Call-By-Unboxed-Value

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Unboxing in Practice

- Good: Unboxed values enables high-performance

- Bad: Low-level code clashes with high-level abstractions (polymorphism)

- <u>Levity Polymorphism</u> and <u>Kinds Are Calling Conventions</u> resolve the tension
  - Ad-hoc restrictions needed for operational meaning & compilability

  - *"If I can't compile it, the type checker rejects it"*

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  - Ad-hoc restrictions needed for operational meaning & compilability

  - *"If I can't compile it, the type checker rejects it"*

- <u>Call-By-Unboxed-Value</u> explains the meaning & implementation of unboxing
  - Logical & semantic foundation ensures meaningful programs

  - *"If I can write it, I can compile & run it"*

COMPILING WITH CALL-BY-UNBOXED-VALUE

- Compiling unboxed polymorphism before:
  - Only compile well-typed source programs; need typing information to generate code

  - Generate ill-typed target programs; compilation can <u>break</u> types

COMPILING WITH CALL-BY-UNBOXED-VALUE

- Compiling unboxed polymorphism before:
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- Compiling unboxed polymorphism with Call-By-Unboxed-Value:
  - Can compile untyped source programs; no typing information needed
  - Compilation preserves typing if the source was well-typed
  - Low-level code can be expressed in a type-safe target language
  - Still support type erasure without changing answers

Unboxed Values

Holding Numbers in Registers

$$sumTo0 \quad :: \mathsf{Int} \to \mathsf{Int}$$
$$sumTo0\ 0 = 0$$
$$sumTo0\ n = n + sumTo0(n - 1)$$

Is *n* an integer register, or a pointer into the heap?

HOLDING NUMBERS IN REGISTERS

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Is n an integer register, or a pointer into the heap?

Accumulator style $\implies$ fast loop

$$sumTo0' \quad :: \text{Int} \to \text{Int}$$
$$sumTo0'\ n = go\ n\ 0$$
$$\textbf{where } go\ 0\ acc = acc$$
$$go\ n\ acc = go\ (n - 1)\ (n + acc)$$

Problems with Polymorphism

Could the polymorphic *a* really be <u>any</u> type?

$$id \quad :: a \rightarrow a$$
$$id \ x = x$$

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$$id \quad :: a \to a$$
$$id \ x = x$$

Need to know *a*'s representation to generate low-level machine code:

- Where does *x* live? (General or specialized register? Heap?)

- How many bits does *x* occupy? (32? 64? 8?)

- How to copy/move *x* from (incoming) parameter to (outgoing) return?

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Polymorphic Ambiguity

Do we need to know *a* and *b*'s representations to compile *app*?

$$app \quad :: (a \to b) \to a \to b$$
$$app\ f\ x = f\ x$$

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$$app\, f\, x = f\, x$$

- a: Yes, to move x
- b: It depends. . .
    - Naïvely yes, to move f's result to $(app\, f\, x)$'s caller
    - But with tail-call optimization, app never handles any b's

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- *a*: Yes, to move *x*
- *b*: It depends...
    - Naïvely yes, to move *f*'s result to $(app\, f\ x)$'s caller
    - But with tail-call optimization, *app* never handles any *b*'s

What about after η-reduction?

$$app' \quad :: (a \to b) \to a \to b$$
$$app'\, f = f$$

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What about after η-reduction?

$$app' \quad :: (a \to b) \to a \to b$$
$$app'\ f = f$$

- a and b's representations are irrelevant!
- Only move $f :: a \to b$, always a pointer

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Higher-Order Ambiguity

What do we need to know about *a* and *b*?

$$
\begin{aligned}
map \quad &:: (a \to b) \to [a] \to [b] \\
map\ f\ [] \quad &= [] \\
map\ f\ (x : xs) &= (f\ x) : (map\ f\ xs)
\end{aligned}
$$

What do we need to know about *a* and *b*?

$$map \qquad :: (a \to b) \to [a] \to [b]$$
$$map\ f\ [] \qquad = []$$
$$map\ f\ (x : xs) = (f\ x) : (map\ f\ xs)$$

- Representations of both *a* and *b*
  - To move $x :: a$ around
  - To store $(f\ x) :: b$ in a list

