

Unified Approach for Hedging Impermanent Loss of Liquidity Provision

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Abstract

We develop static and dynamic approaches for hedging of the impermanent loss (IL) of liquidity provision (LP) staked at Decentralised Exchanges (DEXes) which employ Uniswap V2 and V3 protocols. We provide detailed definitions and formulas for computing the IL to unify different definitions occurring in the existing literature. We show that the IL can be seen a contingent claim with a non-linear payoff for a fixed maturity date. Thus, we introduce the contingent claim termed as IL protection claim which delivers the negative of IL payoff at the maturity date. We apply arbitrage-based methods for valuation and risk management of this claim. First, we develop the static model-independent replication method for the valuation of IL protection claim using traded European vanilla call and put options. We extend and generalize an existing method to show that the IL protection claim can be hedged perfectly with options if there is a liquid options market. Second, we develop the dynamic model-based approach for the valuation and hedging of IL protection claims under a risk-neutral measure. We derive analytic valuation formulas using a wide class of price dynamics for which the characteristic function is available under the risk-neutral measure. As base cases, we derive analytic valuation formulas for IL protection claim under the Black-Scholes-Merton model and the log-normal stochastic volatility model. We finally discuss estimation of risk-reward of LP staking using our results.

Keywords: Automated Market Making, Liquidity Provision, Decentralized Finance, Uniswap, Cryptocurrencies, Impermanent Loss

JEL Classifications: C02, G12, G23

1 Introduction

Decentralised Exchanges (DEXes) play fundamental part in blockchain ecosystem by allowing users to swap digital assets. The functioning of DEXes requires liquidity providers who stake their liquidity to so-called pools, so that traders can use these pools for buying and selling tokens. Automated Market Making (AMM) protocol is a mechanism for settling buy and sell orders at DEXes. An AMM protocol is characterized by a constant function market maker (CFMM) which assigns buy and sell prices for given orders using order sizes and current liquidity of a pool. Uniswap V3 ([Adams et al. \(2021\)](#)) is the most widely employed CFMM which is adopted by many DEXes, in addition to Uniswap DEX itself¹. Uniswap V2 ([Adams et al. \(2020\)](#)) is an earlier AMM protocol employing the constant product CFMM which is less capital efficient than V3 CFMM. Uniswap V2 is still in use for old altcoins pools. For details of the design of various CFMMs, see among others [Angeris et al. \(2019\)](#), [Lipton-Treccani \(2021\)](#), [Mohan \(2022\)](#), [Lipton-Hardjono \(2022\)](#), [Lipton-Sepp \(2022\)](#), [Milionis et al. \(2022\)](#), [Park \(2023\)](#), [Lehar-Parlour \(2024\)](#).

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¹In the second half of year 2024, Uniswap development team is launching Uniswap V4 AMM which has the same CFMM as Uniswap V3.

Uniswap V3 protocol allows liquidity providers to concentrate liquidity in specified ranges. As a result, the liquidity of the pool can be increased in certain ranges (typically around the current price) and the potential to generate more trading fees from the LP is increased accordingly. We illustrate the dynamics of staked LP using ETH/USDT pool as an example. A liquidity provider stakes liquidity to a specific range using initial amount of ETH and USDT tokens as specified by Uniswap V3 CFMM. When the price of ETH falls, traders use the pool to swap USDT by depositing ETH, so that the LP accrues more units of ETH. Thus when ETH falls persistently, the liquidity provider ends up holding more units of the depreciating asset, which is similar to being short a put option. In opposite, when ETH price increases, traders will deplete ETH reserves from the pool by depositing USDT tokens. Thus, the liquidity provider ends up holding less units of the appreciating asset, which is similar to being short a call option. The combined effect of increasing / decreasing the exposure to depreciating / appreciating asset leads to what is known as the impermanent loss in Decentralised Finance (DeFi) applications.

It is clear that the mechanism behind the impermanent loss of LPs is similar to being short a portfolio of call and put options. Accordingly, we can apply well-developed methods from financial engineering for designing the valuation and risk management of LPs using static and dynamic methods for valuation and risk-management of derivative securities. Our contributions include static model-independent replication and dynamic model-dependent replication of IL.

1.1 Literature review and contributions

For the static replication of IL under Uniswap V2 protocol, [Fukasawa et al. \(2023\)](#) derive approximate hedging portfolio using variance and gamma swaps. In addition, [Lipton \(2024\)](#) extends [Lipton-Sepp \(2008\)](#), [Lipton \(2018\)](#), and obtains model-dependent costs of hedging portfolios using values of variance and gamma swaps under Heston model. [Deng et al. \(2023\)](#) and [Maire-Wunsch \(2024\)](#) develop the static replication for Uniswap V3 protocol. We note that their result hinges specifically on the analytical formula of the IL under Uniswap V3 AMM. Also this method requires to hold both call and put options for in-the-money and out-of-the-money strikes, which could be very costly for practical implementation². We provide a generic approach to design cost-efficient replicating portfolio for IL hedging for general CFMM and illustrate our approach for Uniswap V3 protocol.

We note that currently the liquid option market (on Deribit options exchanges and some other centralised exchanges) exists only for Bitcoin and Ether. Thus, hedging of LP stakes for altcoins requires a model-dependent approach for replication of IL by introducing IL protection claim and applying dynamic delta-hedging of this claim. In the current ecosystem, a few market-making and trading companies provide off-chain (over-the-counter) IL protection of claims for a wide range of digital assets. We develop a dynamic model-dependent approach for the valuation of IL protection claims, which appears to be new in the literature.

For valuation purposes under Uniswap V3 protocol, we provide an original result with a decomposition of the IL into components including payoffs of vanilla call and put options, digital options, and an exotic payoff on the square root of the price. This decomposition allows for the valuation of the IL protection claims under a large class of price dynamics, which have solution for their characteristic functions, by utilizing Lipton-Lewis formula. As an important example we derive an analytic solution for IL protecting claim under Black-Scholes-Merton model, which allows to analyze the value of the IL protection claim using a single volatility parameter. As realistic price dynamics including stochastic volatility correlated with the price dynamics, we apply the log-normal stochastic volatility model developed in [Sepp-Rakhmonov \(2023\)](#).

²At major options exchanges for digital and traditional markets, the liquidity is concentrated in call and put options with out-of-the-money and near at-the-money strikes. Options with in-the-money strikes are not liquid with high bid-ask spreads. As a result, a liquid and cost-efficient replicating portfolio should include only out-of-the-money and near at-the-money strikes.

Our paper is organized as follows. In Section 2, we provide definitions and derivations of the IL and payoffs of IL protection claims. In Section 3, we apply these results for Uniswap V2 and V3 protocols. Hereby, we derive the decomposition formula for IL in Uniswap V3 protocol into payoffs of vanilla, digital and square root contracts, which we use further for model-dependent valuation. In Section 4, we develop generic approach for static replication of IL using traded vanilla options. In Section 5, we develop the model-dependent approach for the valuation of protection claim against IL. We conclude in Section 6.

2 Impermanent Loss of Liquidity Provision

2.1 Liquidity Provision

We consider a liquidity pool on a pair of token 1 and token 2. Without loss of generality, we assume that token 1 is a volatile token and token 2 is a stable token with the spot price p of swapping one unit of token 1 to p units of token 2. For concreteness, we fix token 1 to be ETH and token 2 to be USDT with spot price p being ETH/USDT exchange price ($p_0 = 3800$ as of 7th June 2024).

We consider a liquidity provision (LP) provided on x_0 and y_0 units of token 1 (ETH) and token 2 (USDT), respectively. The initial value of the LP in the units of token 2 (USDT) is given by

$$V_0^{(y)} = p_0 x_0 + y_0 \quad (1)$$

The value of the LP at time t is given by

$$V_t^{(y)} = p_t x_t + y_t \quad (2)$$

where x_t and y_t are the current units of ETH and USDT (these units are the outputs from AMM protocol), respectively, in the staked LP and p_t is the current ETH/USDT spot price. We treat accrued LP fees separately in line with Uniswap convention.

The value of the LP in units of token 1 is given by

$$V_t^{(x)} = x_t + p_t^{-1} y_t \quad (3)$$

Our further results can be directly applied for pools with USDT/ETH type of conversion using corresponding inverse prices and ranges. We also note that, in Uniswap V2 and V3, the price is defined on the grid of price ticks which are functions of the pool fee tiers. Price ticks are dense for pools with small fee tiers, so we assume that price range for p_t is continuous (see [Echenim et al. \(2023\)](#) for the analysis using discrete ticks).

2.2 Profit-and-Loss of LP

We consider two types of LP strategies excluding and including static delta hedging of the initial stake.

Definition 2.1 (USDT Funded LP position). *Funded position is created by funding the initial allocation of x_0 and y_0 units with the total capital commitment of $V_0^{(y)}$ USDT. Staking the funded position includes the purchase of x_0 units of ETH token at price p_0 .*

The value of the funded position equals to the value of LP in Eq (2): $V_{funded}^{(y)} = V_t^{(y)}$. As a result, the Profit-and-Loss (P&L) of the funded position in token 2 (USDT) at time t is given by

$$P\&L_{funded}^{(y)} \equiv V_t^{(y)} - V_0^{(y)} = (p_t x_t + y_t) - (p_0 x_0 + y_0) \quad (4)$$

Definition 2.2 (Borrowed LP position). *Borrowed position is created either by borrowing x_0 units of ETH or by purchasing x_0 units of ETH for staking and by simultaneously selling short the perpetual future for hedging the initial stake of x_0 units of ETH³.*

For the borrowed position with hedging, we set the hedge position of selling short x_0 units of token 1 with strike/entry price p_0 . The P&L of the hedge position in units of USDT token at time t is given by

$$Hedge_t^{(y)} = -(p_t - p_0) x_0 \quad (5)$$

We assume that the hedge can be implemented by short selling the perpetual future and we treat the funding cost separately from the LP P&L. The initial value of the staking position is given in Eq (2). The value of the borrowed LP position at time t is given by

$$V_{\text{borrowed}}^{(y)} = p_t x_t + y_t - [p_t - p_0] x_0 \quad (6)$$

The P&L of the borrowed position at time t is given by

$$\begin{aligned} P\&L_{\text{borrowed}}^{(y)} &= (p_t x_t + y_t - [p_t - p_0] x_0) - (p_0 x_0 + y_0) \\ &= (p_t x_t + y_t) - (p_t x_0 + y_0) \end{aligned} \quad (7)$$

2.3 Impermanent Loss

Using the two definitions of LPs in Eqs (4) and (7), we define the quantity known as the impermanent loss in the following three ways.

Definition 2.3 (Impermanent Loss %). *The nominal IL of funded LP is defined for the LP funded with USDT by*

$$IL_{\text{funded}}^{(y)}(p_t) = \frac{(p_t x_t + y_t) - (p_0 x_0 + y_0)}{(p_0 x_0 + y_0)} \quad (8)$$

The nominal IL of borrowed LP is defined for the LP with borrowed ETH by

$$IL_{\text{borrowed}}^{(y)}(p_t) = \frac{(p_t x_t + y_t) - (p_t x_0 + y_0)}{(p_0 x_0 + y_0)} \quad (9)$$

The relative IL of borrowed LP is defined for borrowed LP by

$$Rel\ IL_{\text{borrowed}}^{(y)}(p_t) = \frac{(p_t x_t + y_t) - (p_t x_0 + y_0)}{(p_t x_0 + y_0)} \quad (10)$$

In the literature, all three definitions are being used. Hereby, we clarify the meaning of each definition. The nominal IL of funded LP is applicable when the LP provider funds the position by allocation V_0 USDT tokens and buys the initial stake of x_0 ETH tokens. The nominal IL for borrowed LP is common for LPs accompanied with either borrowing the initial stake of x_0 ETH tokens or with static delta-hedging of the initial stake of x_0 tokens using perpetual futures. The relative IL of borrowed LP defines the P&L of the borrowed LP relative to the buy-and-hold position rather than the initial staked value of the LP.

