Inverse of a Matrix

Please read our [Introduction to Matrices](#) first.

What is the Inverse of a Matrix?

The Inverse of a Matrix is the **same idea** as the [reciprocal](#) of a number:

$$8$$
$$\tfrac{1}{8}$$

Reciprocal   Reciprocal

Reciprocal of a Number

But we don't write $\tfrac{1}{A}$ (because we don't divide by a Matrix!), instead we write A^{-1} for the inverse:

$$A$$
$$A^{-1}$$

Inverse   Inverse

(In fact $\tfrac{1}{8}$ can also be written as 8^{-1})

And there are other similarities:

- When you **multiply a number** by its **reciprocal** you get **1**

$$8 \times (\tfrac{1}{8}) = \mathbf{1}$$

- When you **multiply a Matrix** by its **Inverse** you get the **Identity Matrix** (which is like "1" for Matrices):

$$A \times A^{-1} = \mathbf{I}$$

It also works when the inverse comes first: $(\tfrac{1}{8}) \times 8 = \mathbf{1}$ and $A^{-1} \times A = \mathbf{I}$

Identity Matrix

Note: the "Identity Matrix" is the matrix equivalent of the number "1":

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

A 3x3 Identity Matrix

- It is "square" (has same number of rows as columns),
- It has **1**s on the diagonal and **0**s everywhere else.
- It's symbol is the capital letter **I**.

The Identity Matrix can be 2×2 in size, or 3×3, 4×4, etc ...

Definition

So we have a definition of a Matrix Inverse ...

The Inverse of A is A^{-1} only when:

$$A \times A^{-1} = A^{-1} \times A = I$$

Sometimes there is no Inverse at all.

2x2 Matrix

OK, how do we calculate the Inverse?

Well, for a 2x2 Matrix the Inverse is:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

determinant

In other words: **swap** the positions of a and d, put **negatives** in front of b and c, and **divide** everything by the determinant (ad-bc).

Let us try an example:

$$\begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix}^{-1} = \frac{1}{4 \times 6 - 7 \times 2} \begin{bmatrix} 6 & -7 \\ -2 & 4 \end{bmatrix}$$

$$= \frac{1}{10} \begin{bmatrix} 6 & -7 \\ -2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0.6 & -0.7 \\ -0.2 & 0.4 \end{bmatrix}$$

How do we know this is the right answer?

Remember it must be true that: $A \times A^{-1} = \mathbf{I}$

So, let us check to see what happens when we multiply the matrix by its inverse:

$$\begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix}\begin{bmatrix} 0.6 & -0.7 \\ -0.2 & 0.4 \end{bmatrix} = \begin{bmatrix} 4 \times 0.6 + 7 \times -0.2 & 4 \times -0.7 + 7 \times 0.4 \\ 2 \times 0.6 + 6 \times -0.2 & 2 \times -0.7 + 6 \times 0.4 \end{bmatrix}$$

$$= \begin{bmatrix} 2.4 - 1.4 & -2.8 + 2.8 \\ 1.2 - 1.2 & -1.4 + 2.4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

And, hey!, we end up with the Identity Matrix! So it must be right.

It should **also** be true that: $A^{-1} \times A = \mathbf{I}$

Why don't you have a go at multiplying these? See if you also get the Identity Matrix:

$$\begin{bmatrix} 0.6 & -0.7 \\ -0.2 & 0.4 \end{bmatrix}\begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix} = \begin{bmatrix} & \\ & \end{bmatrix}$$

Why Would We Want an Inverse?

Because with Matrices we **don't divide**! Seriously, there is no concept of dividing by a Matrix.

But we can **multiply by an Inverse**, which achieves the same thing.

Imagine you couldn't divide by numbers, and someone asked "How do I share 10 apples with 2 people?"

But you could take the **reciprocal** of 2 (which is 0.5), so you could answer:

10 × 0.5 = 5

They get 5 apples each

The same thing can be done with Matrices:

Say that you know Matrix A and B, and want to find Matrix X:

$$XA = B$$

It would be nice to divide both sides by A (to get X=B/A), but remember **we can't divide**.

But what if we multiply both sides by A^{-1} ?

$$XAA^{-1} = BA^{-1}$$

And we know that $AA^{-1} = I$, so:

$$XI = BA^{-1}$$

We can remove I (for the same reason we could remove "1" from 1x = ab for numbers):

$$X = BA^{-1}$$

And we have our answer (assuming we can calculate A^{-1})

In that example we were very careful to get the multiplications correct, because with Matrices the order of multiplication matters. AB is almost never equal to BA.

A Real Life Example

A group took a trip on a bus, at $3 per child and $3.20 per adult for a total of $118.40.

They took the train back at $3.50 per child and $3.60 per adult for a total of $135.20.

How many children, and how many adults?

