ASSIGNMENT 2 — OPERATIONAL SEMANTICS

COMP 3010 — ORGANIZATION OF PROGRAMMING LANGUAGES

1. Big-Step Operational Semantics

For exercises 1 to 3, consider this definition of conditional arithmetic in terms of a big-step operational semantics.

Syntax of arithmetic (A) and boolean (B) expressions, and natural number (n) and boolean (b) values:

$$n ::= 0 \mid 1 \mid 2 \mid 3 \mid \ldots$$

$$A ::= \underline{n} \mid \texttt{plus}(A_1, A_2) \mid \texttt{minus}(A_1, A_2) \mid \texttt{times}(A_1, A_2) \mid \texttt{div}(A_1, A_2) \mid \texttt{if}(B, A_1, A_2)$$

$$b ::= \mathit{true} \mid \mathit{false}$$

$$B ::= \underline{b} \mid \texttt{and}(B_1, B_2) \mid \texttt{or}(B_1, B_2) \mid \texttt{zero?}(A)$$

Big-step operational semantics of arithmetic expressions ($A \Downarrow n$):

$$\frac{}{\underline{n} \Downarrow n}$$

$$\frac{A_1 \Downarrow n_1 \quad A_2 \Downarrow n_2 \quad n_1 + n_2 = n}{\texttt{plus}(A_1, A_2) \Downarrow n} \qquad \frac{A_1 \Downarrow n_1 \quad A_2 \Downarrow n_2 \quad n_1 - n_2 = n \quad n_1 \geq n_2}{\texttt{minus}(A_1, A_2) \Downarrow n}$$

$$\frac{A_1 \Downarrow n_1 \quad A_2 \Downarrow n_2 \quad n_1 \times n_2 = n}{\texttt{times}(A_1, A_2) \Downarrow n} \qquad \frac{A_1 \Downarrow n_1 \quad A_2 \Downarrow n_2 \quad n_1 \div n_2 = n \quad n_2 \neq 0}{\texttt{div}(A_1, A_2) \Downarrow n}$$

$$\frac{B \Downarrow \texttt{true} \quad A_1 \Downarrow n_1}{\texttt{if}(B, A_1, A_2) \Downarrow n_1} \qquad \frac{B \Downarrow \texttt{false} \quad A_2 \Downarrow n_2}{\texttt{if}(B, A_1, A_2) \Downarrow n_2}$$

Big-step operational semantics of boolean expressions ($B \Downarrow b$):

$$\frac{}{\underline{\mathit{true}} \Downarrow \mathit{true}} \qquad \frac{}{\underline{\mathit{false}} \Downarrow \mathit{false}}$$

$$\frac{B_1 \Downarrow \mathit{true} \quad B_2 \Downarrow b}{\texttt{and}(B_1, B_2) \Downarrow b} \qquad \frac{B_1 \Downarrow \mathit{false}}{\texttt{and}(B_1, B_2) \Downarrow \mathit{false}}$$

$$\frac{B_1 \Downarrow \mathit{false} \quad B_2 \Downarrow b}{\texttt{or}(B_1, B_2) \Downarrow b} \qquad \frac{B_1 \Downarrow \mathit{true}}{\texttt{or}(B_1, B_2) \Downarrow \mathit{true}}$$

$$\frac{A \Downarrow 0}{\texttt{zero?}(A) \Downarrow \mathit{true}} \qquad \frac{A \Downarrow n \quad n \neq 0}{\texttt{zero?}(A) \Downarrow \mathit{false}}$$

For the natural number division $n_1 \div n_2$ returns only the whole number dividend and drops the remainder, so that $7 \div 2$ is 3 for example.

Exercise 1 (Multiple Choice)**.** Which of the following evaluations of $\texttt{times}(\texttt{if}(\texttt{zero?}(\texttt{minus}(\underline{1}, \underline{1})), \underline{3}, \underline{1}), \underline{2})$ can be derived by the operational semantics?

Date: Spring 2022.

