ASSIGNMENT 1 — INDUCTIVE REASONING

COMP 3010 — ORGANIZATION OF PROGRAMMING LANGUAGES

1. Induction

Exercise 1 (Multiple Choice)**.** Consider the total function

$$f(x) = 4x + 5$$

Which of the following inductive definitions are equivalent to the above f?

(a) $\boxed{f(x+1) = f(x) + 5}$

(b) $\boxed{f(x+1) = f(x) + 4}$

(c) $\boxed{f(0) = 5 \qquad f(x+1) = f(x) + 4}$

(d) $\boxed{f(0) = 4 \qquad f(x+1) = f(x) + 5}$

Exercise 2. $\mathbb{N}$ stands for the set of all natural numbers, and $\mathbb{N}^*$ stands for the set of *all* finite lists of natural numbers of any length. For example, $\mathbb{N}^*$ contains each of the lists $[9, 9, 9]$, $[10, 9, 8, 7, \ldots, 3, 2, 1]$ and $[5, 16, 8, 4, 2, 1]$. $\mathbb{N}^*$ is defined inductively as the smallest set such that:

- $\mathbb{N}^*$ contains the empty list $[]$, and
- given any natural number x_0 from $\mathbb{N}$ and any list $[x_1, \ldots, x_n]$ already in $\mathbb{N}^*$, the set $\mathbb{N}^*$ also contains the list $[x_0, x_1, \ldots, x_n]$.

Give an inductive definition for the function f, which takes a finite list of numbers xs such that $f(xs)$ is three times the sum of the list. Your definition of f should follow the inductive pattern

$$f([]) = \ldots$$
$$f([x_0, x_1, \ldots, x_n]) = \ldots f([x_1, \ldots, x_n]) \ldots$$

so that it gives the same result as:

$$f([x_0, x_1, x_2, \ldots, x_n]) = 3 \times (x_0 + x_1 + x_2 + \cdots + x_n)$$

on any argument. In the case of the empty list $[]$, the sum of $[]$ is 0.

Hint: you might find the fact that

$$3 \times (x_0 + x_1 + \cdots + x_n) = (3 \times x_0) + (3 \times x_1) + \cdots + (3 \times x_n)$$

is useful for writing your inductive definition of f.

Exercise 3 (Multiple Choice)**.** Let X be inductively defined as the smallest set of finite number lists (taken from $\mathbb{N}^*$) satisfying the following closure properties:

- X contains the empty list $[]$.
- Given any natural number n from $\mathbb{N}$, the set X contains the one-element list $[n]$.

Date: Fall 2025 — 3 pages.

- Given any natural number y from $\mathbb{N}$ and any list $[x_1, \ldots, x_n]$ already in X, the set X also contains the list $[y, x_1, \ldots, x_n, y]$.

Which of the following lists is NOT in X?

(a) $[1, 2, 3, 2, 1]$
(b) $[1, 2, 1, 2, 1]$
(c) $[1, 2, 3, 4, 5]$
(d) $[1, 1, 1, 1, 1]$

2. Syntax and Grammars

Exercise 4 (Multiple Choice)**.** Consider the grammar G (note that ε stands for the empty string):

$$S ::= \mathtt{a}S \mid T$$
$$T ::= \mathtt{b}T \mid U$$
$$U ::= \mathtt{c}U \mid \varepsilon$$

(1) Which of the following strings is generated by the grammar G?
   (a) `acb`
   (b) `acc`
   (c) `cba`
   (d) `baa`
(2) Which of the following is a derivation of `bbc` in the grammar G?
   (a) $S \to \mathtt{b}T \to \mathtt{bb}T \to \mathtt{bb}U \to \mathtt{bbc}U \to \mathtt{bbc}$
   (b) $S \to T \to \mathtt{b}T \to \mathtt{b}T\mathtt{b}T \to \mathtt{bb}T \to \mathtt{bb}U \to \mathtt{bbc}U \to \mathtt{bbc}$
   (c) $S \to T \to U \to \mathtt{b}U \to \mathtt{bb}U \to \mathtt{bb}U \to \mathtt{bbc}$
   (d) $S \to T \to \mathtt{b}T \to \mathtt{bb}T \to \mathtt{bb}U \to \mathtt{bbc}U \to \mathtt{bbc}$

Exercise 5 (Short Answer)**.** In your own words, describe what language this grammar generates (remember, ε stands for the empty string):

$$S ::= \mathtt{a}S\mathtt{a} \mid T$$
$$T ::= \mathtt{b}T \mid \varepsilon$$

3. Syntax Trees

Exercise 6. Consider the following grammar for *concrete* syntax of arithmetic expressions:

$$E ::= E{+}T \mid E{-}T \mid T$$
$$T ::= T{\times}F \mid T/F \mid F$$
$$F ::= \mathsf{1} \mid \mathsf{2} \mid \mathsf{3} \mid \mathsf{4} \mid (E)$$

$$\mathsf{1} \times \mathsf{2} - (\mathsf{3}/\mathsf{4})$$

(1) Write a parsing derivation starting from E (looking like $E \to E - T \to \ldots$)
(2) Draw a parse tree (remember that the parentheses in the string should be included somewhere as a node for parentheses in the tree!)

Exercise 7. Consider the following grammar for *abstract* syntax of arithmetic expressions:

$$E ::= E{+}E \mid E{-}E \mid E{\times}E \mid E/E \mid 1 \mid 2 \mid 3 \mid 4$$

with the usual associativity and precedence for the arithmetic operators (all operators are left-associative, $\times$ and $/$ have a higher precedence than $+$ and $-$).

(1) Draw an abstract syntax tree for each of the following strings:
  (a) $\mathtt{1} + (\mathtt{3}/\mathtt{4})$
  (b) $\mathtt{1} \times (\mathtt{2} + \mathtt{3})$
  (c) $(\mathtt{1} \times \mathtt{2}) - (\mathtt{3}/\mathtt{4})$

(2) (Multiple Choice & Short Answer) Which of the following pairs of strings with different parentheses represent the same abstract syntax tree according to the above precedence and associativity? Draw that abstract syntax tree.
  (a) $\mathtt{2} \times \mathtt{4} - \mathtt{3}$ versus $\mathtt{2} \times (\mathtt{4} - \mathtt{3})$
  (b) $\mathtt{1} + \mathtt{2} + \mathtt{3} + \mathtt{4}$ versus $\mathtt{1} + (\mathtt{2} + (\mathtt{3} + \mathtt{4}))$
  (c) $\mathtt{2} + \mathtt{3} \times \mathtt{4}/\mathtt{2}$ versus $\mathtt{2} + ((\mathtt{3} \times \mathtt{4})/\mathtt{2})$