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Testbed Benchmark Engineering Problems and Test Functions for Optimization Algorithms

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1. Introduction

In optimization, interest has rapidly increased in recent years in developing non-gradient based algorithms that can deal with optimization problems that the gradient based optimization techniques often fail to solve. Many optimization algorithms have been introduced or developed and more are being developed. The question is how researchers judge the performance and the robustness of these algorithms. One of the most common ways used to test these algorithms is using benchmark test functions and sometimes engineering problems in which the objective functions, if there are more than one, and the constraints, if there are any, can be formulated and represented mathematically. In this study, a collection of many benchmark test functions are gathered, categorized in different libraries and presented. Continuous, unimodal and multimodal, single objective and multi-objective, constrained and unconstrained test functions are introduced in this report.

In this study, both unconstrained and constrained global optimization problems are considered. Without loss of generality, only minimization problems are studied since maximization problems can be transformed to minimization problems by inverting the sign of their objective functions. The mathematical definitions for the considered problems are given below.

2. Single Objective Optimization Test Problems

The following are the mathematical formula and the analytical global optimum solution of the benchmark test problems that are included in the testbed library.

2.1 Unconstrained benchmark Test Problems

1) **Banana (Rosenbrock) Function**[1] **, n=2:**

   (Small scale, uni-modal, global Optimization (GO), un-constrained, single objective, continuous)

   File name: banana.p.xml

   Rosenbrock's valley or Banana function is a classic test problem for optimization algorithms due to its challenge on the convergence and robustness of the algorithm. The global minimum is inside a long, narrow, parabolic shaped flat valley, and converge to the solution at point (1, 1) is well known to be difficult. The Banana function given by Equation (1) is illustrated in Figure 1.

$$f(x_1, x_2) = 100(x_2 - x_1^2)^2 + (1 - x_1)^2 \tag{1}$$

$$\text{Search Space:} \quad -2 \leq x_i \leq 2, \quad i = 1,2.$$

Global minimum: $x^* = (1, \dots, 1), \;\; f(x^*) = 0$

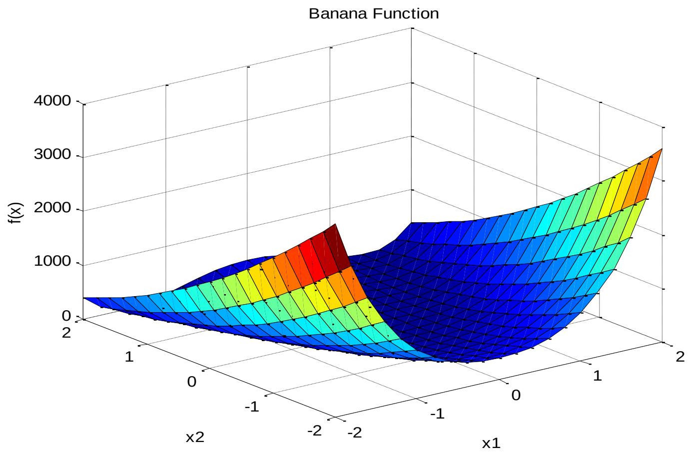

Figure 1: The Objective Function of Banana function

2) Beak Function, n=2:

(small scale, multi-modal, Global Optimization (GO), un-constrained, single objective, continuous)
Note: the reference for this test problem is unknown, so exercise caution when testing.

File name: beak.p.xml

The Beak function is used as a benchmark problem due to the challenge to find its global minimum. The Beak function given by Eq. (2) is illustrated in Figure 2.

$$f(x_1, x_2) = 3(1 - x_1)^2 \mathrm{e}^{(-x_1^2 - (x_2+1)^2)} - 10\left(\frac{x_1}{5} - x_1^3 - x_2^5\right)\mathrm{e}^{(-x_1^2 - x_2^2)} - \frac{1}{3}\mathrm{e}^{(-(x_1+1)^2 - x_2^2)} \quad (2)$$

Search Space: $-3 \leq x_1 \leq 3 \quad -3 \leq x_2 \leq 3.$

Global minimum: $x^* = (0.2283, -1.6255), \;\; f(x^*) = -6.5511$

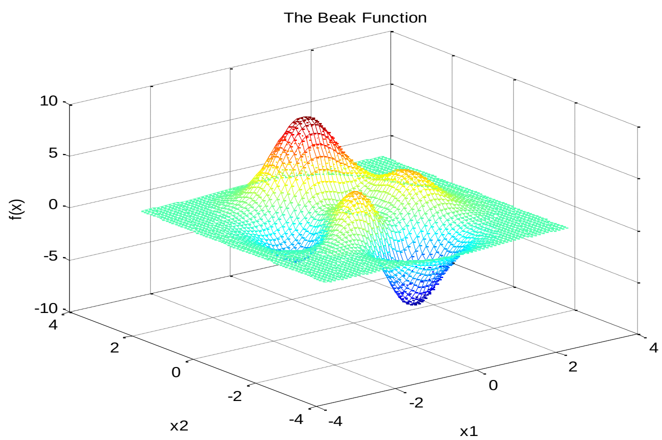

Figure 2: The Objective Function of Beak function

3) **Branin Function** [1]**, n=2:**

(small scale, uni-modal, Global Optimization (GO), un-constrained, single objective, continuous)

File name: branin.p.xml

Branin function is also used as a benchmark function due to the difficulty to find its global minimum; the Branin function given by Eq. (3), is illustrated in Figure 3.

$$f(x_1, x_2) = \left(x_2 - \frac{5.1}{4\pi^2}x_1^2 + \frac{5}{\pi}x_1 - 6\right)^2 + 10\left(1 - \frac{1}{8\pi}\right)\cos x_1 + 10 \qquad (3)$$

$$\text{Search Space: } -5 \leq x_1 \leq 10, \quad 0 \leq x_2 \leq 15.$$

Global minimum: $x^* = (-\pi, 12.275), (\pi, 2.275), (9.42478, 2.475),$

$$f(x^*) = 0.397887$$

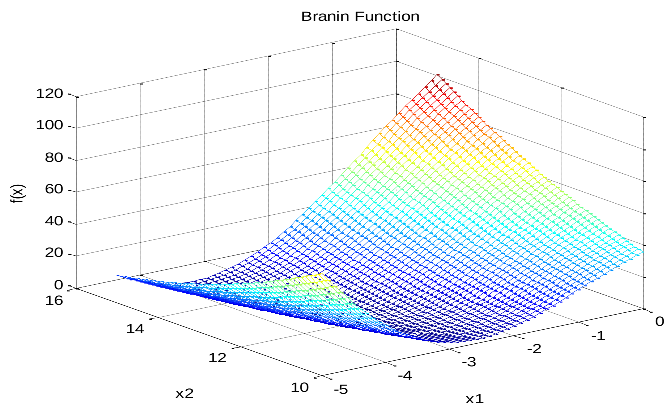

Figure 3: The Objective Function of Branin function

4) Schaffer's F6 Function [1]**, n=2:**

(small scale, uni-modal, Global Optimization (GO), un-constrained, single objective, continuous)

File name: f6.p.xml

Schaffer's function, a nonlinear two variable function, has been considered due to the challenge to converge to its global minimum. The F6 function given by Eq. (4) is illustrated in Figure 4.

$$f(x_1, x_2) = 0.5 + \frac{\sin^2\left(\sqrt{x_1^2 + x_2^2}\right) - 0.5}{(1 + 0.001(x_1^2 + x_2^2))^2} \tag{4}$$

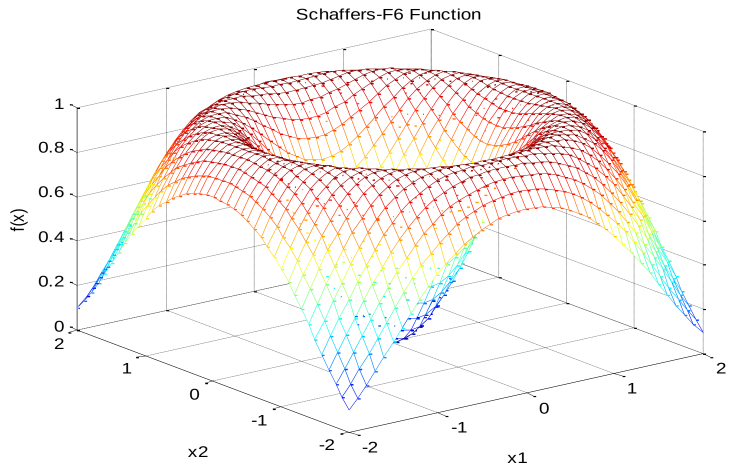

Figure 4: The Objective Function of Benchmark Test 6

$$\text{Search Space: } |x_i| \leq 100.$$

$$\text{Global minimum: } x^* = (0,0), \text{ at } f(x^*) = 0$$

5) **Generalized Polynomial Function, n=2:** (small scale, multi-modal, Global Optimization (GO), un-constrained, single objective, continuous)
Note: the reference for this test problem is unknown, so exercise caution when testing.