The nominal IL can be easily interpreted because P&L of the LP in USDT is the nominal IL multiplied by the initial staked notional so that we obtain

$$\begin{aligned} P\&L_{\text{funded}}^{(y)}(p_t) &= N^{(y)} \times IL_{\text{funded}}^{(y)}(p_t) \\ P\&L_{\text{borrowed}}^{(y)}(p_t) &= N^{(y)} \times IL_{\text{borrowed}}^{(y)}(p_t) \end{aligned} \quad (11)$$

³There is very liquid market for core cryptocurrencies on both on-chain exchanges (such as Hyperliquid, GMX, Aevo) and on off-chain exchanges (such as Binance, Bybit, Deribit), so that hedging of long exposures is possible.

where $N^{(y)}$ is the initial notional in USDT token of the staked LP. In opposite, the relative IL lacks this interpretation. Thus, while relative IL appears in some of the literature to emphasize the IL relative to the buy-and-hold portfolio, its practical application for modelling of the realised P&L from a LP is not obvious. In this paper we focus only on hedging of the nominal IL for funded and borrowed LPs in Eqs (8) and (9), respectively.

Since the tokens can be used interchangeably, our definitions are symmetric. For a position funded in x (ETH) tokens the corresponding P&L is obtained using $p_t^{-1} = 1/p_t$ and Eq (11) becomes

$$\begin{aligned} P\&L \text{ funded}^{(x)}(p_t^{-1}) &= N^{(x)} \times IL \text{ funded}^{(x)}(p_t^{-1}), \\ P\&L \text{ borrowed}^{(x)}(p_t^{-1}) &= N^{(x)} \times IL \text{ borrowed}^{(x)}(p_t^{-1}), \end{aligned} \quad (12)$$

where $N^{(x)}$ is notional in x units and

$$\begin{aligned} IL \text{ funded}^{(x)}(p_t^{-1}) &= \frac{(x_t + p_t^{-1}y_t) - (x_0 + p_0^{-1}y_0)}{(x_0 + p_0^{-1}y_0)} \\ IL \text{ borrowed}^{(x)}(p_t^{-1}) &= \frac{(x_t + p_t^{-1}y_t) - (x_0 + p_t^{-1}y_0)}{(x_0 + p_0^{-1}y_0)} \end{aligned} \quad (13)$$

2.4 Payoff of IL Protection Claim

We fix maturity time T .

Definition 2.4 (Payoff of IL protection claim). *We define the protection claim against IL as a derivative security whose payoff at time T equals to negative value of the IL. For the funded LP, the payoff at time T is defined by*

$$Payoff^{funded}(p_T) = -IL \text{ funded}^{(y)}(p_T) \quad (14)$$

For the borrowed LP, the payoff at time T is defined by

$$Payoff^{borrowed}(p_T) = -IL \text{ borrowed}^{(y)}(p_T) \quad (15)$$

A liquidity provider of staked LP with notional $N^{(y)}$ can buy the IL protection claim to eliminate the impermanent loss from the staked LP. By Eq (11), at time T the P&L of holder's LP will be matched by the payoff of the IL protection claim in Eq (14) or Eq (15). As a result, the liquidity provider can perfectly hedge the IL at time T .

3 Applications to Uniswap AMM Protocol

We now derive explicit formulas for the IL of funded and borrowed LP stakes under Uniswap V2 and V3 protocols.

3.1 Uniswap V2

In Uniswap V2 ([Adams et al. \(2020\)](#)), the CFMM is defined by the constant product rule as follows

$$xy = L^2 \quad (16)$$

where x and y are pool reserves and L is the pool liquidity parameter. In the Uniswap V2 white paper, the constant is defined by k . We use L^2 , $L > 0$, in line with V3 specification.

The pool price is determined by pool reserves as follows

$$p \equiv \frac{y}{x} \quad (17)$$

Thus, we need to solve Eq (16) and (17) in the two unknowns x and y . Substituting $y = px$ from Eq (17) into Eq (16), we obtain that the LP stakes are given as follows

$$x = \sqrt{\frac{L^2}{p}}, \quad y = \sqrt{pL^2}. \quad (18)$$

From (18) the value of LP position is given by⁴

$$V_t^{(y)} = p_t x_t + y_t = 2L\sqrt{p_t}. \quad (19)$$

Proposition 3.1 (Funded LP). *The funded P&L in Eq (4) is computed by:*

$$P\&L \text{ funded}^{(y)}(p_t) = 2L\sqrt{p_0} \left(\sqrt{\frac{p_t}{p_0}} - 1 \right) \quad (20)$$

The nominal IL for funded LP in Eq (8) is computed by

$$\text{Nom IL funded}^{(y)}(p_t) = \sqrt{\frac{p_t}{p_0}} - 1 \quad (21)$$

Proof. Using Eq (19), we obtain

$$P\&L \text{ funded}^{(y)}(p_t) = 2L\sqrt{p_t} - 2L\sqrt{p_0} = 2L\sqrt{p_0} \left(\sqrt{\frac{p_t}{p_0}} - 1 \right) \quad (22)$$

Given an initial notional of the stake such as $N^y = V_0 = 2L\sqrt{p_0}$, we obtain the nominal IL. \square

Proposition 3.2 (IL for Borrowed LP in Uniswap V2). *The borrowed P&L in Eq (7) is given by:*

$$P\&L \text{ borrowed}^{(y)}(p_t) = -L\sqrt{p_0} \left(\sqrt{\frac{p_t}{p_0}} - 1 \right)^2 \quad (23)$$

The nominal IL for borrowed LP in Eq (9) is given by

$$\text{Nom IL borrowed}^{(y)}(p_t) = -\frac{1}{2} \left(\sqrt{\frac{p_t}{p_0}} - 1 \right)^2 \quad (24)$$

Relative IL in Eq (10) is given by

$$\text{Rel IL borrowed}^{(y)}(p_t) = -\frac{\left(\sqrt{\frac{p_t}{p_0}} - 1 \right)^2}{\frac{p_t}{p_0} + 1} \quad (25)$$

Proof. From Eq (18) we note that

$$p_t x_0 + y_0 = p_t \sqrt{\frac{L^2}{p_0}} + \sqrt{p_0 L^2} = \sqrt{p_0} L \left(\frac{p_t}{p_0} + 1 \right) \quad (26)$$

⁴This relationship follows from the condition that the internal price in Eq (17) inferred by pool reserves follows an external price p_t , observed on other DEXes and centralised exchanges. In practice, reserves of liquidity pools are balanced so that the internal price in Eq (17) follows external price feeds withing tight bands most of times due to arbitrage operations of multiple arbitrages in blockchain ecosystem. For details of such arbitrages see among others Milionis *et al.* (2022), Park (2023), Lehar-Parlour (2024), Cartea *et al.* (2023), Cartea *et al.* (2024). Same considerations apply for the internal price in Eq (31) implied by pool reserves for Uniswap V3 pools.

Thus, we obtain

$$\begin{aligned}
P\&L \text{ borrowed}^{(y)}(p_t) &= (p_t x_t + y_t) - (p_t x_0 + y_0) \\
&= 2\sqrt{p_t L^2} - \sqrt{p_0} L \left(\frac{p_t}{p_0} + 1 \right) \\
&= -L\sqrt{p_0} \left(\sqrt{\frac{p_t}{p_0}} - 1 \right)^2
\end{aligned} \tag{27}$$

□

It is clear that the minimum is 0 at $p_t = p_0$ and otherwise the borrowed P&L is negative for any value of p_1 .

Corollary 3.1 (Payoff of claim for IL protection for Uniswap V2 AMM). *Using definitions in Eq (14) and Eq (15) for payoffs of protection claim against funded and borrowed LP, respectively, along with respective Eqs (21) and (24), we obtain*

$$\begin{aligned}
\text{Payoff } f^{\text{funded}}(p_T) &= 1 - \sqrt{\frac{p_t}{p_0}}, \\
\text{Payoff } f^{\text{borrowed}}(p_T) &= \frac{1}{2} \left(\sqrt{\frac{p_t}{p_0}} - 1 \right)^2
\end{aligned} \tag{28}$$

In subplot (A) of Figure (1), we show ETH units (left y-axis) and USDT units (right y-axis) for LP Uniswap V2 with 1m USDT notional and $p_0 = 2000$ ETH/USDT price. The initial LP units of (ETH, USDT) are (250, 500000). The red bar at $p = 1500$ shows LP units of (289, 433013) with the higher allocation to ETH units as ETH/USDT price falls. The green bar at $p = 2500$ shows corresponding LP units of (224, 559017) with the higher allocation to USDT units as ETH/USDT price rises. In subplot (B), we show USDT values of 50%/50% ETH/USDT portfolio, Funded LP position and Borrowed LP position. The funded LP underperforms the 50%/50% portfolio on both the upside (because LP position reduces ETH units) and on the downside (because LP position reduces ETH units). The value of the borrowed LP has zero first-order beta to ETH with negative quadratic convexity to ETH/USDT changes.

3.2 Application to Uniswap V3

In Uniswap V3 protocol (see [Adams et al. \(2021\)](#)), the CFMM is defined by

$$x_v y_v = L^2 \tag{29}$$

where x_v and y_v are termed as virtual reserves:

$$x_v \equiv x + \frac{L}{\sqrt{p_b}}, \quad y_v \equiv y + L\sqrt{p_a} \tag{30}$$

with the liquidity amount L provided in the price range $[p_a, p_b]$.

For ETH/USDT pool, the price p is set by:

$$p \equiv \frac{y_v}{x_v} = \frac{y + L\sqrt{p_a}}{x + \frac{L}{\sqrt{p_b}}} \tag{31}$$

Eqs (29) and (31) are viewed as two equations in four unknowns (x, y, L, p):

$$\begin{aligned}
\left(x + \frac{L}{\sqrt{p_b}} \right) (y + L\sqrt{p_a}) &= L^2 \\
p &= \frac{y + L\sqrt{p_a}}{x + \frac{L}{\sqrt{p_b}}}.
\end{aligned} \tag{32}$$

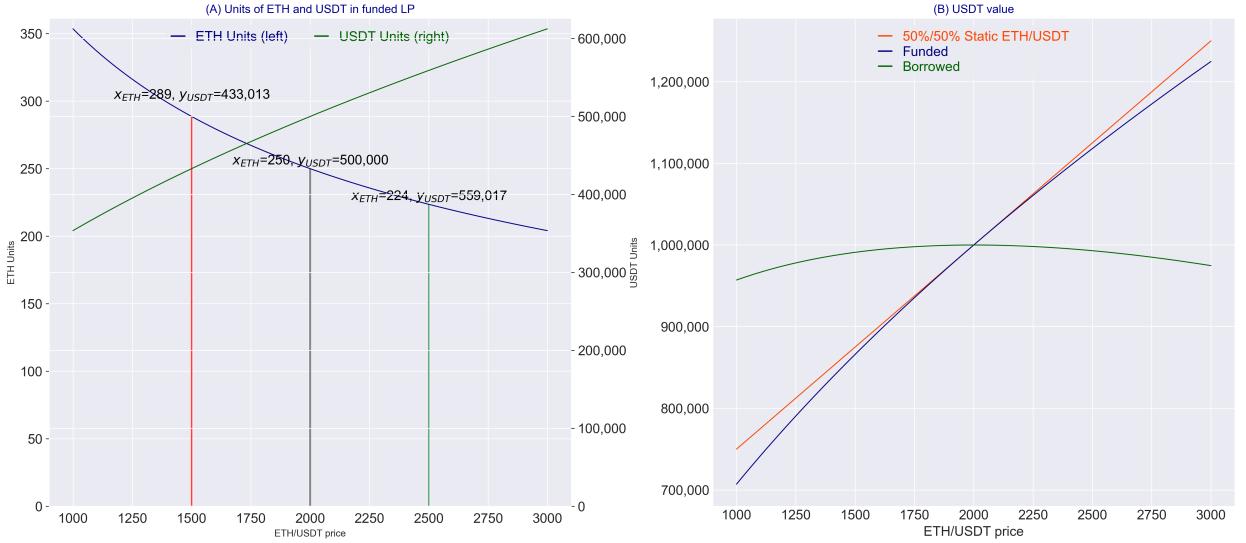


Figure 1: (A) ETH units (left y-axis) and USDT units (right y-axis) for LP Uniswap V2. (B) USDT value of 50%/50% ETH/USDT portfolio, Funded LP position and Borrowed LP position. Uniswap V2 LP position is constructed using 1m USDT notional with $p_0 = 2000$ ETH/USDT price.

