School of Mechanical and Aerospace Engineering

MAE 6263 Course Project III

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1 Introduction

The purpose of this course project is to document the development of numerical routine and its solution for the 2-D Navier-Stokes equations for a flow configuration given by the classical 2-D channel flow problem. An explicit iterative scheme shall be devised with two different methods used to treat the pressure Poisson's equation. Through this process, it is endeavoured to understand the importance of the use of a pressure solver that ensures a divergence free field throughout the solution process. A simple compact stencil for the pressure is used to demonstrate how the divergence increases throughout the system and a fractional step method is used to ensure a divergence free field.

1.1 Problem Statement

The 2-D Navier-Stokes (NS) equations can be written in their primitive variable formulation as:

$$\frac{\partial u}{\partial t} + \bar{u}.\nabla \bar{u} = \nu \Delta \bar{u} - \frac{1}{\rho}\nabla p. \tag{1.1}$$

The evolution equation for pressure to complete the set of equations required to solve for the fluid flow solution in a given domain is obtained by the divergence of this equation and is detailed later on in the document.

The domain to be solved for is a rectangular domain of unit width and length two units which have initial and boundary conditions in velocity specified by:

$$
\begin{aligned}
u(x,y) &= 0 \;\text{ for }\; y = 1;\\
u(x,y) &= 0 \;\text{ for }\; y = 0;\\
u(x,y) &= u(x-2,y) \;\text{ for }\; x = 2;\\
u(x,y) &= u(x+2,y) \;\text{ for }\; x = 2;
\end{aligned}
\tag{1.2}
$$

The pressure boundary conditions are Dirichlet in the streamwise (or x) direction and Neumann in the y direction. These can be shown by -

$$
\begin{aligned}
p(x,y) &= p_{outlet} \text{ for } x = 2\\
p(x,y) &= p_{inlet} \text{ for } x = 0\\
\frac{\partial p}{\partial y} &= 0 \;\text{ for }\; y = 0,1\\
\frac{\partial p}{\partial y} &= 0 \;\text{ for }\; y = 0,1
\end{aligned}
\tag{1.3}
$$

The exact solution of this system is given by the Poiseuille equation which prescribes a parabolic flow profile for the x component of the velocity on variation of the wall normal displacement. The exact solution of this domain and boundary condition is given by -

$$
u_x = -\frac{1}{2\mu}\frac{\partial p}{\partial x}(h^2 - y^2), \tag{1.4}
$$

where h is the height of the channel from the center-line and y is the distance of the point in question from the center-line. A figure depicting the steady state profile of the flow is shown in Fig. (1.1). The outlet pressure can thus be calculated from this expression as follows. We first have, for the conditions given, a straight-line velocity for the centerline from the channel obtained from the Reynolds number.

$$
\frac{2hu_c}{\nu} = 100 \tag{1.5}
$$

The above expression gives us a centerline velocity of $u_c = 1$ cm/s. Substituting this value into Eq. (1.4) gives us the following expression for the pressure gradient -

$$
-\frac{2\mu u_c}{h^2} = \frac{\partial p}{\partial x} \tag{1.6}
$$

Noting that the height from the centerline $h = 0.5$ cm, $\mu = \nu/\rho = 0.01$ cm^2/s and u_c found previously, our pressure gradient becomes -

$$
\frac{\partial p}{\partial x} = -0.08. \tag{1.7}
$$

Therefore our outlet pressure can be expressed as -

$$
p_{outlet} = p_{inlet} - 0.016 \tag{1.8}
$$

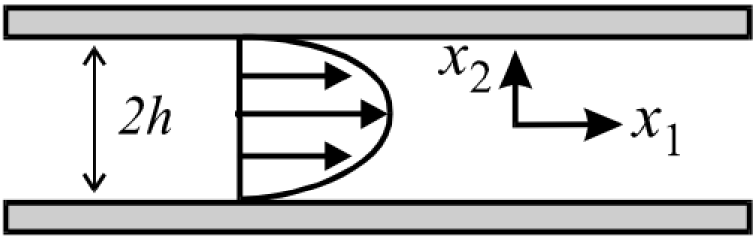


Figure 1.1: Expected steady state profile for velocity

1.2 Derivation of Pressure Poisson Equation

The pressure Poisson equation may be obtained by taking the divergence of the momentum equation. This process can be detailed in the following procedure. The momentum equation for the NS equations are given by -

$$\frac{\partial u}{\partial t} + \bar{u}.\nabla \bar{u} = \nu \Delta \bar{u} - \frac{1}{\rho} \nabla p. \tag{1.9}$$