Filename: genpol.p.xml

Generalized polynomial function is used in the test also due to the challenge of convergence to the global minimum. The GF function given by Eq. (5) is illustrated in Figure 5.

$$f(x_1, x_2) = (1.5 - x_1(1 - x_2))^2 + (2.25 - x_1(1 - x_2^2))^2 + (2.625 - x_1(1 - x_2^3))^2 \quad (5)$$

$$\text{Search Space: } -2 \leq x_1, x_2 \leq 2$$

Global minimum: $x^* = (2.0000,\ 0.1700),\ f(x^*) = 0.5233$

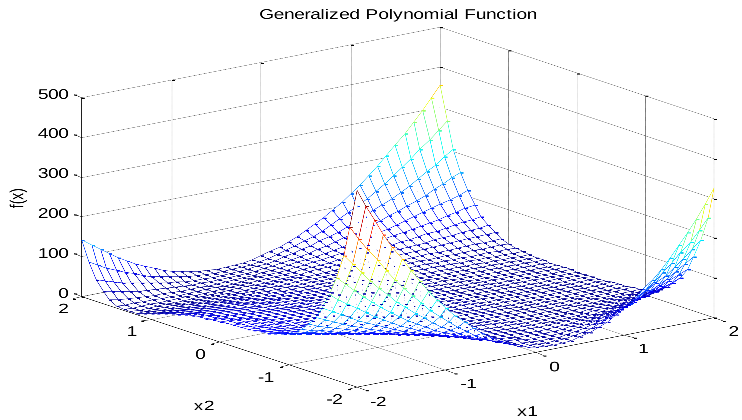

Figure 5: The Objective Function of Generalized Polynomial Function

6) **Goldstein and Price Function (GP)** [1]**, n=2:**
(small scale, multi-modal, Global Optimization (GO), un-constrained, single objective, continuous)

File name: gp.p.xml

Goldstein and Price function is often used as a benchmark function due to the difficulty to find its global minimum; the GP function given by Eq. (6), is illustrated in Figure 6.

$$f(x_1, x_2) = \begin{matrix} [1 + (x_1 + x_2 + 1)^2(19 - 14x_1 + 3x_1^2 - 14x_2 + 6x_1x_2 + 3x_2^2)] \\ \times [30 + (2x_1 - 3x_2)^2(18 - 32x_1 + 12x_1^2 + 48x_2 - 36x_1x_2 + 27x_2^2)] \end{matrix} \quad (6)$$

$$\text{Search Space: } -2 \le x_1, x_2 \le 2$$

Global minimum: $x^* = (0, -1),\ f(x^*) = 3$

Figure 6: The Objective Function of Goldstein and Price function

7) **Griewank Function** [1] **, n=2;** (small scale, uni-modal, Global Optimization (GO), un-constrained, single objective, continuous)

File name: griewank.p.xml

$$f(x_1, x_2) = \frac{x_1^2 + x_2^2}{2000} - \cos x_1 \cos\left({x_2}/{\sqrt{2}}\right) + 1 \qquad (7)$$

$$\text{Search Space: } -100 \le x_1, x_2 \le 100$$

Global minimum: $x^* = (0,0),\ f(x^*) = 0$

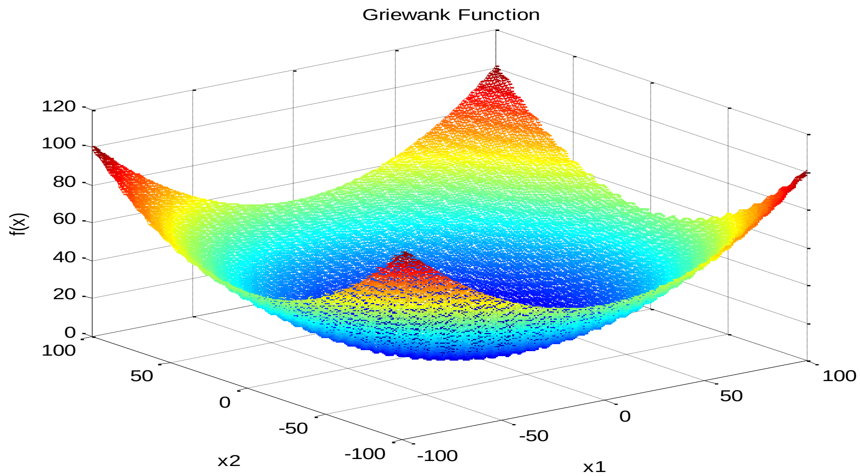

Figure 7: The Objective Function of Griewank Function

8) **Griewank Functions** ($\boldsymbol{G_n}$) [1]**:** *n* is scalable, (large scale, multi-modal, Global Optimization (GO), un-constrained, single objective, continuous)

File name: griewank1.p.xml

Definition: $\boldsymbol{f}(x) = \sum_{i=1}^{n} \frac{x_i^2}{4000} - \prod_{i=1}^{n} \cos\left(x_i/\sqrt{i}\right) + 1$

Search space: $-600 \leq x_i \leq 600,\ i = 1, \dots, n.$

Global minimum: $x^* = (0, 0); \boldsymbol{f}(x^*) = 0$

9) **Hartmann Function** $\boldsymbol{H_3}$ [1] **:** *n*=3**,** (small scale, multi-modal, Global Optimization (GO), un-constrained, single objective, continuous)

File name: hartmann_h3.p.xml

Definition: $H_3(x) = -\sum_{i=1}^{4} \alpha_i\, exp\left[-\sum_{j=1}^{3} A_{ij}\left(x_j - P_{ij}\right)^2\right],$

$$\alpha = [1, 1.2, 3, 3.2]^T,\ A = \begin{bmatrix} 3.0 & 10 & 30 \\ 0.1 & 10 & 35 \\ 3.0 & 10 & 30 \\ 0.1 & 10 & 35 \end{bmatrix}, P = 10^{-4} \begin{bmatrix} 3689 & 1170 & 2673 \\ 4699 & 4387 & 7470 \\ 1091 & 8732 & 5547 \\ 381 & 5743 & 8828 \end{bmatrix}$$

Search space: $0 \leq x_j \leq 1,\ j = 1,2,3.$

Global minimum: $x^* = (0.114,\ 0.556,\ 0.852);\ H_3(x^*) = -3.86$

10) Hartman Function $\boldsymbol{H_6}$ [1] **, n=6: (**large scale, multi-modal, Global Optimization (GO), un-constrained, single objective, continuous)

File name: hartmann_h6.p.xml

Definition: $H_6(x) = -\sum_{i=1}^{4} \alpha_i\, exp\left[-\sum_{j=1}^{6} B_{ij}\left(x_j - Q_{ij}\right)^2\right],$

$$\alpha = [1, 1.2, 3, 3.2]^T,\ B = \begin{bmatrix} 10 & 3 & 17 & 3.5 & 1.7 & 8 \\ 0.05 & 10 & 17 & 0.1 & 8 & 14 \\ 3 & 3.5 & 1.7 & 10 & 17 & 8 \\ 17 & 8 & 0.05 & 10 & 0.1 & 14 \end{bmatrix},$$

$$Q = \begin{bmatrix} 0.1312 & 0.1696 & 0.5569 & 0.0124 & 0.8283 & 0.5886 \\ 0.2329 & 0.4135 & 0.8307 & 0.3736 & 0.1004 & 0.9991 \\ 0.2348 & 0.1451 & 0.3522 & 0.2883 & 0.3047 & 0.6650 \\ 0.4047 & 0.8828 & 0.8732 & 0.5743 & 0.1091 & 0.0381 \end{bmatrix}$$

Search space: $0 \le x_j \le 1,\ j = 1, \ldots, 6.$

Global minimum:

$x^* = (0.201, 0.150, 0.477, 0.275, 0.311, 0.657);\ H_6(x^*) = -3.32$

11) Hartmann Function with 16 Design Variables (H16) [2]**, n=16:** (large scale, multi-modal, GO, un-constrained, single objective, continuous)

File name: hartmann_h16.p.xml

$$f(x) = \sum_{i=1}^{16} \sum_{j=1}^{16} a_{ij}\,(x_i^2 + x_i + 1)\left(x_j^2 + x_j + 1\right), \qquad i, j = 1,2,\cdots,16$$

$$[a_{ij}]_{row1-8} = \begin{bmatrix} 1&0&0&1&0&0&1&1&0&0&0&0&0&0&0&1\\ 0&1&1&0&0&0&1&0&0&1&0&0&0&0&0&0\\ 0&0&1&0&0&0&1&0&1&1&0&0&0&1&0&0\\ 0&0&0&1&0&0&1&0&0&0&1&0&0&0&1&0\\ 0&0&0&0&1&1&0&0&0&1&0&1&0&0&0&1\\ 0&0&0&0&0&1&0&1&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&1&0&0&0&1&0&1&0&0&0\\ 0&0&0&0&0&0&0&1&0&1&0&0&0&0&1&0 \end{bmatrix}$$

$$[a_{ij}]_{row9-16} = \begin{bmatrix} 0&0&0&0&0&0&0&0&1&0&0&1&0&0&0&1\\ 0&0&0&0&0&0&0&0&0&1&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&0&0&1&0&1&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&1&0&1&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&1&1&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\ 0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1 \end{bmatrix},$$

$$\text{Search Space: } -1 \le x_i \le 1, \quad i = 1, \cdots, 16$$

Global minimum: $f(x^*) = 25.875$

12) Levy Function [1] **, n=4:** (small scale, multi-modal, Global Optimization (GO), un-constrained, single objective, continuous)

File name: levy.p.xml

This is “mvfLevy8” in the reference

$$f(x) = \sin^2(\pi y_1) + \sum_{i=1}^{n-1}\left((y_i - 1)^2(1 + 10\sin^2(\pi y_i + 1))\right) + (y_{n-1} - 1)^2(1 + \sin^2(2\pi x_{n-1}))\,, y_i = 1 + \frac{x_i - 1}{4}, i = 1, \cdots, n$$

$$\text{Search Space: } -10 \le x_i \le 10, \qquad i = 1, \cdots, n.$$

Global minimum: $x^* = (1, \ldots, 1)$, $f(x^*) = 0$

13) Shekel Function [1] **, n=4, m = 10:** (small scale, multi-modal, Global Optimization (GO), un-constrained, single objective, continuous)

File name: shekel.p.xml

Definition: $f(x) = -\sum_{i=1}^{m} \frac{1}{(x - A_i)^t \cdot (x - A_i) + c_i}$

$c = \frac{1}{10}[1, 2, 2, 4, 4, 6, 3, 7, 5, 5]^t$, and

$$A = \begin{bmatrix} 4 & 4 & 4 & 4 \\ 1 & 1 & 1 & 1 \\ 8 & 8 & 8 & 8 \\ 6 & 6 & 6 & 6 \\ 3 & 7 & 3 & 7 \\ 2 & 9 & 2 & 9 \\ 5 & 5 & 3 & 3 \\ 8 & 1 & 8 & 1 \\ 6 & 2 & 6 & 2 \\ 7 & 3.6 & 7 & 3.6 \end{bmatrix}$$

Note that x is also a column vector.

