All Trig Identity Derivations

Hamza Hafeez

November 2022

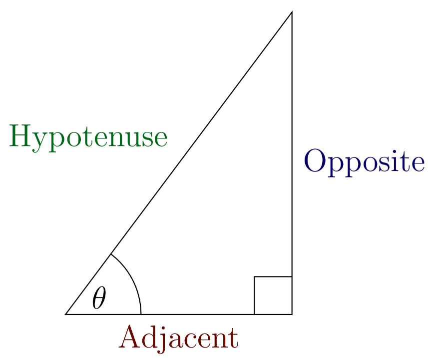
1 Basis Trig

$$\sin(\theta) = \frac{O}{H} \qquad \cos(\theta) = \frac{A}{H} \qquad \tan(\theta) = \frac{O}{A}$$

$$\csc(\theta) = \frac{1}{\sin(\theta)} = \frac{H}{O} \qquad \sec(\theta) = \frac{1}{\cos(\theta)} = \frac{H}{A} \qquad \cot(\theta) = \frac{1}{\tan(\theta)} = \frac{A}{O}$$

$$\frac{\sin(\theta)}{\cos(\theta)} = \frac{\frac{O}{H}}{\frac{A}{H}} = \frac{O}{A}$$

$$\boxed{\frac{\sin(\theta)}{\cos(\theta)} = \tan(\theta)}$$



2 Pythagorean Identities

$$\begin{aligned}
\sin^2(\theta) + \cos^2(\theta) &= \left(\frac{o}{h}\right)^2 + \left(\frac{a}{h}\right)^2 \\
&= \frac{o^2}{h^2} + \frac{a^2}{h^2} \\
&= \frac{o^2 + a^2}{h^2}
\end{aligned}$$

Since $o^2 + a^2 = h^2$ from the Pythagorean theorem.

$$\begin{aligned}
&= \frac{h^2}{h^2} \\
&\boxed{\sin^2(\theta) + \cos^2(\theta) = 1}
\end{aligned}$$

Dividing both sides by $\sin^2(\theta)$ and $\cos^2(\theta)$ yields two more versions.

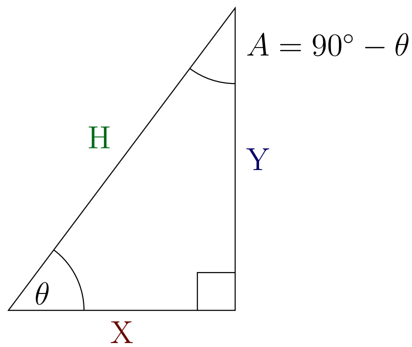
$$\begin{aligned}
\sin^2(\theta) + \cos^2(\theta) &= 1 \\
\frac{\sin^2(\theta) + \cos^2(\theta)}{\sin^2(\theta)} &= \frac{1}{\sin^2(\theta)} \\
\frac{\sin^2(\theta)}{\sin^2(\theta)} + \frac{\cos^2(\theta)}{\sin^2(\theta)} &= \frac{1}{\sin^2(\theta)} \\
\boxed{1 + \cot^2(\theta) = \csc^2(\theta)}
\end{aligned}$$

$$\begin{aligned}
\sin^2(\theta) + \cos^2(\theta) &= 1 \\
\frac{\sin^2(\theta) + \cos^2(\theta)}{\cos^2(\theta)} &= \frac{1}{\cos^2(\theta)} \\
\frac{\sin^2(\theta)}{\cos^2(\theta)} + \frac{\cos^2(\theta)}{\cos^2(\theta)} &= \frac{1}{\cos^2(\theta)} \\
\boxed{\tan^2(\theta) + 1 = \sec^2(\theta)}
\end{aligned}$$

3 Cofunctions

To find the missing angle in the top, named A, use the fact all angles in a triangle add to 180°.

$$\begin{aligned} 90^\circ + \theta + A &= 180^\circ \\ \theta + A &= 90^\circ \\ A &= 90^\circ - \theta \end{aligned}$$



$$\begin{aligned} \sin(\theta) &= \frac{\text{O}}{\text{H}} = \frac{\text{Y}}{\text{H}} \\ \cos(90^\circ - \theta) &= \frac{\text{A}}{\text{H}} = \frac{\text{Y}}{\text{H}} \end{aligned}$$

