THE UNIVERSITY OF ARIZONA

Introduction to Galerkin's Method

Term Project

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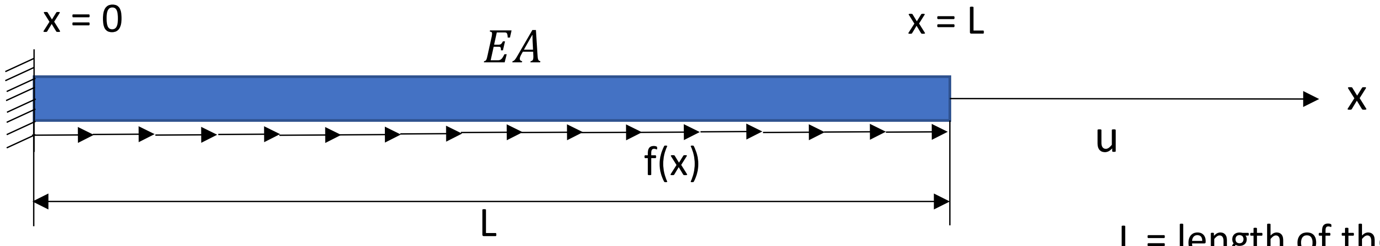
Introduction to Finite Element Method

1. Strong Formulation
2. Weak Formulation

One Dimensional Stress Deformation



L = length of the bar
u(x) = bar deformation
f(x) = traction along the bar

Governing equation,

$$\mathrm{EA}\frac{\mathrm{d}^2\mathrm{u}}{\mathrm{dx}^2} = -f(x)$$

Strong form solution,

High complexity

$$\iint \mathrm{EA}\frac{\mathrm{d}^2\mathrm{u}}{\mathrm{dx}^2}\, dx = \iint -f(x)$$

$$u(x) = \frac{fL^2}{\mathrm{EA}}\left[\frac{x}{L} - \frac{1}{2}\left(\frac{x}{L}\right)^2\right] \quad \text{and} \quad \frac{du}{\mathrm{d}x} = \frac{fL}{\mathrm{EA}}\left[1 - \frac{x}{L}\right]$$

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Introduction to Finite Element Method

2. Weak Formulation

Lower complexity

Variational Formulation

Approximation is based on the minimization of a functional.

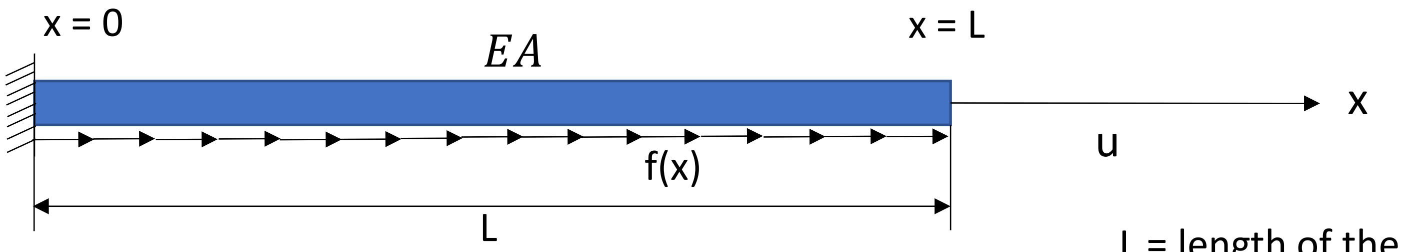
Weighted Residual Method

Weighted Residual Methods start with an estimate of the solution and demand that its weighted average error is minimized:

- Collocation Method
- Subdomain Method
- Least Squares Method
- Pseudo-spectral Methods
- **Galerkin's Method**

Credited to Boss Galerkin

Weighted Residual Method (WRM)



L = length of the bar
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Governing equation,

$$EA\frac{d^2u}{dx^2} = -f(x)$$

Residual Function,

$$R(x) = EA\frac{d^2u}{dx^2} + f(x)$$

Weak formulation

Allow error to occur

Minimization of the error function

WRM

$$\int_0^L R(x)W(x)dx = 0$$

$W(x)$ is the weighted function

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Assumption 1: The weighted average error of the approximation should be zero

$$\int_{0}^{L} \mathrm{R(x)}w(x) = 0$$

Assumption 2: The approximate solution must satisfy the boundary conditions

The approximate solution

$\mathrm{u}(x) = A_i \emptyset_i$

Assumption 3: The weight function is approximated using the same scheme as for the solution