Search Space: $0 \leq x_i \leq 10, \;\; i = 1, \dots, 4.$

Global minimum: $x^* = (4, 4, 4, 4),\;\; f(x^*)_{m=5} = -10.1532,\;\; f(x^*)_{m=7} = -10.4029, f(x^*)_{m=10} = -10.5364.$

m	**Points**
5	(4.00004, 4.00013, 4.00004, 4.00013)
7	(4.00057, 4.00069, 3.99949, 3.99961)
10	(4.00075, 4.00059, 3.99966, 3.99951)

14) Shubert Function [3] **, *n*=2:** (small scale, multi-modal, Global Optimization (GO), un-constrained, single objective, continuous)

File name: shubert.p.xml

$$f(x_1, x_2) = \left(\sum_{i=1}^{5} i\cos\big((i+1)x_1 + i\big)\right)\left(\sum_{i=1}^{5} i\cos\big((i+1)x_2 + i\big)\right),$$

Search Space: $-10 \le x_i \le 10, \quad i = 1,2$

Global minimum: $x^* = (-1.4252, 5.4829),\ f(x^*) = -186.7309$

15) Sphere Function, *n*=10 [1] [2]**:** (large scale, multi-modal, Global Optimization (GO), un-constrained, single objective, continuous)

File name: sphere.p.xml

$$f(x) = \sum_{i=1}^{n} x_i^2,$$

Search space $|x_i| \le 5.12$

Global minimum: $x^* = (0, \dots, 0), \quad f(x^*) = 0$

16) Six-hump Camel-back Function (SC) [1]**,** *n*=2**:** (small scale, muli-modal, Global Optimization (GO), un-constrained, single objective, continuous)

File name: hump.p.xml

$$f(x_1, x_2) = 4x_1^2 - 2.1x_1^4 + \frac{1}{3}x_1^6 + x_1x_2 - 4x_2^2 + 4x_2^4$$

Search Space: $-5 \le x_1, x_2 \le 5,$

Global minimum: $x^* = (0.08983, -0.7126), x^* = (-0.08983, 0.7126)$

$$f(x^*) = -1.0316$$

17) Trid Function [1] **,** *n*=6 and 10: (large scale, multi-modal, Global Optimization (GO), un-constrained, single objective, continuous)

File name: trid.p.xml

Definition: $f(x) = \sum_{i=1}^{n}(x_i - 1)^2 - \sum_{i=2}^{n} x_i x_{i-1}$,

Search Space: $-n^2 \leq x_i \leq n^2, \qquad i = 1, \dots, n$

Global minimum: $n = 6, \quad x_i^* = i(7-i), \quad i = 1, \dots, n, \quad f(x^*) = -50$

$n = 10, \qquad x_i^* = i(11-i), \quad i = 1, \dots, n, \quad f(x^*) = -210$

18) Ackley Function AK [1] : *n*= scalable, (large scale, multi-modal, Global Optimization (GO), un-constrained, single objective, continuous)

File name: ackley.p.xml

Definition: $f(x) = 20 + e - 20e^{-\frac{1}{5}\sqrt{\frac{1}{n}\sum_{i=1}^{n} x_i^2}} - e^{\frac{1}{n}\sum_{i=1}^{n}\cos(2\pi x_i)}$

Search space: $-30 \leq x_i \leq 30, \ i = 1,2, \dots, n$

Global minimum: $x^* = (0, \dots, 0); f(x^*) = 0$

19) Bohachevsky Function [1] : *n*=2, (low scale, multi-modal, Global Optimization (GO), un-constrained, single objective, continuous)

File name: bohachecsky.p.xml

Definition: $f(x) = x_1^2 + 2x_2^2 - 0.3\cos(3\pi x_1) - 0.4\cos(4\pi x_2) + 0.7$

Search space: $-50 \leq x_i \leq 50, \ i = 1,2.$

Global minimum: $x^* = (0, 0); f(x^*) = 0$

20) Beale Function [1]**:** *n*=2**,** (small scale, multi-modal, Global Optimization (GO), un-constrained, single objective, continuous)

File name: beale.p.xml

Definition: $f(x) = (1.5 - x_1 + x_1x_2)^2 + (2.25 - x_1 + x_1x_2^2)^2$

$$+(2.625 - x_1 + x_1x_2^3)^2$$

Search space: $-4.5 \leq x_i \leq 4.5,\ i = 1,2.$

Global minimum: $x^* = (3, 0.5); f(x^*) = 0$

21) Booth Function [1]**:** *n*=2**,** (small scale, multi-modal, Global Optimization (GO), un-constrained, single objective, continuous)

File name: booth.p.xml

Definition: $f(x) = (x_1 + 2x_2 - 7)^2 + (2x_1 + x_2 - 5)^2$

Search space: $-10 \leq x_i \leq 10,\ i = 1,2.$

Global minimum: $x^* = (1, 3); f(x^*) = 0$

22) Colville Function [1] **:** *n*=4**,** (small scale, multi-modal, Global Optimization (GO), un-constrained, single objective, continuous)

File name: colville.p.xml

Definition: $f(x) = 100(x_1 - x_2^2)^2 + (x_1 - 1)^2 + (x_3 - 1)^2 + 90(x_3^2 - x_4)^2 + 10.1(x_2 - 1)^2 + 10.1(x_4 - 1)^2 + 19.8(x_2 - 1)(x_4 - 1)$

Search space: $-10 \leq x_i \leq 10,\ i = 1, \ldots, 4.$

Global minimum: $x^* = (1, 1,1,1); f(x^*) = 0$

23) De Joung Function [4]**:** n=3**,** (small scale, multi-modal, Global Optimization (GO), un-constrained, single objective, continuous)

File name: dejoung.p.xml

Definition: $f(x) = x_1^2 + x_2^2 + x_3^2$

Search space: $-5 \le x_i \le 5,\ i = 1,2,3.$

Global minimum: $x^* = (0, 0, 0); f(x^*) = 0$

24) Dixon&Price Functions : n=25, (large scale, multi-modal, Global Optimization (GO), un-constrained, single objective, continuous)

Note: the reference for this test problem is unknown, so exercise caution when testing.

File name: dixon.p.xml

Definition: $f(x) = (x_1 - 1)^2 + \sum_{i=2}^{n} i(2x_i^2 - x_{i-1})^2$

Search space: $-10 \le x_i \le 10,\ i = 1, \ldots, n.$

Global minimum: $x_i^* = 0,\ i = 1, \ldots, n;\ f(x^*) = 0$

25) Easom Function [1] **:** n=2**,** (small scale, multi-modal, Global Optimization (GO), un-constrained, single objective, continuous)

File name: easom.p.xml

Definition: $f(x) = -\cos(x_1)\cos(x_2)\, exp(-(x_1 - \pi)^2 + (x_2 - \pi)^2)$

Search space: $-100 \le x_i \le 100,\ i = 1,2.$

Global minimum: $x^* = (\pi, \pi); f(x^*) = -1$

26) $\mathbf{F_1}$ Function [4]: *n*=2**,** (small scale, multi-modal, Global Optimization (GO), un-constrained, single objective, continuous)

File name: f1.p.xml

Definition: $F_1(x_1, x_2) = x_1^2 + x_2^2 - \cos(18x_1) - \cos(18x_2)$

Search space: $-1 \leq x_j \leq 1,\ i = 1,2.$

Global minimum: $x^* = (0, 0);\ F_1((x_1, x_2)^*) = -2$

27) $\mathbf{F_2}$ Function : *n*=10**,** (large scale, multi-modal, Global Optimization (GO), un-constrained, single objective, continuous)

Note: the reference for this test problem is unknown, so exercise caution when testing.