(a) $\texttt{times}(\texttt{if}(\texttt{zero?}(\texttt{minus}(\underline{2},\texttt{plus}(\underline{1},\underline{1}))),\underline{3},\underline{1}),\underline{2}) \Downarrow 3$
(b) $\texttt{times}(\texttt{if}(\texttt{zero?}(\texttt{minus}(\underline{2},\texttt{plus}(\underline{1},\underline{1}))),\underline{3},\underline{1}),\underline{2}) \Downarrow 1$
(c) $\texttt{times}(\texttt{if}(\texttt{zero?}(\texttt{minus}(\underline{2},\texttt{plus}(\underline{1},\underline{1}))),\underline{3},\underline{1}),\underline{2}) \Downarrow 2$
(d) $\texttt{times}(\texttt{if}(\texttt{zero?}(\texttt{minus}(\underline{2},\texttt{plus}(\underline{1},\underline{1}))),\underline{3},\underline{1}),\underline{2}) \Downarrow 6$
(e) $\texttt{times}(\texttt{if}(\texttt{zero?}(\texttt{minus}(\underline{2},\texttt{plus}(\underline{1},\underline{1}))),\underline{3},\underline{1}),\underline{2}) \Downarrow \mathit{true}$
(f) $\texttt{times}(\texttt{if}(\texttt{zero?}(\texttt{minus}(\underline{2},\texttt{plus}(\underline{1},\underline{1}))),\underline{3},\underline{1}),\underline{2}) \Downarrow \mathit{false}$

Exercise 2 (This or That)**.** An arithmetic expression A *returns* if there is some number n such that $A \Downarrow n$, and *diverges* if there is no such n. For example, $\texttt{div}(\underline{1},\underline{0})$ and $\texttt{minus}(\underline{0},\underline{1})$ both diverge, and $\texttt{div}(\underline{0},\underline{1})$ and $\texttt{minus}(\underline{1},\underline{0})$ both return (since $\texttt{div}(\underline{0},\underline{1}) \Downarrow 0$ and $\texttt{minus}(\underline{1},\underline{0}) \Downarrow 1$). Similarly, a boolean expression B *returns* if there is some boolean value $b = \mathit{true}$ or $b = \mathit{false}$ such that $B \Downarrow b$, and *diverges* otherwise.

For each of the following arithmetic and boolean expressions, say if that expression returns or diverges.

(a) $\texttt{minus}(\texttt{plus}(\underline{3},\underline{1}),\underline{2})$
(b) $\texttt{plus}(\underline{3},\texttt{minus}(\underline{1},\underline{2}))$
(c) $\texttt{if}(\texttt{zero?}(\texttt{minus}(\underline{2},\underline{2})),\underline{0},\texttt{div}(\underline{3},\texttt{minus}(\underline{2},\underline{2})))$
(d) $\texttt{and}(\texttt{zero?}(\texttt{div}(\underline{0},\underline{0})),\underline{false})$
(e) $\texttt{and}(\underline{false},\texttt{zero?}(\texttt{div}(\underline{0},\underline{0})))$

Exercise 3 (Show Your Work)**.** Determine the number following expression evaluates to by drawing a derivation tree of the big-step evaluation semantics:

$$\texttt{if}(\texttt{zero?}(\texttt{minus}(\texttt{plus}(\underline{1},\underline{1}),\underline{2})),\texttt{div}(\underline{4},\underline{2}),\texttt{div}(\underline{4},\texttt{minus}(\texttt{plus}(\underline{1},\underline{1}),\underline{2})))$$

2. Small-Step Operational Semantics

For exercises 4 to 6, consider this definition of a small-step operational semantics for the same conditional arithmetic language used previously in section 1.

Small-step reduction rules:

$$\begin{array}{rlrl}
\texttt{plus}(\underline{n_1},\underline{n_2}) \mapsto \underline{n} & (n = n_1 + n_2) & \texttt{minus}(\underline{n_1},\underline{n_2}) \mapsto \underline{n} & (n = n_1 - n_2,\ n_1 \geq n_2) \\
\texttt{times}(\underline{n_1},\underline{n_2}) \mapsto \underline{n} & (n = n_1 \times n_2) & \texttt{div}(\underline{n_1},\underline{n_2}) \mapsto \underline{n} & (n = n_1 \div n_2,\ n_2 \neq 0) \\
\texttt{if}(\underline{\mathit{true}},A_1,A_2) \mapsto A_1 & & \texttt{if}(\underline{\mathit{false}},A_1,A_2) \mapsto A_2 & \\
\texttt{and}(\underline{\mathit{true}},B) \mapsto B & & \texttt{and}(\underline{\mathit{false}},B) \mapsto \underline{\mathit{false}} & \\
\texttt{or}(\underline{\mathit{false}},B) \mapsto B & & \texttt{or}(\underline{\mathit{true}},B) \mapsto \underline{\mathit{true}} & \\
\texttt{zero?}(\underline{0}) \mapsto \underline{\mathit{true}} & & \texttt{zero?}(\underline{n}) \mapsto \underline{\mathit{false}} & (n \neq 0)
\end{array}$$

