University of Science and Technology of Hanoi

Solar System and Celestial Mechanics

Exercise 1

ORBITAL MECHANICS

Authors:
Binh Nguyen
email: binhnty.bi12-060@st.usth.edu.vn

Instructor:
Dr. Phan Thanh Hien

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1 Exercise 1

1.1 Elapsed time between two positions on an orbit:

1.1.1 Kepler's method:

We can use the Kepler's method to tackle this problem. First, we can consider we solving this problem in a Polar coordinate where we define the position by (r, θ).

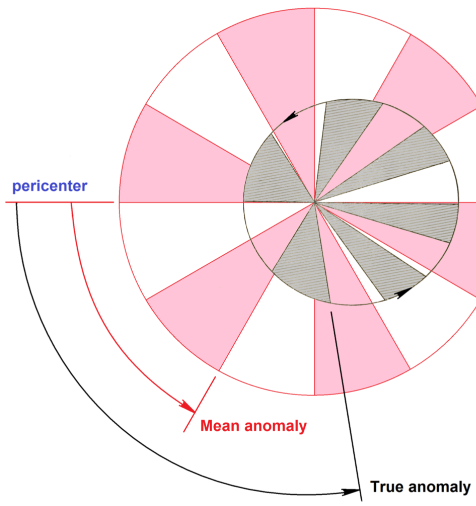

Figure 1: Area swept out per unit time (gray) by an object in an elliptical orbit, and (pink) by an imaginary object in a circular orbit with the same orbital period. Both sweep out equal areas in equal times, but the angular rate of sweep varies for the elliptical orbit and is constant for the circular orbit.

First, we need to define the **mean anomaly**, which an angular distance from the pericenter at arbitrary time t, with dimensions of radians or degrees:

$$M = n(t - \tau) \tag{1}$$

where $n = \frac{2\pi}{T} = \sqrt{\frac{GM}{a^3}}$ is the mean angular motion and τ is the the time at which the body is at the pericenter.

Second, we have the Kepler's equation which relates the mean anomaly to the eccentric anomaly:

$$M = E - esin(E) \tag{2}$$

Third, you need is a way to relate the eccentric anomaly with the true anomaly, which is expressed in this formula:

$$tan(\frac{E}{2}) = \sqrt{\frac{1-e}{1+e}} tan(\frac{\theta}{2}) \tag{3}$$

We can compute $e = \frac{r_a - r_p}{r_a + r_p}$, then compute the eccentric anomaly using the equation 3 and calculate the M value using the equation 2. Finally the elapsed time can be calculate using the equation 1:

$$\Delta_t = \frac{M}{n} \tag{4}$$

1.1.2 Calculus method

The solution in section 1.1.1 is truly a genius way of solving such a problem of Kepler, especially given the context that calculus was not fully developed at that time. With modern mathematics, is there any other way to solve this problem without defining the mean anomaly? Yes, let use Kepler's second law, which states that a line segment joining a planet (point) and the Sun (focus) sweeps out equal areas during equal intervals of time:

$$\frac{dA}{dt} = \frac{A_{total}}{T} = \frac{\pi ab}{T} \tag{5}$$

where $T = 2\pi\sqrt{\frac{a^3}{GM}}$ can be derive from the Kepler's third law.

From the equation 5, we can derive the equation for elapsed time as:

$$t = \frac{T}{\pi ab} A \tag{6}$$

Now, we only need to solve for the area of A using calculus. Now let's define the equation of the ellipse in a Cartesian coordinate:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{7}$$

We need to solve for the area sweep by F_1A, begin from the point P (periapsis). Note that any area can be determined by subtraction of two area sweep from the periapsis, and because of the symmetric, we will only solve for the area above the horizontal axis, the area below can be solved using the same technique.