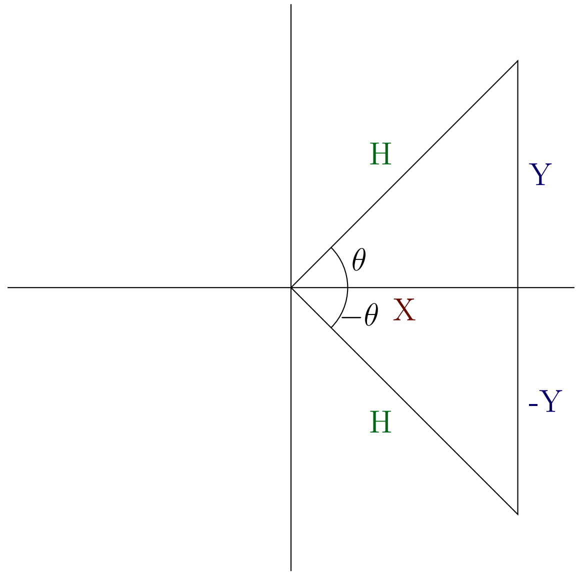
Since both equations equal $\frac{\text{Y}}{\text{H}}$

$$\sin(\theta) = \cos(90^\circ - \theta)$$

The same argument can be made for all the trig functions. These are easy to remember, just add or remove "co" from the beginning of the function name to get the cofunction.

$\sin(\theta) = \cos(90^\circ - \theta)$	$\sec(\theta) = \csc(90^\circ - \theta)$	$\tan(\theta) = \cot(90^\circ - \theta)$
$\cos(\theta) = \sin(90^\circ - \theta)$	$\csc(\theta) = \sec(90^\circ - \theta)$	$\cot(\theta) = \tan(90^\circ - \theta)$

4 Other Identities



$$\sin(\theta) = \frac{\text{Y}}{\text{H}}$$
$$\sin(-\theta) = \frac{\text{-Y}}{\text{H}}$$
$$-\sin(-\theta) = \frac{\text{Y}}{\text{H}}$$

Setting both equations equal to each other.

$$\sin(\theta) = -\sin(-\theta)$$
$$\boxed{-\sin(\theta) = \sin(-\theta)}$$

$$\cos(\theta) = \frac{\text{X}}{\text{H}}$$
$$\cos(-\theta) = \frac{\text{X}}{\text{H}}$$

Setting both equations equal to each other.

$$\boxed{\cos(\theta) = \cos(-\theta)}$$

5 Sum and Difference Identities

This is the only trig identity you need to memorize. All other ones can be derived using this and the trig rules above.

$$\boxed{\sin(A + B) = \sin(A)\cos(B) + \cos(A)\sin(B)}$$

To derive $\sin(A-B)$, change it to $\sin(A+(-B))$. Now the $\sin(A+B)$ identity can be used.

$$\begin{aligned}\sin(A - B) &= \sin(A + (-B)) \\ &= \sin(A)\cos(-B) + \cos(A)\sin(-B)\end{aligned}$$

Remember, $\cos(\theta) = \cos(-\theta)$ and $-\sin(\theta) = \sin(-\theta)$.

$$= \sin(A)\cos(B) + \cos(A) \cdot - \sin(B)$$

$$\boxed{\sin(A - B) = \sin(A)\cos(B) - \cos(A)\sin(B)}$$

To derive $\cos(A + B)$ change it using $\cos(\theta) = \sin(90^\circ - \theta)$.

$$\begin{aligned}\cos(A + B) &= \sin(90^\circ - (A + B)) \\ &= \sin((90^\circ - A - B) \\ &= \sin((90^\circ - A) - B)\end{aligned}$$

Using the sin(A-B) identity,

$$= \sin(90^\circ - A)\cos(B) - \cos(90^\circ - A)\sin(B)$$

$$\boxed{\cos(A + B) = \cos(A)\cos(B) - \sin(A)\sin(B)}$$

To derive $\cos(A-B)$, change it to $\cos(A+(-B))$. Now the $\cos(A+B)$ identity can be used.

$$\begin{aligned}\cos(A - B) &= \cos(A + (-B)) \\ &= \cos(A)\cos(-B) - \sin(A)\sin(-B) \\ &= \cos(A)\cos(B) - \sin(A) \cdot - \sin(B)\end{aligned}$$

$$\boxed{\cos(A - B) = \cos(A)\cos(B) + \sin(A)\sin(B)}$$

To derive $\tan(A + B)$, change it using $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$.

$$\begin{aligned}\tan(A + B) &= \frac{\sin(A + B)}{\cos(A + B)} \\ &= \frac{\sin(A)\cos(B) + \cos(A)\sin(B)}{\cos(A)\cos(B) - \sin(A)\sin(B)}\end{aligned}$$

Our answer should be in term of $\tan(\theta)$ instead of $\cos(\theta)$ or $\sin(\theta)$. Dividing each term by $\cos(A)\cos(B)$ accomplishes this.