The weighting function,

$$w(x) = \emptyset_i$$

For Example:

- u(x) = Asinx, Ø = sinx
- u(x) = ax+ bx^2+ cx^3, $\emptyset_1$ = x, $\emptyset_2$ = x^2, $\emptyset_3$ = x^3

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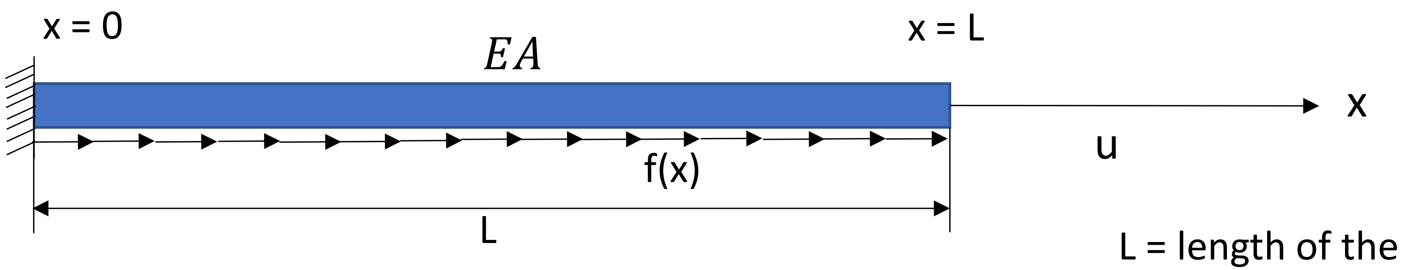
$$\int_0^L \left[\mathrm{EA}\frac{\mathrm{d}^2\mathrm{u}}{\mathrm{dx}^2} + \mathrm{f(x)} \right] \emptyset_i dx = 0$$



L = length of the bar
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- **Case 1**

  $u(x) = ax,\ \emptyset = x$

- **Case 2**

  $u(x) = ax + bx^2,\ \ \emptyset_1 = x,\ \emptyset_2 = x^2$

- **Solution**

$$u(x) = \frac{fL^2}{2\mathrm{EA}}\left[\frac{x}{L}\right]$$

$$\frac{\mathrm{du}}{\mathrm{d}x} = \frac{fL}{2\mathrm{EA}}$$

- **Solution**

$$u(x) = \frac{fL^2}{\mathrm{EA}}\left[\frac{x}{L} - \frac{1}{2}\left(\frac{x}{L}\right)^2\right]$$

$$\frac{\mathrm{du}}{\mathrm{d}x} = \frac{fL}{\mathrm{EA}}\left[1 - \frac{x}{L}\right]$$