- Calling convention of *b*
  - What if $f :: \text{Int} \to \text{Int} \to \text{Int}$?
  - $b = \text{Int} \to \text{Int}$ is a function, needs 1 more argument
  - $(f\ x) :: \text{Int} \to \text{Int}$ might be a partial application, can't jump to *f*'s body
  - To generate code, need to distinguish partial applications from real calls

A First Taste of Call-By-Unboxed-Value

The Two Axes of Unboxing

- Familiar: Values versus computations
    - Values = being
    - Computations = doing

- New: Complexity versus Atomicity
    - Atomic = singular
    - Complex = multiple (parts, choices, …)

Call-By-Unboxed-Value

- Functions are called with complex unboxed values
  - Only atomic values are first class, can be named
  - Complex values are second class, <u>must</u> be matched

- Functions themselves are complex computations
  - Only atomic computations can be run directly
  - Complex computations are inert on their own, <u>must</u> match their context (η-long)

Elaborating Functional Code to Call-By-Unboxed-Value

Haskell

$$sumTo0 \ :: \ \mathsf{Nat} \to \mathsf{Nat}$$
$$sumTo0 \ 0 = 0$$
$$sumTo0 \ n = n + sumTo0(n - 1)$$

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CBPV
$$sumTo0 :: \mathsf{Nat} \to \mathsf{F\,Nat}$$
$$sumTo0 = \lambda n. \; \textbf{if } n == 0 \textbf{ then return } 0$$
$$\textbf{else do } x \leftarrow n - 1;$$
$$\textbf{do } y \leftarrow sumTo0 \; x;$$
$$n + y$$

Elaborating Functional Code to Call-By-Unboxed-Value

Haskell
```
sumTo0   :: Nat → Nat
sumTo0 0 = 0
sumTo0 n = n + sumTo0(n − 1)
```

CBPV
```
sumTo0 :: Nat → F Nat
sumTo0 = λn. if n == 0 then return 0
                        else do x ← n − 1;
                             do y ← sumTo0 x;
                             n + y
```

CBUV
```
sumTo0 :: Val Nat → Eval(Ret(Val Nat))
sumTo0 = {val int n · eval → if n == 0 then ret 0
                              else do val int x ← n − 1;
                                   do val int y ← sumTo0 (val x) . eval;
                                   n + y}
```

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CBUV
$$sumTo0 :: \text{Val Nat} \to \text{Eval}(\text{Ret}(\text{Val Nat}))$$
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$$\textbf{else do } \text{val int } x \leftarrow n - 1;$$
$$\textbf{do } \text{val int } y \leftarrow sumTo0 \ (\text{val } x) \ . \ \text{eval};$$
$$n + y\}$$

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$$sumTo0 \ :: \ \mathsf{Nat} \to \mathsf{Nat}$$
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CBUV
$$sumTo0 :: \mathsf{Val \ Nat} \to \mathsf{Eval(Ret(Val \ Nat))}$$
$$sumTo0 = \{\mathsf{val \ int} \ n \cdot \mathsf{eval} \to \textbf{if } n == 0 \textbf{ then ret } 0$$
$$\textbf{else do } \mathsf{val \ int} \ x \leftarrow n-1;$$
$$\textbf{do } \mathsf{val \ int} \ y \leftarrow sumTo0 \ (\mathsf{val} \ x) \ . \ \mathsf{eval};$$
$$n + y\}$$

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ELABORATING FUNCTIONAL CODE TO CALL-BY-UNBOXED-VALUE

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$$sumTo0 \ :: Nat \rightarrow Nat$$
$$sumTo0 \ 0 = 0$$
$$sumTo0 \ n = n + sumTo0(n - 1)$$

CBPV
$$sumTo0 :: Nat \rightarrow F\,Nat$$
$$sumTo0 = \lambda n. \ \textbf{if } n == 0 \ \textbf{then return } 0$$
$$\textbf{else do } x \leftarrow n - 1;$$
$$\textbf{do } y \leftarrow sumTo0 \ x;$$
$$n + y$$