File name: f2.p.xml

Definition: $F_2(x) = \sum_{j=1}^{10} min\{|x_j - 0.2| + a|x_j - 0.4|, |x_j - 0.7| + a\}, a = 0.05$

Search space: $0 \leq x_j \leq 1,\ j = 1, \ldots, 10.$

Global minimum: $x^* = (0.4, 0.4, \ldots, 0.4);\ F_2(x^*) = 2$

28) De Joung Function $\mathbf{F_5}$ [1]: *n*=2**,** (small scale, multi-modal, Global Optimization (GO), un-constrained, single objective, continuous)

File name: f5.p.xml

Definition: $F_5(x_1, x_2) = \left(0.002 + \sum_{j=1}^{25} \frac{1}{j + \sum_{i=1}^{2} (x_i - a_{ij})^6}\right)^{-1}$

$a_{1j} = -32, -16, 0, 16, 32$ for $j = 1,2, \ldots, 5$

$a_{1k} = a_{1j}$ for $k = j+5, j+10, j+15, j+20, and\ j = 1,2,\dots,5$

$a_{2j} = -32, -16, 0, 16, 32$ for $j = 1, 6, 11, 16, 21$

$a_{2k} = a_{2j}$ for $k = j+1, j+2, j+3, j+4, and\ j = 1, 6, 11, 16, 21$

Search space: $-65.536 \leq x_i \leq 65.536,\ \ j = 1{,}2.$

Global minimum: $(x_1, x_2)^* = (-32, -32); F5((x_1, x_2)^*) = 0.998004$

29) **Matyas Function** [1] **:** *n*=2**,** (small scale, multi-modal, GO, un-constrained, single objective, continuous)

File name: matyas.p.xml

Definition: $f(x_1, x_2) = 0.26(x_1^2 + x_2^2) - 0.48x_1x_2$

Search space: $-10 \leq x_j \leq 10,\ \ i = 1{,}2.$

Global minimum: $x^* = (0, 0);\ f((x_1, x_2)^*) = 0$

30) **Michalewicz Function** [1] [5]**:** *n*=2**,** (small scale, multi-modal, GO, un-constrained, single objective, continuous)

File name: michalewicz.p.xml

Definition: $f(x_1, x_2) = -\sum_{j=1}^{2} \sin\left(x_j\right)\left(\sin\left(jx_j^2/\pi\right)\right)^{2m}; m = 10$

Search space: $0 \leq x_j \leq \pi,\ \ i = 1{,}2.$

31) Global minimum:$x^* = (2.20319{,}1.57049)\ \ f((x_1, x_2)^*) = -1.8013$ **Perm Function** [1]**:** *n* is scalable, (large scale, multi-modal, GO, un-constrained, single objective, continuous)

File name: perm.p.xml

Referred to as “mvfNeumaierPerm” in the reference

Definition: $f(x) = \sum_{k=1}^{n}[\sum_{i=1}^{n}(i^k + \beta)((x_i/i)^k - 1)]^2$;

Search space: $-n \leq x_i \leq n,\ i = 1, \dots, n.$

Global minimum: $x^* = (1, 2, \dots, n);\ f(x^*) = 0$

32) Power Sum Functions [1] **:** *n* = 4, (large scale, multi-modal, GO, un-constrained, single objective, continuous)

File name: power.p.xml

Referred to as “mvfNeumaierPerm” in the reference

Definition: $f(x) = \sum_{k=1}^{n}\left[\left(\sum_{i=1}^{n} x_i^k\right) - b_k\right]^2$

Search space: $0 \leq x_j \leq n,\ i = 1, \dots, n.$

Global minimum: $b = (8,18,44,114)\ x^* = (1,2,2,3);\ f((x_1, x_2)^*) = 0$

33) Powell Functions [1] [6] **:** *n* is a multiple of 4, (large scale, multi-modal, GO, un-constrained, single objective, continuous)

File name: powell.p.xml

Definition: $f(x) = \sum_{i=1}^{n/4}(x_{4i-3} + 10x_{4i-2})^2 + 5(x_{4i-1} - x_{4i})^2 + (x_{4i-2} - 2x_{4i-1})^4 + 10(x_{4i-3} - x_{4i})^4$

Search space: $-4 \leq x_j \leq 5,\ i = 1, \dots, n.$

Global minimum: $x^* = [0\,..\,0];\ f((x)^*) = 0$

34) Rosenbrock Functions [1] : *n* is scalable**:** (large scale, multi-modal, GO, un-constrained, single objective, continuous)

File name: rosenbrock1.p.xml

Definition: $f(x) = \sum_{i=1}^{n-1}[100(x_i^2 - x_{i+1})^2 + (x_i - 1)^2]$

Search space: $-10 \leq x_j \leq 10,\ \ i = 1,2, \dots, n.$

Global minimum: $x^* = (1, \dots, 1);\ f((x)^*) = 0$

35) Rastrigin Function [1]**:** *n* is scalable**,** (large scale, multi-modal, GO, un-constrained, single objective, continuous)

File name: rastrigin1.p.xml

Definition: $f(x) = 10n + \sum_{i=1}^{n}\left(x_i^2 - 10\cos(2\pi x_i)\right)$

Search Space: $-5.12 \leq x_i \leq 5.12,\ \ i = 1, \dots, n.$

Global minimum: $x^* = (0, \dots, 0),\ f(x^*) = 0$

36) Schwefel Functions [1] **:** *n* is scalable**,** (large scale, multi-modal, GO, un-constrained, single objective, continuous)

File name: schwefel.p.xml

Definition: $f(x) = 418.9829n - \sum_{i=1}^{n}\left(x_i \sin\sqrt{|x_i|}\right)$

Search Space: $-500 \leq x_i \leq 500,\ \ i = 1,2, \dots, n.$

Global minimum: $x^* = (420.9687, \dots, 420.9687),\ f(x^*) = 0$

37) Sum Squares Function [1] **:** *n* is scalable, (large scale, multi-modal, GO, un-constrained, single objective, continuous)

Filr name: sum_squares.p.xml

Definition: $f(x) = \sum_{i=1}^{n} ix_i^2$,

Search Space: $-10 \leq x_i \leq 10, \quad i = 1, \cdots, n$

Global minimum: $x^* = (0, \ldots, 0), \quad f(x^*) = 0$

2.2 Constrained Benchmark Test Problems

1) **Problem G1** [3] **:** (large scale, multi-modal, GO, constrained, single objective, continuous)

   File name: g1.p.xml

$$min\, f(x_1, x_2) = 5\sum_{i=1}^{4} x_i - 5\sum_{i=1}^{4} x_i^2 - \sum_{i=5}^{13} x_i$$

Subject to

$$g_1(x) = 2x_1 + 2x_2 + x_{10} + x_{11} - 10 \leq 0,$$
$$g_2(x) = 2x_1 + 2x_3 + x_{10} + x_{12} - 10 \leq 0,$$
$$g_3(x) = 2x_2 + 2x_3 + x_{11} + x_{12} - 10 \leq 0,$$
$$g_4(x) = -8x_1 + x_{10} \leq 0,$$
$$g_5(x) = -8x_2 + x_{11} \leq 0,$$
$$g_6(x) = -8x_3 + x_{12} \leq 0,$$
$$g_7(x) = -2x_4 - x_5 + x_{10} \leq 0,$$
$$g_8(x) = -2x_6 - x_7 + x_{11} \leq 0,$$
$$g_8(x) = -2x_8 - x_9 + x_{12} \leq 0,$$
$$x_i \geq 0, i = 1, \ldots, 13,$$
$$x_i \leq 1, i = 1, \ldots, 9, 13.$$

The bounds: $U = (1,1,1,1,1,1,1,1,1,100,100,100,1)$ and $L = (0, \dots, 0)$,

Global minimum: $x^{*} = (1,1,1,1,1,1,1,1,1,3,3,3,1)$, $f(x^{*}) = -15$

2) **Problem G2** [3] **:** (large scale, multi-modal, GO, constrained, single objective, continuous)

File name: g2.p.xml

$$maxf(x) = \left| \frac{\sum_{i=1}^{n} cos^{4}(x_i) - 2\prod_{i=1}^{n} cos^{2}(x_i)}{\sqrt{\sum_{i=1}^{n} i x_i^2}} \right|$$

Subject to

$$g_1(x) = -\prod_{i=1}^{n} x_i + 0.75 \le 0,$$

$$g_2(x) = \sum_{i=1}^{n} x_i - 7.5n \le 0,$$

The bounds: $U = (10, \dots, 10)$ and $L = (0, \dots, 0)$,

Global minimum: $f(x^{*}) = 0.803619$, for $n = 20$.

3) **Problem G3** [3] **:** (large scale, multi-modal, GO, constrained, single objective, continuous)

File name: g3.p.xml

$$max\ f(x) = \left(\sqrt{n}\right)^{n} \prod_{i=1}^{n} x_i$$

Subject to:

$$h_1(x) = \sum_{i=1}^{n} x_i^2 - 1 = 0$$

The bounds: $U = (1, \dots, 1)$ and $L = (0, \dots, 0)$,

Global minimum: $x^* = \left(\frac{1}{\sqrt{n}}, \dots, \frac{1}{\sqrt{n}}\right)$; $f(x^*) = 1$.

