 1 illustrates the proposition on the diagram of the parameter space (w_h^s, w_l^s). Along the the diagonal no type has the comparative advantage, since ratios of shadow and formal productivity of the two types are equal. The optimal shadow economy requires that type l has the comparative advantage in shadow labor, so all the interesting action happens above the diagonal. Inequality (10) determines 'Necessary condition' curve. All points above this line satisfy (10) and only in this region type l may optimally supply shadow labor. Condition (10) is both necessary and sufficient for the optimal shadow economy to the right of Threshold curve. To the left of it, the sufficiency region is determined by inequality (11). Note that 'Necessary condition' curve crosses the vertical axis at the value $\mu_l w_l^f$. As the proportion of type l decreases toward zero, the region where shadow economy is optimal increases, in the limit encompassing all the points where type l has the comparative advantage over h in shadow labor.

Now we know when type l optimally works in the shadow economy. Proposition 5 tells us, how much shadow labor should type l supply in this case.

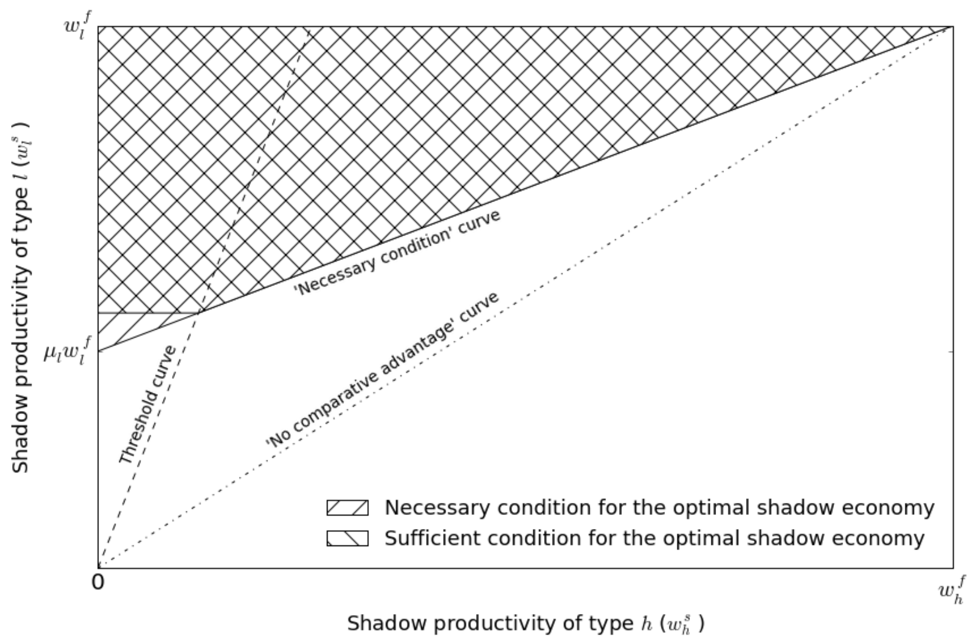
Proposition 5. *Suppose that type l optimally works in the shadow economy. Type l works only in the shadow economy if $w_h^s \geq w_l^s$. Type l works in both sectors simultaneously if $w_h^s < w_l^s$.*

When type l is more productive in the shadows than h and works only in the shadow economy, then by $IC_{h,l}$ the utility of type l will be greater than the utility of h. Since the planner is Rawlsian, the utility levels of both types will be equalized by making type l work partly in the formal economy. On the other hand, when type h is more productive informally, $IC_{h,l}$ means that the utility of type l will be always lower. Then if the shadow economy benefits type l, the planner will use it as much as possible.

2.2.2 Shadow economy and welfare

In order to examine the welfare implications of the shadow economy, we compare social welfare of the two allocations: the optimum of the standard Mirrlees model (noted with a superscript ^{M}) and

Figure 1: The optimal shadow economy



the optimum of the shadow economy model (noted with a superscript ^{SE}). We can think about the standard Mirrlees model as a special case of our model, in which both w_l^s and w_h^s are equal 0. We are going to use the utility of type l as a measure of welfare. Hence, we implicitly assume that in the optimum the planner does not equalize utilities of two types. If the planner is able and willing to do so, the existence of the shadow economy is clearly welfare improving in comparison to the Mirrlees model where type h has always greater utility than type l.

Let's consider the case in which the existence of shadow economy improves welfare. It can happen only if the shadow economy is used in the optimum - otherwise the planner could implement the same allocation in the standard Mirrlees model. Moreover, if in the optimum type l works in the shadow economy, this type will supply only shadow labor.[6] We can decompose the difference in welfare between the two allocations in the following way

$$\underbrace{U\left(c_l^{SE}, n_l^{SE}\right) - U\left(c_l^{M}, n_l^{M}\right)}_{\text{total welfare gain}} = \underbrace{U\left(w_l^s n_l^{SE}, n_l^{SE}\right) - U\left(w_l^f n_l^{M}, n_l^{M}\right)}_{\text{efficiency gain}} + \underbrace{T_l^M - T_l^{SE}}_{\text{redistribution gain}}. \tag{12}$$

The efficiency gain measures the difference in distortions imposed on type l, while the redistribution gain describes the change in the level of transfer type l receives. Thanks to the quasilinear preferences, we can decompose these two effects additively.

[6] Recall that we ruled out cases in which the planner equalizes the utility of both types in the optimum (by Proposition 5 it happens when the shadow economy is optimal and $w_l^s \geq w_h^s$). Only in such cases it may be necessary for optimality that type l optimally supplies both formal and shadow labor.

Efficiency gain The distortion imposed on type l in the shadow economy arise from the productivity loss $w_l^f - w_l^s$. By varying w_l^s, this distortion can be made arbitrarily small. On the other hand, the distortion of the standard Mirrlees model is implied by the marginal tax rate on formal income. Given redistributive social preferences, it is always optimal to impose a positive tax rate on type l. The efficiency gain, which captures the difference in distortions between two regimes, is strictly increasing in w_l^s.

Redistribution gain The shadow economy improves redistribution if the planner is able to give higher transfer to type l (or equivalently raise higher tax from type h). The difference in transfers can be expressed as

$$T_l^M - T_l^{SE} = \mu_h \left(U\left(w_l^f n_l^M, \frac{w_l^f}{w_h^f} n_l^M \right) - U\left(w_h^s n_h^{SE}, n_h^{SE} \right) \right).$$

What determines the magnitude of redistribution is the possibility of production of type h after misreporting. In the standard Mirrlees model deviating type h uses formal productivity and can produce only as much output as type l. In the allocation where type l works only informally, type h cannot supply any formal labor, but is unconstrained in supplying informal labor. Hence, the redistribution gain is strictly decreasing in w_h^s.

Proposition 6 uses the decomposition into the efficiency and redistribution gains in order to derive threshold values for shadow productivity of each type. Depending on which side of the thresholds the productivities are, the existence of the shadow economy improves or deteriorates social welfare in comparison to the standard Mirrlees model.

Proposition 6. *Define an increasing function $H(w^s) = U(w^s g(w^s), g(w^s))$ and the following threshold values*

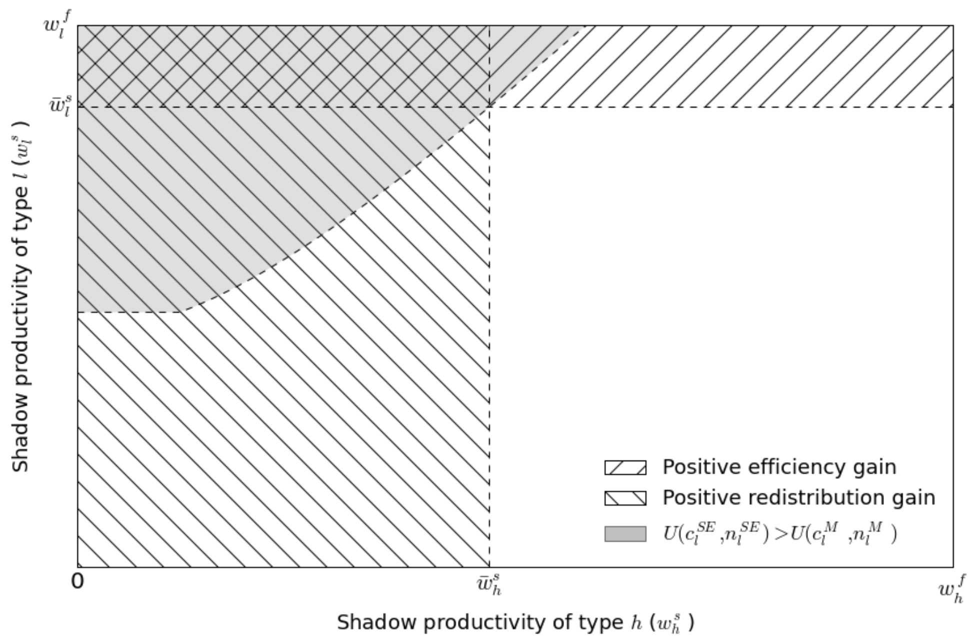
$$\bar{w}_l^s = H^{-1}\left(U\left(w_l^f n_l^M, n_l^M \right) \right) \in \left(0, w_l^f\right), \qquad \bar{w}_h^s = H^{-1}\left(U\left(w_l^f n_l^M, \frac{w_l^f}{w_h^f} n_l^M \right) \right) \in \left(0, w_h^f\right). \tag{13}$$

If $w_l^s \geq \bar{w}_l^s$ and $w_h^s \leq \bar{w}_h^s$, where at least one of these inequalities is strict, the existence of the shadow economy improves welfare in comparison to the standard Mirrlees model.

If $w_l^s \leq \bar{w}_l^s$ and $w_h^s \geq \bar{w}_h^s$, where at least one of these inequalities is strict, the existence of the shadow economy deteriorates welfare in comparison to the standard Mirrlees model.

The proposition is illustrated on the Figure 2. When the shadow productivity of type l is above $\bar{w}_l^s$, the efficiency gain is positive. The redistribution gain is positive when the shadow productivity of type h is below $\bar{w}_h^s$. Obviously, when both gains are positive, the shadow economy benefits welfare. On the other hand, when both gains are negative, the shadow economy is a burden and decreases welfare. Intuitively, the shadow economy does not have to strengthen both redistribution and efficiency simultaneously to be welfare improving. Particularly interesting is the region where the redistribution gain is negative, but the efficiency gain is sufficiently high such that the welfare is higher with the shadow economy. In this case the optimum of the shadow economy model Pareto dominates the optimum of the Mirrlees model. Type l gains, since the welfare is higher with the shadow economy. Type h benefits as well, as the negative redistribution gain implies a lower tax of this type.

Figure 2: Shadow economy and welfare



2.2.3 General social preferences

In this short section we will derive some properties of the whole Pareto frontier of the two-types model. We consider the planner that maximizes the general utilitarian social welfare function

$$\lambda_l \mu_l U(c_l, n_l) + \lambda_h \mu_h U(c_h, n_h),$$

where the two Pareto weights are non-negative and sum up to 1. The maximization is subject to the constraints (5) - (9).

From the Rawlsian case we know that the comparative advantage of type l in shadow labor is necessary for this type to work in the shadows. Proposition 7 generalizes this observation.

Proposition 7. *Type $i \in \{l, h\}$ may optimally work in the shadow economy only if $\frac{w_i^s}{w_i^f} > \frac{w_{-i}^s}{w_{-i}^f}$ and $\lambda_i > \lambda_{-i}$.*

In order to optimally work in the shadow economy, any type $i \in \{l, h\}$ has to satisfy two requirements. First, type i needs to have the comparative advantage in the shadow labor over the other type. Otherwise, shifting labor from formal to shadow sector does not relax the incentive constraints. Second, the planner has to be willing to redistribute resources to type i - the Pareto weight of this type has to be greater than the weight of the other type. The shadow economy can be beneficial only when it relaxes the binding incentive constraints, and the incentive constraint $IC_{-i,i}$ binds if $\lambda_i > \lambda_{-i}$. Intuitively, if the planner prefers to tax rather than support some agents, it is suboptimal to let them evade taxation.

When will type i optimally work in the shadow economy? Let's compare the welfare of two allocations. In the first allocation (denoted by superscript ^{SE}) type i works exclusively in the shadow economy. It provides the lower bound on welfare when type i is employed informally. The second allocation (denoted by ^{M}) is the optimum of the standard Mirrlees model, or equivalently the optimum of the shadow economy model where $w_i^s = w_{-i}^s = 0$. It is the upper bound on welfare when type i is employed only in the formal sector. We can decompose the welfare difference between these two allocations in the familiar way

$$\underbrace{W^{SE} - W^{M}}_{\text{total welfare gain}} = \underbrace{\mu_i \lambda_i \left(U\left(w_i^s n_i^{SE}, n_i^{SE}\right) - U\left(w_i^f n_i^M, n_i^M\right)\right)}_{\text{efficiency gain}} + \underbrace{\mu_i \left(\lambda_i - \lambda_{-i}\right)\left(T_i^M - T_i^{SE}\right)}_{\text{redistribution gain}}. \tag{14}$$

The welfare difference can be decomposed into the difference in effective distortions imposed on type i and the difference in transfers received by this type. The only essential change in comparison to the simpler Rawlsian case given by (12) comes from the Pareto weights. The more the planner cares about type $-i$, the less valuable are gains in redistribution in comparison to the gains in efficiency.

Proposition 8. *Suppose that $\lambda_i > \lambda_{-i}$ for some $i \in \{l, h\}$. Define the following thresholds*

$$\bar{w^s}_i = H^{-1}\left(U\left(w_i^f n_i^M, n_i^M\right)\right) \in \left(0, w_i^f\right), \qquad \bar{w^s}_{-i} = H^{-1}\left(U\left(w_i^f n_i^M, \frac{w_i^f}{w_{-i}^f} n_i^M\right)\right) \in \left(0, w_{-i}^f\right).$$

If $w_i^s \geq \bar{w^s}_i$ and $w_{-i}^s \leq \bar{w^s}_{-i}$, where at least one of these inequalities is strict, then type i optimally works in the shadow economy and the optimum welfare is strictly higher than in the standard Mirrlees model.

Proposition 8 generalizes the thresholds from Proposition 6. Interestingly, when the planner cares more about the more productive formally type h, these agents may end up working in the shadow economy. It may be surprising, since in the standard Mirrlees model the formal labor supply of this type is optimally either undistorted, or distorted upwards, while supplying shadow labor requires a downwards distortion. Nevertheless, if shadow economy magnifies productivity differences between types, it may be in the best interest of type h to supply only informal labor and enjoy higher transfer financed by the other type. The shadow economy in such allocation works as a tax haven, accessible only to the privileged.

3 Full model

In this section we describe the optimal tax schedule in the economy with a large number of types. Below we introduce a general taxation problem. Then we examine the requirements of incentive compatibility, which will involve the standard monotonicity condition. We proceed to characterize the optimal income tax. First we derive optimality conditions (which we call the *interior optimality conditions*) under the assumption that the monotonicity condition holds. It is a common practice in the literature on Mirrleesian taxation to stop here and verify the monotonicity numerically ex post. It is justified, since in the standard Mirrlees model the violation of the monotonicity requires rather unusual assumptions. On the other hand, the shadow economy provides an environment where the monotonicity condition is much more likely to be violated. We discuss in detail why it is the case and carry on to the optimality conditions when the monotonicity constraint is binding. The optimal allocation in this case involves bunching, i.e. some types are pooled together at the kinks of the tax schedule. We derive the optimal bunching condition with an intuitive variational method.[7] In the last subsection we summarize the main results from the full model.

3.1 The planner's problem

Workers are distributed on the type interval $[0,1]$ according to a density μ_i and a cumulative density M_i. The density μ_i is atomless. We assume that formal and informal productivities (w_i^f and w_i^s) are differentiable with respect to type and denote these derivatives by $\dot{w}_i^f$ and $\dot{w}_i^s$. It will be useful to denote the growth rates of productivities by $\rho_i^x = \frac{\dot{w}_i^x}{w_i^x}$, $x \in \{f,s\}$. Types are sorted such that the formal productivity is increasing: $\dot{w}_i^f > 0$. We will use the dot notation to write derivatives with respect to type of other variables as well. For instance, $\dot{y}_i^f$ stands for the derivative of formal income with respect to type, evaluated at some type i.