CBUV
$$sumTo0 :: Val\,Nat \rightarrow Eval(Ret(Val\,Nat))$$
$$sumTo0 = \{ \text{val int } n \cdot \text{eval} \rightarrow \textbf{if } n == 0 \ \textbf{then ret } 0$$
$$\textbf{else do } \text{val int } x \leftarrow n - 1;$$
$$\textbf{do } \text{val int } y \leftarrow sumTo0 \ (\text{val } x) \cdot \text{eval};$$
$$n + y \}$$

Passing & Returning Multiple Arguments

$$quotRem : \mathsf{Val}\,\mathsf{Nat} \to \mathsf{Val}\,\mathsf{Nat} \to \mathsf{Eval}(\mathsf{Ret}(\mathsf{Val}\,\mathsf{Nat} \times \mathsf{Val}\,\mathsf{Nat}))$$

Complex answers must be immediately destructed in place at the call site

OK   **do** $(\mathsf{val}\,\mathsf{int}\,q, \mathsf{val}\,\mathsf{int}\,r) \leftarrow quotRem\,(\mathsf{val}\,12)\,(\mathsf{val}\,5)$. eval

Illegal   **do** $\mathsf{val}\,?\,qr \leftarrow quotRem\,(\mathsf{val}\,12)\,(\mathsf{val}\,5)$. eval

PASSING & RETURNING MULTIPLE ARGUMENTS

$$quotRem : \mathsf{Val\ Nat} \to \mathsf{Val\ Nat} \to \mathsf{Eval}(\mathsf{Ret}(\mathsf{Val\ Nat} \times \mathsf{Val\ Nat}))$$

Complex answers must be immediately destructed in place at the call site

OK	**do** (val int q, val int r) $\leftarrow$ *quotRem* (val 12) (val 5) . eval
Illegal	**do** val ? qr $\leftarrow$ *quotRem* (val 12) (val 5) . eval

$$distance : (\mathsf{Val\ Float}, \mathsf{Val\ Float}) \to \mathsf{Eval}(\mathsf{Ret}(\mathsf{Val\ Float}))$$

Complex arguments must be immediately constructed in place at the call site

OK	*distance* (val 3.14, val 2.71)
OK	*distance* (val x, val y)
Illegal	*distance* xy
Illegal	*distance* $(f\ x)$

Polymorphic Code

$id \quad : \forall a : \text{Type ref } \textbf{val} \, . \, \text{Val } a \rightarrow \text{Eval}(\text{Ret}(\text{Val } a))$

$id \quad = \{ \, \text{ty } a \cdot \text{val ref}(x : a) \cdot \text{eval} \rightarrow \textbf{ret} \, \text{val } x \, \}$

$id2 : \forall a : \text{Type int } \textbf{val} \, . \, (\text{Val } a \times \text{Val Float}) \rightarrow \text{Eval}(\text{Ret}(\text{Val } a \times \text{Val Float}))$

$id2 = \{ \, \text{ty } a \cdot (\text{val int}(x : a), \text{val flt}(y : \text{Float})) \cdot \text{eval} \rightarrow \textbf{ret} \, (\text{val } x, \text{val } y) \, \}$

$$id \; : \forall a : \text{Type ref } \textbf{val} \, . \; \text{Val } a \rightarrow \text{Eval}(\text{Ret}(\text{Val } a))$$

$$id \; = \{ \, \text{ty } a \cdot \text{val ref}(x : a) \cdot \text{eval} \rightarrow \textbf{ret } \text{val } x \, \}$$

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id (Val Int $\times$ Val Float) ill-kinded, but id (Box(Val Int $\times$ Val Float)) is OK because

$$\text{Box} : \textbf{cplx val} \rightarrow \text{ref } \textbf{val}$$

Polymorphic Code

id : $\forall a$: Type ref **val** . Val $a \to$ Eval(Ret(Val a))

id = { ty $a \cdot$ val ref$(x : a) \cdot$ eval $\to$ **ret** val x }

$id2$: $\forall a$: Type int **val** . (Val $a \times$ Val Float) $\to$ Eval(Ret(Val $a \times$ Val Float))