From the equation 8, we can derive y as the function of x for the upper part of the ellipse:

$$y = \frac{b}{a}\sqrt{a^2 - x^2} \tag{8}$$

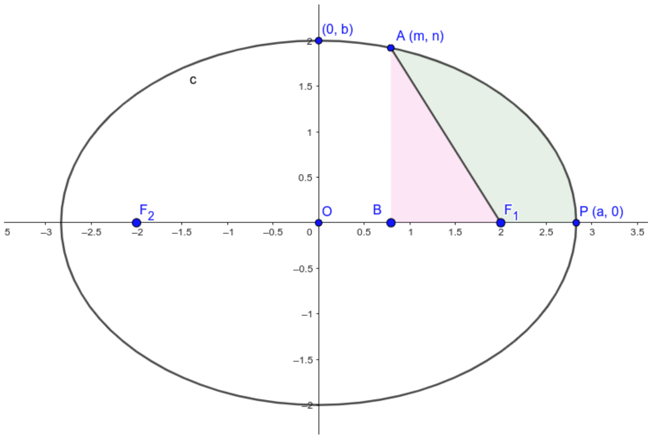

Figure 2: Ellipse

The area of the section F_1AP (green area) is what we need to find, we can calculate it by subtracting the area of BAP - the area of triangle BAF_1 (pink area).

The area of the segment BAP is simply the integral of 8 from x = m to x = a:

$$A_1 = \frac{b}{a}\int_m^a \sqrt{a^2 - x^2}dx \tag{9}$$

Introduce a new variable θ where $x = asin(\theta)$ where $x \in [-a, a]$ so $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. We obtain a new integral as the function of θ:

$$A_1 = \frac{b}{a} \times a^2 \int_{x=m}^{x=a} cos^2(\theta)d\theta \tag{10}$$

We know that $cos^2(\theta) = \dfrac{1 - cos(2\theta)}{2}$. This integral now become really easy to solve, by some mathematical manipulation, we obtain the final expression for A_1:

$$A_1 = \frac{ab}{2} \times (\frac{\pi}{2} - arcsin(\frac{m}{a}) + \frac{m}{a}\sqrt{1 - \frac{m^2}{a^2}}) \tag{11}$$

The area of the triangle BAP is really simple to solve:

$$A_0 = \frac{1}{2}n(c - m) \tag{12}$$

where $c = \sqrt{a^2 - b^2}$.

Finally the sweep area in green can be calculated as:

$$A = A_1 - A_0 \tag{13}$$

Using the equation 6, we obtain the elapsed time between two position on an elliptical orbit.

1.2 Position of an orbit over a period of time:

For a small eccentricities a good approximation of true anomaly can be obtained by the following formula:

$$\nu \approx M + 2esin(M) + 1.25e^2 sin(2M) \tag{14}$$

We can calculate the distance r from based on the value of ν:

$$r = \frac{a(1-e^2)}{1+ecos(\nu)} \tag{15}$$

where the semi-major axis $a = \dfrac{r_a + r_p}{2}$ and eccentricity $e = \dfrac{r_a - r_p}{r_a + r_p}$.

Based on the conservation law of energy and angular momentum, we can obtain the relation between r and θ as following:

$$r = \frac{L^2}{\mu m(1+ecos(\theta))} \tag{16}$$

where $\mu = GM$.

Due to the conservation of angular momentum, the relation of L at the periapsis and apoapsis can be expressed as:

$$r_a v_a = r_p v_p \tag{17}$$

Apply the conservation of energy, we have:

$$T_1 + V_1 = T_2 + V_2 \tag{18}$$

$$\Rightarrow \frac{mv_1^2}{2} - \frac{GMm}{r_1} = \frac{mv_2^2}{2} - \frac{GMm}{r_2} \tag{19}$$

$$\Rightarrow v_2^2 - v_1^2 = 2GM(\frac{1}{r_2} - \frac{1}{r_1}) \tag{20}$$

From the equation 17 and 20, we obtain:

$$v_a = \sqrt{\frac{2GMr_p}{r_p(r_a+r_p)}} \tag{21}$$

Therefore, we can calculate:

$$L = mr_a v_a = mr_a \sqrt{\frac{2GMr_p}{r_p(r_a+r_p)}} \tag{22}$$

With the value of M we can calculate the value of r following the equation 14. Then from the equation 16 and 22, we can find the value of θ.