We focus on preferences without wealth effects. Agents' utility function is $U(c,n) = c - v(n)$, where v is increasing, strictly convex and twice differentiable function. We denote the inverse function of the marginal disutility from labor v' by g and the elasticity of labor supply of type i by ζ_i.[8] Let $V_i\left(y^f, T\right)$ be the indirect utility function of an agent of type i whose reported formal income is y^f and who pays a tax T:

$$V_i\left(y^f, T\right) \equiv \max_{n^s \geq 0} y^f + w_i^s n^s - T - v\left(\frac{y^f}{w_i^f} + n^s\right). \tag{15}$$

In addition to earning the formal income, the agent is optimally choosing the amount of informal labor. Due to concavity of the problem, the choice of n^s is pinned down by the familiar first order

[7] Ebert (1992) relies on the optimal control theory to derive the optimal tax when the monotonicity condition is binding. We use the more transparent variational method and develop the optimal bunching condition in the spirit of the Diamond (1998) tax formula.

[8] Since we abstract from wealth effects, the compensated and uncompensated elasticities coincide. Note that the elasticity is in general an endogenous object, as it depends on labor supply: $\zeta_i = \frac{v'(n_i)}{n_i v''(n_i)}$.

condition, modified to allow for the corner solution

$$\min \left\{ v' \left(\frac{y^f}{w_i^f} + n^s \right) - w_i^s,\ n_i^s \right\} = 0. \tag{16}$$

Whenever the formal income y^f is sufficiently high, no shadow labor is supplied. Conversely, sufficiently low formal income leads to informal employment.

The planner chooses a formal income schedule y^f and a tax schedule T in order to maximize a general social welfare function

$$\max_{\left(y_i^f, T_i\right)_{i\in[0,1]}} \int_0^1 \lambda_i G\left(V_i\left(y_i^f, T_i\right)\right) d\mu_i, \tag{17}$$

where G is an increasing and differentiable function and the Pareto weights $\lambda \in [0,1] \to \mathbb{R}_+$ integrate to 1.[9] The budget constraint is the following

$$\int_0^1 T_i d\mu_i \geq E, \tag{18}$$

where the net tax revenue needs to cover some fixed expenditures E. Moreover, the tax schedule has to satisfy incentive compatibility

$$\forall_{i,j\in[0,1]} V_i\left(y_i^f, T_i\right) \geq V_i\left(y_j^f, T_j\right), \tag{19}$$

which means that no agent can gain by mimicking any other type. The allocation which solves (17) subject to (18) and (19) is called the second-best or the optimum.

We will describe the optimum by specifying the marginal tax rate of each type. The marginal tax rate is given by the ratio of slopes of the total tax schedule and the formal income schedule

$$t_i = \frac{\dot{T}_i}{\dot{y}_i^f}. \tag{20}$$

Intuitively, it describes the fraction of a marginal formal income increase that is claimed by the planner.

3.2 Incentive-compatibility

The single crossing property allows the planner in the standard Mirrlees model to focus only on local incentive compatibility constraints. Intuitively, the single-crossing means that, given a constant tax rate, a higher type is willing to earn more than a lower type. The single-crossing in our model means that, holding the tax rate constant, the higher type is willing to earn *formally* more than the lower type.

[9] It's easy to relax the assumption of a finite Pareto weight on each type and we are going to do it in the quantitative section, where we consider, among others, the Rawlsian planner.

Assumption 1. *A comparative advantage in shadow labor is decreasing with type:* $\frac{d}{di}\left(\frac{w_i^s}{w_i^f}\right) < 0$.

Lemma 1. *Under Assumption 1, the indirect utility function* V *has the single crossing property.*

The single-crossing holds when the agents with lower formal productivity have a comparative advantage in working in the informal sector. The single-crossing allows us to replace the general incentive compatibility condition (19) with two simpler requirements.

Proposition 9. *Under Assumption 1, the allocation* $\left(y_i^f, T_i\right)_{i\in[0,1]}$ *is incentive-compatible if and only if the two conditions are satisfied:*

1. y_i^f *is non-decreasing in type.*
2. *If* $\dot{y}_i^f$ *exists, then the local incentive-compatibility condition holds:* $\frac{d}{dj}V_i\left(y_j^f, T_j\right)\Big|_{j=i} = 0$.

The utility schedule $V_i\left(y_i^f, T_i\right)$ *of an incentive compatible allocation is continuous everywhere, differentiable almost everywhere and for any* $i < 1$ *can be expressed as*

$$V_i\left(y_i^f, T_i\right) = V_0\left(y_0^f, T_0\right) + \int_0^i \dot{V}_j\left(y_j^f, T_j\right) dj, \tag{21}$$

where

$$\dot{V}_j\left(y_j^f, T_j\right) \equiv \left(\rho_j^f n_j^f + \rho_j^s n_j^s\right) v'\left(n_j\right). \tag{22}$$

The single crossing implies that for any tax schedule the level of formal income chosen by a worker is weakly increasing in the worker's type. Hence, assigning a lower income to a higher type would violate incentive compatibility. It is enough to focus just on local deviations: no agent should be able to improve utility by marginally changing the formal earnings. This local incentive-compatibility constraint is equivalent to the familiar condition for the optimal choice of the formal income given the marginal tax rate t_i, allowing for the corner solution

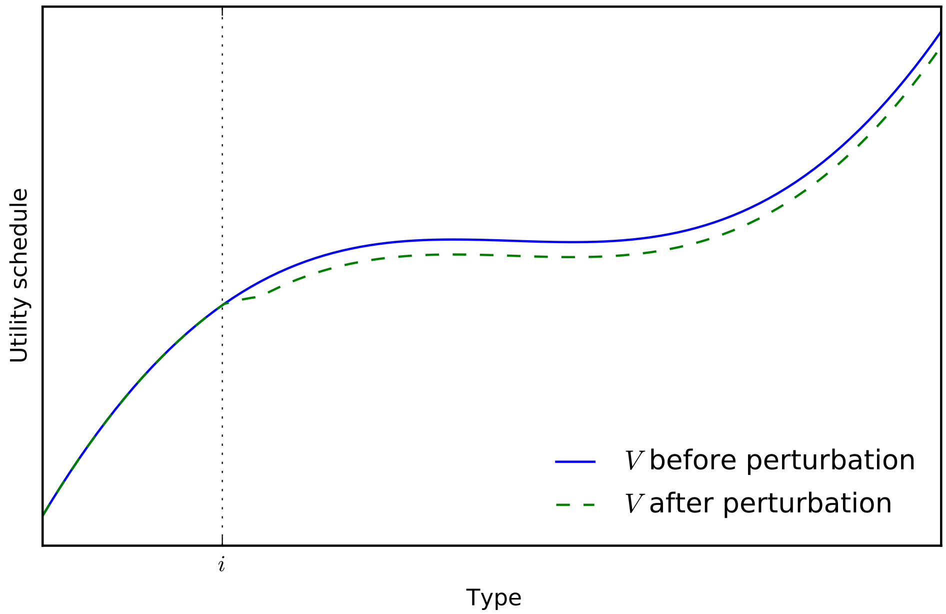
$$\min\left\{v'\left(\frac{y_i^f}{w_i^f} + n_i^s\right) - (1-t_i)\, w_i^f,\ y_i^f\right\} = 0. \tag{23}$$

Note that the formal income may be, and sometimes will be, discontinuous in type. Nevertheless, the indirect utility function preserves some smoothness and can be expressed as an integral of its marginal increments.

Let's call $\dot{V}_i\left(y_i^f, T_i\right)$ the *marginal information rent* of type i. It describes how the utility level changes with type. The higher the average rate of productivity growth, weighted by the labor inputs in two sectors, the faster utility increases with type. We will use perturbations in the marginal information rent to derive the optimal tax schedule.

In what follows we will economize on notation of the utility schedule and its slope by supressing the arguments: $V_i \equiv V_i\left(y_i^f, T_i\right)$ and $\dot{V}_i \equiv \dot{V}_i\left(y_i^f, T_i\right)$.

Figure 3: Decreasing the marginal information rent of type i



3.3 Optimality conditions

First, we solve for the optimum under assumption that the resulting formal income schedule is non-decreasing. Second, we examine when this assumption is justified and show that the existence of the shadow economy make it's violation more likely. Finally, we derive the optimality conditions in the general case.

3.3.1 Interior optimality conditions

We obtain the interior optimality conditions by making sure that the social welfare cannot be improved by perturbing the marginal information rent of any type.[10] A marginal information rent is a slope of the utility schedule at some type i. It can be reduced by increasing tax distortions of this type, which is costly for the budget. On the other hand, by (21) such perturbation shifts downwards the entire utility schedule above type i (see Figure 3). This shift is a uniform increase of a non-distortionary tax of all types above i. The interior optimality conditions balance the cost of distortions with gains from efficient taxation for each type. Below we present terms that capture the marginal costs and benefits of such perturbations. We derive them in detail in the proof of Theorem 1. The shadow economy enters the picture by affecting the cost of increasing tax distortions.

The benefit of shifting the utility schedule of type j without affecting its slope is given by the standard expression

$$N_j \equiv (1-\omega_j)\,\mu_j, \text{ where } \omega_j = \frac{\lambda_j}{\eta} G'(V_j)\,. \tag{24}$$

[10] To the best of our knowledge, Brendon (2013) was the first to use this approach in the Mirrlees model. He also inspired us to express the optimality conditions with endogenous cost terms, although our notation differs from his.

A marginal increase of non-distortionary taxation of type j leads to one-to-one increase of tax revenue. On the other hand, it reduces the social welfare, since the utility of type j falls. Following Piketty and Saez (2013) we call this welfare impact the *marginal welfare weight* and denote it by ω_j. Note that welfare impact is normalized by the Lagrange multiplier of the resource constraint η. It allows us to express changes in welfare in the unit of resources. We multiply the whole expression by the density of type j in order to include all agents of this type. We assumed that there are no wealth effects, so the non-distortionary tax does not affect the labor choice of agents. Consequently, the term N_j does not depend on whether type j works informally.

The cost of decreasing some agent's marginal information rent depends on the involvement of this agent in the shadow activity. Types can be grouped into three sets:

$$
\begin{aligned}
&\text{formal workers:} && \mathcal{F} \equiv \left\{ i \in [0,1] : v'\left(n_i^f\right) > w_i^s \right\}, \\
&\text{marginal workers:} && \mathcal{M} \equiv \left\{ i \in [0,1] : v'\left(n_i^f\right) = w_i^s \right\}, \\
&\text{shadow workers:} && \mathcal{S} \equiv \left\{ i \in [0,1] : v'\left(n_i^f\right) < w_i^s \right\}.
\end{aligned}
$$

The formal workers supply only formal labor: their marginal disutility from working is strictly greater than their shadow productivity. The marginal workers also supply only formal labor, but their marginal disutility from work is exactly equal to their shadow productivity. A small reduction of formal labor supply of these agents would make them work in the informal sector. Finally, the shadow workers are employed informally, although they can also supply some formal labor.

The formal workers act exactly like agents in the standard Mirrlees model. By increasing distortions, the planner is reducing their total labor supply. The cost of increasing distortions is given by

$$
D_i^f \equiv \frac{t_i}{1-t_i} \left(\rho_i^f \left(1 + \frac{1}{\zeta_i} \right) \right)^{-1} \mu_i. \tag{25}
$$

The cost depends positively on the marginal tax rate. The marginal tax rate tell us how strongly a reduction of the formal income influences the tax revenue. Moreover, the cost increases with the elasticity of labor supply ζ_i and is proportional to the density of the distorted type. D_i^f is endogenous, as it depends on the marginal tax rate.

The perturbation of the marginal information rent works differently for the shadow workers. They supply shadow labor in the quantity that satisfies $v'\left(n_i^f + n_i^s\right) = w_i^s$, which means that their total labor supply n_i is constant. By distorting the formal income, the planner simply shift their labor from the formal to the informal sector. As a result, the cost of increasing distortions does not depend on the elasticity of labor supply, but rather on the sectoral productivity differences,

$$
D_i^s \equiv \frac{w_i^f - w_i^s}{w_i^s} \left(\rho_i^f - \rho_i^s \right)^{-1} \mu_i. \tag{26}
$$

The first term is the relative productivity difference between formal and informal sector. Actually, it's also equal to $\frac{t_i}{1-t_i}$, since the marginal tax rate of these types equalizes the return to labor in both sectors: $(1-t_i)\, w_i^f = w_i^s$. Hence, as in the case of formal workers, the first term corresponds to the direct tax revenue cost of reduced formal labor supply. The second term describes how effectively the planner can manipulate the agent's marginal information rent by discouraging the formal labor. By the single-crossing assumption, this term is always positive. Again, the density μ_i aggregates the expression to include all agents of type i. Note that D_i^s is exogenous, as it depends only on the fundamentals of the economy.

The marginal workers are walking a tightrope between their formal and shadow colleagues. If the planner marginally reduces their income, they become the shadow workers. If the planner lifts distortions, they join the formal workers. The cost of changing distortions of these types depends on the direction of perturbation and is equal to either D_i^f or D_i^s.

Having all the cost and benefit terms ready, we can derive the interior optimality conditions. Recall, that by varying the distortions imposed on some type, the planner changes a non-distortionary tax of all types above. In the optimum, the planner cannot increase the social welfare by such perturbations. For the formal workers, this means that

$$\forall_{i\in\mathcal{F}}\ D_i^f = \int_i^1 N_j dj. \tag{27}$$

It is a standard optimality condition from the Mirrlees model, derived first in the quasilinear case by Diamond (1998). The shadow economy does not affect the marginal tax rate of formal agents directly. It may influence them only indirectly, by changing the marginal welfare weights of types above.

For the marginal workers it must be the case that increasing tax distortions is beneficial as long as they work only formally, but it is too costly when they start to supply the shadow labor.

$$\forall_{i\in\mathcal{M}}\ D_i^s \geq \int_i^1 N_j dj \geq D_i^f \text{ and } y_i^f = w_i^f g\left(w_i^s\right). \tag{28}$$

The marginal workers do not supply informal labor, but in their case the shadow economy constitutes a binding constraint for the planner. Absent the shadow economy, the marginal tax rates would be set at a higher level. In our model the planner is not willing to do it, because it would push the marginal workers to informal jobs, which is too costly. Formal labor supply of the marginal workers is fixed at the lowest level that leaves them no incentives to work informally.

Recall that the cost of distorting the shadow worker is fixed by the parameters of the economy. Moreover, the benefit of distorting one particular worker, given by (24), is fixed as well, since the perturbation of the marginal information rent of i has an infinitesimal effect on the utility of types above. If the planner finds it optimal to decrease the formal income of agent i so much that i starts supplying informal labor, it will be optimal to decrease the formal income all the way to zero, when i works only in the shadow economy:

$$\forall_{i\in\mathcal{S}}\ \int_i^1 N_j dj > D_i^s \text{ and } y_i^f = 0. \tag{29}$$

Note that according to this condition all shadow workers are bunched together at zero formal income.[11]

The optimality conditions (27)-(29) determine the slope of the utility schedule at each type. What is left is finding the optimal level. Suppose that the planner varies the tax paid by the lowest type, while keeping all the marginal rates fixed. Optimum requires that such perturbation cannot improve welfare:

$$\int_0^1 N_j dj = 0. \tag{30}$$

Definition. The conditions (27)-(30) are called the *interior optimality conditions.* The allocation $\left(y^f, T\right)$ consistent with the interior optimality conditions is called the *interior allocation.* Specifically, y^f is called the *interior formal income schedule.*

The interior conditions are necessary for the optimum as long as they don't imply a formal income schedule which is locally decreasing. They become sufficient, if they pin down a unique allocation. This happens when the cost of distortions is increasing in the amount of distortions imposed. When that is the case, the planner's problem with respect to each type becomes concave. Theorem 1 provides regularity conditions which guarantee it.

Assumption 2. *(i) The elasticity of labor supply* $\frac{v'(n)}{nv''(n)}$ *is non-increasing in* n*. (ii) The ratio of sectoral growth rates is bounded below* $\forall_i \frac{\rho_i^s}{\rho_i^f} > -\zeta_i^{-1}$.