$id2$ = { ty $a \cdot$ (val int$(x : a)$, val flt$(y :$ Float)) $\cdot$ eval $\to$ **ret** (val x, val y) }

id (Val Int $\times$ Val Float) ill-kinded, but id (Box(Val Int $\times$ Val Float)) is OK because

$$\text{Box} : \textbf{cplx val} \to \text{ref } \textbf{val}$$

Unboxed code still has well-defined operational meaning after type erasure!

id = { ty $a \cdot$ val ref $x \cdot$ eval $\to$ **ret** val x }

$id2$ = { ty $a \cdot$ (val int x, val flt y) $\cdot$ eval $\to$ **ret** (val x, val y) }

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Fusing Values and Calling Conventions

Curried & Uncurried Functions

Unboxed tuples are flattened at compile time (a, b, c : ref **val**; $x : a, y : b, z : c$):

$$
\begin{array}{ccccc}
(\text{Val } a \times \text{Val } b) \times \text{Val } c & \approx & \text{Val } a \times (\text{Val } b \times \text{Val } c) & \approx & \text{Val } a \times \text{Val } b \times \text{Val } c \\
((\text{val } x, \text{val } y), \text{val } z) & \approx & (\text{val } x, (\text{val } y, \text{val } z)) & \approx & \text{val } x, \text{val } y, \text{val } z
\end{array}
$$

CURRIED & UNCURRIED FUNCTIONS

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((\text{val } x, \text{val } y), \text{val } z) & \approx & (\text{val } x, (\text{val } y, \text{val } z)) & \approx & \text{val } x, \text{val } y, \text{val } z
\end{array}
$$

(Un)Curried functions are compiled to the same code $(a, b : \text{ref } \textbf{val}; c : \text{sub } \textbf{comp})$:

$$
\begin{array}{ccc}
f : (\text{Val } a \times \text{Val } b) \to \text{Eval } c & \approx & g : \text{Val } a \to (\text{Val } b \to \text{Eval } c) \\
f = \{ (\text{val ref } x, \text{val ref } y) \cdot \text{eval} \to \dots \} & \approx & g = \{ \text{val ref } x \cdot (\text{val ref } y \cdot \text{eval}) \to \dots \}
\end{array}
$$

CURRIED & UNCURRIED FUNCTIONS

Unboxed tuples are flattened at compile time (a, b, c : ref **val**; $x : a, y : b, z : c$):

$$(\mathsf{Val}\, a \times \mathsf{Val}\, b) \times \mathsf{Val}\, c \quad \approx \quad \mathsf{Val}\, a \times (\mathsf{Val}\, b \times \mathsf{Val}\, c) \quad \approx \quad \mathsf{Val}\, a \times \mathsf{Val}\, b \times \mathsf{Val}\, c$$
$$((\mathsf{val}\, x, \mathsf{val}\, y), \mathsf{val}\, z) \quad \approx \quad (\mathsf{val}\, x, (\mathsf{val}\, y, \mathsf{val}\, z)) \quad \approx \quad \mathsf{val}\, x, \mathsf{val}\, y, \mathsf{val}\, z$$

(Un)Curried functions are compiled to the same code (a, b : ref **val**; c : sub **comp**):

$$f : (\mathsf{Val}\, a \times \mathsf{Val}\, b) \to \mathsf{Eval}\, c \qquad \approx \qquad g : \mathsf{Val}\, a \to (\mathsf{Val}\, b \to \mathsf{Eval}\, c)$$
$$f = \{\, (\mathsf{val\, ref}\, x, \mathsf{val\, ref}\, y) \cdot \mathsf{eval} \to \dots \} \quad \approx \quad g = \{\, \mathsf{val\, ref}\, x \cdot (\mathsf{val\, ref}\, y \cdot \mathsf{eval}) \to \dots \}$$

Safe due to second-class status of complex values & computations

OK	$f\,(\mathsf{val}\, x, \mathsf{val}\, y)\,.\,\mathsf{eval}$	$\approx$	$g\,(\mathsf{val}\, x)\,(\mathsf{val}\, y)\,.\,\mathsf{eval}$	OK
Illegal	$f\,xy\,.\,\mathsf{eval}$	$\not\approx$	**let** $g' = g\,(\mathsf{val}\, x)$ **in** $\dots$	Illegal