A divergence operation of the momentum equation given above gives us -

$$\nabla . \left(\frac{\partial u}{\partial t} + \bar{u}.\nabla \bar{u} = \nu \Delta \bar{u} - \frac{1}{\rho} \nabla p. \right). \tag{1.10}$$

Let us assume the following -

$$\bar{H} = -\bar{u}.\nabla \bar{u} + \nu \Delta \bar{u}. \tag{1.11}$$

Our divergence of the momentum equation can then be represented as -

$$\nabla^2 p = \rho \nabla . \bar{H}. \tag{1.12}$$

Note that the unsteady term drops out due to the assumption of a divergence free field and that the temporal and spatial derivatives commute. The above equation is the pressure Poisson equation that needs to be solved to (its) steady state within each time step of the explicit time marching. In case of the 2D channel flow problem the following relations can be derived for the pressure Poisson's equation. Continuity is determined by -

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0. \tag{1.13}$$

The x momentum equation is given by -

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}) \tag{1.14}$$

and the y momentum equation is given by -

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}). \tag{1.15}$$

Taking a divergence of the momentum equation implies taking the x derivative of equation (1.14) and y derivative of equation (1.15) and adding them together. We get from the above operations the following equation -

$$-\frac{1}{\rho}\nabla^2 p = (\frac{\partial u}{\partial x})^2 + 2\frac{\partial u}{\partial y}\frac{\partial v}{\partial x} + (\frac{\partial v}{\partial y})^2. \tag{1.16}$$

Note that this is the continuous form of the pressure Poisson's equation for the 2D problem. A considerable amount of simplification is achieved by enforcing the continuity condition.

1.3 Explicit Solution of 2D Navier-Stokes equations

The semi-discrete form of the explicit formulation for this problem can be depicted as -

$$\bar{u}^{n+1} = \bar{u}^n + \Delta t\left[\bar{H} - \nabla p\right] \tag{1.17}$$

The above equation is a vector equation and is thus composed of two scalar equations in this 2D case.

$$u^{n+1} = u^n + \Delta t(H_x - \nabla_x p) \tag{1.18}$$

and

$$v^{n+1} = v^n + \Delta t(H_y - \nabla_y p) \tag{1.19}$$

1.4 Conservation of Kinetic Energy

The numerical schemes used for the gradient and divergence scheme are a central difference scheme for both. In order to prove kinetic energy conservation, let us assume a hypothetical 1D domain with nodes numbered from 0 to 5 (i.e. nodes 0 and 5 are at the surface of the domain). Kinetic energy conservation can be confirmed if the following expression holds true.

$$\sum_{i=1}^{4}\left(u_i G_i p + p_i D_i u_i\right)\Delta\Omega = \text{surface terms only} \tag{1.20}$$

Assuming a central difference scheme for both gradient and divergence operations we can obtain the following extended expression for the assumed 1D domain -

$$\begin{aligned} u_1(p_2-p_0)+p_1(u_2-u_0)+u_2(p_3-p_1)+p_2(u_3-u_1)+u_3(p_4-p_2)+p_3(u_4-u_2) \\ +u_4(p_5-p_3)+p_4(u_5-u_3) = -u_1p_0-p_1u_0+p_4u_5+u_4p_5. \end{aligned} \tag{1.21}$$

From the above equation it can be seen that only surface terms are left behind and so the choice of gradient and divergence schemes for this project will conserve kinetic energy.

1.5 Compact stencil pressure solver

A compact stencil may be used to solve the pressure Poisson equation as given in equation 27 of the handouts. The explicit scheme to solve the momentum equation for the x component of the velocity is shown below -

$$\frac{u_{ij}^{n+1}-u_{ij}^{n}}{\Delta t} = -u_{ij}^{n}\frac{u_{ij}^{n}-u_{i-1,j}^{n}}{\Delta x} - \frac{p_{i+1,j}^{n}-p_{i-1,j}^{n}}{\Delta x} + \frac{1}{\mathrm{Re}}\frac{u_{i+1,j}^{n}+u_{i-1,j}^{n}-2u_{ij}^{n}}{\Delta x^2}. \tag{1.22}$$

Similarly we have -

$$\frac{v_{ij}^{n+1}-v_{ij}^{n}}{\Delta t} = -u_{ij}^{n}\frac{v_{ij}^{n}-v_{i,j-1}^{n}}{\Delta y} - \frac{p_{i,j+1}^{n}-p_{i,j-1}^{n}}{\Delta y} + \frac{1}{\mathrm{Re}}\frac{v_{i,j+1}^{n}+v_{i,j-1}^{n}-2v_{ij}^{n}}{\Delta y^2}. \tag{1.23}$$