Theorem 1. *Under Assumption 1, if all interior formal income schedules are non-decreasing, the interior optimality conditions are necessary for the optimum. Under Assumptions 1 and 2, there is a unique interior formal income schedule. If it is non-decreasing, the interior optimality conditions are both necessary and sufficient for the optimum.*

3.3.2 When do the interior conditions fail?

The interior allocation is incentive-compatible and optimal if it leads to formal income that is non-decreasing in type. In the standard Mirrlees model formal income is decreasing if the marginal tax rate increases too quickly with type. However, in virtually all applications of the standard Mirrlees model this is not a problem, as the conditions under which the interior tax rate increases that fast are rather unusual.[12] The shadow economy gives rise to another reason for non-monotone interior formal income. In the interior allocation all shadow workers have zero formal income. Hence, if there is any worker with positive formal income with a type lower than some shadow worker, the formal income schedule will be locally decreasing. It turns out that this second reason makes the failure of the interior allocation much more likely. In Proposition 10 below we provide the sufficient conditions for the formal income to be non-decreasing. Then we discuss the two cases in which the shadow economy leads to the failure of the interior optimality conditions.

[11] Notice that we could replace the strict inequality with a weak one in (29), and conversely regarding the left inequality in (28). In words, when the cost of distorting some marginal worker is exactly equal to the benefit, then this worker could equally well be a shadow worker, with no change in the social welfare. It means that whenever the curves D_i^s and $\int_i^1 N_j dj$ cross, the optimum is not unique, since we could vary allocation of the type at the intersection. Since such a crossing is unlikely to happen more than a few times, we do not consider this as an important issue. We sidestep it by assuming that the planner introduces distortions only when there are strictly positive gains from doing so. Consequently, our notion of uniqueness of optimum should be understood with this reservation.

[12] Probably simplest way to construct an example of locally decreasing formal income schedule is to assume a bimodal productivity distribution, with very low density between the modes.

Assumption 3. *(i) The social welfare function is such that* $G(V) = V$, λ_i *is non-decreasing in type for* $i > 0$. *(ii) The ratio* $\frac{1}{\rho_i^f}\frac{\mu_i}{1-M_i}$ *is non-decreasing in type. (iii) The elasticity of labor supply is constant:* $\forall_i \zeta_i = \zeta$. *(iv) The ratio of sectoral growth rates* $\frac{\rho_i^s}{\rho_i^f}$ *is non-decreasing in type.*

Proposition 10. *Under Assumptions 1, 2 and 3, the unique interior formal income schedule is non-decreasing.*

First, notice that we make sure that the interior formal income schedule is unique (Assumption 2). Simultaneously, it implies that the formal income of the marginal workers is non-decreasing. Assumptions 3(i) - 3(iii) make sure that the marginal tax rate of formal workers is non-increasing in type, which in turn implies that the formal income of these workers is non-decreasing. These conditions are familiar from the standard Mirrlees model. Assumption 3(i) is satisfied by the utilitarian or Rawlsian social welfare function, while Assumption 3(ii) is a weaker counterpart of the usual monotone hazard ratio requirement.[13]

Finally, we have to make sure that all shadow workers, if there are any, are at the bottom of the type space. By (29) it means that the marginal cost of distorting the shadow worker D_i^s can cross the marginal benefit $\int_i^1 N_j dj$ at most once and from below. It is guaranteed jointly by conditions 3(i), 3(ii) and the new requirement 3(iv) which says that the ratio of sectoral productivity growth rates is non-decreasing. In addition to assuring the optimality of the interior allocation, Assumption 3 imply also that sets $\mathcal{S}$,$\mathcal{M}$ and $\mathcal{F}$, if non-empty, can be ordered: the bottom types are the shadow workers, above them are the marginal workers, and the top types are formal.

Assumption 2 makes sure that the D_i^s curve crosses the $\int_i^1 N_j dj$ curve at most once. Let's see how the relaxation of some of its elements make these curves cross more than once. In Example 1 we relax the assumpion on the social welfare function and in Example 2 we allow the non-monotone ratio of sectoral growth rates.

Example 1. (i) The social welfare function is such that $G(V) = V$, the Pareto weights λ_i are continuous in type and satisfy $\lambda_0 > 2$. (ii) The distribution of types is uniform. (iii) The elasticity of labor supply is constant: $\forall_i \zeta_i = \zeta$ and $v'(0) = 0$. (iv) The ratio of sectoral growth rates $\frac{\rho_i^s}{\rho_i^f}$ is fixed. (v) Assumptions 1 and 2 are satisfied.

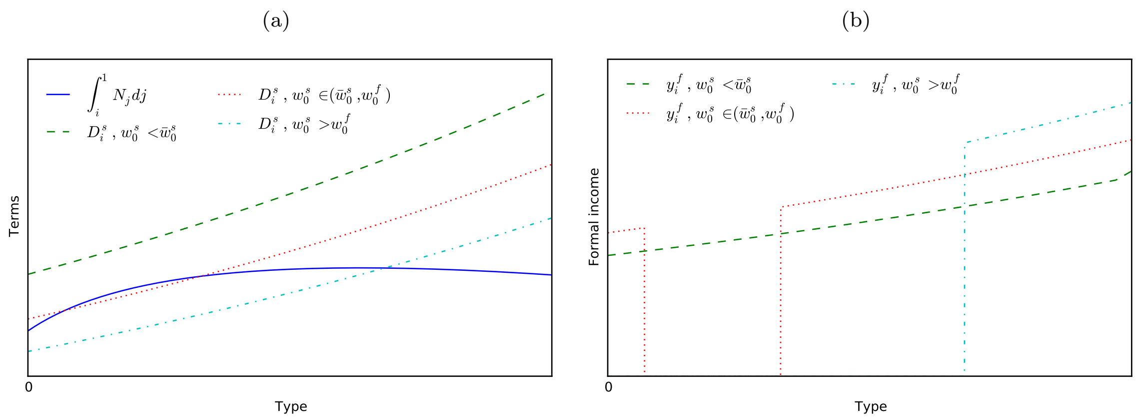
Lemma 2. *In Example 1 there is a threshold* $\bar{w}_0^s \in \left(0, w_0^f\right)$ *such that if* $w_0^f > w_0^s > \bar{w}_0^s$ *the interior formal income schedule* ***is not*** *non-decreasing.*

Example 1 violates Assumption 3 (i), which allows the $\int_i^1 N_j dj$ term to be initially increasing in type.[14] Both terms D_i^s and $\int_i^1 N_j dj$ are increasing at 0, but $\int_i^1 N_j dj$ term increases faster. If $w_0^f > w_0^s$, then the distortion cost at type 0 is greater than the benefit and the bottom type works formally. If the gap between w_0^f and w_0^s is sufficiently small (smaller than $w_0^f - \bar{w}_0^s > 0$), D_i^s curve

[13]We can express the distribution of types as a function of formal productivity rather than type. Then the density is $\bar{\mu}\left(w_i^f\right) = \frac{\mu_i}{\dot{w}_i^f}$ and cumulative density is $\bar{M}\left(w_i^f\right) = M_i$. Hence, assumption 3(ii) means that $\frac{w^f \bar{\mu}\left(w^f\right)}{1-\bar{M}\left(w^f\right)}$ is non-decreasing. For instance, any Pareto distribution of formal productivity satisfies this assumption.

[14]The Pareto weights integrate to 1 over the type space, so they have to be lower than or equal to 1 for some types above 0. Since these weights are continuous and $\lambda_0 > 2$, they will be decreasing for some type above 0, violating 3(i).

Figure 4: A failure of the interior allocation due to increasing benefit of distortions $\int_i^1 N_j dj$ (Example 1).



will cross the benefit curve at some positive type (see Figure 4). Consequently, the agents above the intersection will work in the shadow economy. Since these agents have no formal income, the formal income schedule is locally decreasing.

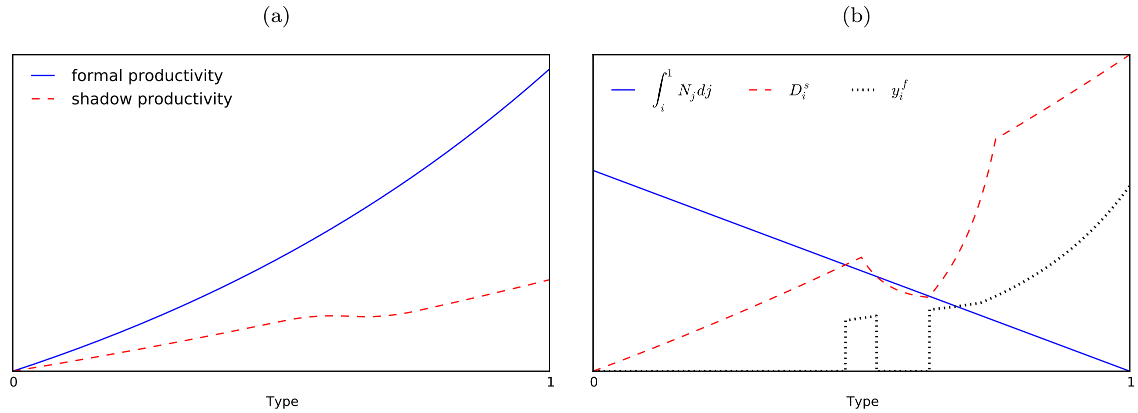
Example 2. (i) The social welfare function is Rawlsian: $\forall_{i>0}\lambda_i = 0$. (ii) The distribution of types is uniform. (iii) The elasticity of labor supply is constant: $\forall_i \zeta_i = \zeta$. (iv) The growth rate of formal productivity is fixed, while the growth rate of shadow productivity is decreasing for some types. (v) Assumptions 1 and 2 are satisfied.

Example 2 satisfies all the requirements of Proposition 10 apart from the non-decreasing sectoral growth rates ratio assumption. In panel (a) of Figure 5 we can see that the growth rate of shadow productivity decreases around the middle type and then bounces back. It is reflected in the marginal cost of distorting shadow workers D_i^s (panel (b)). We chose the parameters such that the fall is substantial, making the D_i^s curve cross the $\int_i^1 N_j dj$ curve three times. Consequently, the formal income first increases, then decreases to 0 once the D_i^s crosses $\int_i^1 N_j dj$ for the second time. This example shows that even minor irregularities in the distribution of productivities can make the interior allocation not implementable.

3.3.3 Optimal bunching

Whenever the interior formal income schedule is decreasing for some types, the interior allocation is not incentive-compatible and hence is not optimal. Ebert (1992) and Boadway, Cuff, and Marchand (2000) applied the optimal control theory to overcome this problem. In contrast to these papers, we derive the optimal bunching condition with the intuitive variational argument and express it in the spirit of the Diamond (1998) optimal tax formula. What we are going to do is essentially "ironing" the formal income schedule whenever it is locally decreasing (see Figure 6). The ironing was originally introduced by Mussa and Rosen (1978) in a solution to the monopolistic pricing problem when the monotonicity condition is binding.

Figure 5: A failure of the interior allocation due to non-monotone ratio of productivity growth rates (Example 2).



Suppose that the interior formal income schedule $\bar{y}^f$ is decreasing on some set of types, beginning with $\bar{a}$. Decreasing formal income is incompatible with the incentive-compatibility. We can regain incentive-compatibility by lifting the schedule such that it becomes overall non-decreasing and flat in the interval $\left[\bar{a}, \bar{b}\right]$ (see Figure 6). Since types $\left[\bar{a}, \bar{b}\right]$ have the same formal income, they are bunched and cannot be differentiated by the planner. Such bunching is implemented by a discontinuous jump of the marginal tax rate.

The flattened schedule is incentive-compatible. However, generally it is not optimal. By marginally decreasing formal income of type $\bar{a}$ the planner relaxes the binding monotonicity constraint and can marginally decrease the formal income of all types in the interval $\left(\bar{a}, \bar{b}\right)$. This perturbation closes the gap between the actual formal income and its interior value for the positive measure of types. On the other hand, the cost of perturbation is infinitesimal: it is a distortion of one type $\bar{a}$. This perturbation is clearly welfare-improving, starting from the flattened interior schedule. Below we find the optimal bunching condition by making sure that the perturbation is not beneficial at the optimal income schedule.

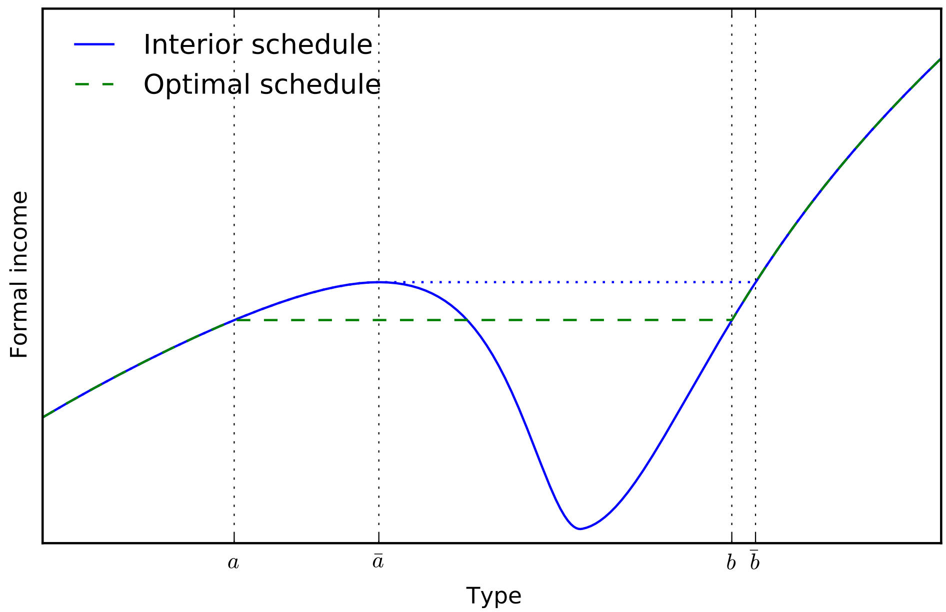
Suppose that an interval of agents $[a, b]$ is bunched. Let's marginally decrease the formal income of agents $[a, b)$ and adjust their total tax paid such that the utility of type a is unchanged. In this way we preserve the continuity of the utility schedule. However, since the other bunched agents have a different marginal rate of substitution between consumption and income, this perturbation will decrease their utility. We normalize the perturbation such that we obtain a unit change of the utility of the highest type in the bunch. The total cost of this perturbation is given by

$$D_{a,b} \equiv \left(t_a + \mathbb{E}\left\{\Delta MRS_i \omega_i \middle| \, b > i \geq a\right\}\right) \frac{M_b - M_a}{t_{b^+} - t_{a^-}}, \tag{31}$$

$$\text{where } \Delta MRS_i = \frac{v'\left(n_a\right)}{w_a^f} - \frac{v'\left(n_i\right)}{w_i^f}.$$

The expression within the brackets is an average impact of a unit perturbation of the formal income. The brackets contain two components: a fiscal and a welfare loss. The fiscal loss from reducing the

Figure 6: Ironing the formal income schedule



formal income of each bunched agent is the marginal tax rate below the kink. The welfare loss is an average marginal welfare weight in the bunch corrected by a discrepancy of the marginal rate of substitution of a given type from type a. The larger ΔMRS_j is, the more type j suffers from the perturbation. Note that ΔMRS_b is just equal $t_{b^+} - t_{a^-}$.[15] Hence, in order to normalize the perturbation to have a unit impact on utility of type b, we divide the brackets by $t_{b^+} - t_{a^-}$. We aggregate this average effect by multiplying it by the mass of bunched types.

The benefit of this perturbation comes from the reduced utility of types above b and is the same as in the interior case. The optimality requires that

$$\min\left\{\int_b^1 N_j dj - D_{a,b},\ y_a^f\right\} = 0. \tag{32}$$

Note that the optimality condition involves a corner solution when $y_a^f = 0$. It corresponds to the situation in which the bunched workers don't work formally at all.