4) **Problem G4** [3] **:** (large scale, multi-modal, GO, constrained, single objective, continuous)
File name: g4.p.xml

$$min f(x) = 5.3578547x_3^2 + 0.8356891x_1x_5 + 37.293239x_1 - 40792.141$$

Subject to:

$$h_1(x) = u(x) - 92 \leq 0,$$

$$h_2(x) = -u(x) \leq 0,$$

$$h_3(x) = v(x) - 110 \leq 0,$$

$$h_4(x) = -v(x) + 90 \leq 0,$$

$$h_5 = w(x) - 25 \leq 0,$$

$$h_6(x) = -w(x) + 20 \leq 0,$$

where

$$u(x) = 85.334407 + 0.0056858x_2x_5 + 0.0006262x_1x_4 - 0.0022053x_3x_5,$$

$$v(x) = 80.51249 + 0.0071317x_2x_5 + 0.0029955x_1x_2 + 0.002183x_3^2,$$

$$w(x) = 9.300961 + 0.0047026x_3x_5 + 0.0012547x_1x_3 + 0.0019085x_3x_4.$$

The bounds: $U = (102,45,45,45,45)$ and $L = (78,33,27,27,27)$,

Global minimum: $x^* =$
$(78, 33, 29.995256025682, 45, 36.775812905788)$; $f(x^*) = -30.665.539$

5) **Problem G5** [3] **: (**large scale, multi-modal, GO, constrained, single objective, continuous)
File name: g5.p.xml

$$min\ f(x) = 3x_1 + 10^{-6}x_1^3 + 2x_2 + \frac{2}{3} \times 10^{-6}x_2^3,$$

Subject to:

$$g_1(x) = x_3 - x_4 - 0.55 \leq 0,$$

$$g_2(x) = x_4 - x_3 - 0.55 \leq 0,$$

$$h_1(x) = 1000[sin(-x_3 - 0.25) + sin(-x_4 - 0.25)] + 894.8 - x_1 = 0,$$

$$h_2(x) = 1000[sin(x_3 - 0.25) + sin(x_3 - x_4 - 0.25)] + 894.8 - x_2 = 0,$$

$$h_3(x) = 1000[sin(x_4 - 0.25) + sin(x_4 - x_3 - 0.25)] + 1294.8 = 0,$$

The bounds: $U = (1200,1200, 0.55,0.55)$ and $L = (0,0, -0.55, -0.55)$,

Global minimum: $x^* = (679.9453,1026,0.118876, -0.3962336)$,

$$f(x^*) = 5126.4981.$$

6) **Problem G6** [2]**:** (small scale, multi-modal, GO, constrained, single objective, continuous)
File name: g6.p.xml

$$min\, f(x_1, x_2) = (x_1 - 10)^3 + (x_2 - 20)^3$$

subject to:

$$h_1(x) = -(x_1 - 5)^2 - (x_2 - 5)^2 + 100 \leq 0,$$

$$h_2(x) = (x_1 - 6)^2 + (x_2 - 5)^2 - 82.81 \leq 0.$$

Search Space: $13 \leq x_1 \leq 100;\ \ 0 \leq x_2 \leq 100.$

The bounds: $U = (100,100), L = (13,0)$

Global minimum: $x^* = (14.095,\ 0.84296),\quad f(x^*) = -6961.81388$

7) **Problem G7** [3] **:** (small scale, multi-modal, GO, constrained, single objective, continuous)
File name: g7.p.xml

$min\ f(x) = x_1^2 + x_2^2 + x_1x_2 - 14x_1 - 16x_2 + (x_3 - 10)^2 + 4(x_4 - 5)^2 + (x_5 - 3)^2 + 2(x_6 - 1)^2 + 5x_7^2 + 7(x_8 - 11)^2 + 2(x_9 - 10)^2 + (x_{10} - 7)^2 + 45,$

Subject to:

$$g_1(x) = 4x_1 + 5x_2 - 3x_7 + 9x_8 - 105 \leq 0,$$

$$g_2(x) = 10x_1 - 8x_2 - 17x_7 + 2x_8 \leq 0,$$

$$g_3(x) = -8x_1 + 2x_2 + 5x_9 - 2x_{10} - 12 \leq 0,$$

$$g_4(x) = 3(x_1 - 2)^2 + 4(x_2 - 3)^2 + 2x_3^2 - 7x_4 - 120 \leq 0,$$

$$g_5(x) = 5x_1^2 + 8x_2 + (x_3 - 6)^2 - 2x_4 - 40 \leq 0,$$

$$g_6(x) = 0.5(x_1 - 8)^2 + 2(x_2 - 4)^2 + 3x_5^2 - x_6 - 30 \leq 0,$$

$$g_7(x) = x_1^2 + 2(x_2 - 2)^2 - 2x_1x_2 + 14x_5 - 6x_6 \leq 0,$$

$$g_8(x) = -3x_1 + 6x_2 + 12(x_9 - 8)^2 - 7x_{10} \leq 0,$$

The bounds: $U = (10, \dots, 10)$ and $L = (-10 \dots, , -10)$,

Global minimum: $x^* =$

$$\begin{pmatrix} 2.171996, & 2.363683, & 8.773926, 5.095984, 0.9906548, 1.430574, \\ & 1.321644, 9.828726, 8.280092, 8.375927 & \end{pmatrix},$$

$$f(x^*) = 24.3062091.$$

8) Problem G8 [3] **:** (small scale, multi-modal, GO, constrained, single objective, continuous)
File name: g8.p.xml

$$max\ f(x) = \frac{sin^3(2\pi x_1)\sin(2\pi x_2)}{x_1^3(x_1 + x_2)}$$

subject to:

$$g_1(x) = x_1^2 - x_2 + 1 \leq 0,$$

$$g_2(x) = 1 - x_1 + (x_2 - 4)^2 \leq 0.$$

The bounds: $U = (10,10)$ and $L = (0,0)$, Global minimum: $x^* = (1.2279713, 4.2453733)$, $f(x^*) = 0.095825$

9) **Problem G9** [3] **:** (large scale, multi-modal, GO, constrained, single objective, continuous)
File name: g9.p.xml

$$min\, f(x) = (x_1 - 10)^2 + 5(x_2 - 12)^2 + x_3^4 + 3(x_4 - 11)^2 + 10x_5^6 + 7x_6^2 + x_7^4 - 4x_6x_7 - 10x_6 - 8x_7$$

Subject to:

$$g_1(x) = 2x_1^2 + 3x_2^4 + x_3 + 4x_4^2 + 5x_5 - 127 \leq 0,$$

$$g_2(x) = 7x_1 + 3x_2 + 10x_3^2 + x_4 - x_5 - 282 \leq 0,$$

$$g_3(x) = 23x_1 + x_2^2 + 6x_6^2 - 8x_7 - 196 \leq 0,$$

$$g_4(x) = 4x_1^2 + x_2^2 - 3x_1x_2 + 2x_3^2 + 5x_6 - 11x_7 \leq 0,$$

The bounds: $U = (10, \ldots, 10)$ and $L = (-10, \ldots, -10)$,

Global minimum: $x^* =$

$$(2.330499, 1.951372, -0.4775414, 4.365726, -0.6244870, 1.038131, 1.594227),$$

$$f(x^*) = 680.6300573.$$

10) **Problem G10** [3] **:** (large scale, multi-modal, GO, constrained, single objective, continuous)
File name: g10.p.xml

$$min\, f(x) = x_1 + x_2 + x_3$$

Subject to:

$$g_1(x) = -1 + 0.0025(x_4 + x_6) \leq 0,$$

$$g_2(x) = -1 + 0.0025(-x_4 + x_5 + x_7) \leq 0,$$

$$g_3(x) = -1 + 0.01(-x_5 + x_8) \leq 0,$$

$$g_4(x) = 100x_1 - x_1x_6 + 833.33252x_4 - 83333.333 \leq 0,$$

$$g_5(x) = x_2x_4 - x_2x_7 - 1250x_4 + 1250x_5 \leq 0,$$

$$g_6(x) = x_3x_5 - x_3x_8 - 2500x_5 + 1250000 \leq 0,$$

The bounds: $U = (10000,10000,10000,1000,1000,1000,1000,1000)$ and $L = (100, 1000, 1000, 10, 10, 10, 10, 10)$,

Global minimum:

$x^* =$

$(579.3167, 1359.943, 5110.071, 182.0174, 295.5985, 217.9799, 286.4162, 395.5979)$,

$$f(x^*) = 7049.3307.$$

11) Problem G11 [3] **:** (large scale, multi-modal, GO, constrained, single objective, continuous)
File name: g11.p.xml

$$min\ f(x) = x_1^2 + (x_2 - 1)^2$$

Subject to:

$$h_1(x) = x_2 - x_1^2 = 0,$$

The bounds: $U = (1, 1)$ and $L = (-1, -1)$,

Global minimum: $x^* = \left(\pm\frac{1}{\sqrt{2}}, \frac{1}{2}\right)$, $f(x^*) = 0.75$

12) Problem G12 [3] **:** (large scale, multi-modal, GO, constrained, single objective, continuous)
File name: g12.p.xml

$$min\ f(x) = 1 - 0.01[(x_1 - 5)^2 + (x_2 - 5)^2 + (x_3 - 5)^2]$$

Subject to:

$$g_{ijk}(x) = (x_1 - i)^2 + (x_2 - j)^2 + (x_3 - k)^2 - 0.0625 \leq 0,$$

For some $i, j, k = 1,2, \dots ,9$

The bounds: $U = (10, 10, 10)$ and $L = (0, 0, 0)$,

Global minimum: $x^* = (5, 5, 5), f(x^*) = 1, i = 5, j = 5, k = 5.$

13) Problem G13 [3] **:** (large scale, multi-modal, GO, constrained, single objective, continuous)
File name: g13.p.xml

$$minf(x_1, x_2, x_3, x_4, x_5) = e^{x_1 x_2 x_3 x_4 x_5}$$

subject to:

$$h_1(x) = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 - 10 = 0,$$

$$h_2(x) = x_2 x_3 - 5x_4 x_5 = 0,$$

$$h_3(x) = x_1^3 + x_2^3 + 1 = 0.$$

The bounds: $U = (2.3, 2.3, 3.2, 3.2, 3.2)$ and $L = (-2.3, -2.3, -3.2, -3.2, -3.2)$,

Search Space: $-2.3 \le x_1, x_2 \le 2.3$; $-3.2 \le x_3, x_4, x_5 \le 3.2$

Global minimum: $x^* = (-1.717143, 1.5957109, 1.827247, -0.7636413, -0.763645)$;

$$f(x^*) = 0.0539498$$

2.3 Practical Engineering Problems (ENGGO library)

2.3.1 Tension compression string problem

(small scale, multi-modal, GO, constrained, single objective, continuous)

File name: engapp1.p.xml

This problem is originally from Arora (1989) [7] and can be found in [8], which needs to minimize the weight (i.e. $f(x)$) of a tension/compression spring (as shown in Figure 8) subject to constraints on minimum deflection, shear stress, surge frequency, limits on outside diameter and on design variables. The design variables are the mean coil diameter D (x_2), the wire diameter d (x_1) and the number of active coils P (x_3).