The compact stencil implementation of the pressure Poisson equation is given by -

$$\frac{p_{i+1,j}^{n}+p_{i-1,j}^{n}-2p_{ij}^{n}}{2\Delta x^2} + \frac{p_{i,j+1}^{n}+p_{i,j-1}^{n}-2p_{ij}^{n}}{2\Delta x^2} = \frac{Hx_{i+1,j}^{n}-Hx_{i-1,j}^{n}}{2\Delta y} + \frac{Hy_{i,j+1}^{n}-Hy_{i,j-1}^{n}}{2\Delta y}. \tag{1.24}$$

The above equation simplifies considerably due to the selection of an equal grid spacing in both x and y directions. It should be noted that the above method is not at all ideal to solve the NS equations with due to the fact that there is no mechanism to ensure the successively evolved fields through the explicit algebraic scheme are divergence free. This would inevitably lead to an increase (or decrease) in the mass of the system eventually leading to a physically incorrect solution.

1.6 Ensuring divergence free evolution of velocity field

There are several techniques in the CFD community to ensure that the velocity fields are divergence free, the problem statement asks us to use the momentum interpolation method for this purpose. The momentum interpolation method is a combination of the compact and non-compact schemes used for the pressure Poisson solver with the advantage of ensured divergence and no checker-boarding. It is utilized by adding a third order dissipation term to the pressure Poisson solver by means of a cell face interpolation carried out in a slightly counter-intuitive manner. This is done through the direct computation of the cell face values instead of the standard linear interpolation.

This project uses the fractional step method to solve the Pressure Poisson equation. The fractional step method can be detailed in the following manner:

For the incompressible NS equations given by

$$\frac{\partial \bar{u}}{\partial t} + (\bar{u.}\nabla)\bar{u} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\bar{u}, \tag{1.25}$$

can be depicted explicitly as -

$$\bar{u}^{n+1} = \bar{u}^n + \Delta t\left(-\bar{u}^n.\nabla\bar{u}^n - \frac{1}{\rho}\nabla^2 p^{n+1} + \nu\nabla^2\bar{u}^n\right). \tag{1.26}$$

In the numerical method, it is needed for $\nabla.\bar{u}^{n+1} = 0$, but there is no guarantee that $\nabla.\bar{u}^n = 0$. This issue can be countered by using the fractional step method. Once we have force the divergence free condition at the $n+1$ time step our explicit equation can be given by -

$$\nabla^2 p^{n+1} = \rho\frac{\nabla.\bar{u}^n}{\Delta t} + \left(-\rho\nabla.(\bar{u}^n.\nabla\bar{u}^n) + \mu\nabla^2(\nabla.\bar{u}^n)\right) \tag{1.27}$$

In essence, it may be considered that the above step is a *correction* to the pressure field to ensure that the velocity field evolved from the NS momentum equations remains divergence field. This newly evolved pressure may now be used again in the momentum equations without any fear of divergence issues. Noting the expression given in Eq. (1.16) we can obtain the following expression for the pressure correction step -

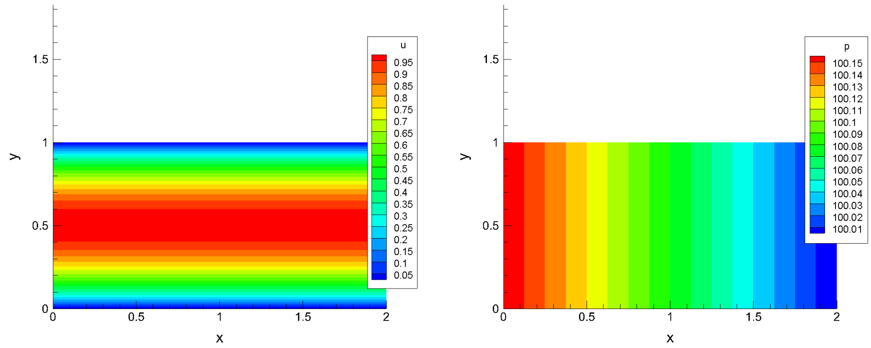
$$\nabla^2 p^{n+1} = \rho\frac{\nabla.\bar{u}^n}{\Delta t} - \rho\left((\frac{\partial u}{\partial x})^2 + 2\frac{\partial u}{\partial y}\frac{\partial v}{\partial x} + (\frac{\partial v}{\partial y})^2\right). \tag{1.28}$$

A short note here is that the first term on the right hand side of the aforementioned equation is left behind because of the assumption that the velocity field evolved at the previous timestep may not be divergence free.