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Unboxed Sums

Invariant: all complex patterns can be fully enumerated at compile time

Unboxed sums are also flattened at compile time (a, b, c : ref **val**; $x : a, y : b, z : c$):

$$
\begin{array}{rcl r}
(\text{Val } a + \text{Val } b) + \text{Val } c & \approx & \text{Val } a + (\text{Val } b + \text{Val } c) & \\
(0, (0, \text{val } x)) & \approx & (0, \text{val } x) & \text{Choice \#0} \\
(0, (1, \text{val } y)) & \approx & (1, (0, \text{val } y)) & \text{Choice \#1} \\
(1, \text{val } z) & \approx & (1, (1, \text{val } z)) & \text{Choice \#2}
\end{array}
$$

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(0, (0, \text{val } x)) & \approx & (0, \text{val } x) & \text{Choice \#0} \\
(0, (1, \text{val } y)) & \approx & (1, (0, \text{val } y)) & \text{Choice \#1} \\
(1, \text{val } z) & \approx & (1, (1, \text{val } z)) & \text{Choice \#2}
\end{array}
$$

Unboxed tuples distribute over unboxed sums (a, b, c : ref **val**; $x : a, y : b, z : c$):

$$
\begin{array}{rcl r}
(\text{Val } a + \text{Val } b) \times \text{Val } c & \approx & (\text{Val } a \times \text{Val } c) + (\text{Val } b \times \text{Val } c) & \\
((0, \text{val } x), \text{val } z) & \approx & (0, (\text{val } x, \text{val } z)) & \text{Choice \#0} \\
((1, \text{val } y), \text{val } z) & \approx & (1, (\text{val } y, \text{val } z)) & \text{Choice \#1}
\end{array}
$$

Choice Fusion

$$maybeAdd \text{ Nothing } y = y \qquad\qquad maybeAdd \text{ (Just } x) \text{ } y = x + y$$

Invariant: mandatory pattern-matching on complex values

Choice Fusion

$$maybeAdd \; \text{Nothing} \; y = y \qquad\qquad maybeAdd \; (\text{Just} \; x) \; y = x + y$$

Invariant: mandatory pattern-matching on complex values

Two equivalent versions (Maybe $a = 1 + a$; Nothing $= (0, ())$; Just $x = (1, x)$):

$$maybeAdd_1 : (1 + \text{Val Int}) \to \text{Val Int} \to \text{Eval}(\text{Ret}(\text{Val Int}))$$

$$
\begin{aligned}
maybeAdd_1 = \{ (0, ()) \quad &\cdot \; (\text{val int } y) \cdot \text{eval} \; \to \textbf{ret} \; \text{val } y &\text{(Choice \#0)} \\
(1, \text{val int } x) \cdot &\; (\text{val int } y) \cdot \text{eval} \; \to x + y &\text{(Choice \#1)} \}
\end{aligned}
$$

$$maybeAdd_2 : (\text{Val Int} \to \text{Eval}(\text{Ret}(\text{Val Int}))) \; \& \; (\text{Val Int} \to \text{Val Int} \to \text{Eval}(\text{Ret}(\text{Val Int})))$$

$$
\begin{aligned}
maybeAdd_2 = \{ 0 \cdot \quad\quad\quad &(\text{val int } y) \cdot \text{eval} \; \to \textbf{ret} \; \text{val } y &\text{(Choice \#0)} \\
1 \cdot \; (\text{val int } x) \cdot &(\text{val int } y) \cdot \text{eval} \; \to x + y &\text{(Choice \#1)} \}
\end{aligned}
$$

$$maybeAdd \text{ Nothing } y = y \qquad\qquad maybeAdd \text{ (Just } x) \; y = x + y$$

Invariant: mandatory pattern-matching on complex values

Two equivalent versions (Maybe $a = 1 + a$; Nothing $= (0, ())$; Just $x = (1, x)$):

$maybeAdd_1 : (1 + \text{Val Int}) \rightarrow \text{Val Int} \rightarrow \text{Eval(Ret(Val Int))}$
$maybeAdd_1 = \{(0, ()) \qquad \cdot ((\text{val int } y) \cdot \text{eval}) \rightarrow \textbf{ret val } y \qquad$ (Choice #0)
$\qquad\qquad (1, \text{val int } x) \cdot ((\text{val int } y) \cdot \text{eval}) \rightarrow x + y \qquad$ (Choice #1)$\}$