$$
\begin{aligned}
&= \frac{\sin(A)\cos(B) + \cos(A)\sin(B)}{\cos(A)\cos(B) - \sin(A)\sin(B)} \left(\frac{\frac{1}{\cos(A)\cos(B)}}{\frac{1}{\cos(A)\cos(B)}} \right) \\
&= \frac{\frac{\sin(A)\cos(B)}{\cos(A)\cos(B)} + \frac{\cos(A)\sin(B)}{\cos(A)\cos(B)}}{\frac{\cos(A)\cos(B)}{\cos(A)\cos(B)} - \frac{\sin(A)\sin(B)}{\cos(A)\cos(B)}}
\end{aligned}
$$

$$
\boxed{\tan(A+B) = \frac{\tan(A) + \tan(B)}{1 - \tan(A)\tan(B)}}
$$

The identity for $\tan(A - B)$ is derived the same way.

$$
\begin{aligned}
\tan(A - B) &= \frac{\sin(A - B)}{\cos(A - B)} \\
&= \frac{\sin(A)\cos(B) - \cos(A)\sin(B)}{\cos(A)\cos(B) + \sin(A)\sin(B)} \\
&= \frac{\sin(A)\cos(B) - \cos(A)\sin(B)}{\cos(A)\cos(B) + \sin(A)\sin(B)} \left(\frac{\frac{1}{\cos(A)\cos(B)}}{\frac{1}{\cos(A)\cos(B)}} \right) \\
&= \frac{\frac{\sin(A)\cos(B)}{\cos(A)\cos(B)} - \frac{\cos(A)\sin(B)}{\cos(A)\cos(B)}}{\frac{\cos(A)\cos(B)}{\cos(A)\cos(B)} + \frac{\sin(A)\sin(B)}{\cos(A)\cos(B)}}
\end{aligned}
$$

$$
\boxed{\tan(A-B) = \frac{\tan(A) - \tan(B)}{1 + \tan(A)\tan(B)}}
$$

6 Double Angle Identities

These are way easier than the last section.

$$
\begin{aligned}
\sin(2A) &= \sin(A + A) \\
&= \sin(A)\cos(a) + \cos(A)\sin(A) \\
\Aboxed{\sin(2A) &= 2\sin(A)\cos(A)}
\end{aligned}
$$

There are three version of the $\cos(2A)$ identity.

$$
\begin{aligned}
\cos(2A) &= \cos(A + A) \\
&= \cos(A)\cos(A) - \sin(A)\sin(A) \\
\Aboxed{\cos(2A) &= \cos^2(A) - \sin^2(A)}
\end{aligned}
$$

Change $\cos^2(A)$ and $\sin^2(A)$ using the Pythagorean identity to obtain the two other versions

$$
\begin{aligned}
\cos(2A) &= \cos^2(A) - \sin^2(A) \\
&= (1 - \sin^2(A)) - \sin^2(A) \\
\Aboxed{\cos(2A) &= 1 - 2\sin^2(A)}
\end{aligned}
$$

$$
\begin{aligned}
\cos(2A) &= \cos^2(A) - \sin^2(A) \\
&= \cos^2(A) - (1 - \cos^2(A)) \\
\Aboxed{\cos(2A) &= 2\cos^2(A) - 1}
\end{aligned}
$$

$$
\begin{aligned}
\tan(2A) &= \tan(A + A) \\
&= \frac{\tan(A) + \tan(A)}{1 - \tan(A)\tan(A)} \\
\Aboxed{\tan(2A) &= \frac{2\tan(A))}{1 - \tan^2(A)}}
\end{aligned}
$$

7 Half Angle Identities

These are a bit harder.
To derive $\cos(\frac{A}{2})$, start with version of $\cos(2A)$ that contains $\cos(A)$.

$$\cos(2A) = 2\cos^2(A) - 1$$

Divide all angles by half and solve for $\cos(\frac{A}{2})$

$$\cos(A) = 2\cos^2\left(\frac{A}{2}\right) - 1$$
$$\frac{\cos(A)+1}{2} = \cos^2\left(\frac{A}{2}\right)$$
$$\boxed{\pm\sqrt{\frac{1+\cos(A)}{2}} = \cos\left(\frac{A}{2}\right)}$$

To derive $\sin(\frac{A}{2})$, start with version of $\cos(2A)$ that contains $\sin(A)$

$$\cos(2A) = 1 - 2\sin^2(A)$$
$$\cos(A) = 1 - 2\sin^2\left(\frac{A}{2}\right)$$
$$\frac{\cos(A)-1}{-2} = \sin^2\left(\frac{A}{2}\right)$$
$$\left(\frac{-1}{-1}\right)\frac{\cos(A)-1}{-2} = \sin^2\left(\frac{A}{2}\right)$$
$$\boxed{\pm\sqrt{\frac{1-\cos(A)}{2}} = \sin\left(\frac{A}{2}\right)}$$