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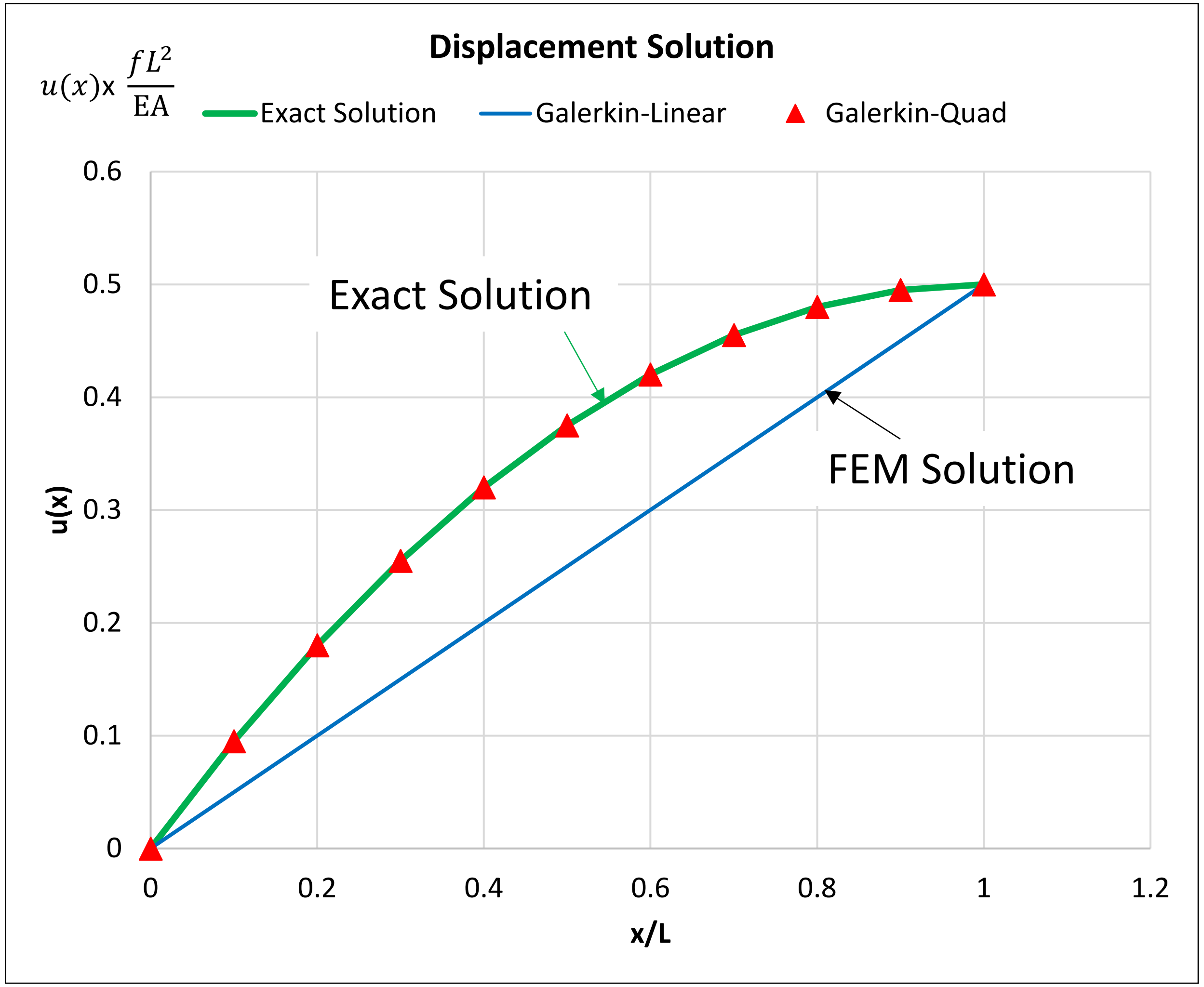