$maybeAdd_2 : (\text{Val Int} \rightarrow \text{Eval(Ret(Val Int))}) \; \& \; (\text{Val Int} \rightarrow \text{Val Int} \rightarrow \text{Eval(Ret(Val Int))})$
$maybeAdd_2 = \{0 \cdot \qquad\qquad ((\text{val int } y) \cdot \text{eval}) \rightarrow \textbf{ret val } y \qquad$ (Choice #0)
$\qquad\qquad 1 \cdot ((\text{val int } x) \cdot (\text{val int } y) \cdot \text{eval}) \rightarrow x + y \qquad$ (Choice #1)$\}$

$maybeAdd_1$ takes a Maybe argument; $maybeAdd_2$ gives a product of 2 functions

Stopping Fusion

Putting complex values in a Box pauses pattern-matching.

$$maybeAdd_3 : \mathsf{Val}(\mathsf{Box}(1 + \mathsf{Val}\,\mathsf{Int})) \rightarrow \mathsf{Val}\,\mathsf{Int} \rightarrow \mathsf{Eval}(\mathsf{Ret}(\mathsf{Val}\,\mathsf{Int}))$$

$$maybeAdd_3 = \{\ \mathsf{val}\ \mathsf{ref}\ x \cdot \mathsf{val}\ \mathsf{int}\ y \cdot \mathsf{eval} \rightarrow$$

$$\textbf{unbox}\ x\ \textbf{as}\ \{$$

$$(0, ()) \qquad \rightarrow \textbf{ret}\ \mathsf{val}\ y$$

$$(1, \mathsf{val}\ \mathsf{int}\ x) \rightarrow x + y$$

$$\}$$

$$\}$$

Putting complex values in a Box pauses pattern-matching.

$$maybeAdd_3 : \mathsf{Val}(\mathsf{Box}(1 + \mathsf{Val\,Int})) \to \mathsf{Val\,Int} \to \mathsf{Eval}(\mathsf{Ret}(\mathsf{Val\,Int}))$$

$$maybeAdd_3 = \{\; \text{val ref } x \cdot \text{val int } y \cdot \text{eval} \to$$

$$\textbf{unbox } x \textbf{ as } \{$$

$$(0, ()) \qquad \to \textbf{ret val } y$$

$$(1, \text{val int } x) \to x + y$$

$$\}$$

$$\}$$

$$maybeAdd_3 \not\approx maybeAdd_1 \qquad\qquad maybeAdd_3 \not\approx maybeAdd_2$$

Foundations for Unboxing

The Usual Division of Types

Call-By-Push-Value

$$\textit{ValueType} \ni A ::= A_0 \times A_1 \mid A_0 + A_1 \mid \mathsf{U}\,\underline{B}$$
$$\textit{ComputationType} \ni \underline{B} ::= A \to \underline{B} \mid \underline{B}_0 \,\&\, \underline{B}_1 \mid \mathsf{F}\,A$$

Focusing & Polarity

$$\textit{PositiveType} \ni P^+ ::= P_0^+ \otimes P_1^+ \mid P_0^+ \oplus P_1^+ \mid {\downarrow}Q^-$$
$$\textit{NegativeType} \ni Q^- ::= P^+ \to Q^- \mid Q_0^- \,\&\, Q_1^- \mid {\uparrow}P^+$$

- *Value = Positive*
- *Computation = Negative*

The Usual Division of Types

Call-By-Push-Value

$$ValueType \ni A ::= A_0 \times A_1 \mid A_0 + A_1 \mid \mathsf{U}\,\underline{B}$$
$$ComputationType \ni \underline{B} ::= A \to \underline{B} \mid \underline{B}_0\,\&\,\underline{B}_1 \mid \mathsf{F}\,A$$

Focusing & Polarity

$$PositiveType \ni P^+ ::= P_0^+ \otimes P_1^+ \mid P_0^+ \oplus P_1^+ \mid {\downarrow}Q^-$$
$$NegativeType \ni Q^- ::= P^+ \to Q^- \mid Q_0^-\,\&\,Q_1^- \mid {\uparrow}P^+$$
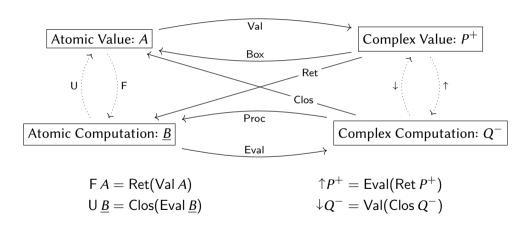
- *Value = Positive*?
- *Computation = Negative*?
- Right?