To derive $\tan(\frac{A}{2})$, change it using $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$.

$$\tan\left(\frac{A}{2}\right) = \frac{\sin(\frac{A}{2})}{\cos(\frac{A}{2})}$$
$$= \frac{\pm\sqrt{\frac{1-\cos(A)}{2}}}{\pm\sqrt{\frac{1+\cos(A)}{2}}}$$
$$= \pm\sqrt{\frac{\frac{1-\cos(A)}{2}}{\frac{1+\cos(A)}{2}}}$$
$$\boxed{\tan\left(\frac{A}{2}\right) = \pm\sqrt{\frac{1-\cos(A)}{1+\cos(A)}}}$$

Multiply both sides by the top to get another version

$$\begin{aligned}
\tan\left(\frac{A}{2}\right) &= \frac{\sqrt{1-\cos(A)}}{\sqrt{1+\cos(A)}} \\
&= \pm\frac{\sqrt{1-\cos(A)}}{\sqrt{1+\cos(A)}}\frac{\sqrt{1-\cos(A)}}{\sqrt{1-\cos(A)}} \\
&= \frac{1-\cos(A)}{\sqrt{(1-\cos(A))(1+\cos(A))}} \\
&= \frac{1-\cos(A)}{\sqrt{1-\cos^2(A)}} \\
&= \frac{1-\cos(A)}{\sqrt{\sin^2(A)}}
\end{aligned}$$

$$\boxed{\tan\left(\frac{A}{2}\right) = \frac{1-\cos(A)}{\sin(A)}}$$

Multiply both sides by the bottom to get another version

$$\begin{aligned}
\tan\left(\frac{A}{2}\right) &= \frac{\sqrt{1-\cos(A)}}{\sqrt{1+\cos(A)}} \\
&= \pm\frac{\sqrt{1-\cos(A)}}{\sqrt{1+\cos(A)}}\frac{\sqrt{1+\cos(A)}}{\sqrt{1+\cos(A)}} \\
&= \frac{\sqrt{(1-\cos(A))(1+\cos(A))}}{1+\cos(A)} \\
&= \frac{\sqrt{1-\cos^2(A)}}{1+\cos(A)} \\
&= \frac{\sqrt{\sin^2(A)}}{1+\cos(A)}
\end{aligned}$$

$$\boxed{\tan\left(\frac{A}{2}\right) = \frac{\sin(A)}{1+\cos(A)}}$$

8 Summary

$$\csc(\theta) = \frac{1}{\sin(\theta)} \qquad \sec(\theta) = \frac{1}{\cos(\theta)} \qquad \cot(\theta) = \frac{1}{\tan(\theta)}$$

$$\begin{aligned}
\sin^2(\theta) + \cos^2(\theta) &= 1 \\
1 + \cot^2(\theta) &= \csc^2(\theta) \\
\tan^2(\theta) + 1 &= \sec^2(\theta)
\end{aligned}$$

$$\begin{aligned}
-\sin(\theta) &= \sin(-\theta) \\
\cos(\theta) &= \cos(-\theta) \\
\sin(\theta) &= \cos(90^\circ - \theta) \\
\cos(\theta) &= \sin(90^\circ - \theta)
\end{aligned}$$

$$\begin{aligned}
\sin(A \pm B) &= \sin(A)\cos(B) \pm \cos(A)\sin(B) \\
\cos(A \pm B) &= \cos(A)\cos(B) \mp \sin(A)\sin(B) \\
\tan(A \pm B) &= \frac{\tan(A) \pm \tan(B)}{1 \mp \tan(A)\tan(B)}
\end{aligned}$$

$$\begin{aligned}
\sin(2A) &= 2\sin(A)\cos(A) \\
\cos(2A) &= \cos^2(A) - \sin^2(A) \\
\cos(2A) &= 1 - 2\sin^2(A) = 2\cos^2(A) - 1 \\
\tan(2A) &= \frac{2\tan(A))}{1 - \tan^2(A)}
\end{aligned}$$

$$\begin{aligned}
\cos\left(\frac{A}{2}\right) &= \pm\sqrt{\frac{1 + \cos(A)}{2}} \\
\sin\left(\frac{A}{2}\right) &= \pm\sqrt{\frac{1 - \cos(A)}{2}} \\
\tan\left(\frac{A}{2}\right) &= \pm\sqrt{\frac{1 - \cos(A)}{1 + \cos(A)}} \\
\tan\left(\frac{A}{2}\right) &= \frac{1 - \cos(A)}{\sin(A)} = \frac{\sin(A)}{1 + \cos(A)}
\end{aligned}$$