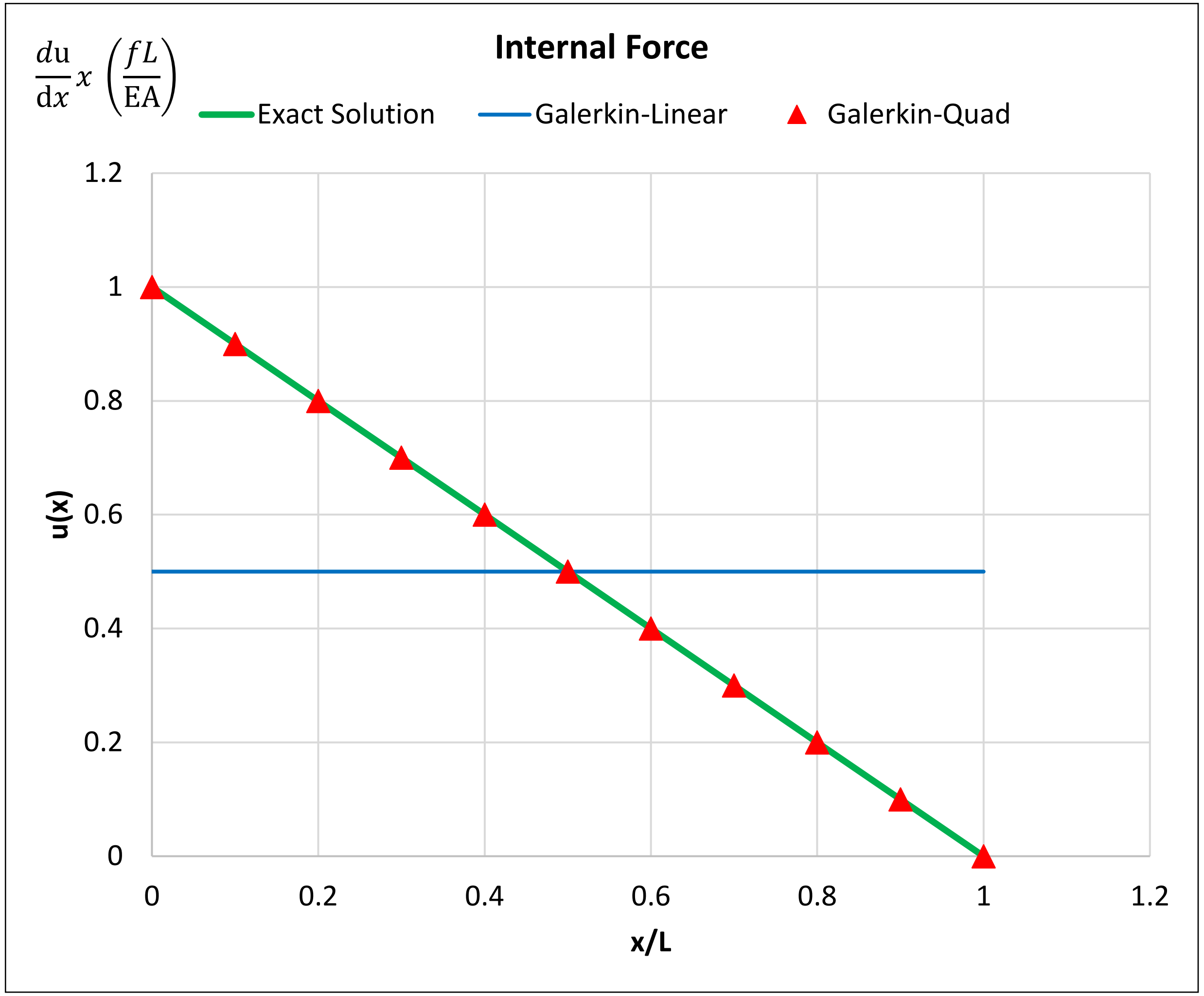

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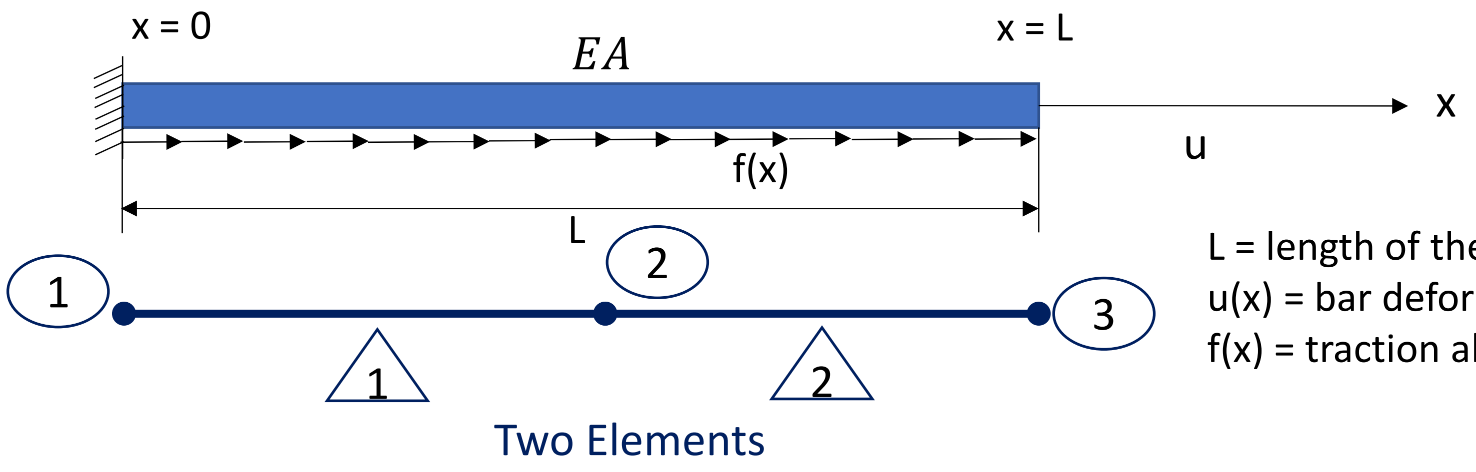
$$\int_0^L \left[EA\frac{d^2u}{dx^2} + f(x) \right] \emptyset_i dx = 0$$