The above equations have to hold at any point in time, with internal price p tracking an internal price observed on other trading venues within tight bounds due to arbitragers adjust pool reserves accordingly to eliminate arbitrage opportunities. Adding (removing) liquidity amounts to increasing (decreasing) L via increasing x and y , while keeping p constant. Swapping (trading) tokens amounts to keeping changing x , y and p , while keeping L the same.

Importantly, we solve for x and y given L and p as independent variables. For a given position with L and p as external parameters, this solution provides how much units x and y are assigned to the LP position.

Lemma 3.1 (Solution for x and y). *Thus for $p \in (p_a, p_b)$, the LP units x and y are given by:*

$$x = L \left(\frac{1}{\sqrt{p}} - \frac{1}{\sqrt{p_b}} \right), \quad y = L (\sqrt{p} - \sqrt{p_a}) \quad (33)$$

For $p \leq p_a$, the position is fully in token 1:

$$x = L \left(\frac{1}{\sqrt{p_a}} - \frac{1}{\sqrt{p_b}} \right), \quad y = 0 \quad (34)$$

For $p \geq p_b$, the position is fully in token 2:

$$x = 0, \quad y = L (\sqrt{p_b} - \sqrt{p_a}) \quad (35)$$

Proof. We substitute the second equation in Eq (32)

$$(y + L\sqrt{p_a}) = p \left(x + \frac{L}{\sqrt{p_b}} \right) \quad (36)$$

into the first one to obtain

$$\left(x + \frac{L}{\sqrt{p_b}} \right)^2 = \frac{L^2}{p} \Rightarrow x = L \left(\frac{1}{\sqrt{p}} - \frac{1}{\sqrt{p_b}} \right) \quad (37)$$

and

$$(y + L\sqrt{p_a}) = L\sqrt{p} \Rightarrow y = L (\sqrt{p} - \sqrt{p_a}) \quad (38)$$

□

We obtain that the initial value of LP using Eq (33) for $p_t \in (p_a, p_b)$ is given by

$$V_0 \equiv p_0 x_0 + y_0 = L \left(2\sqrt{p_0} - \frac{p_0}{\sqrt{p_b}} - \sqrt{p_a} \right) \quad (39)$$

Corollary 3.2 (Initial notional N^y). *Given an initial notional of the stake such as $N^y = V_0$, using Eq (39) we obtain that the provided liquidity L is set by*

$$\Rightarrow L = \frac{N^y}{2\sqrt{p_0} - \frac{p_0}{\sqrt{p_b}} - \sqrt{p_a}} \quad (40)$$

3.2.1 Impermanent Loss

Proposition 3.3 (Funded P&L and IL in Uniswap V3). *The P&L of the funded position in Eq (4) at time t with current price p_t is given by*

$$P\&L \text{ funded}^{(y)} = \begin{cases} L \left[2 \left(\sqrt{p_t} - \sqrt{p_0} \right) + \frac{p_0 - p_t}{\sqrt{p_b}} \right] & p_t \in (p_a, p_b) \\ L \left[p_t \left(\frac{1}{\sqrt{p_a}} - \frac{1}{\sqrt{p_b}} \right) + \frac{p_0}{\sqrt{p_b}} - 2\sqrt{p_0} + \sqrt{p_a} \right] & p_t \leq p_a \\ L \left[\sqrt{p_b} + \frac{p_0}{\sqrt{p_b}} - 2\sqrt{p_0} \right] & p_t \geq p_b \end{cases} \quad (41)$$

where

$$L = \frac{N^y}{2\sqrt{p_0} - \frac{p_0}{\sqrt{p_b}} - \sqrt{p_a}} \quad (42)$$

and N^y is USDT notional.

The nominal IL for funded position defined in Eq (8) is computed by

$$\text{Nom IL funded}^{(y)} = \frac{P\&L \text{ funded}^{(y)}}{L \left(2\sqrt{p_0} - \frac{p_0}{\sqrt{p_b}} - \sqrt{p_a} \right)} \quad (43)$$

Corollary 3.3 (The payoff of IL protection claim for funded LP). *The payoff of IL protection claim at maturity date T for funded LP in Eq (14) is given by the following compact formula*

$$\text{Payoff funded}(p_t) = - \frac{\frac{p_t}{\sqrt{f(p_t; p_a, p_b)}} + \sqrt{f(p_t; p_a, p_b)} - \frac{p_t}{\sqrt{p_b}} - \sqrt{p_a}}{\frac{p_0}{\sqrt{f(p_0; p_a, p_b)}} + \sqrt{f(p_0; p_a, p_b)} - \frac{p_0}{\sqrt{p_b}} - \sqrt{p_a}} + 1 \quad (44)$$

where $f(x; p_a, p_b) = \max(\min(x, p_b), p_a)$.

Proof. See Appendix 6.1. □

Proposition 3.4 (P&L of borrowed LP). *The P&L of the borrowed position in Eq (7) is given by*

$$P\&L \text{ borrowed}^{(y)}(p_t) = \begin{cases} -L\sqrt{p_0} \left(\sqrt{\frac{p_t}{p_0}} - 1 \right)^2 & p_t \in (p_a, p_b) \\ L \left[p_t \left(\frac{1}{\sqrt{p_a}} - \frac{1}{\sqrt{p_0}} \right) - (\sqrt{p_0} - \sqrt{p_a}) \right] & p_t \leq p_a \\ L \left[(\sqrt{p_b} - \sqrt{p_0}) - p_t \left(\frac{1}{\sqrt{p_0}} - \frac{1}{\sqrt{p_b}} \right) \right] & p_t \geq p_b \end{cases} \quad (45)$$

Nominal borrowed impermanent loss in Eq (9) is given by

$$\text{Nom IL borrowed}^{(y)}(p_t) = \frac{P\&L \text{ borrowed}^{(y)}(p_t)}{L \left(2\sqrt{p_0} - \frac{p_0}{\sqrt{p_b}} - \sqrt{p_a} \right)} \quad (46)$$

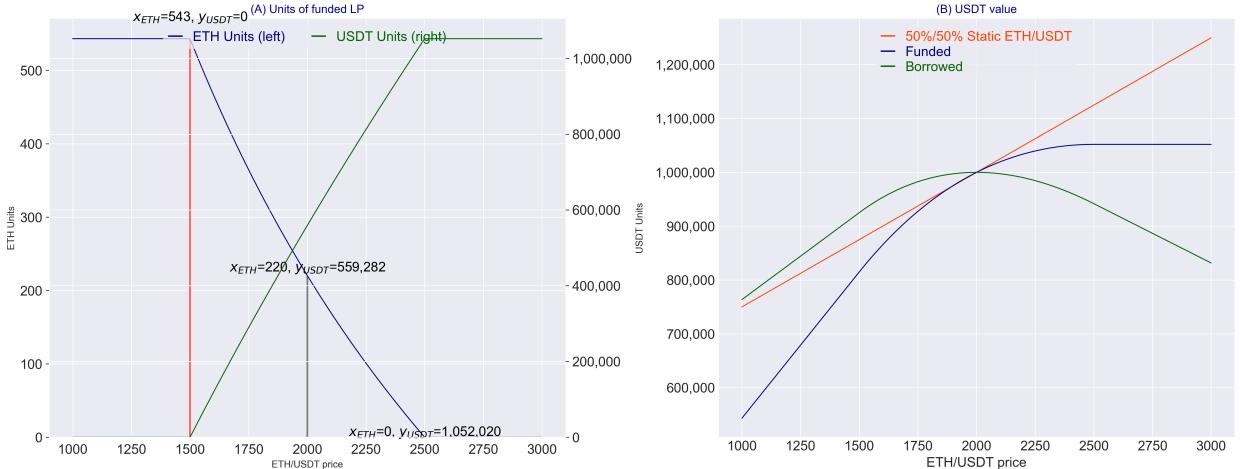


Figure 2: (A) ETH units (left y-axis) and USDT units (right y-axis) for LP on Uniswap V3. (B) USDT value of 50%/50% ETH/USDT portfolio, Funded LP position and Borrowed LP position. Uniswap V3 LP position is constructed using 1m USDT notional with $p_0 = 2000$, $p_a = 1500$, $p_b = 2500$.

Proof. See Appendix 6.2. \square

Corollary 3.4 (Payoff of IL protection claim for borrowed LP). *The payoff of IL protection claim against for funded LP in Eq (14) is given by the following compact formula*

$$Payoff^{borrowed}(p_t) = \frac{\frac{p_t}{\sqrt{f(p_t; p_a, p_b)}} + \sqrt{f(p_t; p_a, p_b)} - \frac{p_t}{\sqrt{f(p_0; p_a, p_b)}} - \sqrt{f(p_0; p_a, p_b)}}{\frac{p_0}{\sqrt{f(p_0; p_a, p_b)}} + \sqrt{f(p_0; p_a, p_b)} - \frac{p_0}{\sqrt{p_b}} - \sqrt{p_a}} \quad (47)$$

where $f(x; p_a, p_b) = \max(\min(x, p_b), p_a)$.

In subplot (A) of Figure (2), we show ETH units (left y-axis) and USDT units (right y-axis) for LP on Uniswap V3 with 1m USDT notional and $p_0 = 2000$, $p_a = 1500$, $p_b = 2500$. The initial LP units of (ETH, USDT) are (220, 559282). The red bar at $p = 1500$ shows LP units of (543, 0) with LP fully in ETH units when price falls below lower threshold p_a . The green bar at $p = 2500$ shows corresponding LP units of (0, 1052020) with LP fully in USDT units when price rises above upper threshold p_b . In subplot (B), we show USDT values of 50%/50% ETH/USDT portfolio, Funded LP positions and Borrowed LP positions. The value profile of funded LP resembles the profile of a covered call option (long ETH and short out-of-the-money call). The value of the borrowed LP resembles the payoff of a short straddle (short both at-the-money call and put).

In Figure (3) we show P&L profiles of borrowed and funded LPs as functions of ranges for Uniswap V3 and full range for Uniswap V2. For borrowed LPs, narrow ranges result in higher losses at same price levels. For funded LPs, narrower ranges result in higher downside losses and smaller upside potential.

3.3 Decomposition of the IL under Uniswap V3 into Simple Payoffs

First we derive the model independent decomposition of IL into payoffs of vanilla and “exotic” options. We will further apply this decompositions for model-based valuation of IL protection claims in Uniswap V3.