2 Exercise 2

Retrograde motion is a phenomenon in which a planet appears to move "backwards" in the sky as observed from the Earth. This illusion occurs because the Earth is also orbiting around the Sun, and as it moves, it periodically passes by a planet, causing it to appear to move in the opposite direction against the background of stars. This can be compared to when driving a car and passing another car, which may appear to move backwards in your reference frame.

However, planets do not actually move backward in their orbits. Retrograde motion is an optical illusion caused by differences in the speeds and distances of the planets as they orbit around the Sun. Ancient astronomers were the first to observe this phenomenon, and it presented a significant challenge for early models of the Solar System. It was not until the 16th century that the heliocentric model of the Solar System was developed by Copernicus and elaborated on by Kepler, providing a more comprehensive explanation for retrograde motion.

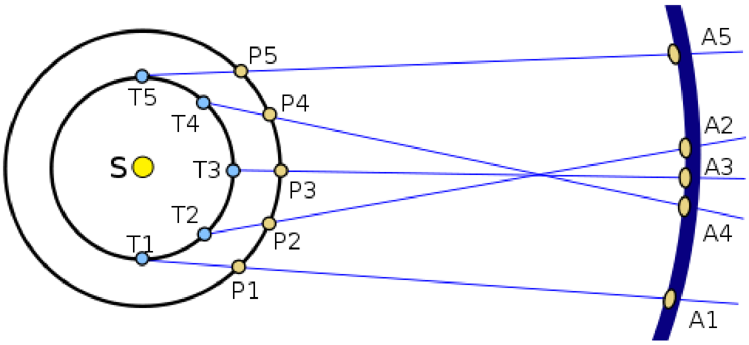

Figure 3: Retrograde motion

3 Exercise 3

3.1 Hohmann Transfer

In order to solve this problem, we need to understand the Hohmann transfer. A Hohmann transfer is an two-impulse elliptical transfer between two co-planar circular orbits. This transfer can be applied to interplanetary transfers from Earth to Venus, in which the transfer consist of an elliptical orbit with an aphelion at the outer orbit to a perihelion at the inner orbit.

The Hohmann transfer for satellite orbits assume the satellite is in a circular orbit about a central body and desires to transfer to another circular and co-planar orbit about the central body. It also assumes that no other gravitational influence is nearby.

This transfer is usually optimal, as it requires the minimum Δ_v to perform a transfer between two circular orbits, which in turn conserve a lot of energy needed for impulse.

3.1.1 Circular orbit

To find out the velocity needed for the satellite to orbit around a central body at a certain distance, we can use the Newton's Second Law as follow:

$$\sum F = ma \tag{23}$$

with the assumption that the only force exerted on the satellite is the gravitational pull from the central body

$$F_g = \frac{GMm}{r^2} \tag{24}$$

where G is the gravitation constant, M is the mass of central body, m is the mass of the satellite and r is the distance between two body.

$$F_c = F_g \tag{25}$$

$$\Rightarrow \frac{mv^2}{r} = \frac{GMm}{r^2} \tag{26}$$

$$\Rightarrow v = \sqrt{\frac{GM}{r}} \tag{27}$$

3.1.2 Semi-major axis

Because the transfer was taken from aphelion at the departure planet to perihelion at the target planet which is half of the elliptical orbit, there for the semi-major axis can be calculated as:

$$a = \frac{r_1 + r_2}{2} \tag{28}$$

where r_a and r_p are the distance from the central body to each circular orbits.