A Distinction between the Distinctions

- In Call-By-Push-Value, "value types" are the denotable values
  - Only value types are first class, can be named
  - Computation types are second class, cannot be named unless "thunked"

- With strict focusing, pattern matching is mandatory
  - Positive types are second class, must be matched instead of named
  - Negative types are first class, cannot be matched so they are named

- Opposite sides of the complex vs atomic divide:
  - Call-By-Push-Value talks about atomic values and computations
  - Focusing talks about complex values and computations

$$F\,A = \mathsf{Ret}(\mathsf{Val}\,A)$$
$$U\,\underline{B} = \mathsf{Clos}(\mathsf{Eval}\,\underline{B})$$

$$\uparrow P^+ = \mathsf{Eval}(\mathsf{Ret}\,P^+)$$
$$\downarrow Q^- = \mathsf{Val}(\mathsf{Clos}\,Q^-)$$

Sound & Complete w.r.t. Call-By-Push-Value!

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- Default "uniform" atomic representations / calling conventions:
  - Atomic value: ref = "reference" (i.e., pointer to value)
  - Atomic computation: sub = "subroutine" (i.e., pointer to top call stack frame)

- First-class closure values built by Clos : **cplx comp** → ref **val**
  - Closure introduced by clos { ... } around copattern-matching code
  - Closure f : Clos a eliminated with f. call operation

$app = \lambda f\, x.\, (f\, x)$

$app : \forall a : \text{Type ref } \mathbf{val}.\forall b : \text{Type sub } \mathbf{comp}\,.\, \downarrow(\text{Val } a \rightarrow \text{Eval } b) \rightarrow \text{Val } a \rightarrow \text{Eval } b$

$app = \{\text{ty } a \cdot \text{ty } b \cdot \text{val ref } f \cdot \text{val ref } x \cdot \text{eval sub} \rightarrow f.\text{call}(\text{val } x).\text{eval sub}\}$

HIGHER-ORDER CALLING CONVENTIONS

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Important Application: representation-polymorphic (type class) operator overloading

$$\textbf{class } \mathsf{Num} \; a \; \textbf{where } (+) \quad :: a \to a \to a$$
$$negate :: a \to a$$

What can we do without explicit representation polymorphism?

Stress Test: Representation-Polymorphic Overloading

Important Application: representation-polymorphic (type class) operator overloading

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$$\textbf{type } \mathsf{Num}(a : \textbf{cplx val}) : \textbf{cplx val} = \mathsf{Clos}(a \to a \to \uparrow a) \times \mathsf{Clos}(a \to \uparrow a)$$

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$$(+) \quad = \{ \text{ ty } a \cdot (\text{val ref } f, \text{val ref } g) \cdot \text{eval} \to \textbf{ret } \text{val } f \}$$

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What can we do without explicit representation polymorphism?

After type erasure, still get well-defined, operational code

$$(+) \quad = \{\ \mathsf{ty}\ a \cdot (\mathsf{val\ ref}\ f, \mathsf{val\ ref}\ g) \cdot \mathsf{eval} \to \textbf{ret}\ \mathsf{val}\ f\ \}$$

$$negate = \{\ \mathsf{ty}\ a \cdot (\mathsf{val\ ref}\ f, \mathsf{val\ ref}\ g) \cdot \mathsf{eval} \to \textbf{ret}\ \mathsf{val}\ g\ \}$$