The optimality condition (32) is influenced by the shadow economy again through the cost of distortion. If some worker i in the bunch $[a, b)$ supplies shadow labor, then the difference in the marginal rate of substitution for this worker is given by $\Delta MRS_i = \frac{v'(n_a)}{w_a^f} - \frac{w_i^s}{w_i^f}$.

Theorem 2 combines all the optimality conditions.

Theorem 2. *Under Assumption 1, the optimal allocation satisfies (30) and at each level of formal income one of the three mutually exclusive alternatives hold:*

[15] The marginal tax rate discontinuously increases at the kink. By t_{a^-} we denote the tax rate below the kink and by t_{b^+} the tax rate above the kink.

- *there is no type that reports such formal income,*
- *there is a unique type whose allocation satisfies the interior optimality conditions (27)-(29),*
- *there is a bunch of types whose allocation satisfy the optimal bunching condition (32).*

Although we managed to characterize the full set of optimality conditions, the interior conditions are generally easier to use. Below we show that the interior allocation, even if not incentive-compatible, are a good predictor of which agents optimally work in the shadow economy.

Assumption 4. *(i) G is a concave function. (ii) ρ_i^f, ρ_i^s, μ_i and λ_i are continuous in type.*

Proposition 11. *Under Assumptions 1, 2 and 4, all the types that supply shadow labor in the interior allocation remain the shadow workers in the optimum.*

3.4 Summary of results

Which agents should work in the shadow economy?

Corollary 2. *Suppose that $v'(0) = 0$. Under Assumptions 1, 2 and 4 type i optimally works in the shadow economy if*

$$\mathbb{E}\left\{1-\omega_j \middle| j>i\right\} \geq \frac{w_i^f - w_i^s}{w_i^f}\left(-\frac{d}{di}\left(\frac{w_i^s}{w_i^f}\right)\right)^{-1}\frac{\mu_i}{1-M_i}. \tag{33}$$

This condition is both necessary and sufficient if the interior allocation is incentive-compatible.

The inequality (33) compares the gains from efficient taxation of all types above i with the cost of distorting type i, when this type is at the edge of joining the shadow economy. A type i is likely to optimally work in the shadow economy if the planner on average puts a low marginal welfare weights on the types above i, the relative productivity loss from moving to informal employment is low and the density of distorted types is low in comparison to the fraction of types above. Finally, the shadow employment is more likely if the comparative advantage of working in the shadow sector $\frac{w_i^s}{w_i^f}$ is quickly decreasing with type. It means that higher types have less incentives to follow type i into the shadow economy. We assume $v'(0) = 0$ so that we do not have to worry about some types not supplying any labor at all.

Note that with the Rawlsian planner the inequality (33) is just a continuous equivalent of the condition (10) from the simple model.

The optimal tax rates. Let's focus on agents that supply some formal labor and are not bunched at the kinks of the tax schedule. These types never supply informal labor. The optimal tax formula is

$$\frac{t_i}{1-t_i} = \min\left\{\frac{w_i^f - w_i^s}{w_i^s}, \rho_i^f\left(1+\frac{1}{\zeta_i}\right)\frac{1-M_i}{\mu_i}\mathbb{E}\left(1-\omega_j \middle| j>i\right)\right\}. \tag{34}$$

The shadow economy imposes an upper bound on the marginal tax rate. The bound (the left term in the min operator of (34)) is such that the tax rate equalizes the return from formal and informal labor - it is the highest tax rate consistent with agents working in the formal sector.

If the bound is not constraining the planner, then the tax rate should be set according to Diamond (1998) formula (the right term in the min operator of (34)). The expectations describe the average social preferences towards all types above i. In general, the less the planner cares about increasing utility of the types above i, the higher t_i will be. If the Pareto weights increase with type or G is a strictly convex function, this term may become negative, leading to negative marginal tax rates, as explained by Choné and Laroque (2010). Since the sign of the tax rate is ambiguous, below we describe how the other terms influence its absolute value. The optimal tax rate increases in absolute value when the growth rate of formal productivity with respect to type is high. If the planner is redistributive and types above i are much more productive than types below, it is optimal to set a high tax rate. The tax rate decreases with elasticity of labor supply ζ_i, as it makes workers more responsive to the tax changes. The ratio $\frac{1-M_i}{\mu_i}$ tells us how many agents will be taxed in a non-distortionary manner relative to the density of distorted agents. If this ratio is high, the gain from increasing tax rates relative to the cost will be high as well.

Optimal bunching. Bunching may arise at the bottom of the formal income distribution, resulting in de facto exclusion from the formal labor market. Bunching may also appear at a positive level of formal income, which implies a kink in a tax schedule. All workers who supply shadow labor are subject to bunching, though not necessarily at the same tax kink. Some workers supplying only formal labor can be found at the kinks as well. The formal income schedule at which the kink is located is determined by

$$\frac{t_{a^-}}{t_{b^+} - t_{a^-}} = \frac{1 - M_b}{M_b - M_a}\mathbb{E}\left\{1 - \omega_j \middle| j \geq b\right\} - \mathbb{E}\left\{\Delta MRS_i w_i \middle| b > i \geq a\right\}, \tag{35}$$

where a and b are respectively the lowest and the highest type bunched at the kink. Note that both t_{a^-} and t_{b^+}, the tax rates below and above the kink, are set according to (34). The location of the kink is determined by the trade-off between tax and welfare losses from the bunched agents and the tax revenue gains from the efficient taxation of agents above the kink.

4 Shadow and formal productivities in the data

In this section we estimate the empirical counterparts of the three key objects of the model: the formal productivity (w_i^f), the informal productivity (w_i^s) and the distribution of types (μ_i). The estimation is conducted with data from Colombia, a country with a large shadow economy.[16] We also recover the current labor tax scheme for Colombia and use it to construct the pre-labor tax

[16] 58% of the workers are part of the shadow economy according to our estimates. Estimates of the size of the shadow economy by the official statistical agency in Colombia are close to ours but the definition is different as they use the scale of the firm or business to determine informality instead of the taxation criteria we use here.

income from reported income. In section 5 we go further and analyze the differences between the current and the optimal tax scheme obtained with the estimated model.

In the model, w_i^f and w_i^s correspond to the pre-tax (real) income for one unit of labor for individual i in each sector. In the data this corresponds to the wage that a given individual can get in each sector of the economy.[17] If the wage every individual can get in each sector were observable then it would be enough to order individuals according to their formal (pre-tax) wage, such that w_i^f is increasing in i. With this order of individuals and the observed wages it is straightforward to compute w_i^f, w_i^s and μ_i.

Extracting these theoretical objects from the data is difficult because workers endogenously sort themselves into each sector. This implies that we only observe the wage of a worker at one sector and the distribution of workers across sectors depends on the wages itself. Therefore, our sample of formal and informal wages is not randomly drawn from the population, hindering the use of simple identifying assumptions to recover the productivity profiles. If the workers with high formal productivity tend to be in the formal sector such approach would lead to a positive bias on the estimated difference in productivities between sectors; a selection bias.

To solve these difficulties, we estimate a model with a factor that can explain most of the variability of wages in both sectors and use this factor to order individuals in the population. The factor we use is a linear combination of worker characteristics and job characteristics, such as the education level and the task done in the job. The estimated formal and shadow productivities correspond to the predicted wage in each sector by the factor model.

The factor cannot account for the income dispersion of the top earners and the gap with respect to the rest of the population. We extend our identification strategy estimating a Pareto distribution for the wages of top earners in the formal sector. We assume that the relation between formal and shadow productivity in the main part of the distribution of types holds also at the upper tail. Specifically, it means that if the single crossing property holds (w_i^f/w_i^s is increasing in type) for non-top earners then it also holds for top earners.

We find that both productivity estimates are increasing in type and that the single-crossing property is satisfied. Specifically, the wedge between the productivity levels of each sector is almost zero for the least productive agents and increases rapidly as the formal productivity increases. The main novelty of this section is that we assess the differences between the formal and the shadow economy at the worker level; controlling for the sorting of workers. Productivity as measured in La Porta and Shleifer (2008) can come also from the worker characteristics and not only from the type of firms or jobs in each sector. With our approach we are able to discuss the wage differential across sectors for a given worker and job. On the other hand, the mapping of our estimates to productivity levels depends on the structure of the labor and goods market, because we rely on data on wages rather than quantities produced or profits of the firm; as those other studies do. For the purposes of this paper this is not important since our object of interest is the income of the worker in each sector. Our results can shed light on the productive structure of the two sectors once the link between wages and productivity is specified.[18]

[17] Or equivalently income if she is a self-account worker.

[18] For example, if the perfect competition in the labor market is assumed, then our measure corresponds directly

The remaining of the section is organized as follows: first, we present the data and show how we identify informal workers. Second, the empirical specification is presented and last, the results are shown and discussed.

4.1 Data

All the information we use in this section is obtained from the household survey done by the official statistical agency in Colombia (DANE). Our sample is for the year 2013 and we have 170.000 observations of workers. The sample includes personal information such as age, gender, years of education and also labor market related variables including hours worked, number of jobs, type of job, income sources and social security affiliation. All of the information is reported by the worker.

The variables we use from the survey can be grouped in 4 categories: worker characteristics, job characteristics, worker-firm relationship and social security status. A linear combination of the variables in the first three categories is used to construct a factor that captures the variability of wages. The fourth set category is used to classify individuals as formal or informal workers. Below we provide a brief description of the variables included in each category, for more detailed information see Appendix B.

Worker characteristics capture the type of worker irrespectively of the job he is currently doing. They include: age, gender, education level and work experience in previous jobs.

Job characteristics characterize the type of job and task that the worker does irrespectively of the characteristics of the worker. The variables included are: number of workers in the firm (size), industry to which the firm belongs, geographical location of the firm and the task the worker has to do.

Worker-firm relationship characterize the information about the type of contract and the wage determination. The variables included here are: The wage of the worker, number of working hours, the length of the match, if the worker is hired through an intermediary firm and if the worker belongs to a union.

Social security status determine if the worker is affiliated to social security in its different dimensions and the type of affiliation. The variables included are: affiliation to the health system, the pension system and the labor accidents insurance; also who pays for the affiliation to each component.

We use the sampling weights, that correspond to each observation in the survey, in the estimation of productivities and the density of types to make our estimates representative of the Colombian economy.

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to the worker marginal productivity. If a production function with constant returns to scale is also assumed, then our measure also reflects the average productivity of the worker.

Classification of workers into formal and shadow workers

Workers are not directly asked whether they belong to the formal or the shadow economy, or alternatively whether they pay or not the labor income taxes. Therefore, we have to rely on indirect measures. We use the survey questions related to the compliance with the labor regulation. Specifically, the affiliation to the health security system, the pension system and the accidents insurance policy. The criteria we use to identify a formal worker is the affiliation to the three components through his work (not as a beneficiary of other worker). The affiliation status of a worker, in these three components of the social security system, is expected to be highly effective to identify which workers do not comply with labor taxes. The affiliation fees are linked administratively with the payment of the payroll tax for employees and the income tax for self account workers.

4.2 Empirical specification

The logarithm of both productivities (w_i^f and w_i^s) can be written as a function of a single factor F_i as follows

$$\log\left(w_i^f\right) = \gamma_0^f + F_i \tag{36}$$
$$\log\left(w_i^s\right) = \gamma_0^s + \gamma_1^s F_i \tag{37}$$

where γ_0^j, γ_1^j characterize the linear function in sector $j \in \{f, s\}$. We have set $\gamma_1^f = 1$ without loss of generality, given that this will just rescale the factor.

The factor is a linear combination of a set of n variables contained in vector X_i with weights given by the vector β. Then we have that

$$F_i = \beta X_i \tag{38}$$

The proxy we have for the model productivities are the wages of workers $\tilde{w}_i^j$ in each sector j, then we have that[19]

$$\log\left(\tilde{w}_i^f\right) = \log\left(w_i^f\right) + u_i^f \tag{39}$$
$$\log\left(\tilde{w}_i^s\right) = \log\left(w_i^s\right) + u_i^s \tag{40}$$

where u_i^f and u_i^s are random variables with mean zero. Wages are drawn from a probability distribution where the key location parameters are w_i^f and w_i^s,the theoretical concepts in our analysis. In the theoretical analysis we abstract from the underlying variance of the distribution and focus on the limit when it tends to zero. The model is a static economy so we are not concerned with short term variations of wages but rather on the distribution of the location parameters across the population.

Combining equations (36) to (40) we get the specification of the empirical model that corresponds

[19]Note that, as discussed earlier, w_i^j is only observed if type i works in sector j.

to

$$\log\left(\tilde{w}_i\right) = \gamma_0^f + I_i\left(\gamma_0^s - \gamma_0^f\right) + \left(1 + I_i\left(\gamma_1^s - 1\right)\right)\beta X_i + u_i \tag{41}$$

where I_i is an indicator function that takes the value of 1 if type i works in the shadow economy and $u_i = I_i u_i^s + u_i^f$. We estimate 41 by non-linear least squares.

Ordering of agents and estimated productivities

Note the estimate of parameter a as $\hat{a}$. We proceed to order the individuals in our sample with indexes $i \in [0,1]$ such that $i < i' \iff \hat{\beta}X_i < \hat{\beta}X_{i'}$. We compute the index of each individual using the following formula

$$i = \frac{\hat{\beta}X_i - min_{i'}\{\hat{\beta}X_{i'}\}}{max_{i'}\{\hat{\beta}X_{i'}\}}$$

that is just rescaling the factor using the minimum and the maximum values it takes in the sample. The estimated productivities of each type i then correspond to

$$\hat{w}_i^f = \exp\left\{\hat{\gamma}_0^f + \hat{\beta}X_i\right\} \tag{42}$$

$$\hat{w}_i^s = \exp\left\{\hat{\gamma}_0^s + \hat{\gamma}_1^s\hat{\beta}X_i\right\}. \tag{43}$$

Single-crossing condition

The single-crossing condition states that the ratio w_i^f / w_i^s is increasing in type. Using (42) and (43) this ratio can be written as

$$\frac{\hat{w}_i^f}{\hat{w}_i^s} = \exp\left\{\hat{\gamma}_0^f - \hat{\gamma}_0^s\right\}\exp\left\{\left(1 - \hat{\gamma}_1^s\right)\hat{\beta}X_i\right\}.$$

Then, if $\hat{\gamma}_1^s < 1$ holds, the single-crossing condition is satisfied. Recall that we standardized to 1 the marginal (percentile) increase of formal productivity to a marginal increase in the factor. Therefore, this condition states that a marginal increase in the factor has to imply a lower marginal increase in shadow than in formal productivity.

Top income earners

Note that since $\tilde{w}_i$ is in units of year income for full time work, then $\hat{w}_i^f$ corresponds (on average) to the maximum income that type i can achieve. Nevertheless, some income observations are above the maximum value implied by the factor for the most productive worker working full time. That is, there could be yearly labor income observations y_i that satisfy

$$y_i > max_{i'}\{\hat{w}_{i'}^f\} = \hat{w}_1^f \tag{44}$$

We classify the individuals that satisfy this criterion as top earners. These are individuals with a very large wage premium that cannot be accounted for with our benchmark specification and for which the wage does not seem to have the same relationship with the factor as for the rest of the population.

To characterize with more accuracy this behavior at the top of the income distribution we estimate the upper tail of the productivity distribution by fitting a Type I Pareto distribution for the gross wage $\tilde{w}$ of top earners. The support of the distribution is given by $\left[\hat{w}_1^f, \infty\right)$ and the shape parameter is estimated by maximum likelihood.

A final adjustment has to be made to the index of agents. To fit the top earners in the type space $[0, 1]$ we compress the indexes on non-top earners to the interval $[0, k]$ and top earners are assigned to $[k, 1]$ and ordered by their gross wage.

Distribution of types

The assignment of indexes for each observation and their corresponding sampling weights implies a discrete distribution of workers (non-top earners). The continuous distribution of types is obtained by a kernel density estimation with a linear interpolation at the evaluation points. The estimated kernel distribution gives us the distribution of types in the interval $[0, k]$.

There are many (potentially a continuum) probability distributions and formal productivity profiles for the types in the interval $[k, 1]$ (top earners). For top earners we have a Pareto distribution for productivities with the support $[max_{i\prime}\{\hat{w^f}_{i\prime}\}, \infty)$ but this distribution can be replicated by different types distributions in $[k, 1]$ at the types space, provided that the formal productivities w_i^f for $i \in [k, 1]$ are adjusted accordingly. This phenomenon does not occur with non-top earners because their productivity profiles are given by our parametric model.