Proposition 3.5 (Decomposition of IL for Funded LP). *IL of funded LP can be decomposed into the three parts as follows:*

$$IL_{funded}^{(y)}(p_t) = u_0(p_t) + u_{1/2}(p_t) + u_1(p_t), \quad (48)$$

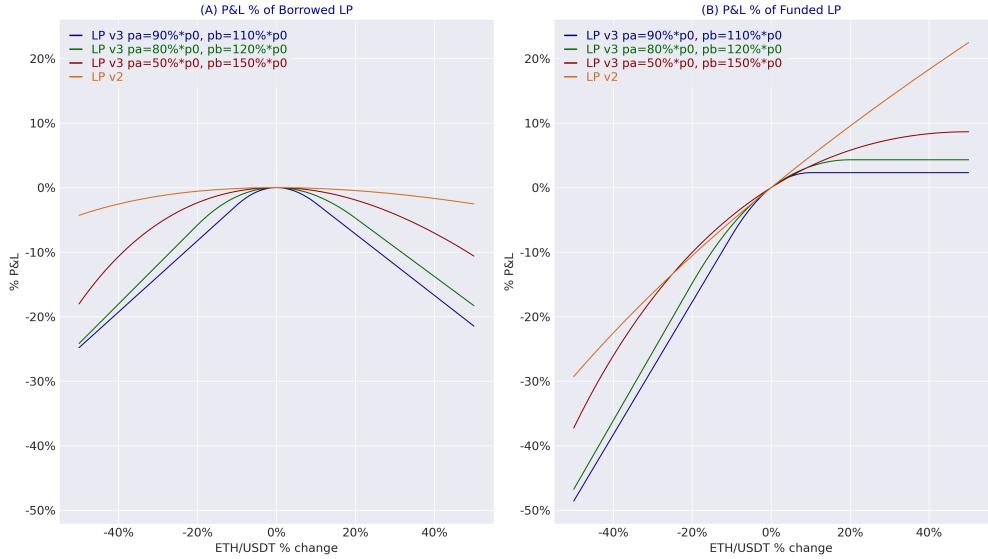


Figure 3: P&L % of hedged and funded LPs as functions of ranges for Uniswap V3 and full range for Uniswap V2. (A) P&L % of borrowed LP; (B) P&L % of funded LP. LP positions are constructed using 1m USDT notional with $p_0 = 2000$.

where $u_0(p_t)$, $u_{1/2}(p_t)$, and $u_1(p_t)$ are linear part, (exotic) square root price, and (vanilla) option part defined as follows

$$\begin{aligned} u_0(p_t) &= -\frac{1}{\sqrt{p_b}} p_t + \left(\frac{p_0}{\sqrt{p_b}} - 2\sqrt{p_0} \right) \\ u_{1/2}(p_t) &= \sqrt{p_t} \mathbb{1} \{p_a < p_t < p_b\} \\ u_1(p_t) &= -\frac{1}{\sqrt{p_a}} \max \{p_a - p_t, 0\} + \frac{1}{\sqrt{p_b}} \max \{p_t - p_b, 0\} + 2\sqrt{p_a} \mathbb{1} \{p_t \leq p_a\} + 2\sqrt{p_b} \mathbb{1} \{p_t \geq p_b\}. \end{aligned} \quad (49)$$

Proposition 3.6 (Decomposition of IL for Borrowed LP). *IL of borrowed LP can be decomposed into the three parts as follows:*

$$IL \text{ borrowed}^{(y)}(p_t) = u_0(p_t) + u_{1/2}(p_t) + u_1(p_t), \quad (50)$$

where $u_0(p_t)$, $u_{1/2}(p_t)$, and $u_1(p_t)$ are linear part, (exotic) square root price, and (vanilla) option part defined as follows

$$\begin{aligned} u_0(p_t) &= -\frac{1}{\sqrt{p_0}} p_t - \sqrt{p_0} \\ u_{1/2}(p_t) &= \sqrt{p_t} \mathbb{1} \{p_a < p_t < p_b\} \\ u_1(p_t) &= -\frac{1}{\sqrt{p_a}} \max \{p_a - p_t, 0\} + \frac{1}{\sqrt{p_b}} \max \{p_t - p_b, 0\} + 2\sqrt{p_a} \mathbb{1} \{p_t \leq p_a\} + 2\sqrt{p_b} \mathbb{1} \{p_t \geq p_b\}. \end{aligned} \quad (51)$$

Proof. See Appendix 6.4. □

To summarize, we note that the only difference between the decomposition of IL of funded LP in Eq (48) and borrowed LP in Eq (50) is given by the linear term $u_0(p_t)$ with the square root and option terms being the same. As a result, we can unify our results for ILs of funded LP and borrowed LP as defined in Eq (8) and (9), respectively, for Uniswap V3 for as follows.

Corollary 3.5 (Decomposition of IL for Funded and Borrowed LPs). *Using Eq (48) for IL of funded LP and Eq (50) for IL of borrowed LP we obtain*

$$\begin{aligned} IL \text{ funded}^{(y)}(p_t) &= u_0^{\text{funded}}(p_t) + u_{1/2}(p_t) + u_1(p_t), \\ IL \text{ borrowed}^{(y)}(p_t) &= u_0^{\text{borrowed}}(p_t) + u_{1/2}(p_t) + u_1(p_t), \end{aligned} \quad (52)$$

where

$$\begin{aligned} u_0^{\text{funded}}(p_t) &= -\frac{1}{\sqrt{p_b}} p_t + \left(\frac{p_0}{\sqrt{p_b}} - 2\sqrt{p_0} \right) \\ u_0^{\text{borrowed}}(p_t) &= -\frac{1}{\sqrt{p_0}} p_t - \sqrt{p_0} \\ u_{1/2}(p_t) &= \sqrt{p_t} \mathbb{1}\{p_a < p_t < p_b\} \\ u_1(p_t) &= -\frac{1}{\sqrt{p_a}} \max\{p_a - p_t, 0\} + \frac{1}{\sqrt{p_b}} \max\{p_t - p_b, 0\} + 2\sqrt{p_a} \mathbb{1}\{p_t \leq p_a\} + 2\sqrt{p_b} \mathbb{1}\{p_t \geq p_b\}. \end{aligned} \quad (53)$$

Corollary 3.6 (Payoff of IL protection claim for Uniswap V3 AMM). *Using definitions in Eq (14) and Eq (15) for payoffs of IL protection claim of funded and borrowed LP, respectively, at maturity time T along with decomposition of IL in Eq (52), we obtain*

$$\begin{aligned} Payoff^{\text{funded}}(p_T) &= -[u_0^{\text{funded}}(p_T) + u_{1/2}(p_T) + u_1(p_T)], \\ Payoff^{\text{borrowed}}(p_T) &= -[u_0^{\text{borrowed}}(p_T) + u_{1/2}(p_T) + u_1(p_T)], \end{aligned} \quad (54)$$

where functions u are defined in Eq (53).

4 Static Replication of Impermanent Loss with Vanilla Options

We derive a static replication of the IL at the fixed maturity time T using a portfolio of European call and put options. Accordingly, the IL can be hedged by buying a portfolio of traded options.

We introduce payoff functions of call and put options as follows⁵:

$$u^{\text{call}}(p, k) = (p - k)^+, \quad u^{\text{put}}(p, k) = (k - p)^+ \quad (55)$$

where k is the strike price k and p is the current price.

We note that [Deng et al. \(2023\)](#) and [Maire-Wunsch \(2024\)](#) obtain the following replication formula for the replication of funded P&L as defined in Eq (4) and its analytic expression for Uniswap V3 defined in Eq (41) (expressed using our notation)

$$\begin{aligned} \frac{1}{L} P\&L \text{ funded}^{(y)}(p_t) + 1 &= -\frac{1}{4} \int_{p_a}^{p_b} k^{-3/2} (u^{\text{put}}(p_t, k) + u^{\text{call}}(p_t, k)) dk \\ &\quad + \frac{1}{2\sqrt{p_a}} (u^{\text{call}}(p_t, p_a) - u^{\text{put}}(p_t, p_a)) + \frac{1}{2\sqrt{p_b}} (u^{\text{put}}(p_t, p_b) - u^{\text{call}}(p_t, p_b)) \end{aligned} \quad (56)$$

[Deng et al. \(2023\)](#) derive the replication formula (56) using the following representation of IL under Uniswap V3

$$\begin{aligned} \frac{1}{L} P\&L \text{ funded}^{(y)}(p_t) + 1 &= p_t \left(\frac{1}{\sqrt{p_t}} - \frac{1}{\sqrt{p_b}} \right)^+ - p_t \left(\frac{1}{\sqrt{p_t}} - \frac{1}{\sqrt{p_a}} \right)^+ \\ &\quad + (\sqrt{p_t} - \sqrt{p_a})^+ - (\sqrt{p_t} - \sqrt{p_b})^+ \end{aligned} \quad (57)$$

⁵We note that on crypto exchanges most options are the so-called inverse options with the payoff paid in the underlying token. There is a direct arbitrage-based equivalence between vanilla and inverse options, see [Alexander et al. \(2023\)](#) and [Lucic-Sepp \(2024\)](#) for details, so that our analysis follows the same logic when using inverse options.

where we apply our definition of nominal IL in Eq (43) and Carr-Madan representation (Carr-Madan (2001)). We note that, since the function on the left-hand side in (57) is not twice differentiable, strictly speaking the Carr-Madan representation does not apply. For completeness, in Appendix 6.7 we provide a derivation which only relies of existence of the generalized derivatives.

Decomposition in Eq (57) includes four exotic payoffs on the square root of the price. In contrast, we derive an alternative decomposition of IL of the funded LP in Eq (48) which decomposes the IL into one exotic payoff on the square root of the price, two payoffs of vanilla call and put options, and two payoffs of digital options. It is clear that the decomposition of the IL is not unique and can be done with different base payoff function. Our decomposition for IL of funded and borrowed LPs in Eq (48) and (50), respectively, is most suited for the model-dependent valuation of the claims for IL protection.

We note that there are two complications with replication formula (56) for practical usage. First, formula (56) assumes that there are strikes corresponding to lower and upper levels p_a and p_b . In practice, on Deribit exchange, BTC and ETH options are traded with the strike width of 1000 and 50 USD, respectively. Therefore, the practical application of formula (56) is very limited because it only applicable to LPs with ranges contained in strikes of traded options. Second, replication formula (56) requires to buy both calls and puts at the same strikes. Thus, for strikes smaller than p_t , both out-the-money puts and in-the-money calls are purchased and, for strikes above p_t , both in-the-money puts and out-the-money calls are purchased. In practice on Deribit exchange, the liquidity for in-the-money calls and puts is limited with much wider bid-ask spreads, so that implementation of formula (56) can be too cost-inefficient in reality.

Instead, we derive an alternative replication formula which only purchases out-out-the money puts for strikes below the current price and out-out-the money calls for strikes above the current price. We also note that our replication formula for a generic CFMM of AMM protocols in addition to CFMMs of Uniswap V2 or V3 protocols.

4.1 Replication of IL with Vanilla Options

We derive a replication portfolio for IL under a generic AMM protocol. We assume that IL $IL(p)$ is a function of the current price p and specified by Eq (4) for funded LP or by Eq (7) for borrowed LP. In particular, the IL for funded and borrowed LPs in Uniswap V2 is obtained using Eq (21) and (24), respectively. For Uniswap V3, the IL for funded and borrowed LPs is obtained using Eq (43) and (46), respectively. The corresponding IL for all these specifications is denoted by $IL(p)$.

Proposition 4.1 (Replication portfolio for generic LP). *We consider IL of a generic AMM as function of price $IL(p)$. We fix option maturity time T and consider a set of call and put options traded for this maturity time.*

Put side. *We assume a discrete grid of strikes \mathcal{K}^{put} and corresponding payoffs of put options \mathcal{U}^{put} :*

$$\mathcal{K}^{put} = (k_1, k_2, \dots, k_N), \quad \mathcal{U}^{put} = (u_1, u_2, \dots, u_N) \quad (58)$$

where $u_n = (k_n - p)^+$, $n = 1, \dots, N$, $k_{n-1} < k_n$ and $k_N \leq p_0$ with p_0 being T -forward price. We consider the replication portfolio of puts with weights w_n :

$$\Pi^{put} = \sum_{n=1}^N w_n u_n^{put} \quad (59)$$

We define the first-order derivative of the IL function at discrete strike points as follows:

$$\delta IL(k_n) = \frac{IL(k_n) - IL(k_{n-1})}{k_n - k_{n-1}} \quad (60)$$

Then the weights of put options for the replication portfolio in Eq (59) are computed by:

$$w_{n-1} = -(\delta IL(k_n) - \delta IL(k_{n-1})), \quad n = N, \dots, 3 \quad (61)$$

with $w_N = \delta IL(k_N)$ and with $w_1 = 0$.