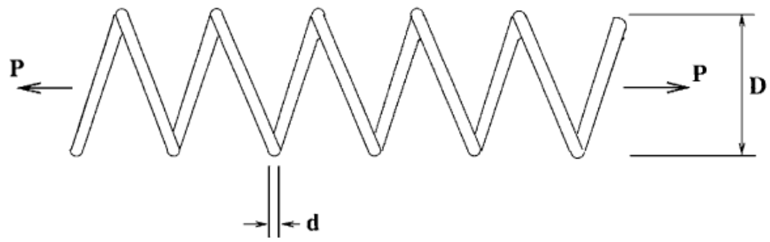

Figure 8: Tension/Compression string design problem

The mathematical formulation of this problem can be described as follows:The problem of minimizing the weight of a tension-compression string [8] can be expressed as the following optimization problem with three design variables $x = (x_1, x_2, x_3)$;

$$\min f(x_1, x_2, x_3) = x_1^2 x_2 (x_3 + 2),$$

subject to:

$$g_1(x) = 1 - \frac{x_2^3 x_3}{71785\, x_1^4} \leq 0,$$

$$g_2(x) = \frac{4x_2^2 - x_1 x_2}{12566 x_1^3 (x_2 - x_1)} + \frac{1}{5108 x_1^2} - 1 \leq 0,$$

$$g_3(x) = 1 - \frac{140.45 x_1}{x_3 x_2^2} \leq 0,$$

$$g_4(x) = \frac{x_1 + x_2}{1.5} - 1 \leq 0.$$

Global minimum: $x^* = (0.051989, 0.363965, 10.890522)$; $f(x^*) = 0.012681$

2.3.2 Welded beam design problem

(small scale, multi-modal, GO, constrained, single objective, continuous)

File name: engapp2.p.xml

The welded beam design problem is taken from Rao [9] and Cagrina [10] , in which a welded beam is designed for minimum cost subject to constraints on shear stress (τ), bending stress in the beam (θ), buckling load on the bar (P_c), end deflection of the beam δ, and side constraints. There are four design variables as shown in Figure 9, i.e. $h(x_1)$, $l(x_2)$, $t(x_3)$, and $b(x_4)$.

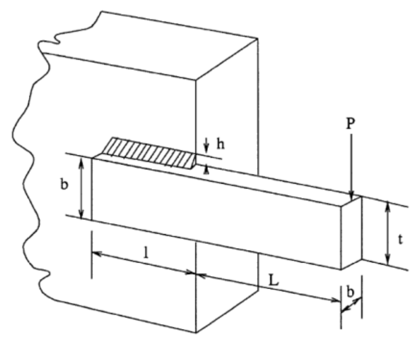

Figure 9: Welded beam design problem

The welded beam design problem yields an optimization problem which has four design variables $x = (x_1, x_2, x_3, x_4,)$ and takes the following form:

$$\min f(x) = 1.1047x_1^2x_2 + 0.04811x_3x_4(14 + x_2)$$

subject to:

$$g_1(x) = \tau(x) - 13600 \le 0,$$

$$g_2(x) = \sigma(x) - 30000 \le 0,$$

$$g_3(x) = x_1 - x_4 \le 0,$$

$$g_4(x) = 0.10471x_1^2 + 0.04811x_3x_4(14.0 + x_2) - 5.0 \le 0,$$

$$g_5(x) = 0.125 - x_1 \le 0,$$

$$g_6(x) = \delta(x) - 0.25 \le 0,$$

$$g_7(x) = 6000 - P_c(x) \le 0,$$

Where

$$\tau(x) = \sqrt{(\tau')^2 + 2\tau'\tau''\frac{x_2}{2R} + (\tau'')^2},$$

$$\tau' = \frac{6000}{\sqrt{2}x_1x_2},$$

$$\tau'' = \frac{MR}{J},$$

$$M = 6000\left(14 + {}^{x_2}/_{2}\right),$$

$$R = \sqrt{\frac{x_2^2}{4} + \left(\frac{x_1 + x_3}{2}\right)^2};$$

$$J = 2\left\{\sqrt{2}x_1x_2\left[\frac{x_2^2}{12} + \left(\frac{x_1 + x_3}{2}\right)^2\right]\right\},$$

$$\sigma(x) = \frac{504000}{x_4x_3^2},$$

$$\delta(x) = \frac{2.1952}{x_3^3x_4},$$

$$P_c(x) = 102372.4(1 - 0.0282346x_3)x_3x_4^3,$$

$$0.1 \le x_1, x_4 \le 2; \qquad 0.1 \le x_2, x_3 \le 10,$$

$$x^* = (0.2455, 6.1960, 8.2730, 0.2455)\ f(x^*) = 2.386$$

Rao keeps the units for x^*in inches, and $f(x^*)$ dollars.

2.3.3Pressure Vessel Design Problem

(small scale, multi-modal, GO, constrained, single objective, continuous)

File name: engapp3.p.xml

In this problem, the objective is to minimize the total cost ($f(x)$), including the cost of the material, forming and welding. A cylindrical vessel is capped at both ends by hemispherical heads as shown in Figure 10. There are four design variables: T_s (x_1, thickness of the shell), T_h (x_2, thickness of the head), R (x_3, inner radius) and L (x_4, length of the cylindrical section of the vessel, not including the head). Among the four variables, T_s and T_h are integer (real numbers rounded up to the closest integer) multiples of 0.0625in that are the available thicknesses of rolled steel plates, and R and L are continuous variables.

The problem can be formulated as follows [8]:

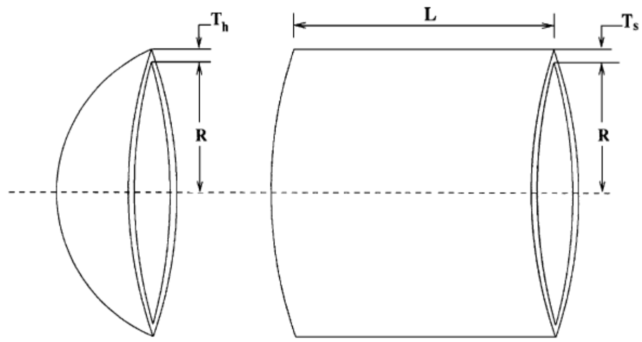

Figure 10: Center and end section of pressure vessel design problem [8]

$$\min f(x) = 0.6224x_1x_3x_4 + 1.7781x_2x_3^2 + 3.1661x_1^2x_4 + 19.84x_1^2x_3$$

Subject to:

$$g_1(x) = -x_1 + 0.0193x_3 \leq 0,$$

$$g_2(x) = -x_2 + 0.00954x_3 \leq 0,$$

$$g_3(x) = -\pi x_3^2x_4 - \frac{4}{3}\pi x_3^3 + 1296000 \leq 0,$$

$$g_4(x) = x_4 - 240 \leq 0.$$

$$25 \leq x_3 \leq 150;\ 1.0 \leq x_1 \leq 1.375;\ 25 \leq x_4 \leq 240;\ 0.625 \leq x_2 \leq 1.0$$

$$1 \leq x_1 \leq 99, 1 \leq x_2 \leq 99, 10.0 \leq x_3 \leq 200.0, 10.0 \leq x_4 \leq 200$$

$$x_1 = T_s$$

$$x_2 = T_h$$

$$x_3 = R$$

$$x_4 = L$$

Global minimum:

$$x^* = (T_s, T_h, R, L)^* = (0.812500, 0.437500, 42.097398, 176.654047),$$
$$f(x^*) = 6059.946341$$

2.3.4 Structure design of a stepped cantilever beam with rectangular shape

File name: engapp4.p.xml

Hsu et al. describe the problem as below:

"[Figure 11] shows a five-stepped cantilever beam with total length L of 500 cm, and the material elasticity modulus E is 200GPa. A concentrated load of 50,000N is applied at the tip end of the beam. This problem was presented by Thanedar and Vanderplaats [11]. In this problem, the height and width of the beam in all five steps of the cantilever beam are design variables. The volume of the beam is to be minimized. There are 11 constraints in this problem, 5 bending stress constraints in all five steps of the beam to be less than the allowable stress (14,000N/cm^2), and one displacement constraint on the tip deflection is to be less than the allowable deflection (2.7cm). A specified aspect ratio 20 has to be maintained between the height and width of all 5 cross sections of the beam. Substituting the parameters, this problem can be formulated as follows" [12] **:**

$$\min V = D(b_1 h_1 l_1 + b_2 h_2 l_2 + b_3 h_3 l_3 + b_4 h_4 l_4 + b_5 h_5 l_5)$$