There are two requirements that the distribution of types and productivity profiles of top earners satisfy always: the total mass of the distribution has to coincide with the mass of top earners and that $\lim_{i \to 1} w_i^f = \infty$.

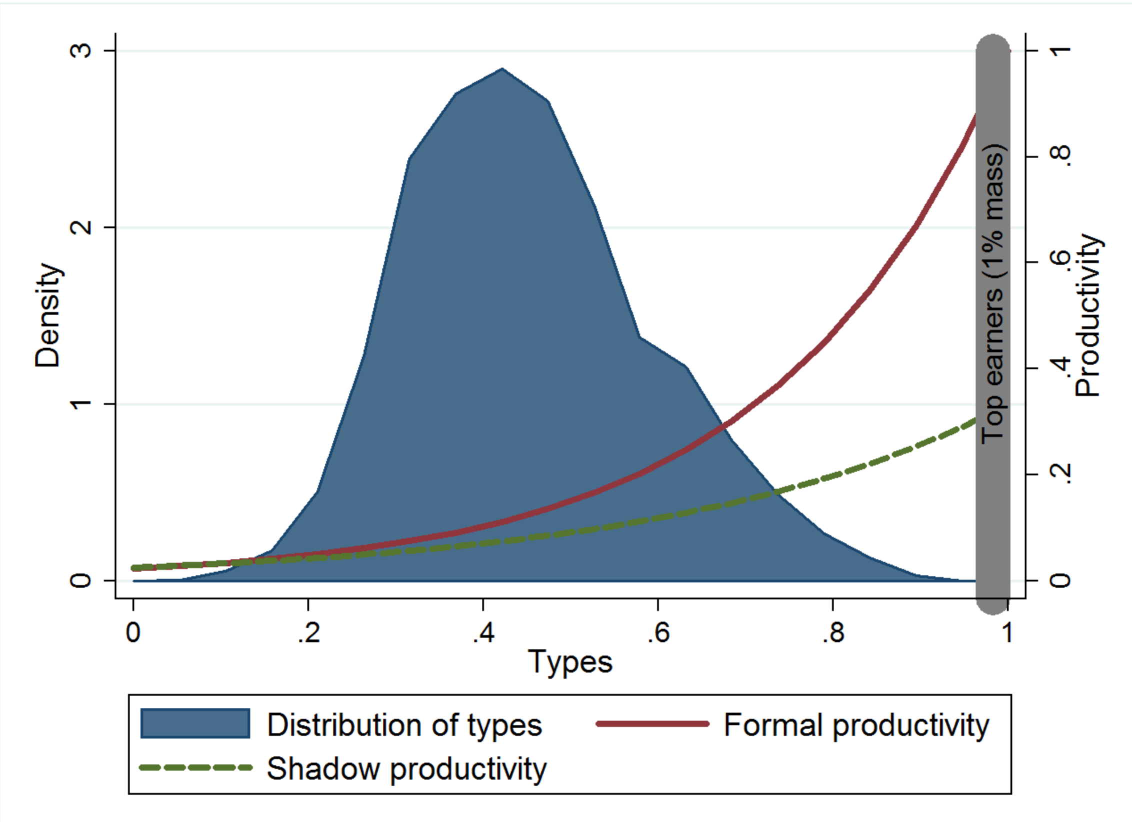
4.3 Estimation results

Here we discuss the results of the estimation of the formal productivity (w_i^f), the informal productivity (w_i^s) and the distribution of types (μ_i). Parameter estimates for β and the detailed description of the variables included in X_i are presented in Appendix B.

Figure 7 presents the estimated productivities and the types distribution for non-top earners. The estimated values of γ_0^f and γ_0^s are almost identical with $\hat{\gamma}_0^s$ slightly greater so type 0 is slightly more productive in the shadow economy. The single-crossing condition is supported by the data since the hypothesis $\gamma_1^s < 1$ is not rejected at a 1% confidence level. The most productive individual among non-top earners is almost three times more productive in the formal economy than in the shadow economy.

Top earners are assigned to the set $[0.98, 1]$, the estimated value of the shape parameter of the Pareto distribution is 1.81 and comprise a mass of about 1% of the total population (details of the estimation are presented in Appendix B). The shaded region in Figure 7 corresponds to the top earners. We do not plot their productivity profiles and density. Recall that what is identified is the distribution of formal productivities at the top with support $[max_{i\prime}\{\hat{w^f}_{i\prime}\}, \infty)$ and this can be matched with many different combinations of formal productivity and probability density specifications in the types space; all of them equivalent for the optimal taxation problem that solves the planner. We assume that the relation between the shadow and the formal productivity from the main part of the distribution of types holds also for the top earners.

Figure 7: Estimated productivities and types distributions



5 Calibrated exercise

Given the productivity schedules estimated in the previous section, we calibrate the utility function and derive the optimal allocations for Colombia.

5.1 Calibration of the utility function

We assume that the agents' utility function is

$$U(c, n) = \log\left(c - \Gamma\frac{n^{1+\frac{1}{\zeta}}}{1+\frac{1}{\zeta}}\right),\ n \in [0, 1]. \tag{45}$$

The parameter ζ is the elasticity of labor supply. Since we consider a permanent tax reform, the relevant notion is the steady-state intensive margin elasticity. We fix ζ at different values and find Γ which minimizes the deviation of selected K model moments $\left(m_k^{model}\left(\zeta,\Gamma\right)\right)_{k=1}^{K}$ from the corresponding data moments $\left(m_k^{data}\right)_{k=1}^{K}$ according to the loss function

$$L\left(\zeta,\Gamma\right)=\sum_{k=1}^{K}\left(\frac{m_k^{model}\left(\zeta,\Gamma\right)-m_k^{data}}{m_k^{data}}\right)^2. \tag{46}$$

We use three moments: the share of shadow workers in total employment, the share of shadow income in total income and the average total income. The first two moments capture the relative size of the shadow economy, while the third one controls for the total production of Colombia. Chetty, Guren, Manoli, and Weber (2011) recommend using the steady-state intensive elasticity of 0.33, which we treat as a benchmark. However, the estimates behind this number implicitly incorporate responses on multiple margins, possibly also shifting labor to the shadow economy. Since we model this response explicitly, the correct value of elasticity could lower. Hence, we consider also the values of 0.2 and 0.1. Table 1 shows the matched moments for different values of the elasticity of labor supply.

Table 1

Moments	**Actual economy**	**Model economy for different values of elasticity ζ**		
		$\zeta=0.33$	$\zeta=0.2$	$\zeta=0.1$
share of shadow workers	57.99%	64.51%	62.12%	60.53%
share of shadow income	30.94%	23.25%	25.24%	26.64%
mean total income [USD]	7166	6673	6659	6677

The model replicates well the magnitude of the shadow economy for a range of elasticities of labor supply. We conclude that the empirical distribution of productivities and the actual tax schedule can explain the high level of informality in Colombia.

5.2 Optimal allocations

We find the optimum for the two social welfare functions. First, we use the Rawlsian welfare criterion, which puts all the weight on the individual with the lowest utility level. Since both formal and shadow productivities are increasing with type, the Rawlsian planner cares only about the lowest type. Second, we derive the Utilitarian optimum with the planner that maximizes the average utility level in the economy. In each case we require that the planner obtains the same net tax revenue as the actual tax schedule.

The optimal allocations are described in Table 2. The Rawlsian planner would displace approximately the quarter of the workforce to informality. The share of shadow income falls even more, since only the least productive workers end up in the shadow economy. The Utilitarian planner is less redistributive and would cut the size of the informal sector even more. For lower values of elasticity, less than 2% of workers are employed informally.

As labor becomes more elastic, the least productive workers work relatively less, and the highly productive workers relatively more. On the one hand, it means that the share of shadow workers increases with elasticity, as the least productive workers are less inclined to earn formal income high enough to compensate for the lost transfers. On the other hand, the share of shadow income decreases with elasticity, as the shadow workers with low productivity supply much less labor than the highly productive formal workers.[20]

The welfare gains from implementing the optimum are large. The Rawlsian planner manages to increase the transfers to the workers with no formal income by 85% in comparison to the actual tax and transfer system. It translates into welfare gains of 40% to 50% in consumption equivalent terms. The Utilitarian planner takes into consideration the welfare cost of increased taxation of the high types and expands the redistribution less. Nevertheless, the transfers received by the bottom types increase by more than 55% in comparison to the actual tax system in Colombia, which yields welfare gains of 20% in terms of consumption. In order to make sure that the welfare gains are not driven by a thick Pareto tail at the top, we recompute the optima without the top tail (see the last row of Table 2).[21] The welfare gains are naturally smaller, since the top earners constitute a sizable source of tax revenue. However, it is clear that the vast majority of welfare gains come from the efficient taxation of the ordinary workers and not from the very rich.

Table 2

Moments	Actual economy	Optimal Rawlsian allocation			Optimal Utilitarian allocation		
		$\zeta = 0.33$	$\zeta = 0.2$	$\zeta = 0.1$	$\zeta = 0.33$	$\zeta = 0.2$	$\zeta = 0.1$
share of shadow workers	57.99%	26.1%	22.24%	22.48%	6.52%	1.89%	1.36%
share of shadow income	30.94%	5.59%	6.33%	6.98%	0.07%	0.09%	0.1%
mean total income [USD]	7165	6672	6968	7113	6820	7087	7246
welfare (cons. equiv.)	100%	151.8%	147.1%	142%	121.3%	120.9%	119.7%
welfare w/o top (cons. equiv.)	100%	144.6%	141.8%	139.3%	119.6%	119.48%	119.6%

Figure 8 demonstrates how the optimal tax schedule is determined. Recall that the shadow economy imposes an upper bound on the tax rate. If the tax rate of type i exceeds $1 - w_i^s/w_i^f$, the return to shadow labor is strictly greater than the return to formal labor. No agent of type i would be willing to supply formal labor at such terms. As is evident from the figure, all bottom types face tax rate above the upper bound. Hence, they are bunched together at the zero formal income. By equation (34) we know that for types with positive income that are not bunched the optimal marginal tax

[20] Recall that we simultaneously change two parameters: the elasticity ζ and the weight Γ. However, relative changes in labor supply between types cannot be driven by Γ, since Γ affects all types uniformly. Γ is adjusted merely to counter the impact of changes in the elasticity on the average labor supply.

[21] In this case the distribution of types has finite support. The mass of the excluded tail is 0.0045.

rate is the minimum of the two expressions: the standard Mirrleesian tax rate given by a Diamond (1998) and the upper bound $1 - w_i^s/w_i^f$. In all our calibrations the upper bound plays a dominant role (see Figure 8). For the Utilitarian planner with elasticity of 0.33 the standard Mirrleesian tax rate dives under the upper bound just for some high types. For the Rawlsian planner, as well as in the cases of lower elasticity of labor supply, the Mirrleesian tax rate does not intercept the upper bound below the upper tail and hence does not influence the optimal tax in the main part of distribution. In contrast, in all our calibrations some of the upper tail workers are taxed according to the Diamond (1998) formula (the upper tail is not represented on Figure 8). We conclude that the optimal tax schedule of workers below the upper tail is predominantly determined by the shadow economy considerations. However, the usual labor supply responses are important for taxing very productive workers.

Figure 8: Determining the optimal tax schedule

(a) Rawlsian planner ($\zeta = 0.33$) (b) Utilitarian planner ($\zeta = 0.33$)

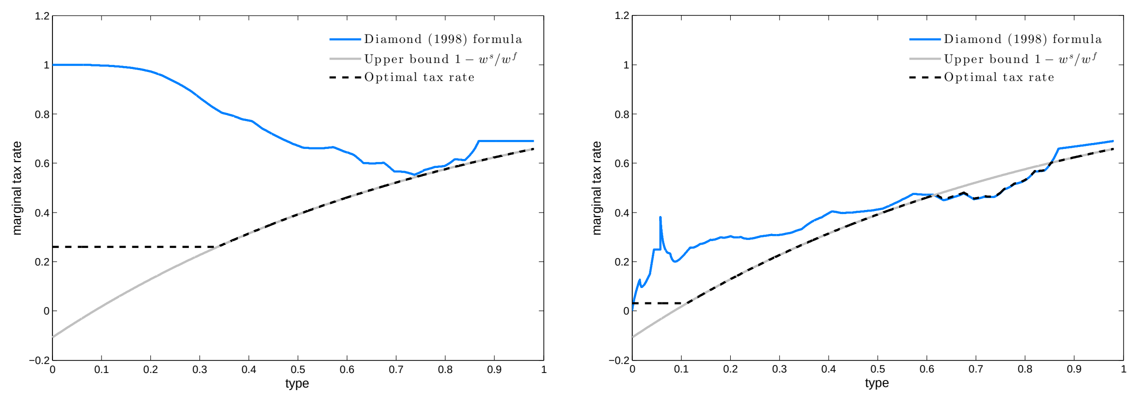


Figure 8 informs us also what would happen if the shadow economy was neglected and the standard Mirrleesian tax was implemented. All the types for which the tax rate exceeds the upper bound would be displaced to the shadow economy. Moreover, many types for which the Mirrleesian tax rate is below the upper bound are likely to move to the shadows as well.[22] Careless implementation of the classical tax formula would lead to a dramatic fall in tax revenue.

How does the optimal tax schedule compares with the one implemented at the time in Colombia? The actual tax schedule involves high 45% marginal rate at low levels of income, implied by phasing-out of transfers (see Figure 9). As income increases the rate drops to 22% and remains flat - workers with this income pay only the flat payroll tax. The progressive income tax starts at the high income level and gradually increases the marginal tax, reaching 49% for the top earners (at income levels not represented at Figure 9)).[23]

In comparison to the actual tax rate, the optimal tax rates are lower at low levels of income and much higher elsewhere. Lower marginal rates at the bottom mean transfers are phased-out more

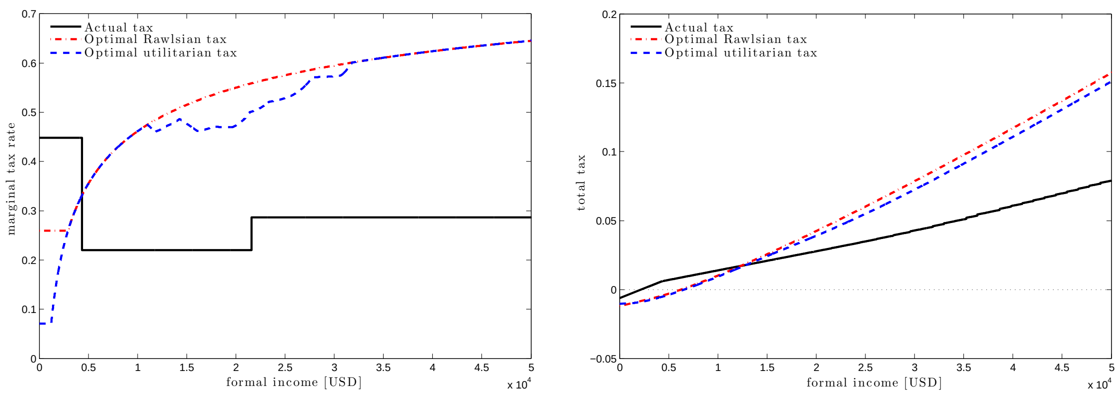
[22]The tax burden accumulated at the low income levels is likely to outweigh the gain from higher return to formal labor at the high income levels.

[23]The progressive tax is a step function with more than 80 steps of varying width and Figure 9 (a) shows its smoothly approximation. The true tax involves 0 rate at the interior of each step and an unbounded rate between steps, hence it cannot be represented on such graph.

slowly, so less productive workers have less incentives to move to the informal sector. Higher marginal tax rates elsewhere imply that the richest agents pay much higher total tax than in the actual economy, which allows the planner to finance the generous transfer (Figure 9 (b)). The tax rates at lower elasticities have very similar shape, as they are determined by the upper bound.

