Problem Set 6

Due Monday, October 30, 2023 at 8pm

1. Prove that for all integers $n \geqslant 1$,
$$\sum_{j=1}^{n}(4j+1) = n(2n+3).$$

Solution:

1 We will give a proof by mathematical induction.
For all integers $n \geqslant 1$,
$$\sum_{j=1}^{n}(4j+1) = n(2n+3).$$

Proof. Let $P(n)$ be the following statement:
$P(n)$: For all integers $n \geqslant 1$,
$$\sum_{j=1}^{n}(4j+1) = n(2n+3).$$

Base Case. $P(1)$ is the equation $4(1)+1 = 1(2(1)+3)$
$5 = 1(5)$
$5 = 5$
Therefore the base case is true.

Induction Step. Assume that $P(n)$ is true. We can show that this holds for $n+1$
We can rewrite
$$\sum_{j=1}^{n+1}(4j+1) = (n+1)(2(n+1)+3).$$
as the following, $\displaystyle\sum_{j=1}^{n}(4j+1) + 4(n+1) + 1.$
We can substitute the induction hypothesis such that $n(2n+3)+4(n+1)+1$
We can simplify the expression into the following, $2n^2+3n+4n+4+1$
$= 2n^2+7n+5$
$= (n+1)(2n+5)$, by factoring
Finally, we can write the right side of our equality such that $(n+1)(2(n+1)+3) = (n+1)(2n+5)$.
Thus, the sides are $(n+1)(2n+5) = (n+1)(2n+5)$, which are equal to each other, proving the statement true.
By the Principal of Mathematical Induction, $P(n)$ is true for all integers $n \geqslant 1$,
$$\sum_{j=1}^{n}(4j+1) = n(2n+3).$$

$\square$

2. Use mathematical induction to prove the following:

Theorem 1. *Let x be a real number with $x > 0$. Then for each natural number $n \geqslant 2$,*
$$(1+x)^n > 1+nx.$$

Make sure you note where you have used the assumption that $x > 0$.
Solution:

2 Let x be a real number with $x > 0$. Then for each natural number $n \geqslant 2$,

$$(1 + x)^n > 1 + nx.$$

Proof. Let $P(n)$ be the following statement: $P(n)$: Let x be a real number with $x > 0$. Then for each natural number $n \geqslant 2$,

$$(1 + x)^n > 1 + nx.$$

Base Case. For the base case, we consider $n = 2$. We want to show that $(1 + x)^2 > 1 + 2x$.

$$(1 + x)^2 = 1 + 2x + x^2$$

$$\text{Since } x > 0, \text{ we know } x^2 \text{ is positive.}$$

$$\text{So, } 1 + 2x + x^2 > 1 + 2x.$$

Therefore, the base case holds.
We must have that $x > 0$ since this could cause our base case to be incorrect.
Inductive Step. Assume that for some natural number $k \geq 2$,

$$(1 + x)^k > 1 + kx.$$

We need to prove that the statement is true for $n = k + 1$

$$(1 + x)^{k+1} > 1 + (k + 1)x.$$

Starting with $(1+x)^{k+1}$, we can rewrite it as $(1+x)^k(1+x)$. By our induction hypothesis, we know that $(1 + x)^k > 1 + kx$. So,

$$(1 + x)^{k+1} > (1 + kx)(1 + x).$$

Expanding this, we get

$$1 + (k + 1)x + kx^2.$$

Since $k \geq 2$ and $x > 0$, kx^2 is also positive. Therefore,

$$1 + (k + 1)x + kx^2 > 1 + (k + 1)x.$$

This proves the induction step.
By mathematical induction, we have shown that for each natural number $n \geq 2$,

$$(1 + x)^n > 1 + nx.$$

□

3. Let $(f_n)_n$ be the sequence of Fibonacci numbers. Prove that for all integers $n \geqslant 1$, f_{5n} is a multiple of 5.
Solution:

3 Let $(f_n)_n$ be the sequence of Fibonacci numbers. Prove that for all integers $n \geq 1$, f_{5n} is a multiple of 5.
Proof. We will prove this statement using mathematical induction.
For the base case, when $n = 1$, we have $f_{5n} = f_{5 \cdot 1} = f_5$.
The Fibonacci sequence is defined as follows:
$f_0 = 0$
$f_1 = 1$
$f_2 = f_0 + f_1 = 1$
$f_3 = f_1 + f_2 = 2$
$f_4 = f_2 + f_3 = 3$
$f_5 = f_3 + f_4 = 5$

As we can see, f_5 is indeed equal to 5, which is a multiple of 5. So, the base case holds.

