ASSIGNMENT 3 — LAMBDA CALCULUS

COMP 3010 — ORGANIZATION OF PROGRAMMING LANGUAGES

1. Syntax

Remember that the abstract syntax of the λ-calculus is defined by the grammar

$$M ::= x \mid M\ M \mid \lambda x.M$$

where x, y, and z stand for *identifier names* (*a.k.a. variables*) used inside expressions in the language of the λ-calculus.

Ambiguities in the grammar, where multiple juxtapositions and λs are mixed, are resolved by these associativity and precedence rules:

$$M_1\ M_2\ \dots\ M_n = ((M_1\ M_2)\ \dots)\ M_n \qquad M_1\ \lambda x.M_2 = M_1\ (\lambda x.M_2)$$
$$\lambda x.M_1\ M_2\ \dots\ M_n = \lambda x.(M_1\ M_2\ \dots\ M_n) \qquad \lambda x_1.\lambda x_2.M = \lambda x_1.(\lambda x_2.M)$$

In other words, application (written as juxtaposition $M_1\ M_2$) associates to the *left*, abstraction (written as $\lambda x.M$) associates to the *right*, a λ to the right of an application (a λ argument $M_1\ (\lambda x.M_2)$) has a *higher precedence* than that application, and an application to the right of a λ (an applied function body $\lambda x.(M_1\ M_2)$) has a *higher precedence* than the λ.

Exercise 1 (Multiple Choice)**.** Which of the following, fully-parenthesized λ-calculus expressions stand for the same syntax tree as $x\ y\ \lambda z.z\ x$?

(a) $(x\ y)\ (\lambda z.(z\ x))$
(b) $x\ (y\ (\lambda z.z))\ x$
(c) $((x\ y)\ \lambda z.z)\ x$
(d) $x\ (y\ \lambda z.(z\ x))$

Exercise 2 (Short Answer)**.** Use the above rules of precedence/associativity to draw the syntax tree represented by the fully-parenthesized version of $\lambda x.(x\ \lambda y.y\ x\ z)$

2. Alpha

Exercise 3 (Short Answer)**.** Remember that the set of *free variables* of an expression M, written as $\mathrm{FV}(M)$, is *inductively defined* on the *syntax of* M like so:

$$\mathrm{FV}(x) = \{x\}$$
$$\mathrm{FV}(M_1\ M_2) = \mathrm{FV}(M_1) \cup \mathrm{FV}(M_2)$$
$$\mathrm{FV}(\lambda x.M) = \mathrm{FV}(M) - \{x\}$$

Give the set of *free variables* for each of the following expressions. Write your answer for each part in set notation (e.g. $\{x, y\}$).

(a) $(\lambda x.\ y\ (\lambda z.\ x\ z))\ (\lambda y.\ y\ w)$
(b) $\lambda y.\ ((\lambda x.\ x\ y)\ (\lambda y.\ x))$

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(c) $(\lambda x.\ \lambda y.\ y\ (\lambda x.\ z\ x))\ (u\ v)$

Exercise 4 (This or That)**.** Remember that the operation which *renames y to x in M*, written $[y/x]M$, is *inductively defined* by transforming the *syntax of M* as:

$$\begin{aligned}
[y/x]x &= y \\
[y/x]z &= z && \text{(if } z \neq x\text{)} \\
[y/x](M_1\ M_2) &= ([y/x]M_1)\ ([y/x]M_2) \\
[y/x](\lambda x.M) &= \lambda x.M \\
[y/x](\lambda z.M) &= \lambda z.([y/x]M) && \text{(if } z \neq x \text{ and } z \neq y\text{)}
\end{aligned}$$

The law of α-equivalence is defined in terms of renaming as:

$$\lambda x.M =_\alpha \lambda y.([y/x]M) \qquad \text{(if } y \notin \mathrm{FV}(M)\text{)}$$

Determine which of the following are valid renamings. In other words, say whether or not each pair of λ-calculus expressions are equal according to α-equivalence.