Figure 9: Determining the optimal tax schedule

(a) Marginal tax rates ($\zeta = 0.33$) (b) Total tax ($\zeta = 0.33$)



6 Conclusions

A large fraction of the economic activity in most countries is informal. This paper incorporates this fact into the optimal income tax theory. We find that the shadow economy puts severe restrictions on the taxes the government can levy, often leading to a welfare loss. However, in some cases the shadow economy can raise welfare by improving both redistribution and efficiency. If the informal sector suppresses productivity differences between workers, the government can tax high earners more when the low productivity workers are employed informally. Furthermore, the shadow economy shelters poor workers from distortions implied by the taxation of the rich, allowing for more efficient allocation of labor.

The mechanism proposed has a quantitatively sizable effect. In the case of Colombia it provides a rationale for a large shadow economy. Nevertheless, the observed levels of informality are even higher and more than double the optimal level. According to our model, the large size of the Colombian shadow economy is explained by high marginal tax rates at low levels of income. The optimal tax schedule features lower rates at the bottom, leading to a smaller informal sector, and higher rates above, raising more revenue from top earners.

This paper suggests that allowing less productive people to collect welfare benefits and simultaneously work in the shadow economy could be desirable. Moreover, policies designed to deter the creation of informal jobs should focus on the jobs taken by the workers with the potential for high formal earnings. It is important to stress that the way the shadow economy is modeled in this paper abstracts from many issues, such as competition between formal and informal firms, lack of regulation and law enforcement, as well as potential negative externalities caused by the informal activity. All those phenomena are likely to reduce the potential welfare gains from exploiting the shadow economy.

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A Proofs from Section 2

Proof of Proposition 1. Omitted. □

Proof of Proposition 2. Note that the first-best allocation is consistent with the additional constraint (5), hence it is the solution to the planner's problem. Essentially, conditional on truthfully revealing type, incentives of the agent and the planner regarding the shadow labor are perfectly aligned. If a given type pays taxes according to the true type, choosing shadow labor in order to maximize utility cannot hurt the social welfare. □

Proof of Proposition 3. Existence follows from the Weierstrass theorem. The objective function is continuous. All constraints consist of equalities or weak inequalities, so the choice set is a closed set. We can ensure boundedness of the choice set by considering the allocations that yield welfare not lower than the allocation without transfers, *laissez-faire*. Since *laissez-faire* satisfies all the constraints, the optimum cannot be worse. Note the welfare of laissez-faire by $W_{lf} > -\infty$. Suppose that the set of allocations with welfare greater than W_{lf} is unbounded. By the resource constraint, it means that the labor supply of some type in some allocations has to diverge to infinity. By $\lim_{n\to\infty} v'(n) = \infty$, it implies that disutility from labor of this type, as well as welfare of the whole allocation, goes to $-\infty$. It contradicts the assumption that the allocation is superior than *laissez-faire*. Finally, notice that our restricted choice set is always non-empty, since it contains *laissez-faire*.

In the first-best, $U(c_l, n_l) \geq U(c_h, n_h)$. By assumption of $v'(0) = 0$, we know that $n_l^f > 0$. Then the utility of h mimicking l is $U\left(c_l, \frac{w_l^f}{w_h^f} n_l^f\right) > U\left(c_l, n_l^f\right) \geq U(c_h, n_h)$, which violates $IC_{h,l}$. Hence, the optimum is not the first-best.

Suppose that at the optimum $IC_{h,l}$ does not bind. First, let's consider the case in which $U(c_h, n_h) > U(c_l, n_l)$. Since $IC_{h,l}$ is slack, the planner may increase transfers from h to l, which raises welfare, so it could not be the optimum in the first place. Second, suppose that $U(c_l, n_l) \geq U(c_h, n_h)$. It can happen only if $n_l^s > 0$. Otherwise, as we have shown above, $IC_{h,l}$ is violated. If $n_l^s > 0$ and $IC_{h,l}$ is slack, the planner can marginally decrease n_l^s and increase n_l^f, which generates free resources. Hence, at the optimum $IC_{h,l}$ has to bind.

Suppose that $IC_{l,h}$ binds. If the resource constraint is satisfied as equality, it may happen only if l type is paying a positive tax, while h type receives a transfer. Then the planner can improve welfare by canceling the redistribution altogether and reverting to *laissez-fare*, where none of the incentive constraints bind. □

Lemma 3. *At the optimum either $U(c_l, n_l) = U(c_h, n_h)$ and $n_l^s > 0$, or the following optimality condition holds*

$$\min\left\{\frac{v'(n_l)}{w_l^f} - \left(\mu_l + \mu_h \frac{v'(n_{h,l})}{w_h^f}\right), n_l^f\right\} = 0, \tag{47}$$

where $n_{i,-i} = \frac{w_{-i}^f}{w_i^f} n_{-i}^f + n_i^s\left(\frac{w_{-i}^f}{w_i^f} n_{-i}^f\right)$ is the total labor supply of type i pretending to be of type $-i$. Suppose that v'' is nondecreasing. If $\frac{w_h^f}{w_l^f} g(w_h^s) \geq g(w_l^s)$ then this optimality condition is sufficient for the optimum.

Proof of Lemma 3. If $U(c_l, n_l) = U(c_h, n_h)$ and $n_l^s = 0$, then such allocation is not incentive compatible. The proof is identical as the proof of the claim that the first-best is not incentive compatible in Proposition 3. Hence, if $U(c_l, n_l) = U(c_h, n_h)$, then $n_l^s > 0$.

Let's consider the case in which $U(c_h, n_h)$ is always greater than $U(c_l, n_l)$. $IC_{h,l}$ has to bind, otherwise the planner could equalize utilities of both types. Consider changing n_l^f by a small amount and adjusting T_l such that $IC_{h,l}$ is satisfied as equality. It means that

$$\frac{dT_l}{dn_l^f} = w_l^f \mu_h \left(1 - \frac{v'(n_{h,l})}{w_h^f}\right).$$

This perturbation affects social welfare by

$$\frac{dU\left(c_l,n_l\right)}{dn_l^f}=w_l^f-\frac{\partial T_l}{\partial n_l^f}-v'\left(n_l\right)=w_l^f\left(\mu_l+\mu_h\frac{v'\left(n_{h,l}\right)}{w_h^f}\right)-v'\left(n_l\right).$$

Optimum requires that either $\frac{dU(c_l,n_l)}{dn_l^f}=0$ or $\frac{dU(c_l,n_l)}{dn_l^f}<0$ and $n_l^f=0$, which results in (47). Sufficiency of this first order condition depends on the second order derivative of welfare with respect to the perturbation. In order to have the second derivative well behaved, we are going to assume that v'' is nondecreasing. Then, we need to consider two cases (see Table 3). If $\frac{w_h^f}{w_l^f}g\left(w_h^s\right)\geq g\left(w_l^s\right)$ holds, then $\frac{dU(c_l,n_l)}{dn_l^f}$ is non-increasing in n_l^f. It means that the optimality condition (47) is sufficient. If $\frac{w_h^f}{w_l^f}g\left(w_h^s\right)<g\left(w_l^s\right)$, then $\frac{dU(c_l,n_l)}{dn_l^f}$ is not monotone in n_l^f and it may be the case that (47) points at either local maximum which is not a global maximum, or at the local minimum.

Table 3: Second order derivative of welfare with respect to the perturbation

The case of $\frac{w_h^f}{w_l^f}g\left(w_h^s\right)\geq g\left(w_l^s\right)$			
	$n_l^f<g\left(w_l^s\right)$	$g\left(w_l^s\right)<n_l^f<\frac{w_h^f}{w_l^f}g\left(w_h^s\right)$	$\frac{w_h^f}{w_l^f}g\left(w_h^s\right)<n_l^f$
$\frac{d^2U(c_l,n_l)}{dn_l^{f2}}=$	0	$-v''\left(n_l^f\right)<0$	$\mu_h\left(\frac{w_l^f}{w_h^f}\right)^2v''\left(\frac{w_l^f}{w_h^f}n_l^f\right)-v''\left(n_l^f\right)<0$
The case of $\frac{w_h^f}{w_l^f}g\left(w_h^s\right)<g\left(w_l^s\right)$			
	$n_l^f<\frac{w_h^f}{w_l^f}g\left(w_h^s\right)$	$\frac{w_h^f}{w_l^f}g\left(w_h^s\right)<n_l^f<g\left(w_l^s\right)$	$g\left(w_l^s\right)<n_l^f$
$\frac{d^2U(c_l,n_l)}{dn_l^{f2}}=$	0	$\mu_h\left(\frac{w_l^f}{w_h^f}\right)^2v''\left(\frac{w_l^f}{w_h^f}n_l^f\right)>0$	$\mu_h\left(\frac{w_l^f}{w_h^f}\right)^2v''\left(\frac{w_l^f}{w_h^f}n_l^f\right)-v''\left(n_l^f\right)<0$

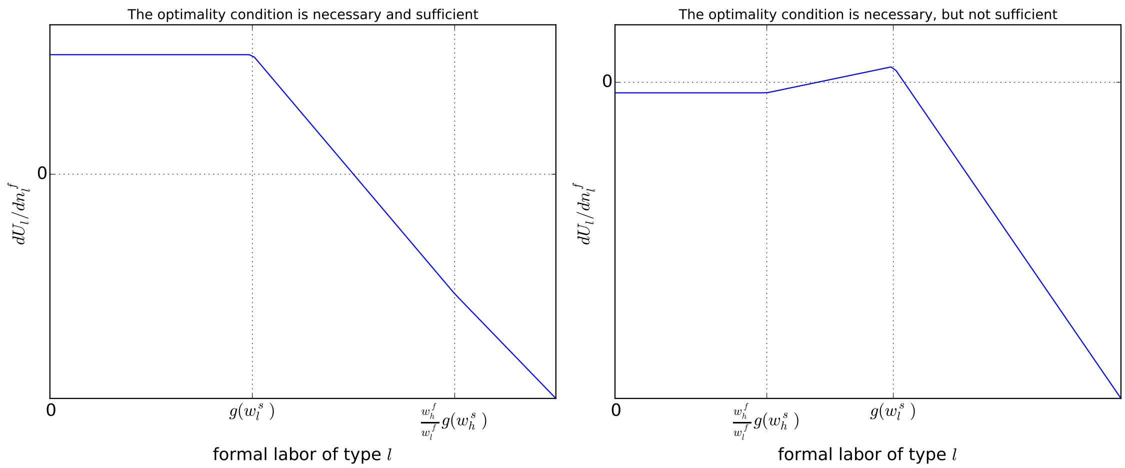
Figure 10 shows these two cases. In the first panel $\frac{w_h^f}{w_l^f}g\left(w_h^s\right)\geq g\left(w_l^s\right)$ holds and the optimality condition (47) always points at the optimum (in this case, the value of n_l^f where $\frac{dU(c_l,n_l)}{dn_l^f}=0$). In the second panel $\frac{w_h^f}{w_l^f}g\left(w_h^s\right)<g\left(w_l^s\right)$ holds and the optimality condition is not sufficient. There are three points that satisfy condition (47): local maximum at $n_l^f=0$, local minimum with $n_l^f\in\left(\frac{w_h^f}{w_l^f}g\left(w_h^s\right),g\left(w_l^s\right)\right)$ and the other local maximum with $n_l^f>g\left(w_l^s\right)$. $\square$

Proof of Proposition 4. In the proof of Lemma 3 above we described the impact of changing formal labor of l on the social welfare, $\frac{dU(c_l,n_l)}{dn_l^f}$. The condition (10) describes situations when the impact of the perturbation is non-positive at $n_l^f=0$. From Figure 10 it is clear that if it is not the case, type l will never optimally work in the shadow economy.

Suppose that $\frac{w_h^f}{w_l^f}g\left(w_h^s\right)\geq g\left(w_l^s\right)$. Condition (10) implies that $\frac{dU(c_l,n_l)}{dn_l^f}$ is always non-positive, so it is optimal to reduce n_l^f as long as $U\left(c_h,n_h\right)>U\left(c_l,n_l\right)$. From Lemma 3 we know also that $U\left(c_h,n_h\right)>U\left(c_l,n_l\right)$ if l works only formally, so it is optimal to place type l in the shadow economy.

Now suppose that $\frac{w_h^f}{w_l^f}g\left(w_h^s\right)<g\left(w_l^s\right)$. Condition (11) means that the maximum of $\frac{dU(c_l,n_l)}{dn_l^f}$ attained at $n_l^f=g\left(w_l^s\right)$ (see Figure 10) is non-positive. Therefore, it is optimal to reduce n_l^f until utilities of

Figure 10: Sufficiency of the optimality condition



both types are equalized, which can happen only when l works in the shadow economy. Condition (11) is sufficient, but not necessary for l to work in the shadow economy, because the social welfare changes in a non-monotone way with n_l^f. If (11) is not satisfied, marginally increasing n_l^s from 0 is bad for welfare, but increasing it further may eventually lead to welfare gains, and the total effect on welfare is ambiguous. $\square$

Proof of Proposition 5. Suppose that optimally $n_l^s > 0$. From Figure 10 it is clear that in such situation it is in the best interest of type l to work exclusively in the shadow economy. However, if $w_l^s > w_h^s$ and $n_l^f = 0$, the incentive compatibility constraint of the type h implies that

$$U(c_l, n_l) = U(w_l^s n_l^s - T_l, n_l^s) > U\left(w_h^s n_{h,l}^s - T_l, n_{h,l}^s\right) = U(c_h, n_h).$$

Since the planner is Rawlsian, such allocation is not desirable. The planner will rather stop decreasing n_l^f at the point where utilities of both types are equal. On the other hand, if $w_l^s \leq w_h^s$ then

$$U(c_l, n_l) = U(w_l^s n_l^s - T_l, n_l^s) \leq U\left(w_h^s n_{h,l}^s - T_l, n_{h,l}^s\right) = U(c_h, n_h),$$

so the planner will optimally decrease n_l^f to zero. $\square$

Proof of Proposition 6. In order to examine when the optimum welfare is strictly higher than in the standard Mirrlees model, we will compare utility of type l in the standard Mirrlees model $\left(U\left(c_l^M, n_l^M\right)\right)$ and in the shadow economy model, when l is working only in the shadow economy $\left(U\left(c_l^{SE}, n_l^{SE}\right)\right)$. Clearly, when the second scenario yields higher utility, the existence of the shadow economy is welfare improving.

In the standard Mirrlees model, the binding constraint is $U\left(w_h^f n_h^M, n_h^M\right) - T_h^M = U\left(w_l^f n_l^M, \frac{w_l^f}{w_h^f} n_l^M\right) - T_l^M$. Together with the resource constraint it means that $T_l^M = \mu_h \left(U\left(w_l^f n_l^M, \frac{w_l^f}{w_h^f} n_l^M\right) - U\left(w_h^f n_h^M, n_h^M\right)\right).$

Now, the utility of type l is

$$U\left(c_l^M, n_l^M\right) = U\left(w_l^f n_l^M, n_l^M\right) - T_l = U\left(w_l^f n_l^M, n_l^M\right) - \mu_h \left(U\left(w_l^f n_l^M, \frac{w_l^f}{w_h^f} n_l^M\right) - U\left(w_h^f n_h^M, n_h^M\right)\right).$$

Using the same steps, we can express the utility of type l working only in the shadow economy as

$$U\left(c_l^{SE}, n_l^{SE}\right) = U\left(w_l^s n_l^{SE}, n_l^{SE}\right) - \mu_h \left(U\left(w_h^s n_{h,l}^{SE}, n_{h,l}^{SE}\right) - U\left(w_h^f n_h^{SE}, n_h^{SE}\right)\right).$$

Since there are no distortions at the top and no wealth effects, $n_h^M = n_h^{SE}$. The shadow economy is welfare improving, $U\left(c_l^{SE}, n_l^{SE}\right) - U\left(c_l^M, n_l^M\right) > 0$, iff

$$U\left(w_l^s n_l^{SE}, n_l^{SE}\right) - U\left(w_l^f n_l^M, n_l^M\right) + \mu_h \left(U\left(w_l^f n_l^M, \frac{w_l^f}{w_h^f} n_l^M\right) - U\left(w_h^s n_{h,l}^{SE}, n_{h,l}^{SE}\right)\right) > 0.$$

The first difference (the efficiency gain) is positive if $w_l^s > \bar{w}_l^s$. The second difference (the redistribution gain) is positive when $w_h^s < \bar{w}_h^s$. Hence, when both inequalities hold weakly and at least one holds strictly, the existence of the shadow economy improves welfare in comparison to the standard Mirrlees model.