3.1.3 Eccentriciy

Eccentricity of an elliptical orbit is the measure of the amount by which it deviates from a circle. The eccentricity can be found knowing the distance from the Sun to each planets is defined as:

$$e = \frac{c}{a} \tag{29}$$

where c is the distance from the Sun to the center of the elliptical orbit, a is the semi-major axis. With some geometry, we can derive the formula for the eccentricity based on the distance from the Sun to perihelion and aphelion as follow:

$$e = \frac{r_1 - r_2}{r_1 + r_2} \tag{30}$$

3.1.4 Velocity needed to change the orbit

Based on the Conservation of Energy, we will try to calculate the velocity needed to kick our satellite from the orbital orbit to the elliptical orbit or vice versa.

The mechanical energy when the satellite is in the initial circular orbit:

$$E = V + U \tag{31}$$

$$E = \frac{1}{2}mv^2 - \frac{GMm}{r} \tag{32}$$

Substitute the velocity from the equation 27, and for elliptical orbit, we change the variable from r to a, we get the equation for energy.

$$E = -\frac{GMm}{2a} \tag{33}$$

We can find the new velocity to transfer the satellite to the new orbit with r is the radius of the circular orbit and a is the semi-major axis of elliptical orbit.

$$-\frac{GMm}{2a} = \frac{1}{2}mv_t^2 - \frac{GMm}{r} \tag{34}$$

$$v_t = \sqrt{GM(\frac{2}{r} - \frac{1}{a})} \tag{35}$$

From the heliocentric velocity and orbital velocity we calculated from the equation 35 and 27, we can calculate the excess velocity that we need to provide to satellite:

$$\Delta_v = |v_t - v| \tag{36}$$

3.1.5 Transfer time

The Hohmann transfer consist of half of an period of the elliptical orbit, we can use Kepler's Third Law to derive the time needed for the transfer:

$$t_{transfer} = \pi\sqrt{\frac{a^3}{GM}} \tag{37}$$

3.2 Results

We can applied the method we introduced in the section 3.1 to calculate the transfer time from Earth to Venus. The results is shown in table 1.

Note: You may find the code for this calculation in the attached Jupyter Notebook.

Definition	Formula	Result
Heliocentric velocity of departure planet	$v_{dep} = \sqrt{\frac{\mu_{sun}}{r_{dep}}}$	29784.48 m/s
Heliocentric velocity of target planet	$v_{tar} = \sqrt{\frac{\mu_{sun}}{r_{tar}}}$	35054.51 m/s
Circular velocity around departure planet	$v_{c0} = \sqrt{\frac{\mu_{earth}}{r_{earth} + h_{earth}}}$	7784.28 m/s
Circular velocity around target planet	$v_{c3} = \sqrt{\frac{\mu_{venus}}{r_{venus} + h_{venus}}}$	6885.43 m/s
Semi-major axis of transfer orbit	$a = \frac{r_{dep} + r_{tar}}{2}$	1.288×10^{11} m
Eccentricity of transfer orbit	$e = \frac{\lvert r_{dep} - r_{tar} \rvert}{r_{dep} + r_{tar}}$	0.16
Heliocentric velocity at departure position	$v_1 = \sqrt{\mu_{sun}(\frac{2}{r_{dep}} - \frac{1}{a})}$	27273.69 m/s
Heliocentric velocity at target position	$v_1 = \sqrt{\mu_{sun}(\frac{2}{r_{tar}} - \frac{1}{a})}$	37779.12 m/s
Excess velocity at departure planet	$v_{\text{inf},1} = \lvert v_1 - v_{dep} \rvert$	2510.79 m/s
Excess velocity at target planet	$v_{\text{inf},2} = \lvert v_2 - v_{dep} \rvert$	2724.60 m/s
Velocity in pericenter of hyperbola around departure planet	$v_0 = \sqrt{\frac{2\mu_{earth}}{r_{earth} + h_{earth}} + V_{\text{inf},1}}$	11458.38 m/s
Velocity in pericenter of hyperbola around target planet	$v_3 = \sqrt{\frac{2\mu_{venus}}{r_{venus} + h_{venus}} + V_{\text{inf},1}}$	10713.36 m/s
Maneuver in pericenter around departure planet	$\Delta_{v_0} = \lvert v_0 - v_{c0} \rvert$	3674.09 m/s
Maneuver in pericenter around target planet	$\Delta_{v_3} = \lvert v_3 - v_{c3} \rvert$	3827.93 m/s
Total velocity increase	$\Delta_{v_{total}} = \Delta_{v_0} + \Delta_{v_3}$	7502.019 m/s
Transfer time	$T = \pi\sqrt{\frac{a^3}{\mu_{sun}}}$	0.399 yr