2 Numerical Experimentation

The effect of the pressure Poisson solver was quantified using two numerical simulations for the steady state solution of the 2D channel flow problem. It was expected that the divergence of the solution field for the compact stencil version would be non zero (and increasing in absolute value) through the simulation whereas the fractional step method would preserve the divergence free condition throughout.

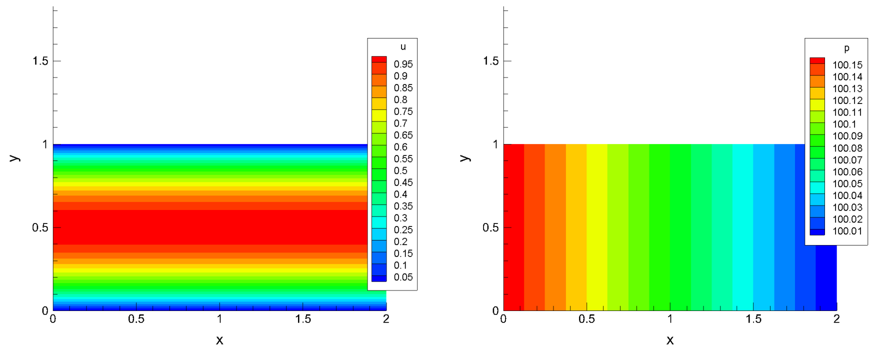
First we compare the steady state isocontours of the pressure and velocity field. For the compact stencil version of the Pressure Poisson solver we obtained the following steady state isocontours.



(a) Steady-state isocontour for x component of velocity

(b) Steady-state isocontour for pressure

Figure 2.1: Steady-state isocontours for the compact stencil pressure Poisson solver.

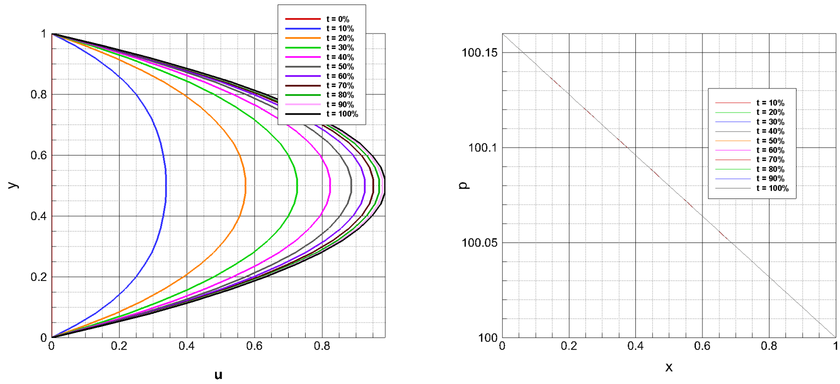


(a) Steady-state isocontour for x component of velocity

(b) Steady-state isocontour for pressure

Figure 2.2: Steady-state isocontours for the fractional step Poisson solver.

As on can see from Figs. (2.1 and 2.2). It is difficult to determine a significant difference in both methods using isocontour data at the steady-state. The difference between both methods can better be elucidated using a divergence tracker. The following lineplots were generated for the steady state profile of the pressure and velocity at the centerline (i.e. $y = 1$). For the compact scheme we had -

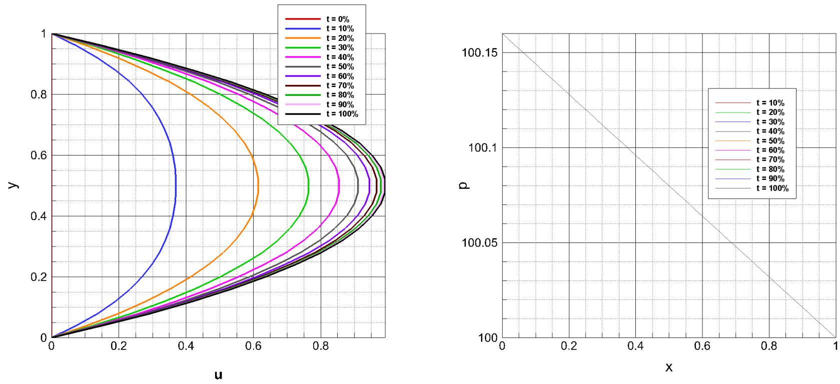


(a) Steady-state line plot for x component of velocity

(b) Steady-state line plot for pressure

Figure 2.3: Steady-state centerline plots for the compact stencil pressure Poisson solver.