Now we will show that when the inequalities hold in the other direction, the shadow economy hurts welfare. Suppose that $w_l^s = \bar{w}_l^s$ and $w_h^s = \bar{w}_h^s$. We will prove that allocation ^{SE} is a unique optimum at this point. First we will show that when the redistribution gain is non-positive, it is true that $n_h^{SE} > \frac{w_l^f}{w_h^f} n_l^M$. Suppose on the contrary that $n_h^{SE} \leq \frac{w_l^f}{w_h^f} n_l^M$. Then we can write the following sequence of inequalities

$$U\left(w_l^f n_l^M, \frac{w_l^f}{w_h^f} n_l^M\right) \geq U\left(w_h^f n_h^{SE}, n_h^{SE}\right) > U\left(w_h^s n_h^{SE}, n_h^{SE}\right).$$

The first inequality comes from the fact that $\frac{w_l^f}{w_h^f} n_l^M$ is below the efficient level of labor supply of type h, so lowering the labor of this type even further to n_h^{SE} will decrease the utility. The second inequality is simply implied by our assumption $w_h^f > w_h^s$. This sequence of inequalities implies that the redistribution gain is strictly positive. Hence, if the redistribution gain is non-positive, $n_h^{SE} > \frac{w_l^f}{w_h^f} n_l^M$ holds.

Note that $n_h^{SE} > \frac{w_l^f}{w_h^f} n_l^M$ means that the optimal allocation of the standard Mirrlees model is not incentive-compatible with the shadow economy - deviating type h would supply some additional shadow labor. Hence, any allocation which yields the social welfare equal or higher than $U\left(c_l^M, n_l^M\right)$ has to involve type l working in the shadow economy.

Let's go back to the optimal allocation with the shadow economy, when $w_l^s = \bar{w}_l^s$ and $w_h^s = \bar{w}_h^s$. From the considerations above we know that the optimum involves some shadow labor. If we sum the efficiency gain and the redistribution gain divided by μ_h and rearrange the terms, we get

$$\left(U\left(w_l^f n_l^M, \frac{w_l^f}{w_h^f} n_l^M\right) - U\left(w_l^f n_l^M, n_l^M\right)\right) - \left(U\left(w_h^s n_h^{SE}, n_h^{SE}\right) - U\left(w_l^s n_l^{SE}, n_l^{SE}\right)\right) = 0.$$

The expression in the first brackets is positive. Hence, the second brackets are positive as well, which means that $w_h^s > w_l^s$. By Proposition 5 type l will work exclusively in the shadow economy.

To sum up, we know that at $(w_l^s, w_h^s) = (\bar{w}_l^s, \bar{w}_h^s)$ the optimum of the shadow economy model is unique and involves type l working entirely in the shadow economy. Consequently, a decrease in the shadow productivity of type l or an increase in the shadow productivity of type h leads to a strict welfare loss, since it either decreases the effective productivity of type l or decreases the transfer type l receives. □

Proof of Proposition 7. Suppose that $\lambda_i \leq \lambda_{-i}$. In this case the $IC_{i,-i}$ may bind (it will if the inequality is strict), while $IC_{-i,i}$ is always slack. The planner will not distort the allocation of type i. Without distortions, this type will never work in the shadow economy.

Suppose that $\lambda_i > \lambda_{-i}$, so that $IC_{-i,i}$ binds. Perturb n_i^f and adjust T_i such that $IC_{-i,i}$ holds as equality:

$$\frac{dT_i}{dn_i^f} = w_i^f \mu_{-i} \left(1 - \frac{v'\left(n_{-i,i}\right)}{w_{-i}^f}\right).$$

This perturbation affects social welfare by

$$\begin{aligned} \frac{dW}{dn_i^f} &= \lambda_i \mu_i \left(w_i^f - \frac{\partial T_i}{\partial n_i^f} - v'\left(n_i\right)\right) + \lambda_{-i} \mu_{-i} \frac{\mu_i}{\mu_{-i}} \frac{\partial T_i}{\partial n_i^f} \\ &= \lambda_i \mu_i w_i^f \left(\left(1 - \frac{v'(n_i)}{w_i^f}\right) + \left(\frac{\lambda_{-i}}{\lambda_i} - 1\right) \mu_{-i} \left(1 - \frac{v'(n_{-i,i})}{w_{-i}^f}\right)\right) \end{aligned} \tag{48}$$

Suppose that $\frac{w_{-i}^s}{w_{-i}^f} \geq \frac{w_i^s}{w_i^f}$ and $n_i^f \leq g\left(w_i^s\right)$, which means that $v'\left(n_i\right) = w_i^s$. Note that $\frac{v'(n_{-i,i})}{w_{-i}^f} \geq \frac{w_{-i}^s}{w_{-i}^f} \geq \frac{w_i^s}{w_i^f}$. Hence

$$1 - \frac{w_i^s}{w_i^f} \geq 1 - \frac{v'\left(n_{-i,i}\right)}{w_{-i}^f} > \left(1 - \frac{\lambda_{-i}}{\lambda_i}\right) \mu_{-i} \left(1 - \frac{v'\left(n_{-i,i}\right)}{w_{-i}^f}\right),$$

which means that $\frac{dW}{dn_i^f} > 0$. Therefore, it is never optimal to decrease n_i^f so much that type i works in the shadow economy. □

Proof of Proposition 8. First we will show how to obtain (14). The efficiency gain is straightforward. In order to obtain the redistribution gain, note that there are no distortions imposed on type $-i$, hence

$$\mu_{-i}\lambda_{-i}\left(U\left(c_{-i}^{SE}, n_{-i}^{SE}\right) - U\left(c_{-i}^M, n_{-i}^M\right)\right) = \mu_{-i}\lambda_{-i}\left(T_{-i}^M - T_{-i}^{SE}\right) = -\mu_i\lambda_{-i}\left(T_i^M - T_i^{SE}\right).$$

Summing up the terms results in (14). In order to derive thresholds, recall that $H(w^s) = U(w^s g(w^s), g(w^s))$. The efficiency gain is given by

$$\mu_i \lambda_i \left(H(w_i^s) - U\left(w_i^f n_i^M, n_i^M \right) \right),$$

it is strictly increasing in w_i^s and positive for $w_i^s > \bar{w^s}_i$. Note that by (48) n_i^M will always be distorted (downwards if $i = l$, upwards if $i = h$). Hence, $U\left(w_i^f n_i^M, n_i^M \right) < H\left(w_i^f \right)$ and the threshold $\bar{w^s}_i$ is strictly lower than w_i^f.

Using the binding $IC_{-i,i}$ constraint, we can express the redistribution gain as

$$\mu_i \mu_{-i} (\lambda_i - \lambda_{-i}) \left(U\left(w_i^f n_i^M, \frac{w_i^f}{w_{-i}^f} n_i^M \right) - H\left(w_{-i}^s \right) \right).$$

It is strictly decreasing in w_{-i}^s and is positive for $w_{-i}^s < \bar{w^s}_{-i}$. Since $\frac{w_i^f}{w_{-i}^f} n_i^M \neq g\left(w_{-i}^f \right)$, it is true that $U\left(w_i^f n_i^M, \frac{w_i^f}{w_{-i}^f} n_i^M \right) < H\left(w_{-i}^f \right)$ and the threshold $\bar{w^s}_{-i}$ is strictly lower than w_{-i}^f. □

A Proofs from Section 3

Proof of Lemma 1. The single-crossing requires that $\frac{d}{di}\left(\frac{\partial V_i(y^f,T)/\partial y^f}{\partial V_i(y^f,T)/\partial T} \right) < 0$. Suppose that $v'\left(\frac{y^f}{\phi_i} \right) < \psi_i$. Then the agent supplies no informal labor and the indirect utility function V is just the utility function U evaluated at the formal allocation. Since v' is increasing, the single crossing holds in this case. When $v'\left(\frac{y^f}{\phi_i} \right) \geq \psi_i$, then the optimal provision of informal labor means that $v'(n_i) = w_i^f$, which implies $\frac{\partial V_i(y^f,T)/\partial y^f}{\partial V_i(y^f,T)/\partial T} = \frac{w_i^s}{w_i^f}$. Therefore the single crossing condition requires that $\frac{d}{di}\left(\frac{w_i^s}{w_i^f} \right) < 0$. □

Proof of Proposition 9. First note that the incentive compatibility requires that if $\frac{d}{dj} V_i\left(y_j^f, T_j \right)\Big|_{j=i}$ exists, it is equal to 0. Otherwise type i can improve welfare by changing income marginally, so the allocation is not incentive compatible. Hence, if $\frac{d}{di} V_i\left(y_i^f, T_i \right) = \frac{d}{dj} V_i\left(y_j^f, T_j \right)\Big|_{j=i} + \frac{d}{di} V_i\left(y_j^f, T_j \right)\Big|_{j=i}$ exists, it is equal to $\frac{d}{di} V_i\left(y_j^f, T_j \right)\Big|_{j=i} = \left(\frac{\dot{w}_i^f}{w_i^f} n_i^f + \frac{\dot{w}_i^s}{w_i^s} n_i^s \right) v'(n_i)$. We call this derivative a marginal information rent and denote it simply by $\dot{V}_i$.

By Milgrom and Segal (2002) (see their 10th footnote and Theorem 2), we can represent the utility schedule for any $i < 1$ as an integral of marginal information rents

$$V_i\left(y_i^f, T_i \right) = V_0\left(y_0^f, T_0 \right) + \int_0^i \dot{V}_j dj,$$

Moreover, the utility schedule V_i is continuous everywhere and differentiable almost everywhere.

Now we will show that the allocation is not incentive compatible if the formal income is decreasing in type. Suppose that the allocation is incentive-compatible and that there are two types $a < b$ such that $y_a^f > y_b^f$. By the incentive compatibility, we have

$$V_a\left(y_a^f, T_a\right) \geq V_a\left(y_b^f, T_b\right). \tag{49}$$

$\frac{d}{di}V_i\left(y^f, T\right)$ is increasing in y^f. To see it, note that

$$\frac{d}{di}V_i\left(y^f, T\right) = \left(\rho_i^f \frac{y^f}{w^f{}_i} + \rho_i^s \max\left\{g\left(w_i^s\right) - \frac{y^f}{w_i^f}, 0\right\}\right) v'\left(n_i\right),$$

where g is the inverse function of v'. The single-crossing implies that $\rho_i^f > \rho_i^s$, so the right hand side is increasing in y^f.

Since $y_a^f > y_b^f$, for each type i it is true that $\frac{d}{di}V_i\left(y_a^f, T_a\right) > \frac{d}{di}V_i\left(y_b^f, T_b\right)$. It implies

$$V_b\left(y_a^f, T_a\right) - V_a\left(y_a^f, T_a\right) = \int_a^b \frac{d}{di}V_i\left(y_a^f, T_a\right) di > \int_a^b \frac{d}{di}V_i\left(y_b^f, T_b\right) di = V_b\left(y_b^f, T_b\right) - V_a\left(y_b^f, T_b\right). \tag{50}$$

Summing (49) and (50) results in

$$V_b\left(y_a^f, T_a\right) > V_b\left(y_b^f, T_b\right),$$

which contradicts the incentive-compatibility. Therefore, a nondecreasing formal income schedule is necessary for incentive compatibility. Conversely, suppose that the local incentive constraints hold and the formal income schedule is nondecreasing. Then for any two types $a < b < 1$

$$V_b\left(y_b^f, T_b\right) - V_a\left(y_a^f, T_a\right) = \int_a^b \frac{d}{di}V_i\left(y_i^f, T_i\right) di \geq \int_a^b \frac{d}{di}V_i\left(y_a^f, T_a\right) di = V_b\left(y_a^f, T_a\right) - V_a\left(y_a^f, T_a\right), \tag{51}$$

which implies

$$V_b\left(y_b^f, T_b\right) \geq V_b\left(y_a^f, T_a\right).$$

We can use the same reasoning, but bound the utility difference on the left hand side of (51) from above by $\int_a^b \frac{d}{di}V_i\left(y_b^f, T_b\right) di$ to get

$$V_a\left(y_a^f, T_a\right) \geq V_a\left(y_b^f, T_b\right).$$

We cannot use this argument when $b = 1$ and w_1^f is unbounded, but then by continuity of V_i we have $\lim_{b \to 1}\left\{V_b\left(y_b^f, T_b\right) - V_b\left(y_a^f, T_a\right)\right\} \geq 0$. □

Proof of Theorem 1. First we will derive D_i^f and D_i^s term. Then we will show that conditions from the theorem are necessary. Finally we will prove sufficiency.

Suppose that $i \in \mathcal{F}$. A perturbation of formal income changes the marginal information rent of type i by

$$\frac{\partial \dot{V}_i}{\partial y_i^f} = (1 - t_i)\, \rho_i^f \left(1 + \frac{1}{\zeta_i}\right). \tag{52}$$

where $\zeta_i = \frac{v'(n_i)}{n_i v''(n_i)}$ is the elasticity of labor supply. This change of income affects the utility level of type i by $\frac{dV_i}{dy_i^f} = 1 - \frac{v'(n_i)}{w_i^f}$. By Proposition 9 the utility schedule has to be continuous, so we have to introduce additional change in tax T_i in order to keep the utility level of type i constant. We adjusts the total tax paid by an agent of type i by $dT_i = 1 - \frac{v'(n_i)}{w_i^f}$. Note that dT_i is just equal the marginal tax rate t_i. This perturbation influences the tax revenue as if we were decreasing the formal income of type i while keeping the marginal tax rate fixed. Since we are interested in the tax revenue impact of the unit perturbation of the marginal information rent, we normalize dT_i by $\frac{\partial \dot{V}_i}{\partial y}$. In order to capture the tax revenue impact of perturbation of all agents of type i, we multiply this expression by μ_i and get

$$D_i^f = dT_i \left(\frac{\partial \dot{V}_i}{\partial y_i^f}\right)^{-1} \mu_i = \frac{t_i}{1 - t_i} \left(\rho_i^f \left(1 + \frac{1}{\zeta_i}\right)\right)^{-1} \mu_i.$$

Suppose that $i \in \mathcal{S}$. Shadow labor is supplied according to $v'(n_i) = w_i^s \implies n_i = g(w_i^s)$. The marginal information rent can be expressed as

$$\dot{V}_i = \left(\frac{\dot{w}_i^f}{\left(w_i^f\right)^2} y_i^f + \frac{\dot{w}_i{}^s}{w_i^s} \left(g(w_i^s) - \frac{y_i^f}{w_i^f}\right)\right) w_i^s. \tag{53}$$

We marginally perturb y_i^f. The impact of the perturbation of the marginal information rent is

$$\frac{d\dot{V}_i}{dy_i^f} = \left(\rho_i^f - \rho_i^s\right) \frac{w_i^s}{w_i^f}.$$

As in the formal workers' case, the perturbation of y_i^f alone changes the utility level of type i. In order to keep the utility schedule continuous at i, we need to adjust the tax T_i such that the utility of this type is unchanged. The required change of the tax is $dT_i = 1 - \frac{v'(n_i)}{w_i^f}$, which for the shadow worker equals $\frac{w_i^f - w_i^s}{w_i^f}$. By multiplying the tax revenue change with μ_i and normalizing it with $\frac{d\dot{V}_i}{dy^f}$, we obtain the tax revenue cost of decreasing the marginal information rent of type i:

$$D_i^s = dT_i \left(\frac{\partial \dot{V}_i}{\partial y_i^f}\right)^{-1} \mu_i = \frac{w_i^f - w_i^s}{w_i^s} \left(\rho_i^f - \rho_i^s\right)^{-1} \mu_i.$$

If the interior formal income is nondecreasing, the interior allocation implied is incentive-compatible. The necessity of the conditions (27)-(30) was demonstrated in the main text. If these conditions do not hold, there exists a beneficial perturbation.