Call side. We assume a discrete grid of strikes \mathcal{K} and corresponding payoffs of call options \mathcal{U}^{call} :

$$\mathcal{K}^{call} = (k_1, k_2, \dots, k_M), \quad \mathcal{U}^{call} = (u_1, u_2, \dots, u_M) \quad (62)$$

where $u_m = (p - k_m)^+$, $m = 1, \dots, M$, with $k_m > k_{m-1}$ and $k_1 \geq p_0$ being T -forward price.. We consider the hedging portfolio of calls with weights w_m :

$$\Pi^{call} = \sum_{m=1}^M w_m u_m^{call} \quad (63)$$

We compute the first-order derivative of the IL function $\delta IL(k_m)$ as in Eq (60).

Then the weights of call options for the replication portfolio in Eq (63) are computed by:

$$w_m = -(\delta IL(k_m) - \delta IL(k_{m-1})), \quad m = 2, \dots, M-1 \quad (64)$$

with $w_1 = \delta IL(k_1)$ and with $w_M = 0$.

Proof. See Appendix 6.5. □

Corollary 4.1 (Static portfolio for replication of the payoff of IL Protection claim). *The option replication portfolio for payoff of IL protection claim is given by:*

$$\Pi \equiv \Pi^{put} + \Pi^{call} = \sum_{n=1}^N w_n U_n^{put} + \sum_{m=1}^M w_m U_m^{call} \quad (65)$$

where the weights of put and call options are computed using Eq (60) and Eq (64), respectively, with IL function $IL(p)$ specified using Eq (28) and (54) for payoffs under Uniswap V2 and V3 respectively.

The cost of the replication portfolio is Π_0 computed using option prices observed at inception of a IL protection claim .

In Figure (4), we illustrate the application of formulas (59) and (63) for replicating of IL for borrowed Uniswap V3 LP using 1m USDT notional, $p_0 = 2000$ ETH/USDT with $p_a = 1500$ and $p_b = 2500$. We use strikes with widths of 50 USDT in alignment with ETH options traded on Deribit exchange (for options with maturity of less than 3 days, Deribit introduces new strikes with widths of 25). In subplot (A), we show the IL of the borrowed LP position, and the payoffs of replicating calls and puts portfolios (with negative signs to align with the P&L). In subplot (B), we show the residual computed as the difference between the IL and the payoff of the replication portfolios. In Subplot (C), we show the number of put and call option contracts for the replication portfolios.

From Eq (152) it is clear that the approximation error is zero at strikes in the grid, which is illustrated in subplot (B). The maximum value of the residual is 0.025% or 2.5 basis points, which is very small. A small approximation error with a similar magnitude will occur in case, p_0, p_a, p_b are not placed exactly at the strike grid.

5 Model-dependent Valuation of Protection Claims against IL

In this section we develop the model-dependent valuation and dynamics hedging of IL protection claims for Uniswap V2 and V3 protocols.

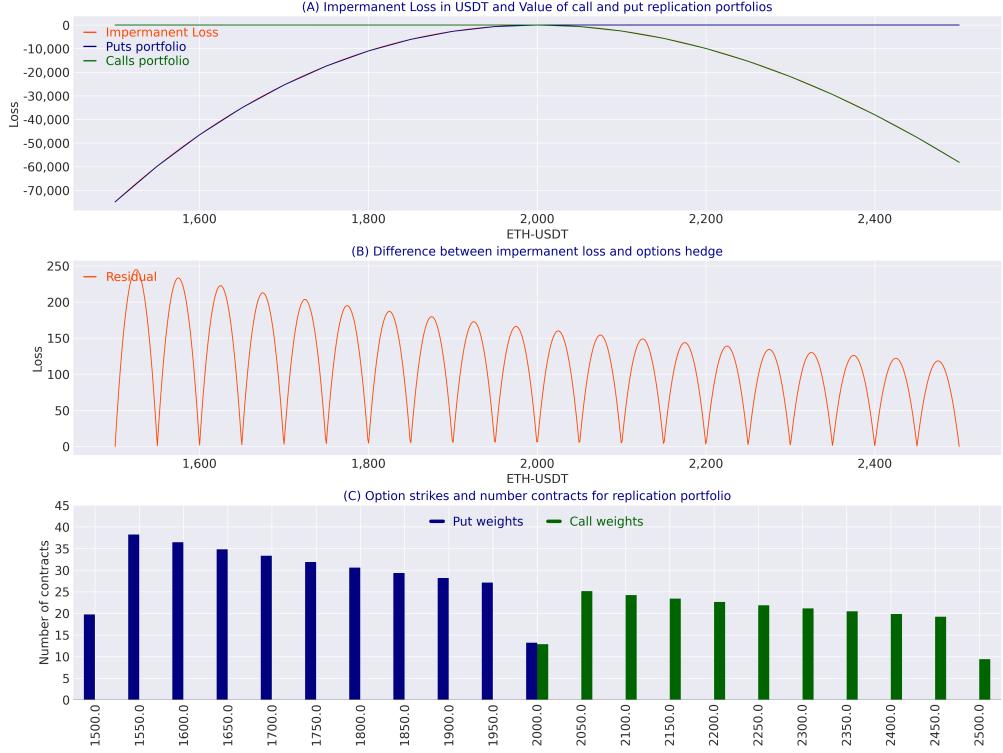


Figure 4: Replication of P&L of borrowed Uniswap V3 LP for allocation of 1m USDT notional, $p_0 = 2000$ ETH/USDT with $p_a = 1500$ and $p_b = 2500$. (A) Impermanent loss in USDT and (negative) values of replicating puts and call portfolios; (B) Residual, which is the spread between LP P&L and options replication portfolios; (C) Number of option contracts for put and calls portfolios.

5.1 Exponential Price Dynamics

We consider a continuous-time market with a fixed horizon date $T^* > 0$ and uncertainty modeled on probability space $(\Omega, \mathbb{F}, \mathbb{P})$ equipped with filtration $\mathbb{F} = \{\mathcal{F}_t\}_{0 \leq t \leq T^*}$. We assume that \mathbb{F} is right-continuous and satisfies usual conditions.

We introduce exponential price dynamics for price p_t :

$$p_t = p_0 e^{x_t}, \quad x_0 = 0, \quad (66)$$

where x_t is a stochastic process driving the log performance. We assume an arbitrage-free market with a risk-neutral measure \mathbb{Q} such that⁶:

$$\mathbb{E}^{\mathbb{Q}}[p_T | \mathcal{F}_t] \equiv p_t \mathbb{E}^{\mathbb{Q}}[e^{x_T} | \mathcal{F}_t] = p_t e^{(r-q)(T-t)} \quad (67)$$

where r is the discount rate and q is the borrow rate.

For hedging a IL protection claim, a trader needs to sell short the underlying token. Short-selling can be readily executed either through a centralized exchange (CEX) using perpetual futures or through borrowing the token on a DeFi protocol using stablecoins as collateral. In the CEX case, the borrow rate q is the negative of the funding rate reported by CEX (by the convention of crypto CEXes, the funding rate is the rate paid by traders with long positions). In the DeFi case, the borrow rate q is the accrued borrow rate.

As for the discount rate r , most DEXes and CEXes (such as Deribit when marking their listed options) assume zero discount rate. We would call r as a low risk opportunity cost available in

⁶This assumption is valid for complete markets. For incomplete markets, e.g. dynamics including stochastic volatility or jumps, we fix a martingale measure using specific risk preferences (see for an example Lewis (2000)). Sepp-Rakhmonov (2023) consider the existence of equivalent risk-neutral measures for stochastic volatility models.

DeFi with the risk being a potential hack of blockchain technology when deposited and staked assets could be appropriated. Staking of high quality stablecoins in top DeFi protocols would yield 1% – 2% in the current environment, which is far less than rates on government short-term bonds in traditional markets (4% – 5% as of June 2024).

When we value issued IL protection claims, we emphasize that the entry price p_0 is fixed at the time of initialising of an LP so that the entry price p_0 becomes a parameter of the IL formula in Eq (50) along with the lower and upper ranges. Once the valuation time advanced to time t and new price is observed we need to compute the expected IL using price p_t . We introduce the following decomposition at time t for the total log-performance x_T

$$x_T = x_t + x_\tau, \quad (68)$$

where x_t is the realised log-performance over period $(0, t]$ and x_τ is the stochastic performance over the period $(t, T]$. Here $\tau, \tau = T - t$, is the time-to-maturity. As a result, we model p_T using Eq (66) with (68) as follows

$$p_T = p_0 e^{x_t + x_\tau}. \quad (69)$$

5.2 Model-dependent Valuation

We consider the valuation of claims for IL protection as defined in Eqs (14) and (15). We focus on the valuation of these payoffs for Uniswap V2 and V3 using the payoff decomposition and by applying the exponential model in Eqs (66) and (69). Hereby, we fix maturity time T .

Corollary 5.1 (Payoff of IL protection claim in Uniswap V2 AMM under exponential model (66)). *Applying exponential model in Eq (69) to Eqs (28), we obtain*

$$\begin{aligned} \text{Payoff}^{funded}(x_T) &= 1 - e^{\frac{1}{2}(x_t + x_\tau)}, \\ \text{Payoff}^{borrowed}(x_T) &= \frac{1}{2} \left(e^{\frac{1}{2}(x_t + x_\tau)} - 1 \right)^2 \end{aligned} \quad (70)$$

Corollary 5.2 (Payoff of claim for IL protection for Uniswap V3 AMM under exponential model (66)). *Applying exponential model in Eq (69) to Eqs (54), we obtain*

$$\begin{aligned} \text{Payoff}^{funded}(x_\tau) &= - \left[u_0^{funded}(x_t + x_\tau) + u_{1/2}(x_t + x_\tau) + u_1(x_t + x_\tau) \right], \\ \text{Payoff}^{borrowed}(x_\tau) &= - \left[u_0^{borrowed}(x_t + x_\tau) + u_{1/2}(x_t + x_\tau) + u_1(x_t + x_\tau) \right], \end{aligned} \quad (71)$$

where the linear part is computed by

$$\begin{aligned} u_0^{funded}(x_t + x_\tau) &= -\frac{p_0}{\sqrt{p_b}} e^{x_t + x_\tau} + \left(\frac{p_0}{\sqrt{p_b}} - 2\sqrt{p_0} \right) \\ u_0^{borrowed}(x_t + x_\tau) &= -\sqrt{p_0} (e^{x_t + x_\tau} + 1), \end{aligned} \quad (72)$$

the square root part is computed by:

$$u_{1/2}(x_t + x_\tau) = \sqrt{p_0} \exp \left\{ \frac{1}{2}(x_t + x_\tau) \right\} \mathbb{1} \{ x_a < x_t + x_\tau < x_b \} \quad (73)$$

with $x_a = \ln(p_a/p_0)$ and $x_b = \ln(p_b/p_0)$.