First, let us set up the matrices (be careful to get the rows and columns correct!):

$$\begin{array}{cc} \text{Child} & \text{Adult} \end{array} \quad \begin{array}{cc} \text{Bus} & \text{Train} \end{array} \qquad\qquad \begin{array}{cc} \text{Bus} & \text{Train} \end{array}$$

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 3 & 3.5 \\ 3.2 & 3.6 \end{bmatrix} = \begin{bmatrix} 118.4 & 135.2 \end{bmatrix}$$

This is just like the example above:

$$XA = B$$

So to solve it we need the inverse of "A":

$$\begin{bmatrix} 3 & 3.5 \\ 3.2 & 3.6 \end{bmatrix}^{-1} = \frac{1}{3\times3.6 - 3.5\times3.2} \begin{bmatrix} 3.6 & -3.5 \\ -3.2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -9 & 8.75 \\ 8 & -7.5 \end{bmatrix}$$

Now we have the inverse we can solve using:

$$X = BA^{-1}$$

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} = \begin{bmatrix} 118.4 & 135.2 \end{bmatrix} \begin{bmatrix} -9 & 8.75 \\ 8 & -7.5 \end{bmatrix}$$

$$= \begin{bmatrix} 118.4\times-9 + 135.2\times8 & 118.4\times8.75 + 135.2\times-7.5 \end{bmatrix}$$

$$= \begin{bmatrix} 16 & 22 \end{bmatrix}$$

There were 16 children and 22 adults!

The answer almost appears like magic. But it is based on good mathematics.

Calculations like that (but using much larger matrices) help Engineers design buildings, are used in video games and computer animations to make things look 3-dimensional, and many other places.

It is also a way to solve Systems of Linear Equations.

The calculations are done by computer, but the people must understand the formulas.

Order is Important

Say that you are trying to find "X" in this case:

$$AX = B$$

This is different to the example above! X is now **after** A.

With Matrices the order of multiplication usually changes the answer. Do not assume that AB = BA, it is almost never true.

So how do we solve this one? Using the same method, but put A^{-1} in front:

$$A^{-1}AX = A^{-1}B$$

And we know that $A^{-1}A = I$, so:

$$IX = A^{-1}B$$

We can remove I:

$$X = A^{-1}B$$

And we have our answer (assuming we can calculate A^{-1})

Why don't we try our example from above, but with the data set up this way around.
(Yes, you can do this, just be careful how you set it up.)

This is what it looks like as $AX = B$:

$$\begin{bmatrix} 3 & 3.2 \\ 3.5 & 3.6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 118.4 \\ 135.2 \end{bmatrix}$$

It looks so neat! I think I prefer it like this.

Also note how the rows and columns are swapped over ("Transposed")
compared to the previous example.

To solve it we need the inverse of "A":

$$\begin{bmatrix} 3 & 3.2 \\ 3.5 & 3.6 \end{bmatrix}^{-1} = \frac{1}{3 \times 3.6 - 3.2 \times 3.5} \begin{bmatrix} 3.6 & -3.2 \\ -3.5 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} -9 & 8 \\ 8.75 & -7.5 \end{bmatrix}$$

It is like the Inverse we got before, but
Transposed (rows and columns swapped over).

Now we can solve using:

$$X = A^{-1}B$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -9 & 8 \\ 8.75 & -7.5 \end{bmatrix} \begin{bmatrix} 118.4 \\ 135.2 \end{bmatrix} = \begin{bmatrix} -9 \times 118.4 + 8 \times 135.2 \\ 8.75 \times 118.4 - 7.5 \times 135.2 \end{bmatrix} = \begin{bmatrix} 16 \\ 22 \end{bmatrix}$$

Same answer: 16 children and 22 adults.

So, Matrices are powerful things, but they do need to be set up correctly!

The Inverse May Not Exist

First of all, to have an Inverse the Matrix must be "Square" (same number of rows and columns).

But also the **determinant cannot be zero** (or you would end up dividing by zero). How about this:

$$\begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}^{-1} = \frac{1}{3 \times 8 - 4 \times 6} \begin{bmatrix} 8 & -4 \\ -6 & 3 \end{bmatrix}$$

$$= \frac{1}{24 - 24} \begin{bmatrix} 8 & -4 \\ -6 & 3 \end{bmatrix}$$

24-24? That equals 0, and **1/0 is undefined**.
We cannot go any further! This Matrix has no Inverse.

Such a Matrix is called "Singular", which only happens when the determinant is zero.

And it makes sense ... look at the numbers: the second row is just double the first row, and does **not add any new information**.

Imagine in our example above that the prices on the train were exactly, say, 50% higher ... we wouldn't be any closer to figuring out how many adults and children ... we need something different.

And the determinant neatly works this out.

Bigger Matrices

The inverse of a 2x2 is **easy** ... compared to larger matrices (such as a 3x3, 4x4, etc).

For those larger matrices there are three main methods to work out the inverse:

- Inverse of a Matrix using Elementary Row Operations (Gauss-Jordan)
- Inverse of a Matrix using Minors, Cofactors and Adjugate
- Use a computer (such as the Matrix Calculator)

Conclusion

- The Inverse of A is A^{-1} only when $A \times A^{-1} = A^{-1} \times A = \mathbf{I}$

- To find the Inverse of a 2x2 Matrix: **swap** the positions of a and d, put **negatives** in front of b and c, and **divide** everything by the determinant (ad-bc).

- Sometimes there is no Inverse at all

Your turn: Question 1 Question 2 Question 3 Question 4 Question 5
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