Induction Hypothesis: Assume that for some positive integer $k \geq 1$, f_{5k} is a multiple of 5.
We need to prove that the statement is true for $n = k+1$, $f_{5(k+1)}$ is a multiple of 5.
Starting with the Fibonacci recurrence relation, we can keep using relations.

$$\begin{aligned} f_{5(k+1)} &= f_{5k+5} = f_{5k+3} + f_{5k+4}. \\ &= 2(f_{5k+3}) + f_{5k+2}. \\ &= 3(f_{5k+2}) + 2(f_{5k+1}). \\ &= 5(f_{5k+1}) + 3(f_{5k}). \end{aligned}$$

Since, (f_{5k+1}) is multiplied by 5, then $5(f_{5k+1})$ is a multiple of 5.
Also, using our induction hypothesis, (f_{5k}) is a multiple of 5, thus $3(f_{5k})$ is a multiple of 5.

Next, since $f_{5(k+1)} = 5(f_{5k+1}) + 3(f_{5k})$. We can show that two multiples of 5 added together can be divided by 5.

Suppose a is an integer such that $a = 5k$ and b is an integer such that $b = 5m$, where k and m are some integer, we can show that $a+b$ is divided by 5.

$a + b = a + b$
$= 5k + 5m$
$= 5(k + m)$
By definition of divides, 5 divides $a + b$ thus it a multiple of 5.

Applying that, (f_{5k+1}) is a multiple of 5

Therefore, we've shown that if f_{5k} is a multiple of 5, then $f_{5(k+1)}$ is also a multiple of 5, which proves the statement for $n = k+1$.
By the Principle of Mathematical Induction, we've demonstrated that for all integers $n \geq 1$, f_{5n} is indeed a multiple of 5. $\square$

4. For each part of this problem, there is a proposed proof of a theorem. However, each theorem **is false**. Therefore, each proposed proof is, of course, incorrect. Your task is to find the error in each proof and clearly explain how the reasoning is flawed.
Warning: *It is not sufficient to only give a counterexample* – that just shows that the statement is false, but does not indicate where the proof logic is flawed.

False Theorem 4.1. All cows in Nebraska are the same color.

Proof. Let $P(n)$ be the following statement:

$$P(n) : \text{All cows in a set of } n \text{ Nebraska cows are the same color.}$$

- *Base Case.* $P(1)$ is true, since a set consisting of 1 cow will all be the same color.
- *Induction Step.* Assume that $P(n)$ is true. That is, assume that all cows in any set of n cows are the same color. Consider any $n+1$ cows; number these as cows $1, 2, 3, \ldots, n, n+1$. The first n of these cows (cows $1, 2, \ldots, n$) all must have the same color by the induction hypothesis, and the last n of these cows (cows $2, 3, \ldots, n+1$) must also have the same color. Since the set of the first n cows and the set of the last n cows overlap, all $n+1$ must be the same color. This verifies that $P(n+1)$ is true.

By the Principle of Mathematical Induction, $P(n)$ is true for all $n \in \mathbb{N}$, and since there are a finite number of cows in Nebraska, the statement is true. $\square$

False Theorem 4.2. All positive integers are equal, i.e. $a = b$ for any positive integers a and b.

Proof. Let $P(n)$ be the following statement:

$$P(n) : \text{Given any two positive integers } a \text{ and } b \text{ with } a \leqslant n \text{ and } b \leqslant n,\ a = b.$$

If we prove this statement for all natural numbers n via the Principle of Mathematical Induction, this would prove the proposition.

- *Base Case.* $P(1)$ is the statement

  $$P(1) : \text{Given any two positive integers } a \text{ and } b \text{ with } a \leqslant 1 \text{ and } b \leqslant 1,\ a = b.$$

  This statement is true, since the only positive integer less that or equal to 1 is 1 itself: $a = b = 1$.
- *Induction Step.* Assume that $P(n)$ is true, and we must show that $P(n + 1)$ is true:

  $$P(n + 1) : \text{Given any two positive integers } a \text{ and } b \text{ with } a \leqslant n + 1 \text{ and } b \leqslant n + 1,\ a = b.$$

  Let a and b be any two positive integers satisfying $a \leqslant n+1$ and $b \leqslant n+1$. Notice that $a-1 \leqslant n$ and $b - 1 \leqslant n$. By $P(n)$, we know that $a - 1 = b - 1$, which implies that $a = b$. □

Solution:

4.1 The logic of the proof is mislead in the base case. The base case can only handle the logic of sets that are relative to a single cow. This fails when performing the induction step when n = 2. In that case you have two sets, $S_0 = C_1$ and $S_1 = C_2$. This is not valid as the induction step states, "Since the set of the first n cows and the set of the last n cows overlap, all $n + 1$ must be the same color". In order to make this proof valid. It is required that the base case would show that for any two cows, they have the same color.

4.2 The logic of the proof is wrong in the induction step. Everything seems to be fine until they try to subtract 1 from a and b. As the assumption says, a and b are two positive integers, this causes issues since there is a possibility that once you subtract 1 from either they are no longer positive, which does not allow for you to use the induction hypothesis anymore. Therefore, the proof is not valid.