Evaluation contexts (E):

$$\begin{aligned}
E ::= {} & \square \mid \texttt{plus}(E,B) \mid \texttt{plus}(\underline{n},E) \mid \texttt{minus}(E,B) \mid \texttt{minus}(\underline{n},E) \\
& \mid \texttt{times}(E,B) \mid \texttt{times}(\underline{n},E) \mid \texttt{div}(E,B) \mid \texttt{div}(\underline{n},E) \mid \texttt{if}(E,A_1,A_2) \\
& \mid \texttt{and}(E,B) \mid \texttt{or}(E,B) \mid \texttt{zero?}(E)
\end{aligned}$$

Exercise 4. What do you get from plugging the expression $\texttt{plus}(\underline{2},\underline{3})$ into the evaluation context $\texttt{if}(\texttt{zero?}(\square),\texttt{times}(\underline{2},\underline{4}),\texttt{minus}(\underline{4},\underline{1}))$

(a) $\texttt{if}(\texttt{zero?}(\underline{5}),\texttt{times}(\underline{2},\underline{4}),\texttt{minus}(\underline{4},\underline{1}))$
(b) $\texttt{if}(\texttt{zero?}(\texttt{plus}(\underline{2},\underline{3})),\texttt{times}(\underline{2},\underline{4}),\texttt{minus}(\underline{4},\underline{1}))$
(c) $\texttt{if}(\texttt{plus}(\underline{2},\underline{3}),\texttt{times}(\underline{2},\underline{4}),\texttt{minus}(\underline{4},\underline{1}))$
(d) $\texttt{if}(\texttt{times}(\underline{2},\underline{4}),\texttt{minus}(\underline{4},\underline{1}),\texttt{plus}(\underline{2},\underline{3}))$

Exercise 5 (This or That)**.** For each of the following pairs of a context and a sub-expression, identify which ones are valid or invalid decompositions of the expression

$$\texttt{if}(\texttt{zero?}(\texttt{times}(\texttt{plus}(\underline{3},\underline{4}),\texttt{minus}(\underline{1},\underline{1}))),\texttt{minus}(\underline{5},\underline{3}),\texttt{div}(\underline{6},\underline{2}))$$

according to the small-step operational semantics. (There may be multiple valid decompositions.)

(a) $\texttt{if}(\Box,\texttt{minus}(\underline{5},\underline{3}),\texttt{div}(\underline{6},\underline{2}))$ **and** $\texttt{zero?}(\texttt{times}(\texttt{plus}(\underline{3},\underline{4}),\texttt{minus}(\underline{1},\underline{1})))$

(b) $\texttt{if}(\texttt{zero?}(\texttt{times}(\texttt{plus}(\underline{3},\underline{4}),\texttt{minus}(\underline{1},\underline{1}))),\Box,\texttt{div}(\underline{6},\underline{2}))$ **and** $\texttt{minus}(\underline{5},\underline{3})$

(c) $\texttt{if}(\texttt{zero?}(\texttt{times}(\texttt{plus}(\underline{3},\underline{4}),\texttt{minus}(\underline{1},\underline{1}))),\texttt{minus}(\underline{5},\underline{3}),\Box)$ **and** $\texttt{div}(\underline{6},\underline{2})$

(d) $\texttt{if}(\texttt{zero?}(\texttt{times}(\Box,\texttt{minus}(\underline{1},\underline{1}))),\texttt{minus}(\underline{5},\underline{3}),\texttt{div}(\underline{6},\underline{2}))$ **and** $\texttt{plus}(\underline{3},\underline{4})$

(e) $\texttt{if}(\texttt{zero?}(\texttt{times}(\texttt{plus}(\underline{3},\underline{4}),\Box)),\texttt{minus}(\underline{5},\underline{3}),\texttt{div}(\underline{6},\underline{2}))$ **and** $\texttt{minus}(\underline{1},\underline{1})$

Exercise 6 (Show Your Work)**.** Write down the sequence of reduction step using the rules of the small-step operational semantics for simplifying

$$\texttt{if}(\texttt{zero?}(\texttt{times}(\texttt{plus}(\underline{3},\underline{4}),\texttt{minus}(\underline{1},\underline{1}))),\texttt{minus}(\underline{5},\underline{3}),\texttt{div}(\underline{6},\underline{2}))$$

to its final result.