Subject to:

$$g_1(x) = \frac{6Pl_5}{b_5 h_5^2} - 14000 \leq 0,$$

$$g_2(x) = \frac{6P(l_5 + l_4)}{b_4 h_4^2} - 14000 \leq 0,$$

$$g_3(x) = \frac{6P(l_5 + l_4 + l_3)}{b_3 h_3^2} - 14000 \leq 0,$$

$$g_4(x) = \frac{6P(l_5 + l_4 + l_3 + l_2)}{b_2 h_2^2} - 14000 \leq 0,$$

$$g_5(x) = \frac{6P(l_5 + l_4 + l_3 + l_2 + l_1)}{b_1 h_1^2} - 14000 \leq 0,$$

$$g_6(x) = \frac{Pl^3}{3E}\left(\frac{1}{I_5} + \frac{7}{I_4} + \frac{19}{I_3} + \frac{37}{I_2} + \frac{61}{I_1}\right) - 2.7 \leq 0,$$

$$g_7(x) = \frac{h_5}{b_5} - 20 \leq 0,$$

$$g_8(x) = \frac{h_4}{b_4} - 20 \leq 0,$$

$$g_9(x) = \frac{h_3}{b_3} - 20 \leq 0,$$

$$g_{10}(x) = \frac{h_2}{b_2} - 20 \leq 0,$$

$$g_{11}(x) = \frac{h_1}{b_1} - 20 \leq 0,$$

Where:

$$I_1 = \frac{b_5 h_5^3}{12}$$

$$I_2 = \frac{b_4 h_4^3}{12}$$

$$I_3 = \frac{b_3 h_3^3}{12}$$

$$I_4 = \frac{b_2 h_2^3}{12}$$

$$I_5 = \frac{b_1 h_1^3}{12}$$

E=2e11 and D=1.

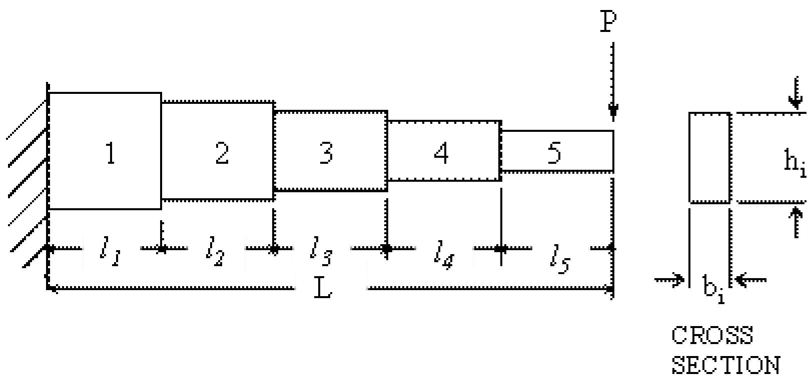

Figure 11: Steeped cantilever beam [12]

The feasible design point is $(b_1, b_2, b_3, b_4, b_5, h_1, h_2, h_3, h_4, h_5)$ = (3.0, 3.4, 2.6, 2.6, 2.0, 60, 56, 50, 46, 37) has the lower objective value of 69,400cm^3. Note: all l_i variables are implemented in the XML with default value of 100, and for the minimum previous mentioned, $D = 1$. In the xml file, x is implemented as: $(x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10})$ = $(b_1, h_1, b_2, h_2, b_3, h_3, b_4, h_4, b_5, h_5)$.

3. Multi-objective Optimization Test Problems

3.1 Unconstrained Multi-objective Benchmark Test Problems

3.1.1 Unconstrained Problem #1 [13]

(Mutli-Objective Optimization (MOO); multimodal; large scale; unconstrained; continuous)

File name: uf1.p.xml

$$minimize\ f_1 = x_1 + \frac{2}{|J_1|}\sum_{j\in J_1}\left[x_j - sin\left(6\pi x_1 + \frac{j\pi}{n}\right)\right]^2$$

$$minimize\ f_2 = 1 - \sqrt{x_1} + \frac{2}{|J_2|}\sum_{j\in J_2}\left[x_j - sin\left(6\pi x_1 + \frac{j\pi}{n}\right)\right]^2$$

Where $J_1 = \{j|j \text{ is odd and } 2 \le j \le n\}$ and $J_2 = \{j|j \text{ is even and } 2 \le j \le n\}$

The search space is $[0,1] \times [-1,1]^{n-1}$

Its PF is $f_2 = 1 - \sqrt{f_1},\ \ 0 \le f_1 \le 1.$

Its PS is $x_j = \sin\left(6\pi x_1 + \frac{j\pi}{n}\right), j = 2, \dots, n,\ \ 0 \le x_1 \le 1.$

3.1.2 Unconstrained Problem #2 [14]

(Multi-Objective Optimization (MOO); multimodal; large scale; unconstrained; continuous)

File name: uf2.p.xml

$$minimize\ f_1 = x_1 + \frac{2}{|J_1|} \sum_{j \in J_1} y_j^2$$

$$minimize\ f_2 = 1 - \sqrt{x_1} + \frac{2}{|J_2|} \sum_{j \in J_2} y_j^2$$

Where $J_1 = \{j | j \text{ is odd and } 2 \leq j \leq n\}$ and $J_2 = \{j | j \text{ is even and } 2 \leq j \leq n\}$, and

$$y_j = \begin{cases} x_j - \left[0.3x_1^2 \cos\left(24\pi x_1 + \frac{4j\pi}{n}\right) + 0.6x_1\right] \cos\left(6\pi x_1 + \frac{j\pi}{n}\right) & j \in j_1 \\ x_j - \left[0.3x_1^2 \cos\left(24\pi x_1 + \frac{4j\pi}{n}\right) + 0.6x_1\right] \sin\left(6\pi x_1 + \frac{j\pi}{n}\right) & j \in j_2 \end{cases}$$

The search space is $[0,1] \times [-1,1]^{n-1}$

Its PF is $f_2 = 1 - \sqrt{f_1},\ \ 0 \leq f_1 \leq 1.$

Its PS is $x_j = \begin{cases} \left\{0.3x_1^2 \cos\left(24\pi x_1 + \frac{4j\pi}{n}\right) + 0.6x_1\right\} \cos\left(6\pi x_1 + \frac{j\pi}{n}\right) & j \in j_1 \\ \left\{0.3x_1^2 \cos\left(24\pi x_1 + \frac{4j\pi}{n}\right) + 0.6x_1\right\} \sin\left(6\pi x_1 + \frac{j\pi}{n}\right) & j \in j_2 \end{cases}$

$$0 \leq x_1 \leq 1.$$

3.1.3 Unconstrained Problem #3 [14]

(Multi-Objective Optimization (MOO); multi-modal; large scale; unconstrained; continuous)

File name: uf3.p.xml

The two objectives to be minimized

$$minimize\ f_1 = x_1 + \frac{2}{|J_1|}\sum_{j\in J_1} h(y_i)$$

$$minimize\ f_2 = 1 - x_1^2 + \frac{2}{|J_2|}\sum_{j\in J_2} h(y_i)$$

where $J_1 = \{j|j \text{ is odd and } 2 \leq j \leq n\}$ and $J_2 = \{j|j \text{ is even and } 2 \leq j \leq n\}$, and

$$y_j = x_j - \sin\left(6\pi x_1 + \frac{j\pi}{n}\right), j = 2, \dots, n.$$

and

$$h(t) = \frac{|t|}{1 + e^{2|t|}}$$

The search space is $[0,1] \times [-2,2]^{n-1}$

Its PF is $f_2 = 1 - f_2^2,\ \ 0 \leq f_1 \leq 1.$

Its PS is $x_j = \sin\left(6\pi x_1 + \frac{j\pi}{n}\right), j = 2, \dots, n.\ \ 0 \leq x_1 \leq 1.$

3.1.4 Unconstrained Problem #4 [14]

(Mutli-Objective Optimization (MOO); multimodal; large scale; unconstrained; continuous)

File name: uf4.p.xml

The two objectives to be minimized

$$f_1 = x_1 + \left(\frac{1}{2N} + \varepsilon\right)|\sin(2N\pi x_1)| + \frac{2}{|J_1|}\sum_{j\in J_1} h(y_i)$$

$$f_2 = 1 - x_1 + \left(\frac{1}{2N} + \varepsilon\right)|\sin(2N\pi x_1)| + \frac{2}{|J_2|}\sum_{j\in J_2} h(y_i)$$

where $J_1 = \{j|j \text{ is odd and } 2 \leq j \leq n\}$ and $J_2 = \{j|j \text{ is even and } 2 \leq j \leq n\}$. N is an integer, $\varepsilon > 0$,

$$y_j = x_j - \sin\left(6\pi x_1 + \frac{j\pi}{n}\right), j = 2, \dots, n.$$

and

$$h(t) = 2t^2 - \cos(4\pi t) + 1.$$

The search space is $[0,1] \times [-1,1]^{n-1}$

Its PF has 2*N*+1 pareto optimal solutions:

$$\left(\frac{i}{2N}, 1 - \frac{i}{2N}\right)$$

for $i = 0,1, \dots, 2N$.