The option part is computed by

$$\begin{aligned} u_1(x_t + x_\tau) &= \frac{1}{\sqrt{p_a}} \max \{ p_a - p_0 e^{x_t + x_\tau}, 0 \} - \frac{1}{\sqrt{p_b}} \max \{ p_0 e^{x_t + x_\tau} - p_b, 0 \} \\ &\quad - 2\sqrt{p_a} \mathbb{1} \{ x_t + x_\tau \leq x_a \} - 2\sqrt{p_b} \mathbb{1} \{ x_t + x_\tau \geq x_b \} \end{aligned} \quad (74)$$

Given payoff function in Eqs (70) and (71), the present value of the IL protection claim at time t is computed under the risk-neutral measure \mathbb{Q} by:

$$PV(t, p_t; T) = e^{-r\tau} \mathbb{E}^{\mathbb{Q}} [\text{Payoff}(x_\tau) | \mathcal{F}_t] \quad (75)$$

5.3 Valuation in Black-Scholes-Merton (BSM) model

We consider the BSM model with the price dynamics under the risk-neutral measure \mathbb{Q} given by

$$dp_t = \mu p_t dt + \sigma p_t dw_t, \quad p_0 = p \quad (76)$$

where $\mu = r - q$ is the risk-neutral drift, w_t is a Brownian motion with $w_0 = 0$. Accordingly, the log-performance $x_t = \log p_t/p_0$ is driven by

$$dx_t = \left(\mu - \frac{1}{2}\sigma^2 \right) dt + \sigma dw_t, \quad x_0 = 0 \quad (77)$$

and the distribution of x_τ is normal with mean $(\mu - \frac{1}{2}\sigma^2)\tau$ and volatility $\sigma\sqrt{\tau}$.

Proposition 5.1 (BSM Value of IL protection claim under Uniswap V2). *Applying model dynamics (76) to payoff functions in Eq (70) using valuation operator in Eq (75), we obtain*

$$\begin{aligned} PV^{funded}(t, p_t) &= e^{-r\tau} \left[1 - e^{\frac{1}{2}x_t} G\left(\tau; -\frac{1}{2}\right) \right], \\ PV^{borrowed}(t, p_t) &= \frac{1}{2}e^{-r\tau} \left[e^{(x_t + \mu\tau)} - 2e^{\frac{1}{2}x_t} G\left(\tau; -\frac{1}{2}\right) + 1 \right] \end{aligned} \quad (78)$$

where

$$G\left(\tau; -\frac{1}{2}\right) = \exp \left\{ \frac{1}{2} \left(\mu - \frac{1}{2}\sigma^2 \right) \tau + \frac{1}{8}\sigma^2\tau \right\} \quad (79)$$

Proof. For the dynamics in Eq (77), we apply the fact that the moment generation function $G(\tau; \phi)$ of x_τ is given by:

$$G(\tau; \phi) \equiv \mathbb{E}^{\mathbb{Q}}[\exp\{-\phi x_\tau\} | \mathcal{F}_t] = \exp \left\{ - \left(\mu - \frac{1}{2}\sigma^2 \right) \phi\tau + \frac{1}{2}\phi^2\sigma^2\tau \right\} \quad (80)$$

□

Proposition 5.2 (BSM value of IL protection claim under Uniswap V3). *The values of IL protection claims with payoff functions in Eq (71) under valuation operator in Eq (75) and BSM dynamics (77) are given by*

$$\begin{aligned} PV^{funded}(t, p_t) &= - \left[U_0^{funded}(p_t) + U_{1/2}(p_t) + U_1(p_t) \right], \\ PV^{borrowed}(t, p_t) &= - \left[U_0^{borrowed}(p_t) + U_{1/2}(p_t) + U_1(p_t) \right], \end{aligned} \quad (81)$$

where the linear part is computed by:

$$\begin{aligned} U_0^{funded}(t, p_t) &= e^{-q\tau} \frac{p_t}{\sqrt{p_b}} - e^{-r\tau} \left(\frac{p_0}{\sqrt{p_b}} - 2\sqrt{p_0} \right) \\ U_0^{borrowed}(t, p_t) &= \sqrt{p_0} \left(\frac{p_t}{p_0} e^{-q\tau} + e^{-r\tau} \right). \end{aligned} \quad (82)$$

Here $U_{1/2}(p_t)$ is the BSM value of the square root payoff in Eq (73) computed by:

$$U_{1/2}(p_t) = 2e^{-r\tau} \sqrt{p_t} \exp \left\{ \frac{1}{2}\mu\tau - \frac{1}{8}\sigma^2\tau \right\} \left(\mathbf{N} \left(\frac{\ln(p_b/p_t) - (r - q)\tau}{\sigma\sqrt{\tau}} \right) - \mathbf{N} \left(\frac{\ln(p_a/p_t) - (r - q)\tau}{\sigma\sqrt{\tau}} \right) \right), \quad (83)$$

where \mathbf{N} is the cpdf of normal random variable.

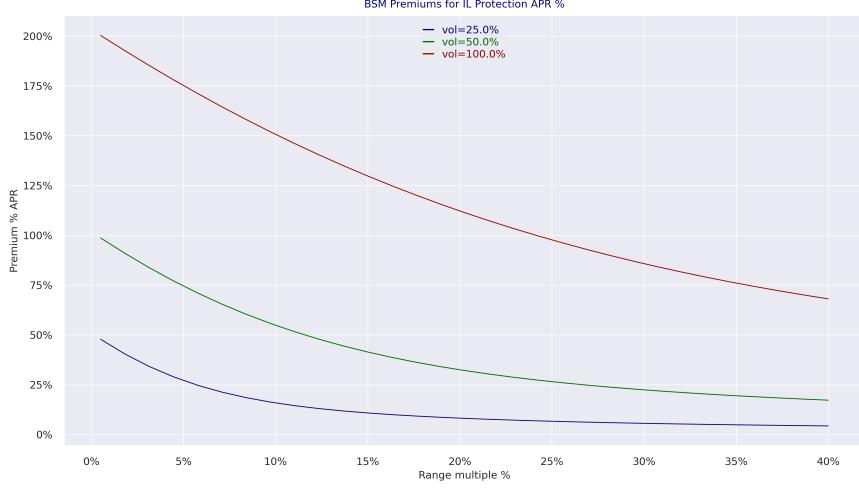


Figure 5: BSM premium annualised ($U^{borrower}(t, p_t)/T$) for borrowed LP computed using Eq (81) with time to maturity of two weeks $T = 14/365$ and notional of 1 USDT as function of the range multiple m such that $p_a(m) = e^{-m}p_0$ and $p_b(m) = e^m p_0$.

The option part is computed by

$$U_1(p_t) = \frac{1}{\sqrt{p_a}} O^{BSM}(p_t; p_a, -1) - \frac{1}{\sqrt{p_b}} O^{BSM}(p_t; p_b, +1) - 2\sqrt{p_a} D^{BSM}(p_t; p_a, -1) - 2\sqrt{p_b} D^{BSM}(p_t; p_b, +1). \quad (84)$$

Here $O^{BSM}(p_t; k, +1)$ and $O^{BSM}(p_t; k, -1)$ are the BSM prices of vanilla call and put with the strike price k and indicator $\omega \in \{+1, -1\}$ respectively:

$$O^{BSM}(p_t; k, \omega) = e^{-q\tau} p_t \mathbf{N}(\omega d_+(p_t, k)) - k e^{-r\tau} \mathbf{N}(\omega d_-(p_t, k)), \quad (85)$$

where

$$d_{\pm}(p_t, k) = \frac{\ln(p_t/k) + (r - q)\tau}{\sigma\sqrt{\tau}} \pm \frac{1}{2}\sigma\sqrt{\tau}, \quad (86)$$

and $D^{BSM}(p_t; k, +1)$ and $D^{BSM}(p_t; k, -1)$ are digital call and put options with strike price k , respectively:

$$D^{BSM}(p_t; k, \omega) = e^{-r\tau} \mathbf{N}(\omega d_-(p_t, k)). \quad (87)$$

Proof. See Appendix 6.6. □

In Figure (5), we show the annualised cost (APR) % for the cost of BSM hedge for the borrowed LP as a function of the range multiple m such that $p_a(m) = e^{-m}p_0$ and $p_b(m) = e^m p_0$. We use two weeks to maturity $T = 14/365$ and different values of log-normal volatility σ . All being the same, it is more expensive to hedge narrow ranges.

The protection seller must hedge the claim dynamically using option delta. As a result, we consider option delta under BSM model.

Corollary 5.3 (The delta of IL protection claim under BSM model).

$$\begin{aligned} \Delta^{funded}(t, p_t) &= - \left[\partial_p U_0^{funded}(p_t) + \partial_p U_{1/2}(p_t) + \partial_p U_1(p_t) \right], \\ \Delta^{borrowed}(t, p_t) &= - \left[\partial_p U_0^{borrowed}(p_t) + \partial_p U_{1/2}(p_t) + \partial_p U_1(p_t) \right], \end{aligned} \quad (88)$$

where

$$\partial_p U^{funded}(t, p_t) = e^{-q\tau} \frac{1}{\sqrt{p_b}}, \quad \partial_p U^{borrowed}(t, p_t) = e^{-q\tau} \frac{1}{\sqrt{p_0}}, \quad (89)$$

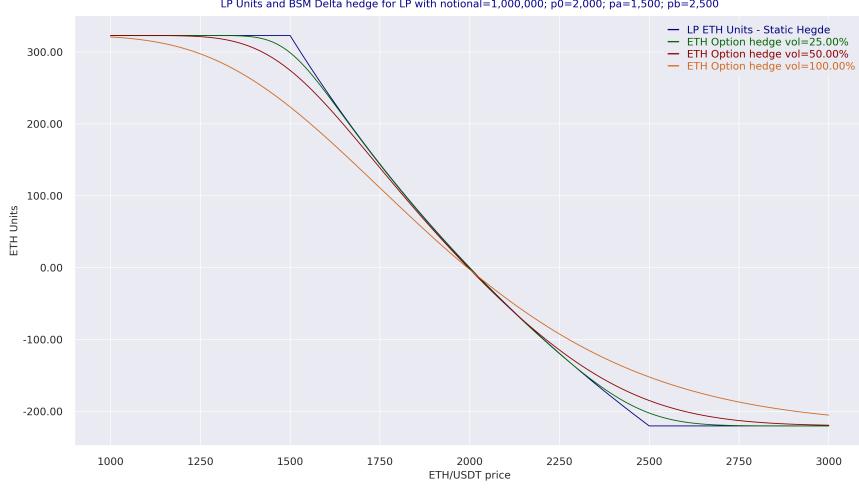


Figure 6: BSM delta for borrowed LP computed using Eq (88) with time to maturity of two weeks $T = 14/365$ for Uniswap V3 LP with $p_0 = 2000$, $p_a = 1500$, $p_b = 2500$, and notional of 1000000 USDT.

and

$$\begin{aligned} \partial_p U_{1/2}(p_t) &= \frac{1}{2\sqrt{p_t}} \exp \left\{ \frac{1}{2}\mu\tau - \frac{1}{8}\sigma^2\tau \right\} \left(\mathbf{N} \left(\frac{\ln(p_b/p_t) - (r-q)\tau}{\sigma\sqrt{\tau}} \right) - \mathbf{N} \left(\frac{\ln(p_a/p_t) - (r-q)\tau}{\sigma\sqrt{\tau}} \right) \right) \\ &\quad - \frac{1}{\sigma\sqrt{\tau}\sqrt{p_t}} \exp \left\{ \frac{1}{2}\mu\tau - \frac{1}{8}\sigma^2\tau \right\} \left(\mathbf{n} \left(\frac{\ln(p_b/p_t) - (r-q)\tau}{\sigma\sqrt{\tau}} \right) - \mathbf{n} \left(\frac{\ln(p_a/p_t) - (r-q)\tau}{\sigma\sqrt{\tau}} \right) \right). \end{aligned} \quad (90)$$

Finally

$$\begin{aligned} \partial_p U_1(p_t) &= \frac{1}{\sqrt{p_a}} \Delta_O(p_t; p_a, -1) - \frac{1}{\sqrt{p_b}} \Delta_O(p_t; p_b, +1) \\ &\quad - 2\sqrt{p_a} \Delta_D(p_t; p_a, -1) - 2\sqrt{p_b} \Delta_{DG}(p_t; p_b, +1) \end{aligned} \quad (91)$$

where $\Delta_O(p_t; k, +1)$ and $\Delta_D(p_t; k, -1)$ are Black-Scholes-Merton deltas of vanilla option and digital option, respectively, given by:

$$\Delta_O(p_t; k, \omega) = \mathbf{N}(\omega d_1), \quad \Delta_D(p_t; k, \omega) = \frac{\omega e^{-r\tau}}{p_t \sigma \sqrt{\tau}} \mathbf{n}(d_2) \quad (92)$$

Proof. By taking the partial derivative wrt p in Eq (81). \square

In Figure 6, we show BSM delta for borrowed LP computed using Eq (88) with time to maturity of two weeks $T = 14/365$ for borrowed LP in Uniswap V3 with $p_0 = 2000$, $p_a = 1500$, $p_b = 2500$, and notional of 1000000 USD. The initial units in the LP is (220.36, 559282.18) units of ETH and USDT, respectively. The static hedge is constructed by shorting 220.36 units of ETH. The first line labeled LP ETH-Units corresponds to the excess ETH units of borrowed LP with zero units at $p = p_0$ and being under-hedged on the downside and over-hedged on the upside. ETH option hedge shows the BSM delta computed using Eq (88) (the hedge for borrower LP is implemented using the negative sign of BSM delta). For high volatilities or large maturity times, BSM delta under-hedges near the range.