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- **Case 3**

  u(x) = ax, ∅ = x

- **Solution**

  For 0 < x < L/2,

  $$u(x) = \frac{3fL^2}{4EA}\left[\frac{x}{L}\right], \qquad \frac{du}{dx} = \frac{3fL}{4EA}$$

  For L/2 < x < L,

  $$u(x) = \frac{fL^2}{4EA}\left[1 + \frac{x}{L}\right], \qquad \frac{du}{dx} = \frac{fL}{4EA}$$

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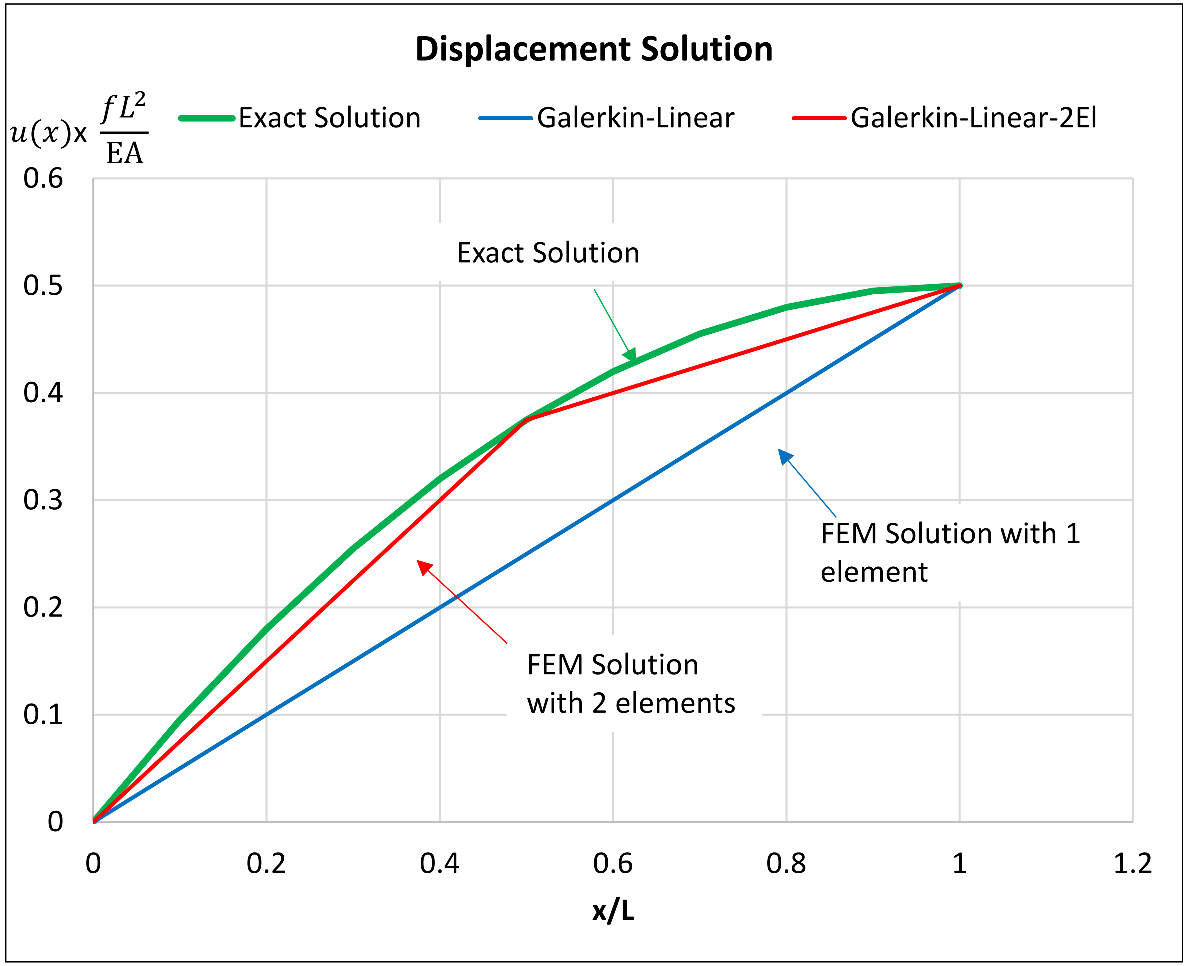