3.1.5 Unconstrained Problem #5 [14]

(Mutli-Objective Optimization (MOO); multimodal; large scale; unconstrained; continuous)

File name: uf5.p.xml

The three objectives to be minimized:

$$f_1 = \cos(0.5x_1\pi)\cos(0.5x_2\pi) + \frac{2}{|J_1|}\sum_{j\in J_1}\left[4y_j^2 - \cos(8\pi y_j) + 1\right]$$

$$f_2 = \cos(0.5x_1\pi)\sin(0.5x_2\pi) + \frac{2}{|J_2|}\sum_{j\in J_1}\left[4y_j^2 - \cos(8\pi y_j) + 1\right]$$

$$f_3 = \sin(0.5x_1\pi) + \frac{2}{|J_3|}\sum_{j\in J_1}\left[4y_j^2 - \cos(8\pi y_j) + 1\right]$$

Where $J_1 = \{\, j|3 \leq j \leq n$ and $j - 1$ is a multiplication of 3$\}$

Where $J_2 = \{\, j|3 \leq j \leq n$ and $j - 2$ is a multiplication of 3$\}$

Where $J_3 = \{\, j|3 \leq j \leq n$ and j is a multiplication of 3$\}$

and

$$y_j = x_j - 2x_2 \sin\left(2\pi x_1 + \frac{j\pi}{n}\right), j = 3, \dots, n.$$

The search space is $[0,1]^2 \times [-2,2]^{n-2}$

Its PF is $f_1^2 + f_2^2 + f_3^2 = 1, 0 \leq f_1, f_2, f_3 \leq 1$

Its PS is $x_j = 2x_2 \sin\left(2\pi x_1 + \frac{j\pi}{n}\right), j = 3, \dots, n.$

3.2 Constrained Multi-Objective, Small Scale Benchmark Test Problems

3.2.1Test Problem 1: Kita's Function [15]

(Mutli-Objective Optimization (MOO); small scale; constrained; continuous)

File name: tf1.p.xml

The problem was proposed by Kita et al. It is a two-objective function problem with two decision variables.

$$Maximize\ \ F = \left(f_1(x_1, x_2), f_2(x_1, x_2)\right)$$

where

$$f_1(x_1, x_2) = -x_1^2 + x_2; \qquad f_2(x_1, x_2) = \tfrac{1}{2}x_1 + x_2 + 1;$$

Subject to

$$\frac{1}{6}x_1 + x_2 - \frac{13}{2} \le 0;\ \frac{1}{2}x_1 + x_2 - \frac{15}{2} \le 0;\ 5x_1 + x_2 - 30 \le 0;\ 0 \le x_1, x_2 \le 10$$

3.2.2 Test Problem 2: Veldhuizen and Lamont's Function [16]

(Mutli-Objective Optimization (MOO); small scale; constrained)

File name: tf2.p.xml

Veldhuizen and Lamont's function is a three-objective function of two decision variables.

$$Minimize\ F = \left(f_1(x_1, x_2), f_2(x_1, x_2), f_3(x_1, x_2)\right)$$

where

$$f_1(x_1, x_2) = 0.5(x_1^2 + x_2^2) + \sin(x_1^2 + x_2^2);$$

$$f_2(x_1, x_2) = \frac{(3x_1 - 2x_2 + 4)^2}{8} + \frac{(x_1 - x_2 + 1)^2}{27} + 15;$$

$$f_3(x_1, x_2) = \frac{1}{(x_1^2 + x_2^2 + 1)} - 1.1\, exp(-x_1^2 - x_2^2);$$

subject to:

$$-3 \le x_1, x_2 \le 3$$

This test function has a disconnected Pareto optimal set, and the Pareto front is a curve following a convoluted path through objective space.

3.2.3 Test Problem 3: Comet Function [17]

(Mutli-Objective Optimization (MOO); small scale; constrained; continuous)

File name: comet.p.xml

This function is known as the Comet function.

$$Minimize\ F = \left(f_1(x_1, x_2, x_3), f_2(x_1, x_2, x_3), f_3(x_1, x_3)\right)$$

where

$$f_1(x_1, x_2, x_3) = (1 + x_3)(x_1^3 x_2^2 - 10x_1 - 4x_2);$$
$$f_2(x_1, x_2, x_3) = (1 + x_3)(x_1^3 x_2^2 - 10x_1 + 4x_2);$$
$$f_3(x_1, x_3) = 3(1 + x_3)x_1^2;$$

subject to:

$$1 \le x_1 \le 3.5; \quad -2 \le x_2 \le 2; \quad x_3 \ge 0$$

3.2.4 Test Problem 4: The Concave Pareto Front Function [18]

(Mutli-Objective Optimization (MOO); small scale; constrained; continuous)

File name: tf4.p.xml

This function is a two-objective function of two decision variables.

$$Maximize\ \ F = \left(f_1(x_1, x_2), f_2(x_1, x_2)\right)$$

Where

$$f_1(x_1, x_2) = (x_1 - 2)^2 + (x_2 - 1)^2;$$

$$f_2(x_1, x_2) = x_1^2 + (x_2 - 6)^2;$$

Subject to

$$x_1 - 1.6 \le 0; \quad 0.4 - x_1 \le 0; \quad x_2 - 5 \le 0; \quad 2 - x_2 \le 0$$

The problem has a concave Pareto front.

3.3Constrained Multi-Objective, Large Scale Benchmark Test Problems

3.3.1 Constrained Problem #1 [14]

(Mutli-Objective Optimization (MOO); multimodal; large scale; constrained; continuous)

File name: cf1.p.xml

$$f_1(x) = x_1 + \frac{2}{|J_1|}\sum_{j\in J_1}\left(x_j - x_1^{0.5\left(1.0+\frac{3(j-2)}{n-2}\right)}\right)^2$$

$$f_2(x) = 1 - x_1 + \frac{2}{|J_2|}\sum_{j\in J_2}\left(x_j - x_1^{0.5\left(1.0+\frac{3(j-2)}{n-2}\right)}\right)^2$$

where $J_1 = \{j | j \text{ is odd and } 2 \le j \le n\}$ and $J_2 = \{j | j \text{ is even and } 2 \le j \le n\}$, and

The constraint is

$$-f_1 - f_2 + a|\sin[N\pi(f_1 - f_2 + 1)]| + 1 \le 0$$

where N is an integer and $a \ge \frac{1}{2N}$

The search space is $[0,1]^n$

The pareto front PF is the objective space consists of $2N$+1points:

$$(i/2N\,,\ 1 - i/2N\ ), i = 0,1, \dots ,2N$$

3.3.2 Constrained problem #2 [14]

(Mutli-Objective Optimization (MOO); multimodal; large scale; constrained; continuous)

File name: cf2.p.xml

$$f_1(x) = x_1 + \frac{2}{|J_1|}\sum_{j\in J_1}\left(x_j - \sin\left(6\pi x_1 + \frac{j\pi}{n}\right)\right)^2$$

$$f_2(x) = 1 - \sqrt{x_1} + \frac{2}{|J_2|}\sum_{j\in J_2}\left(x_j - \cos\left(6\pi x_1 + \frac{j\pi}{n}\right)\right)^2$$

where $J_1 = \{j|j \text{ is odd and } 2 \le j \le n\}$ and $J_2 = \{j|j \text{ is even and } 2 \le j \le n\}$, and

The constraint is

$$\frac{-t}{1+e^{4|t|}} \le 0$$

where

$$t = f_2 + \sqrt{f_1} - a\sin\left[N\pi\left(\sqrt{f_1} - f_2 + 1\right)\right] - 1$$

The search space is $[0,1] \times [-1,1]^{n-1}$

The pareto front PF is the objective space consists of

-an isolated pareto optimal solution (0,1) in the objective space, and

-N disconnected parts, the i-th part is

$$f_2 = 1 - \sqrt{f_1}, \qquad \left(\frac{2i-1}{2N}\right)^2 \le f_1 \le \left(\frac{2i}{2N}\right)^2, \qquad i = 1, \dots, N$$

3.3.3 Constrained problem #3 [14]

(Mutli-Objective Optimization (MOO); multimodal; large scale; constrained; continuous)

File name: cf4.p.xml

$$f_1 = x_1 + \sum_{j \in J_1} h_j(y_j)$$

$$f_2 = 1 - x_1 + \sum_{j \in J_2} h_j(y_j)$$

where $J_1 = \{j | j \text{ is odd and } 2 \leq j \leq n\}$ and $J_2 = \{j | j \text{ is even and } 2 \leq j \leq n\}$

$$y_j = x_j - \sin\left(6\pi x_1 + \frac{j\pi}{n}\right), j = 2, \dots, n.$$

The search space is $[0,1] \times [-2,2]^{n-1}$

$$h_2(t) = \begin{cases} |t| & if\ t < \frac{3}{2}\left(1 - \frac{\sqrt{2}}{2}\right) \\ 0.125 + (t-1)^2 & \text{otherwise} \end{cases}$$

and

$$h_j(t) = t^2$$

For $j = 3,4, \dots, n$.

The constraint is:

$$\frac{-t}{1 + e^{4|t|}} \leq 0$$

where

$$t = x_2 - \sin\left(6\pi x_1 + \frac{2\pi}{n}\right) - 0.5x_1 + 0.25$$

The PF in the objective space is:

$$f_2 = \begin{cases} 1 - f_1 & if\ 0 \le f_1 \le 0.5 \\ -0.5f_1 + \frac{3}{4} & if\ 0.5 < f_1 \le 0.75 \\ 1 - f_1 + 0.125 & if\ 0.75 < f_1 \le 1 \end{cases}$$

4. Conclusions

Benchmark test problem are needed to judge whether optimization algorithms perform well or fail to converge to optimum solutions. In this report, many benchmark test functions were summarized and introduced. Many categories of these benchmark test functions were included in this report. Practical engineering problems and mathematical test functions were presented. Continuous, discrete, unimodal and multimodal, single objective and multi-objective, constrained and unconstrained test functions were also presented. All benchmark test problem shown in this report were coded in Matlab. The library which includes these test problems and their categorization is under construction. More benchmark functions will be added to this library of test functions to make it rich and unique for researchers to use and to test their optimization algorithms.

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