A validation of the methods can be seen by the maximum velocity matching the initial centerline velocity assumed for the calculation of pressure gradient (1cm/s). Similar plots can be seen using the fractional step method -

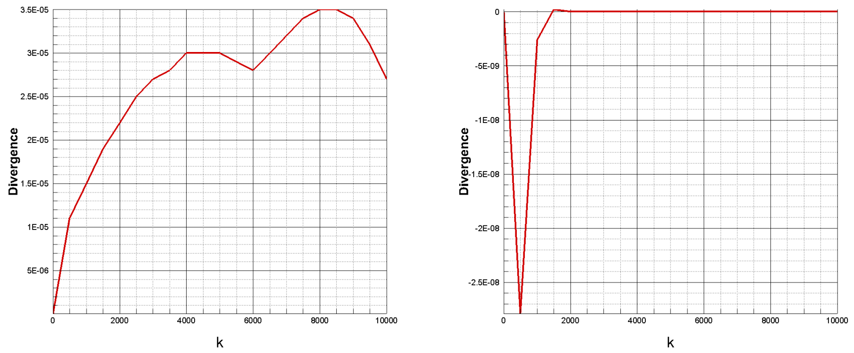


(a) Steady-state line plot for x component of velocity

(b) Steady-state line plot for pressure

Figure 2.4: Steady-state centerline plots for the fractional step pressure Poisson solver.

The mass in the system can be tracked by plotting the divergence with iteration number on progress to the steady state. In the compact scheme, as expected a non-zero divergence was seen as shown in Figure (2.5) whereas the fractional step method displayed a reduction of many orders of magnitude in the divergence in comparison.



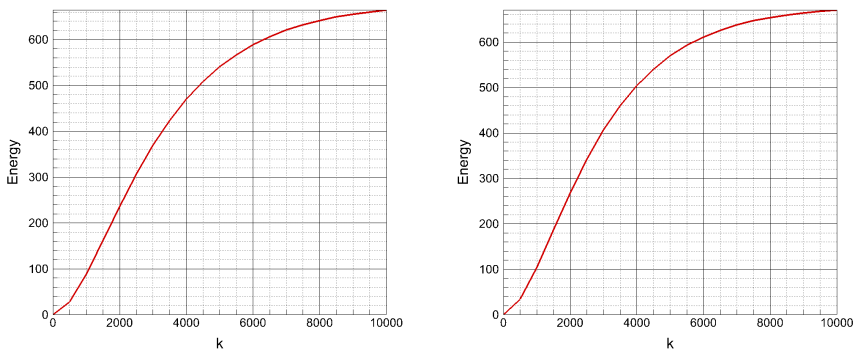
(a) Compact stencil for pressure (b) Fractional step solver for pressure

Figure 2.5: A comparison of pressure solvers for their effect on divergence

The energy in the system is plotted as a function of iteration number as the code converges to steady state. The energy of the system is plotted by the following summation -

$$\sum_{j=0}^{N_y} \sum_{i=0}^{N_x} = \rho u_{ij}^2 \tag{2.1}$$

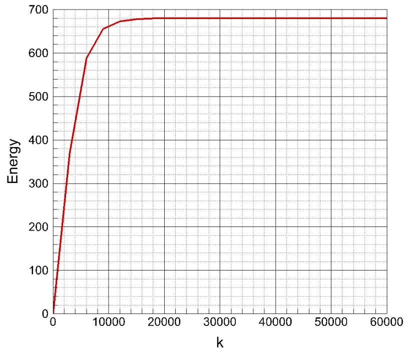
where N_x and N_y are the total number of points in the x and y directions respectively. The following plots for energy were observed -



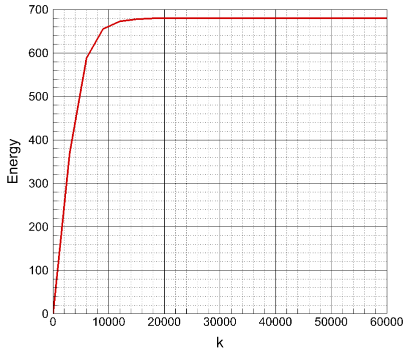
(a) Compact stencil for pressure (b) Fractional step solver for pressure

Figure 2.6: Total energy in the system through iteration k

On increasing the simulation time beyond steady state the following behavior was observed -



(a) Compact stencil for pressure



(b) Fractional step solver for pressure

Figure 2.7: Total energy in the system through iteration k

As observed the energy in the system remains constant after attainment of steady state.