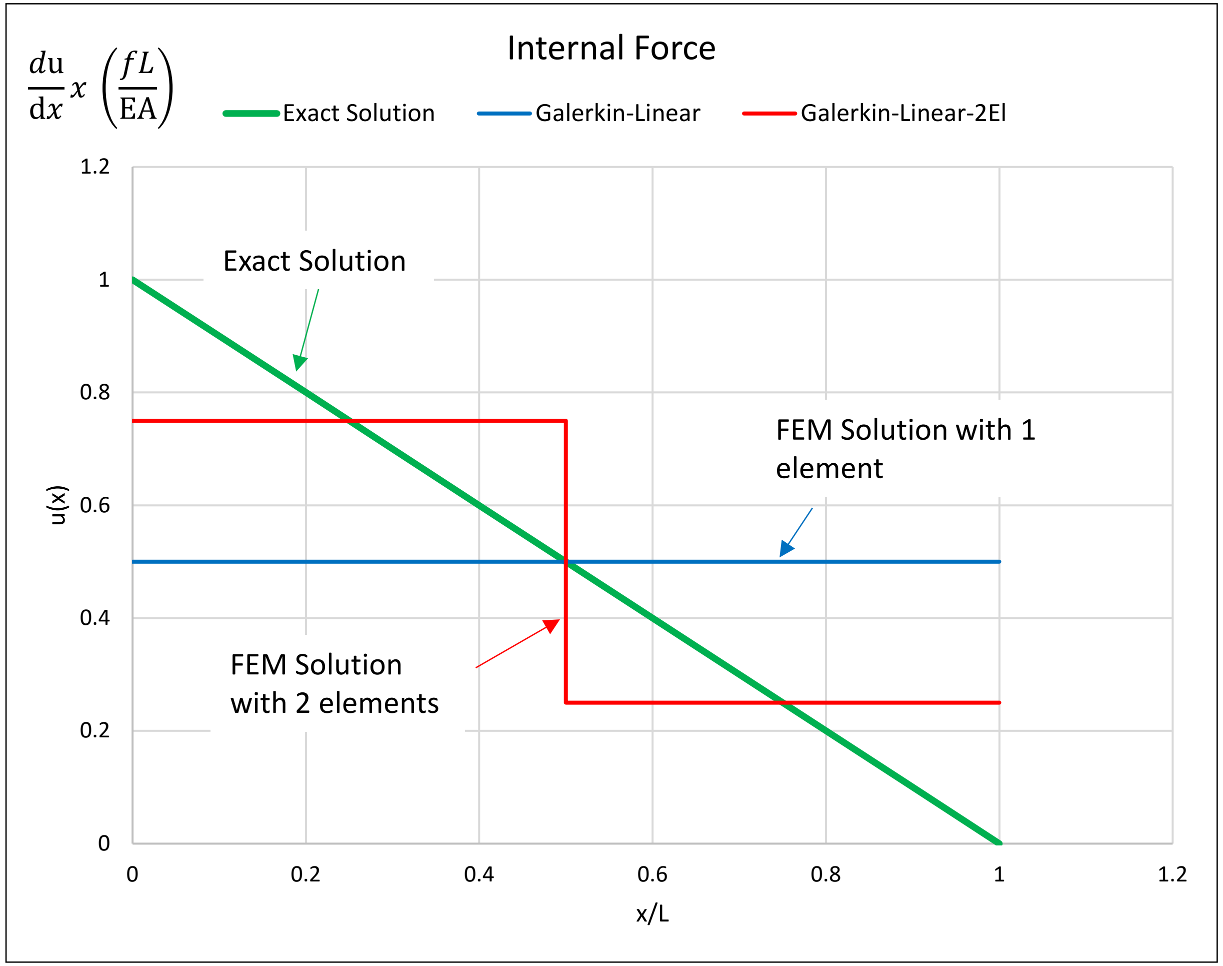

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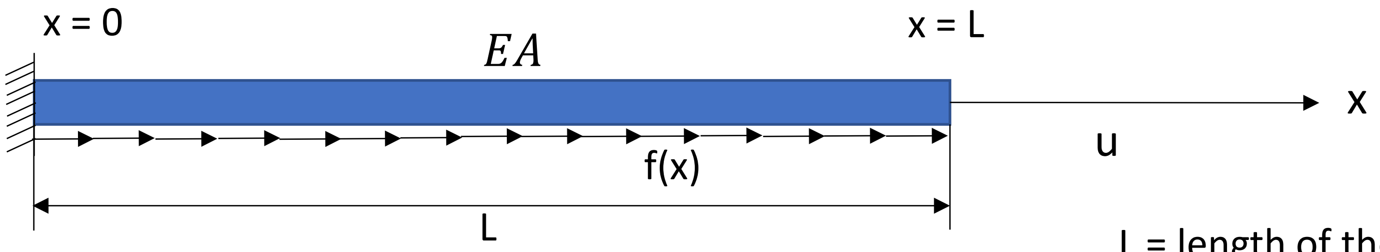

L = length of the bar
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$$\int_{0}^{L} \mathrm{R(x)}W(x)dx = 0$$

$$\int_{0}^{L} \left[\mathrm{EA}\frac{\mathrm{d}^2\mathrm{u}}{\mathrm{dx}^2} + \mathrm{f(x)}\right] W(x)dx = 0$$

$$\underbrace{\mathrm{EA}\frac{d\mathrm{u}}{\mathrm{d}x}\bigg|_{x=L} - \mathrm{EA}\frac{d\mathrm{u}}{\mathrm{d}x}\bigg|_{x=0}}_{\text{Boundary Condition}} \underbrace{- \int_{0}^{L} \mathrm{EA}\frac{d\mathrm{u}}{\mathrm{d}x}\frac{d\mathrm{W}}{\mathrm{d}x}dx}_{\text{Stiffness Matrix}} + \underbrace{\int_{0}^{L} \mathrm{f(x)}W(x)dx}_{\text{Load Vector}} = 0$$

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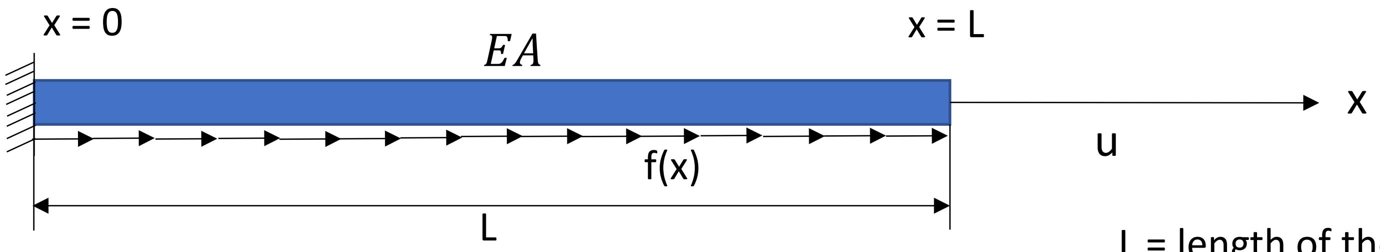

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Stiffness Matrix

$$\int_0^L \mathrm{EA}\frac{d\mathrm{u}}{\mathrm{d}x}\frac{d\mathrm{W}}{\mathrm{d}x}\,dx$$

Load Vector

$$\int_0^L \mathrm{f(x)}W(x)dx$$

Using Interpolation Function

$$\mathrm{u} = \mathrm{N_1 u_1} + \mathrm{N_2 u_2}$$

$$\mathrm{w} = \mathrm{N_1} + \mathrm{N_2}$$

$\mathrm{N_1}$ and $\mathrm{N_2}$ are interpolation functions

Stiffness Matrix

$$[k] = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Same as Variational Procedure

Load Vector,

$$\{Q\} = \frac{fL}{2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

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Comparison to Variational Procedure

- **For most of the problem Galerkin’s method yields results identical to those from variational procedure.**
- **Not all governing equation can be solved by variational procedure in which case WRM (Galerkin’s method) can be handy.**
- **In comparing to variational procedure, variational procedure gives better solution than the Galerkin’s method.**

Introduction to Galerkin's Method

Summary

- Galerkin's method is a most popular Method of Weighted Residuals in finite element method.
- The approximate solution should satisfy the boundary condition
- The weighting functions are chosen to be the same form as each part of the approximate solution.
- The residual function weighted with weighting functions is forced to be zero for the given domain.