(1) $\lambda x.\,(x\ y)\ =_\alpha \lambda z.\,(z\ y)$
(2) $\lambda x.\,(\lambda y.\,x\ y)\ =_\alpha \lambda y.\,(\lambda y.\,y\ y)$
(3) $(\lambda x.\,x)\ x\ =_\alpha (\lambda y.\,y)\ x$
(4) $\lambda x.\,\lambda y.\,x\ =_\alpha \lambda y.\,\lambda x.\,x$
(5) $\lambda x.\,(\lambda x.\,x)\ y\ =_\alpha \lambda z.\,(\lambda x.\,x)\ y$

3. Beta

Exercise 5 (Multiple Choice)**.** Remember that the operation which *substitutes M' for x in M*, written $[M'/x]M$, is *inductively defined* by transforming the *syntax of M* like so:

$$\begin{aligned}
[M'/x]x &= M' \\
[M'/x]z &= z && \text{(if } z \neq x\text{)} \\
[M'/x](M_1\ M_2) &= ([M'/x]M_1)\ ([M'/x]M_2) \\
[M'/x](\lambda x.M) &= \lambda x.M \\
[M'/x](\lambda z.M) &= \lambda z.([M'/x]M) && \text{(if } z \neq x \text{ and } z \notin \mathrm{FV}(M')\text{)}
\end{aligned}$$

What is the result of the substitution

$$[(\lambda a.\ y\ a)/x]\ \Big((\lambda w.\ x\ w)\ (\lambda z.\ x\ z)\Big)\ ?$$

(a) $[(\lambda a.\ y\ a)/x]\Big((\lambda w.\ x\ w)\ (\lambda z.\ x\ z)\Big) = (\lambda w.\ (\lambda a.\ y\ a)\ w)\ (\lambda z.\ (\lambda a.\ y\ a)\ z)$
(b) $[(\lambda a.\ y\ a)/x]\Big((\lambda w.\ x\ w)\ (\lambda z.\ x\ z)\Big) = (\lambda w.\ x\ w)\ (\lambda z.\ (\lambda a.\ y\ a)\ z)$
(c) $[(\lambda a.\ y\ a)/x]\Big((\lambda w.\ x\ w)\ (\lambda z.\ x\ z)\Big) = (\lambda w.\ x\ w)\ (\lambda z.\ x\ z)$
(d) $[(\lambda a.\ y\ a)/x]\Big((\lambda w.\ x\ w)\ (\lambda z.\ x\ z)\Big) = (\lambda w.\ x\ w)\ (\lambda z.\ \lambda a.\ y\ a)$

Exercise 6 (Multiple Choice)**.** Remember that the law of β-reduction is defined in terms of substitution as:

$$(\lambda x.M)\ M' \mapsto_\beta [M'/x]M$$

and the *call-by-name operational semantics* for the λ-calculus is defined by applying the above β-reduction rule inside of these evaluation contexts E defined like so:

$$E ::= \Box \mid E\ M \qquad\qquad \frac{M \mapsto_\beta M'}{E[M] \mapsto_\beta E[M']}$$

Which of the following is the result of evaluating the expression

$$(\lambda x.\lambda y.\lambda z.y\ x\ z)\ (\lambda x.x)\ (\lambda x.\lambda y.x)\ z$$

according to the call-by-name operational semantics (as shown above)? In other words, which of the following do you get by applying β-reduction (as shown above) as many times as possible in a sequence like so: (Note: Show each reduction step)

$$(\lambda x.\lambda y.\lambda z.y\ x\ z)\ (\lambda x.x)\ (\lambda x.\lambda y.x)\ z \mapsto M_1 \mapsto M_2 \mapsto \cdots \mapsto \text{answer}$$

(a) $(\lambda x.x)$
(b) $(\lambda x.\lambda y.x)$
(c) $(\lambda y.z)$
(d) z

Exercise 7 (Short Answer)**.** What happens when you evaluate the λ-calculus expression $(\lambda y.y\ y)\ (\lambda y.y\ y)$? by repeatedly applying β-reduction? If β-reduction eventually stops, what final expression is the result? If you think β-reduction does not stop, explain why not?

4. Eta

Exercise 8 (This or That)**.** Remember that the law of η-reduction is defined in terms of free variables like so:

$$(\lambda x.(M\ x)) \to_\eta M \qquad\qquad (\text{if } x \notin \mathrm{FV}(M))$$

Say which of the following are correct η-reductions (according to the above rule) and which are not. Justify your answer in each case.

(1) $\lambda z.(f\ z) \to_\eta f$
(2) $\lambda x.(x\ x) \to_\eta x$
(3) $\lambda y.((y\ z)\ y) \to_\eta (y\ z)$
(4) $\lambda x.((\lambda z.(g\ z))\ x) \to_\eta \lambda z.(g\ z)$
(5) $\lambda w.((h\ w)\ k) \to_\eta h\ k$
(6) $\lambda x.(\lambda y.(x\ y))\ x \ \to_\eta\ \lambda y.(x\ y)$