Table 1: Hohmann transfer from the Earth to Venus

4 Exercise 4

Using the similar method introduced in section 3.1, we can do the same Hohmann transfer from Venus to Mercury and estimate the transfer time. The result is shown in the table 2.

Definition	**Formula**	**Result**
Heliocentric velocity of departure planet	$v_{dep} = \sqrt{\frac{\mu_{sun}}{r_{dep}}}$	35054.51 m/s
Heliocentric velocity of target planet	$v_{tar} = \sqrt{\frac{\mu_{sun}}{r_{tar}}}$	1513967.44 m/s
Circular velocity around departure planet	$v_{c0} = \sqrt{\frac{\mu_{earth}}{r_{earth} + h_{earth}}}$	6885.43 m/s
Circular velocity around target planet	$v_{c3} = \sqrt{\frac{\mu_{venus}}{r_{venus} + h_{venus}}}$	3004.63 m/s
Semi-major axis of transfer orbit	$a = \frac{r_{dep} + r_{tar}}{2}$	5.402895×10^{10} m
Eccentricity of transfer orbit	$e = \frac{\lvert r_{dep} - r_{tar}\rvert}{r_{dep} + r_{tar}}$	0.99892835
Heliocentric velocity at departure position	$v_1 = \sqrt{\mu_{sun}(\frac{2}{r_{dep}} - \frac{1}{a})}$	1147.55 m/s
Heliocentric velocity at target position	$v_1 = \sqrt{\mu_{sun}(\frac{2}{r_{tar}} - \frac{1}{a})}$	2140499.59 m/s
Excess velocity at departure planet	$v_{\inf,1} = \lvert v_1 - v_{dep}\rvert$	33906.97 m/s
Excess velocity at target planet	$v_{\inf,2} = \lvert v_2 - v_{dep}\rvert$	626532.15 m/s
Velocity in pericenter of hyperbola around departure planet	$v_0 = \sqrt{\frac{2\mu_{earth}}{r_{earth} + h_{earth}} + V_{\inf,1}}$	35454.69 m/s
Velocity in pericenter of hyperbola around target planet	$v_3 = \sqrt{\frac{2\mu_{venus}}{r_{venus} + h_{venus}} + V_{\inf,1}}$	626546.56 m/s
Maneuver in pericenter around departure planet	$\Delta_{v_0} = \lvert v_0 - v_{c0}\rvert$	28569.26 m/s
Maneuver in pericenter around target planet	$\Delta_{v_3} = \lvert v_3 - v_{c3}\rvert$	623541.93 m/s
Total velocity increase	$\Delta_{v_{total}} = \Delta_{v_0} + \Delta_{v_3}$	652111.18 m/s
Transfer time	$T = \pi\sqrt{\frac{a^3}{\mu_{sun}}}$	0.109 yr

Table 2: Hohmann transfer from the Venus to Mercury

References

[1] H. Curtis, *Orbital Mechanics for Engineering Students 3rd.* Elsevier, 2013.

[2] R. A. Braeunig, "Orbital mechanics," 2013. [Online]. Available: http://www.braeunig.us/space/orbmech.htm