NUMERICAL METHODS IN FINANCE

PROBLEM SET

Instructions: You must submit a zip file to jpgaivao@iseg.ulisboa.pt no later than midnight June 02 2019. The zip file must contain all the m-files used and a pdf file reporting all the answers, including the graphs.

Problem 1. Consider the IBVP

$$\begin{cases} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0, & (x,t) \in]a,b[\times]0,T] \\ u(a,t) = u(b,t) = 0, & t \in [0,T] \\ u(x,0) = \sin\left(\frac{\pi(x-a)}{b-a}\right), & x \in [a,b] \end{cases}$$

(1) Use the **ImpHeat.m** developed in class to plot the solution of the IBVP when $a = 0$, $b = 2$ and $T = 2$.

(2) Modify the **ImpHeat.m** developed in class to include the source term $f(x,t)$ in the IBVP

$$\begin{cases} \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = f(x,t), & (x,t) \in]a,b[\times]0,T] \\ u(a,t) = u(b,t) = 0, & t \in [0,T] \\ u(x,0) = \sin\left(\frac{\pi(x-a)}{b-a}\right), & x \in [a,b] \end{cases}$$

(3) Plot the graph of the solution of the previous IBVP using the same values for the parameters in (1) and $f(x,t) = \frac{\pi^2}{4}\sin(\frac{\pi x}{2})$.

(4) Compute and plot the error in absolute value at $T = 2$, knowing that the analytic solution to the IBVP is $u(x,t) = (e^{-\frac{\pi^2 t}{4}} + 1)\sin(\frac{\pi x}{2})$.

Problem 2. Consider the heat equation in two spatial dimensions with the following Dirichlet boundary conditions:

$$\begin{cases} \frac{\partial u}{\partial t} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), & (x,y,t) \in]-1,1[\times]-1,1[\times]0,T] \\ u(-1,y,t) = u(1,y,t) = 0, & (y,t) \in [-1,1] \times [0,T] \\ u(x,-1,t) = u(x,1,t) = 0, & (x,t) \in [-1,1] \times [0,T] \\ u(x,y,0) = (1-x^2)(1-y^2), & (x,y) \in [-1,1] \times [-1,1] \end{cases}$$

(1) Implement a MATLAB function that solves numerically the above problem, using the Peaceman-Rachford scheme. Name the function PRHeat2D.m

(2) Plot the graph of the solution of the problem at $T = 1$ when $c = 1/3$.

Problem 3. It can be shown that, by a suitable variables transformation, the Black-Scholes equation can be cast into a IVP for the heat equation. In the case of, for instance, an European call option $C(S,t)$, we obtain the following IVP (for more details on these variables transformation see lecture notes):

$$\begin{cases} \frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2} = 0, & (x,\tau) \in \mathbb{R}\times]0,\sigma^2 T/2] \\ u(x,0) = \max\left(e^{\frac{1}{2}(k_1+1)x} - e^{\frac{1}{2}(k_1-1)x}, 0\right), & x \in \mathbb{R} \end{cases}$$

where

$$x = \log \frac{S}{K}, \quad \tau = \frac{\sigma^2}{2}(T-t), \quad k_1 = \frac{r}{\sigma^2/2}$$

and the value of the call option is

$$C(S,t) = K \exp\left(-\frac{1}{2}(k_1-1)x - \frac{1}{4}(k_1+1)^2\tau\right) u(x,\tau).$$

(1) Implement a MATLAB function that solves numerically the above problem, using the Crank-Nicolson scheme. Name the function CNHeatBSCall.m
(2) Plot the price curve for a call option for $S \in [0, 3K]$ with the following set of parameters: $S_0 = 40, K = 30, r = 0.1, \sigma = 0.2, t_0 = 0, T = 1$.
(3) Use the Black-Scholes formula as benchmark (BSFormula.m developed in class) and:
   (a) Draw the error function for call prices computed in (1) for $S \in [0, 3K]$;
   (b) determine the L_∞ norm of the error function.

Problem 4. An asset-or-nothing call option is an exotic option whose payoff exhibits the following profile: the holder of the option receives the asset S if the asset price at maturity is higher then the strike price K, otherwise receives nothing. In other words, if $A(S,t)$ is the price of the asset-or-nothing, then A solves the following equation:

$$\begin{cases} \frac{\partial A}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 A}{\partial S^2} + rS\frac{\partial A}{\partial S} - rA = 0 \\ A(S,t) = S, \quad \text{as } S \to +\infty \\ A(0,t) = 0 \\ A(S,T) = \begin{cases} S, & S > K \\ 0, & S \leq K \end{cases} \end{cases}$$

(1) Implement a MATLAB function to solve the previous problem using a θ-scheme. Name the function ThetaSchemeBS_asset_call.m (You may modify the m-file ThetaSchemeBS.m developed in class).
(2) Plot the payoff and the asset-or-nothing call prices calculated using $\theta = 2/3$ and the following parameters: $S \in [0, 30]$, $K = 15$, $r = 0.04$, $\sigma = 0.15$, $t_0 = 0$, $T = 1$.

Problem 5. Most of the Finite Difference Schemes to solve PDE's in Finance can be cast into a problem of solving a set of linear algebraic equations $Mx = b$, where the matrix M is a so called tridiagonal matrix. The LU-decomposition of M can be efficiently implemented using the algorithm described on pages 11 to 16 of the lecture notes "LU decomposition" available on the section Slides of course web page.

(1) Write a MATLAB function that given a tridiagonal matrix M computes L and U with the use of the above mentioned algorithm. Name the function LUtridiag.m
(2) Write a MATLAB function that, given a tridiagonal matrix M and a vector b, solves the system $Mx = b$ using the LU factorization computed in LUtridiag.m. Name the function SolveTridiag.m

(3) Use the SolveTridiag.m function to solve the system

$$\begin{pmatrix} -1 & -3 & & & & \\ 2 & 2 & 3 & & & \\ & 2 & 1 & 4 & & \\ & & 2 & 5 & -2 & \\ & & & 6 & 7 & 2 \\ & & & & 4 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 9 \\ -13 \\ -1 \\ -1 \\ 26 \\ 14 \end{pmatrix}$$