COMPILING TO THE MACHINE

Complex patterns $\implies$ 1 simple switch

$x : \mathsf{Box}((\mathsf{Val\,Int} + \mathsf{Val\,Float} \times \mathsf{Val\,Int}) + 1)$

$$\textbf{unbox } x \textbf{ as } \{\, 0, 0, \mathsf{val\,int}\ y \qquad\qquad \rightarrow M_1;$$
$$0, 1, \mathsf{val\,flt}\ y, \mathsf{val\,int}\ z \rightarrow M_2;$$
$$1, ()\qquad\qquad\qquad\qquad \rightarrow M_3\}$$

```c
struct {
  char tag;
  union { // case 0 = 0, 0, val int
    int zero;
         // case 1 = 0, 1, val flt, val int
    struct { float fst; int snd; } one;
         // empty case 2 = 1, ()
  } body;
} *x;
switch (x->tag) {
  case 0:
    int y = x->body.zero; M1...; break;
  case 1:
    float y = x->body.one.fst;
    int z = x->body.one.snd;
    M2...; break;
  case 2:
    M3...
}
```

Complex variables

$$and\ \mathsf{True}\ x = x \qquad\qquad and\ \mathsf{False}\ x = \mathsf{False}$$

Complex variables $x \in \{\ pattern \dots \}$ match multiple patterns

$$\mathsf{Bool} = 1 + 1 \qquad \mathsf{True} = 1, () \qquad \mathsf{False} = 0, ()$$

$$and \quad : \mathsf{Bool} \to \mathsf{Bool} \to \mathsf{Eval}(\mathsf{Ret\ Bool})$$
$$and \quad = \{\, 1, () \cdot x \in \{\ 1, (); 0, ()\ \} \cdot \mathsf{eval\ sub} \to \mathbf{ret}\ x \in \{\ 1, (); 0, ()\ \}$$
$$\qquad\qquad 0, () \cdot x \in \{\ 1, (); 0, ()\ \} \cdot \mathsf{eval\ sub} \to \mathbf{ret}\ 0, ()\ \}$$

is syntactic shorthand for

$$and : \mathsf{Bool} \to \mathsf{Bool} \to \mathsf{Eval}(\mathsf{Ret\ Bool})$$
$$and = \{\, 1, () \cdot 1, () \cdot \mathsf{eval\ sub} \to \mathbf{ret}\ 1, (); \qquad 0, () \cdot 1, () \cdot \mathsf{eval\ sub} \to \mathbf{ret}\ 0, ();$$
$$\qquad\quad 1, () \cdot 0, () \cdot \mathsf{eval\ sub} \to \mathbf{ret}\ 0, (); \qquad 0, () \cdot 0, () \cdot \mathsf{eval\ sub} \to \mathbf{ret}\ 0, ();\ \}$$

COMPLEX ANSWERS

Complex continuations $more \in \{\ copattern \ldots \}$ match multiple calling conventions

app : $\forall a :$ Type ref **val** . $\forall b :$ Type sub **comp** . $\downarrow$(Val $a \rightarrow$ Eval b) $\rightarrow$ Val $a \rightarrow$ Eval b

app = $\{$ty $a \cdot$ ty $b \cdot$ val ref $f \cdot more \in \{$ val ref $x \cdot$ eval sub $\} \rightarrow f.$ call$\}$

$app2$: $\forall a, b :$ Type ref **val** . $\forall c :$ Type sub **comp** .

$\qquad\qquad \downarrow$(Val $a \rightarrow$ Val $b \rightarrow$ Eval c) $\rightarrow$ Val $a \rightarrow$ Val $b \rightarrow$ Eval c

$app2$ = $\{$ty $a \cdot$ ty $b \cdot$ ty $c \cdot$ val ref $f \cdot more \in \{$ val ref $x \cdot$ val ref $y \cdot$ eval sub $\} \rightarrow f.$ call$\}$

is syntactic shorthand for

$\qquad app$ = $\{$ty $a \cdot$ ty $b \cdot$ val ref $f \cdot$ val ref $x \cdot$ eval sub $\rightarrow f.$ call(val x). eval sub$\}$

$\qquad app2$ = $\{$ty $a \cdot$ ty $b \cdot$ ty $c \cdot$ val ref $f \cdot$ val ref $x \cdot$ val ref $y \cdot$ eval sub

$\qquad\qquad\qquad \rightarrow f.$ call(val x) (val y) . eval sub$\}$