5.4 Valuation using Moment Generating Function

We consider a wide class of Markovian exponential dynamics in Eq (66) for which the moment generating function (MGF) for the log-return x_τ is available in closed-form. The closed-form

solution for the MGF is available under a wide class of models including jump-diffusions and diffusions with stochastic volatility. Thus, we can develop analytic solution for model-dependent valuation of IL protection under various models with analytic MGF.

We denote the MGF by $G(\tau; \phi)$, where ϕ is a complex-valued transform variable such that $\phi = \phi_r + i\phi_i$, $i = \sqrt{-1}$. Formally, the MGF solves the following problem:

$$G(\tau; \phi) = \mathbb{E}^{\mathbb{Q}} \left[e^{-\phi x_{\tau}} \mid \mathcal{F}_t \right] \quad (93)$$

where the expectation is computed using model dynamics under the risk-neutral measure \mathbb{Q} and \mathcal{F}_t is information set.

Using the MGF, the density of $x' = x_{\tau}$ denoted by $P(\tau, x; x')$ is computed by the Fourier inversion

$$P(\tau, x; x') = \frac{1}{\pi} \Re \left[\int_0^{\infty} \exp \{ \phi x' \} G(\tau; \phi) d\phi \right] \equiv \frac{1}{\pi} \Re \left[\int_0^{\infty} \exp \{ \phi(x' - x) \} E(\tau; \phi) d\phi \right] \quad (94)$$

with $d\phi \equiv d\phi_i$ and $G(\tau; \phi) \equiv e^{x\phi} E(\tau; \phi)$.

For valuation of the IL protection claim using MGF $G(\tau; \phi)$, we need to evaluate vanilla and digital options, and the square root payoff in Eq (71). First, we derive generic valuation method for payoff $u(x_{\tau})$. We evaluate the present value $U(\tau, x_t)$ at time t of the payoff function $u(x_{\tau})$ which is given similarly to Eq (75) by

$$U(\tau, x_t) = e^{-r\tau} \mathbb{E}^{\mathbb{Q}}[u(x_{\tau}) \mid \mathcal{F}_t] \quad (95)$$

Using Eq(94) we obtain

$$\begin{aligned} U(\tau, x) &= e^{-r\tau} \int_{-\infty}^{\infty} u(x') P(\tau, x; x') dx' \\ &= e^{-r\tau} \Re \left[\int_{-\infty}^{\infty} u(x') \left[\frac{1}{\pi} \int_0^{\infty} e^{\phi(x' - x)} E(\tau, \phi) d\phi \right] dx' \right] \\ &= \frac{1}{\pi} e^{-r\tau} \Re \left[\int_0^{\infty} \hat{u}(\phi) E(\tau, \phi) d\phi \right], \end{aligned} \quad (96)$$

where we assume that the inner integrals are finite to exchange the order of the integration. Here $\hat{u}(\phi)$ is the transformed payoff function defined by

$$\hat{u}(\phi) = e^{-\phi x} \int_{-\infty}^{\infty} e^{\phi x'} u(x') dx'. \quad (97)$$

Proposition 5.3 (Value of IL protection claim under Uniswap V2 under exponential model dynamics (76) with using the MGF). *Applying dynamics (76) to payoff functions in Eq (70) under expectation operator (75), we obtain*

$$\begin{aligned} PV^{funded}(t, p_t) &= e^{-r\tau} \left[1 - e^{\frac{1}{2}x_t} G\left(\tau; -\frac{1}{2}\right) \right], \\ PV^{borrowed}(t, p_t) &= \frac{1}{2} e^{-r\tau} \left[e^{(x_t + \mu\tau)} - 2e^{\frac{1}{2}x_t} G\left(\tau, -\frac{1}{2}\right) + 1 \right] \end{aligned} \quad (98)$$

Proof. We apply the definition of the MGF in Eq (93) and the martingale condition in Eq (67). \square

Accordingly, the IL under Uniswap V2 can be solved analytically for a wide class of models which is first concluded in Lipton (2024).

Proposition 5.4 (Valuation of IL protection claim under Uniswap V3). *Given the MGF $G(\tau; \phi)$ defined in Eq (94) for log-price x_τ in the exponential model in Eq (66) and payoff functions in Eq (71), we obtain the following valuation formula*

$$\begin{aligned} U^{funded}(t, p_t) &= - \left[U_0^{funded}(p_t) + U_{1/2}(p_t) + U_1(p_t) \right], \\ U^{borrowed}(t, p_t) &= - \left[U_0^{borrowed}(p_t) + U_{1/2}(p_t) + U_1(p_t) \right], \end{aligned} \quad (99)$$

where $U_0^{funded}(t, p_t)$ and $U_0^{borrowed}(t, p_t)$ are model independent linear parts computed as in BSM model in Eq (82)

$$\begin{aligned} U_0^{funded}(t, p_t) &= e^{-q\tau} \frac{p_t}{\sqrt{p_b}} - e^{-r\tau} \left(\frac{p_0}{\sqrt{p_b}} - 2\sqrt{p_0} \right) \\ U_0^{borrowed}(t, p_t) &= \sqrt{p_0} \left(\frac{p_t}{p_0} e^{-q\tau} + e^{-r\tau} \right). \end{aligned} \quad (100)$$

$U_{1/2}(p_t)$ is the square root claim computed using valuation formula (96) with transform in Eq (113).

The option part is computed by

$$\begin{aligned} u_1(p_t) &= \frac{1}{\sqrt{p_a}} O(p_t; p_a, -1) - \frac{1}{\sqrt{p_b}} O(p_t; p_b, +1) \\ &\quad - 2\sqrt{p_a} D(p_t; p_a, -1) - 2\sqrt{p_b} D(p_t; p_b, +1) \end{aligned} \quad (101)$$

where $O(p_t; k, +1)$ and $O(p_t; k, -1)$ are model values of call and put options, respectively, computed in Eq (103); $D(p_t; k, +1)$ and $D(p_t; k, -1)$ are model values of digital options computed in Eq (107).

We emphasize that for the computation of the payoff transform $\hat{u}(\phi)$ of capped option in Eq (106) we use $\phi = iy - 1/2$. Then for call and put digitals we use Eq (108) and (110) also with $\phi = iy - 1/2$. Finally the transform of the square root payoff in Eq (113) can be also evaluated using $\phi = iy - 1/2$. Thus for numerical implementation we fix the grid of $\{y\}$ and set $\phi = iy - 1/2$. We the compute MGF $E(\tau; \phi)$ and 5 transforms of payoffs along $\phi = iy - 1/2$, and compute 5 option values using Eq (96). Accordingly, the numerical implementation of pricing formula in Eq (99) is efficient.

Proof. Vanilla call and put options

We represent the put and call payoff functions using capped payoffs as

$$\begin{aligned} c(p_\tau, k) &= \max \{e^{x_\tau} - k, 0\} = p_\tau - \min \{e^{x_\tau}, k\}, \\ p(p_\tau, k) &= \max \{k - e^{x_\tau}, 0\} = k - \min \{e^{x_\tau}, k\} \end{aligned} \quad (102)$$

Accordingly, we need to evaluate option on the capped payoff using Eq(96) so that we can value vanilla calls and puts using

$$\begin{aligned} O(p_t; k, +1) &= e^{-q\tau} p_t - U(p_t, k) \\ O(p_t; k, -1) &= e^{-r\tau} k - U(p_t, k) \end{aligned} \quad (103)$$

where $U(p_t; k, \omega)$ is the value of the capped payoff.

Applying Eq (97) for capped payoff $u(x') = \min \{e^{x'}, k\}$, we obtain

$$\hat{u}(\phi) = e^{-\phi x} \left(\frac{e^{(\phi+1)k^*}}{\phi+1} - e^{k^*} \frac{1}{\Phi} e^{\phi k^*} \right) = e^{-\phi x} \left(-e^{(\phi+1)k^*} \frac{1}{(\phi+1)\phi} \right) = -ke^{-\phi x^*} \frac{1}{(\phi+1)\phi} \quad (104)$$

where $x^* = \ln(p_t/k) + \mu\tau$ is the log-moneyness, $k^* = \ln k - \mu\tau$ with the first integral being finite for $\Re[\phi] > -1$ and the second integral being finite for $\Re[\phi] < 0$. The integral (104) is finite for $-1 < \phi_r < 0$. Setting $\phi = iy - 1/2$, we derive:

$$\widehat{u}(\phi = iy - 1/2) = -ke^{-(iy-1/2)x^*} \frac{1}{(1/2 + iy)(-1/2 + iy)} = ke^{-(iy-1/2)x^*} \frac{1}{y^2 + 1/4} \quad (105)$$

Finally we obtain the valuation formula for capped payoff known as Lipton-Lewis formula (Lipton (2001), Lewis (2000)) as follows

$$U(p_t, k) = \frac{ke^{-r\tau}}{\pi} \Re \left[\int_0^\infty e^{-(iy-1/2)x^*} \frac{1}{y^2 + 1/4} E(\tau; \phi = iy - 1/2) dy \right], \quad (106)$$

where $x^* = \ln(p_t/k) + \mu\tau$ is log-moneyness.

Digital options

We represent the value of digital calls and puts as follows

$$\begin{aligned} D(p_\tau, k = x_b, +1) &= \mathbb{1}\{x_\tau \geq k\} = 1 - \mathbb{1}\{x_\tau < k\}, \\ D(p_\tau, k = x_a, -1) &= \mathbb{1}\{x_\tau \leq k\} = 1 - \mathbb{1}\{x_\tau > k\} \end{aligned} \quad (107)$$

We compute the transform of the payoff function in (96) for digital call as follows

$$\widehat{u}^c(\phi) = e^{-\phi x} \int_{-\infty}^\infty e^{\phi x'} U^c(x') dx' = e^{-\phi x} \int_{x_b}^\infty \exp\{\phi x'\} dx' = -e^{-\phi x} \frac{1}{\phi} \exp\{\phi x_b\} \quad (108)$$

where the integral converges if $\phi_r < 0$.

We compute the transform of the payoff function in (96) for digital put as follows

$$\widehat{u}^p(\phi) = e^{-\phi x} \int_{-\infty}^\infty e^{\phi x'} U^p(x') dx' = e^{-\phi x} \int_{-\infty}^{x_a} \exp\{\phi x'\} dx' = e^{-\phi x} \frac{1}{\phi} \exp\{\phi x_a\} \quad (109)$$

where the integral converges if $\phi_r > 0$.

We note that using (107), we can evaluate digital put as

$$D(p_\tau, k = x_a, \omega = -1) = 1 - D(p_\tau, k = x_a, \omega = +1) \quad (110)$$

so that we can use call transform in Eq (108) to evaluate both call and put digital with $\phi_r < 0$.