The conditions (27)-(30) are sufficient when they pin down the unique interior allocation. This happens when the cost of distortions is decreasing in the formal income of each type. Then the government's problem of choosing formal income of each type is concave. For formal workers it requires that ζ_i is non-increasing in the labor supply, as then increasing the marginal tax rate t_i leads to an increase in D_i^f. For the marginal workers we need $D_i^s > D_i^f$, which is guaranteed by $\frac{\rho_i^s}{\rho_i^f} > \zeta_i^{-1}$. See the footnote 11 for the comment regarding the uniqueness of allocation for types for which $D_i^s = \int_i^1 N_j dj$ holds. □

Proof of Proposition 10. We will examine the monotonicity of an interior formal income schedule separately for the formal, marginal and shadow workers.

The single-crossing condition implies that if the marginal tax rate is non-increasing in type, the formal income of workers in $\mathcal{F}$ is increasing. By (27) the marginal tax rate satisfies

$$\frac{t_i}{1-t_i} = \rho_i^f \left(1 + \frac{1}{\zeta_i}\right) \frac{1-M_i}{\mu_i} \mathbb{E}\left(1-\omega_j \middle| j > i\right). \tag{54}$$

Assumption 3(i) means that $\mathbb{E}\left(1-\omega_j \middle| j > i\right) = \mathbb{E}\left(1 - \frac{\lambda_j}{\eta} \middle| j > i\right)$ is non-increasing in i. Assumptions 3(ii) and 3(iii) imply that the rest of the right hand side of (54) is non-increasing in i. Hence, t_i is non-increasing and the interior formal income schedule is increasing in $\mathcal{F}$.

For any marginal worker i the formal income is fixed at $w_i^f g\left(w_i^s\right)$. The derivative of formal income with respect to type is

$$\dot{y}_i^f = \frac{dw_i^f g\left(w_i^s\right)}{di} = \dot{w}_i^f g\left(w_i^f\right) + w_i^f \dot{w}_i^s g'\left(w_i^s\right) = w_i^f g\left(w_i^s\right)\left(\rho_i^f + \rho_i^s \frac{w_i^s g'\left(w_i^s\right)}{g\left(w_i^s\right)}\right).$$

Notice that $\frac{w_i^s g'\left(w_i^s\right)}{g\left(w_i^s\right)} = \frac{v'(n_i)}{n_i v''(n_i)} = \zeta_i$. Hence, for any marginal worker $\dot{y}_i^f \geq 0$ if and only if $\rho_i^f + \rho_i^s \zeta_i \geq 0 \Leftrightarrow \frac{\rho_i^s}{\rho_i^f} \geq -\zeta_i^{-1}$.

In the interior allocation all shadow workers have zero formal income. Hence, the formal income schedule is non-decreasing only if shadow workers are present exclusively at the bottom of the type space. According to (29), a worker i belongs to $\mathcal{S}$ in an interior allocation if and only if

$$\frac{w_i^f - w_i^s}{w_i^s} \leq \rho_i^f \frac{1-M_i}{\mu_i}\left(1 - \frac{\rho_i^s}{\rho_i^f}\right) \mathbb{E}\left(1-\omega_j \middle| j > i\right).$$

The left hand side is increasing in i by the single-crossing assumption. The right hand side is non-increasing by assumptions 3(i), 3(ii) and 3(iv).□

Proof of Proposition 2. We will show that under the assumptions made the interior allocation is such that bottom types do not work in the shadow economy, while some types above them do. This leads to the income schedule locally decreasing in type.

Let's compute the term $\int_i^1 N_j dj$. By (30) we know that $\eta = \mathbb{E}\{\lambda_i\} = 1$. Hence $\int_i^1 N_j dj = \int_i^1 (1-\lambda_j)\, dj$ and the derivative of this term is $\frac{\partial \int_i^1 N_j dj}{\partial i} = \lambda_i - 1$.

The term D_i^s is

$$D_i^s = \left(\frac{w_0^f}{w_0^s} e^{\left(\rho^f - \rho^s\right)i} - 1 \right) \left(\rho^f - \rho^s\right)^{-1}.$$

By (29) any type i is a shadow worker in the interior allocation if and only if $\int_i^1 N_j dj \geq D_i^s$. We can rewrite this inequality as

$$w_0^s \geq \frac{e^{\left(\rho^f - \rho^s\right)i}}{1 + \left(\rho^f - \rho^s\right) \int_i^1 N_j dj} w_0^f.$$

Denote the right hand side by X_i. Note that $X_0 = 1$, which together with $w_0^f > w_0^s$ implies that the bottom types do not work in the shadow economy and by the Assumption 1 *(iii)* have a positive formal income.

We define the threshold $\bar{w}_0^s$ as $\min_{i \in [0,1]} X_i$. In order to see that $\bar{w}_0^s < w_0^f$, let's compute the derivative of X_i :

$$\dot{X}_i = \left(\rho^f - \rho^s\right) e^{\left(\rho^f - \rho^s\right)i} \left(2 - \lambda_i + \left(\rho^f - \rho^s\right) \int_i^1 N_j dj \right).$$

Note that $\dot{X}_0 = \left(\rho^f - \rho^s\right)(2 - \lambda_i) < 0$, so X_i is decreasing at the bottom type and $\min_{i \in [0,1]} X_i < X_0 = w_0^f$. Therefore, whenever $w_0^f > w_0^s > \bar{w}_0^s$, the bottom types have a positive formal income, while some types above them work in the shadow economy and have no formal income. □

Proof of Theorem 2.

Proof. There are three cases we should consider, depending on whether the interior formal income is increasing, locally decreasing, or increasing but not strictly. In the first case (strictly increasing schedule) by Theorem 1 the interior allocation is optimal. In the second case (locally decreasing schedule) by Theorem 2 we need to use the optimal bunching condition (32). Below we derive this condition formally. In the third case the interior income schedule is non-decreasing with flat parts. By Theorem 1 the interior allocation is optimal. However, Theorem 2 says that the flat parts of the income schedule should be consistent with the optimal bunching condition (32). We will show that those two approaches are equivalent.

Suppose that the formal income schedule y^f is constant at the segment of types $[a, b]$. Let's marginally decrease the formal income of types $[a, b)$. Since we don't change the allocation of types below a, we have to make sure that V_a is unchanged - otherwise the utility schedule becomes discontinuous. Together with the cut of the formal income, we have to introduce a change in the total tax paid at this income level $dT_a = 1 - \frac{v'(n_i)}{w_i^f} = t_{a^-}$. Since all types $[a, b)$ are affected, the tax revenue loss is equal to

$$t_a \left(M_b - M_a\right). \tag{55}$$

Although this perturbation does not affect the utility of type a, it does affect the utility of all other bunched types. The utility impact of the perturbation of some type $i \in (a, b)$ equals $dU_i =$

$1-\frac{v'(n_i)}{w_i^f}-dT_a=\frac{v'(n_a)}{\phi_a}-\frac{v'(n_i)}{\phi_i}$. The welfare loss of bunched agents due to this utility change is

$$\int_a^b \Delta MRS_i \omega_i d\mu, \tag{56}$$

where $\Delta MRS_i=\frac{v'(n_a)}{\phi_a}-\frac{v'(n_i)}{\phi_i}$. Having the fiscal and welfare loss at the kink, we can add them into a cost of increasing distortions at the bunch $[a,b)$. We normalize the sum by $t_{b^+}-t_{a^-}$, which makes sure that the perturbation results in the unit change of the utility of type b, and we obtain (31). As the perturbation results in a uniform utility change of agents above the bunch, we can use the standard term (24) in order to obtain the optimal bunching condition (32).

Suppose that the interior formal income schedule is flat on the segment $[a,b]$. We will prove the equivalence of the interior optimality conditions and the optimal bunching condition. Let's consider the following sequence of perturbations. First, decrease the marginal information rent of agent a such that the formal income of this type falls by a unit. Take a marginally higher type and again perturb the marginal information rent such that the formal income of this agents is decreased by a unit as well. Do it until you reach type b. Note that incentive compatibility is preserved at each stage, since the formal income is always non-decreasing. The aggregate welfare impact of this sequence of perturbations is

$$W_{interior}=\int_a^b \frac{\partial \dot{V}_i}{\partial y}\left(D_i-\int_i^1 N_j dj\right)di,$$

where $D_i\equiv\begin{cases}D_i^f & \text{if i}\in\mathcal{F}\\ D_i^s & \text{if i}\in\mathcal{S}\end{cases}$. We do not need to consider the marginal workers, because their formal income is increasing (see the proof of Proposition 10), hence they cannot be bunched. We can decompose $W_{interior}$ into three components

$$W_{interior} = \underbrace{\int_a^b \frac{\partial \dot{V}_i}{\partial y} D_i di}_{X_1} - \underbrace{\int_a^b \frac{\partial \dot{V}_i}{\partial y}\int_i^b N_j dj di}_{X_2} - \underbrace{\int_a^b \frac{\partial \dot{V}_i}{\partial y}\int_b^1 N_j dj di}_{X_3}.$$

Note that $D_i=\frac{1-MRS_i}{\frac{\partial \dot{V}_i}{\partial y}}\mu_i$, hence $X_1=\int_a^c\left(1-MRS_i\right)\mu_i di$. We observe that $\frac{\partial \dot{V}_i}{\partial y}=\frac{\partial^2 V_i}{\partial i \partial y}=-M\dot{R}S_i$ and we integrate X_2 by parts

$$X_2=-\int_a^b M\dot{R}S_i\int_i^b N_j dj di=-\left(\left[MRS_i\int_i^b N_j dj\right]_a^b+\int_a^b MRS_i N_i di\right)=-\int_a^b\left(MRS_i-MRS_a\right)N_i di.$$

We simply integrate X_3

$$X_3=-\int_a^b M\dot{R}S_i\int_b^1 N_j dj di=-\left(MRS_b-MRS_a\right)\int_b^1 N_j dj.$$

Now by summing and rearranging the terms we get

$$\begin{aligned}
W_{interior} &= X_1 - X_2 - X_3 \\
&= \int_a^b (1 - MRS_i)\, d\mu + \int_a^b (MRS_i - MRS_a)(1 - \omega_i)\, d\mu + (MRS_b - MRS_a) \int_b^1 N_j dj \\
&= \int_a^b (1 - MRS_a)\, d\mu + \int_a^b (MRS_a - MRS_i)\, \omega_i d\mu + (MRS_b - MRS_a) \int_b^1 N_j dj \\
&= t_{a^-} (M_b - M_a) + \int_a^b \Delta MRS_i \omega_i d\mu + (t_{a^-} - t_{b^+}) \int_b^1 N_j dj = (t_{b^+} - t_{a^-}) \left(D_{a,b} - \int_b^1 N_j dj \right).
\end{aligned}$$

Since $t_{b^+} - t_{a^-} = \frac{v'(n_a)}{\phi_a} - \frac{v'(n_b)}{\phi_b} > 0$, the sequence of interior optimality conditions is equivalent to the optimal bunching condition (32). □

Proof of Proposition 11. If the interior allocation is incentive-compatible, the claim holds. Suppose that it is not the case, i.e. there is a kink in the tax schedule. In this case incentive compatibility constrains the government from reducing the utility of agents above kink as much as in the interior case. Since G is concave, it means that N_j terms for j above the kink is weakly higher and the government's will to impose distortions does not decrease. If there are shadow workers at the bottom and the curve $\int_i^1 N_j dj$ shifts upwards, then even more types will be bunched at zero formal income at the bottom.

Let's think about shadow workers which are not at the bottom of the type space. The continuity assumptions guarantee that D_i^s and $\int_i^1 N_j dj$ terms are continuous in type. It implies that before any set of shadow workers that are not at the bottom of the type space is a marginal worker. Consider an interior allocation with a bunch of shadow workers at some positive formal income level. If we flatten the interior formal income schedule in order to make it non-decreasing (as in Figure 6), the first type in the bunch (type $\bar{a}$) will be a marginal worker ($\frac{v'\left(\frac{y_{\bar{a}}}{w_{\bar{a}}^f}\right)}{w_{\bar{a}}^s} = 1$), while all the other types with this level of formal income will be shadow workers ($\frac{v'\left(\frac{y_{\bar{a}}}{w_i^f}\right)}{w_i^s} < 1, i > \bar{a}$). To see this, note that $\frac{\partial}{\partial_i} \left(\frac{v'\left(\frac{y_{\bar{a}}}{w_i^f}\right)}{w_i^s} \right)$ is negative by $\frac{\rho_i^s}{\rho_s^f} > -\zeta^{-1}$. So far we discussed what happens at the flattened income schedule. The optimal income schedule involves no less distortions, so the shadow workers will not cease to supply shadow labor. □

Proof of Corollary 2. It is just an interior optimality condition for the shadow worker (29). By Lemma 11, all the shadow workers from the interior allocation are shadow workers in the optimum. □

B The estimation of the factor F_i and top earners Pareto distribution.

Here we present the variables included in the vector X_i and the parameter estimates of β and γ obtained from the specification given by (41). Table 4 lists the variables included in X_i with its

Table 4: Variables included in X_i

Variable	Description	Values
Worker characteristics		
Gender	Dummy variable equal to 1 for women	0-1
Age	Age of the worker	16-90
Age^2	Age squared	
Ed years	Number of education years	0-26
Degree	Highest degree achieved	1-5
		1 - no degree
		5 - postgraduate degree
Y work	Number of months worked in the last year	1-12
Experience	Number of months worked in the last job	0-720
First job	Dummy for the first job (1 if it is the first job)	0-1
Production unit (firm) characteristics		
Sector Man	Dummy for the manufacturing sector	0-1
Sector Fin	Dummy for financial intermediation	0-1
Sector ret	Dummy for the sales and retailers sector	0-1
Big city	Dummy for a firm in one of the two largest cities	0-1
Size	Categories for the number of workers	1-9
		1 - One worker
		9 - More than 101 workers
Production unit (Type of job) characteristics		
Lib	Dummy for a liberal occupation	0-1
Admin	Dummy for an administrative task	0-1
Seller	Dummy for sellers and related	0-1
Services	Dummy for a service task (bartender ..)	0-1
Worker-firm relationship		
Union	Dummy for labor union affiliation (1 if yes)	0-1
Agency	Dummy for agency hiring (1 if yes)	0-1
Seniority	Number of months of the worker in the firm	0-720

corresponding description and associated category. The parameter estimates are presented in Table 5. Finally, table 6 presents the estimate of the Pareto distribution for top earners.

Table 5: Estimation results

Parameter	Point estimate	std. error	t-statistic	95% conf. interval	
γ_0^f	6.859	0.033	211.9	6.89	7.02
$\gamma_0^s - \gamma_0^f$	0.102	0.032	-3.2	-0.16	-0.04
γ_1^s	0.682	0.037	12.6	0.648	0.716
β-Gender	-0.077	0.005	-11.6	-0.06	-0.04
β-Age	0.025	0.001	13.1	0.01	0.02
β-Age^2	0.000	0.000	-8.8	0.00	0.00
β-Ed years	0.037	0.002	15.4	0.02	0.03
β-Degree	0.156	0.005	21.1	0.10	0.12
β-Sector Man	-0.098	0.006	-11.9	-0.08	-0.06
β-Sector Fin	0.156	0.015	6.9	0.08	0.14
β-Sector ret	-0.150	0.006	-16.9	-0.11	-0.09
β-Big city	0.010	0.007	1.0	-0.01	0.02
β-Size	0.032	0.001	18.7	0.02	0.02
β-Union	0.126	0.010	8.3	0.07	0.11
β-Agency	-0.144	0.005	-18.3	-0.11	-0.09
β-Seniority	0.001	0.000	17.9	0.00	0.00
β-Y work	0.029	0.001	18.4	0.02	0.02
β-First job	-0.053	0.008	-4.7	-0.05	-0.02
β-Experience	0.000	0.000	5.3	0.00	0.00
β-Lib	0.074	0.013	3.9	0.03	0.08
β-Admin	-0.272	0.009	-19.9	-0.20	-0.17
β-Seller	-0.186	0.014	-9.2	-0.15	-0.10
β-Services	-0.267	0.009	-19.3	-0.20	-0.16

Table 6: Pareto distribution estimates

Parameter	Point estimate	std. error	z-statistic	95% conf. interval	
Shape parameter	1.81	0.0018	953.34	1.806	1.833