Square root payoff

We evaluate the value function in Eq (95) corresponding to the square root payoff in Eq (73) as follows

$$U_{1/2}(\tau, p_t) = e^{-r\tau} \sqrt{p_t} \mathbb{E}[u(x)] \quad (111)$$

where

$$u(x) = \exp\left\{\frac{1}{2}x\right\} \mathbb{1}\{x_a < x_\tau < x_b\} \quad (112)$$

and $x_a = \ln(p_a/p_t)$ and $x_b = \ln(p_b/p_t)$.

We compute the transform of the payoff function in (96) as follows

$$\begin{aligned} \widehat{u}(\phi) &= e^{-\phi x} \int_{-\infty}^\infty e^{\phi x'} u(x') dx' \\ &= e^{-\phi x} \int_{x_a}^{x_b} \exp\left\{\left(\phi + \frac{1}{2}\right)x'\right\} dx' \\ &= e^{-\phi x} \frac{1}{\left(\phi + \frac{1}{2}\right)} \left[\exp\left\{\left(\phi + \frac{1}{2}\right)x_b\right\} - \exp\left\{\left(\phi + \frac{1}{2}\right)x_a\right\} \right] \end{aligned} \quad (113)$$

where the integral converges for $\phi_r \in \mathbf{R}$. For $x_b \rightarrow \infty$ we set $\phi_r < 0$, and for $x_a \rightarrow -\infty$ we set $\phi_r > 0$. □

5.5 Application of Log-normal SV Model

We apply the log-normal SV model which can handle positive correlation between returns and volatility observed in price-volatility dynamics of digital assets (see [Sepp-Rakhmonov \(2023\)](#) for details). We consider price dynamics under the risk-neutral measure \mathbb{Q} for the spot price S_t and the instantaneous volatility σ_t as follows

$$\begin{aligned} dS_t &= r(t)S_t dt + \sigma_t S_t dW_t^{(0)}, \quad S_0 = S, \\ d\sigma_t &= (\kappa_1 + \kappa_2 \sigma_t)(\theta - \sigma_t)dt + \beta \sigma_t dW_t^{(0)} + \varepsilon \sigma_t dW_t^{(1)}, \quad \sigma_0 = \sigma \end{aligned} \quad (114)$$

where $W^{(0)}, W^{(1)}$ are uncorrelated Brownian motions, $\kappa_1 > 0$ and $\kappa_2 \geq 0$ are linear and quadratic mean-reversion rates respectively, $\theta > 0$ is the mean of the volatility, $\beta \in \mathbb{R}$ is the volatility beta which measures the sensitivity of the volatility to changes in the spot price, and $\varepsilon > 0$ is the volatility of residual volatility.

[Sepp-Rakhmonov \(2023\)](#) find the first-order solution to MGF defined in Eq (93) is given as follows

$$G(\tau; \phi) = \exp \{-\phi x\} E^{[1]}(\tau, \phi), \quad (115)$$

where $E^{[1]}$ is the exponential-affine function

$$E^{[1]}(\tau, \phi) = \exp \left\{ \sum_{k=0}^2 A^{(k)}(\tau; \phi)(\sigma - \theta)^k \right\}, \quad (116)$$

where $\vartheta^2 = \beta^2 + \varepsilon^2$ and vector function $\mathbf{A}(\tau) = \{A^{(k)}(\tau, \Phi)\}$, $k = 0, 1, 2$, solve the quadratic differential system as a function of τ :

$$A_\tau^{(k)} = \mathbf{A}^\top M^{(k)} \mathbf{A} + \left(L^{(k)} \right)^\top \mathbf{A} + H^{(k)}, \quad (117)$$

$$\begin{aligned} M^{(k)} &= \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{\theta^2 \vartheta^2}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & \theta \vartheta^2 & \theta^2 \vartheta^2 \\ 0 & \theta^2 \vartheta^2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{\vartheta^2}{2} & 2\theta \vartheta^2 \\ 0 & 2\theta \vartheta^2 & 2\theta^2 \vartheta^2 \end{pmatrix} \right\}, \\ L^{(k)} &= \left\{ \begin{pmatrix} 0 \\ -\theta^2 \beta \phi \\ \theta^2 \vartheta^2 \end{pmatrix}, \begin{pmatrix} 0 \\ -(\kappa_1 + \kappa_2 \theta) - 2\theta \beta \phi \\ 2(\theta \vartheta^2 - \theta^2 \beta \phi) \end{pmatrix}, \begin{pmatrix} 0 \\ -\beta \phi - \kappa_2 \\ \vartheta - 2(\kappa_1 + \kappa_2 \theta) - 4\theta \beta \phi \end{pmatrix} \right\}, \\ H^{(k)} &= \left\{ \frac{1}{2} \theta^2 (\phi^2 + \phi), \theta (\phi^2 + \phi - 2\psi), \frac{1}{2} (\phi^2 + \phi) \right\}, \end{aligned}$$

with the initial condition $\mathbf{A}(0) = (0, 0, 0)^\top$. The second order solution is provided in Theorem 4.6. [Sepp-Rakhmonov \(2023\)](#).

In Subplot (A) of Figure (7)⁷, we show the implied volatilities of the log-normal SV model for a range of volatility of residual volatility ε with $\beta = 0$. In Subplot (B), we show the premium APR for IL protection as a function of range multiple for a range of ε . We see that the model-value of IL protection is not very sensitive to tails of implied distribution (or, equivalently, to the convexity of the implied volatility). The reason is that the most of the value of IL protection is derived from the center of returns distribution.

⁷Github project <https://github.com/ArturSepp/StochVolModels> provides Python code for this computations using the log-normal SV model.

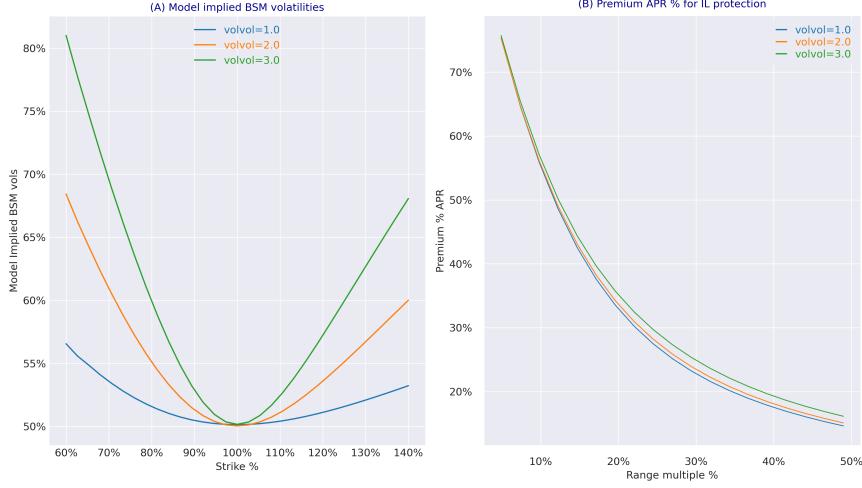


Figure 7: (A) BSM volatilities implied by log-normal SV model as function of volatility-of-volatility parameter ε for $\tau = 14/365$; (B) Premiums APR computed using log-normal SV model for borrowed LP as function of the range multiple m such that $p_a(m) = e^{-m}p_0$ and $p_b(m) = e^m p_0$. Other model parameters include $\sigma_0 = \theta = 0.50$, $\kappa_1 = 2.21$, $\kappa_2 = 2.18$, $\beta = 0.0$.

6 Conclusions

We have developed a unified approach for hedging of Impermanent Loss (IL) which arises when providing liquidity to Automated Market Making (AMM) Pools in blockchain ecosystem. We have introduced the two ways to create a liquidity provision (LP) including a funded LP (with the long initial exposure to underlying token) and a borrowed LP (with the zero initial exposure to the underlying token). We have applied Uniswap V2 and V3 protocols, which are constant function market maker (CFMM) most commonly employed by most of decentralized exchanges. We have shown that the IL can be represented with a non-linear function of the current spot price. As a result, using traditional methods of financial engineering, we can handle the valuation and risk-management of the IL protection claim which delivers the negative of the IL at a fixed maturity date.

First, we have derived a static replication approach for the IL arising from a generic constant function market maker (CFMM) using a portfolio of traded call and put options at a fixed maturity date. This approach allows for model-free replication of the IL when a liquid options market exists, which is the case for core digital assets including Bitcoin and Ethereum.

Second, for digital assets without a liquid options market, we have developed a model-based approach using the decomposition of the IL function into vanilla options, digital options, and an exotic square root payoff. We have derived a closed-form valuation formula for a wide class of price dynamics with tractable characteristic and moment generating functions (MGF) by means of Fourier transform.

Model-based valuation can be employed by a few crypto trading companies that currently sell over-the-counter IL protection claims. When using model-based dynamics delta-hedging for the replication of the payoff of the IL protection claim, the profit-and-loss (P&L) of the dynamic delta-hedging strategy will be primarily driven by the realised variance of the price process. Thus, the total P&L of a trading desk will be the difference between premiums received (from selling IL protection claims) and the variance realised through delta-hedging. Trading desk can employ our results for the analysis of price dynamics and hedging strategies which optimize their total P&L.

For liquidity providers, who buy IL protection claims for their LP position, the total P&L will be driven by the difference between accrued fees from LP positions and costs of IL protection claims. The cost of the IL protection claim can be estimated beforehand using either the cost of static options replicating portfolio or costs of buying IL protection from a trading desk. As a

result, liquidity providers can focus on selecting DEX pools and liquidity ranges where expected fees could exceed hedging costs. Thus, liquidity providers can apply our analysis optimal allocation to LP pools and for creating static replication portfolios using either traded options or assessing costs quoted by providers of IL protection⁸.

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⁸The empirical analysis of the profitability of LP strategies is a hot area in AMM-related literature, see for an example among others Heimbach *et al.* (2022), Cartea *et al.* (2023), Bergault *et al.* (2023), Cartea *et al.* (2024), Li *et al.* (2023), Li *et al.* (2024).

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Appendices

6.1 Proof of Proposition 3.3

Proof. Using Eq(4) with Eq(33) for $p_t \in (p_a, p_b)$:

$$\begin{aligned}
P\&L^{funded} &= (p_t x_t + y_t) - (p_0 x_0 + y_0) \\
&= L \left(p_t \left(\frac{1}{\sqrt{p_t}} - \frac{1}{\sqrt{p_b}} \right) + (\sqrt{p_t} - \sqrt{p_a}) \right) - L \left(p_0 \left(\frac{1}{\sqrt{p_0}} - \frac{1}{\sqrt{p_b}} \right) + (\sqrt{p_0} - \sqrt{p_a}) \right) \\
&= L \left[\left(\sqrt{p_t} - \frac{p_t}{\sqrt{p_b}} \right) + (\sqrt{p_t} - \sqrt{p_a}) - \left(\sqrt{p_0} - \frac{p_0}{\sqrt{p_b}} \right) - (\sqrt{p_0} - \sqrt{p_a}) \right] \\
&= L \left[\left(\sqrt{p_t} - \frac{p_t}{\sqrt{p_b}} \right) + \sqrt{p_t} - \left(\sqrt{p_0} - \frac{p_0}{\sqrt{p_b}} \right) - \sqrt{p_0} \right] \\
&= L \left[2(\sqrt{p_t} - \sqrt{p_0}) + \frac{p_0 - p_t}{\sqrt{p_b}} \right]
\end{aligned} \tag{118}$$

For $p_t \leq p_a$, using (34):

$$\begin{aligned}
P\&L^{funded} &= (p_t x_t + y_t) - (p_0 x_0 + y_0) \\
&= L \left[\left(p_t \left(\frac{1}{\sqrt{p_a}} - \frac{1}{\sqrt{p_b}} \right) + 0 \right) - \left(p_0 \left(\frac{1}{\sqrt{p_0}} - \frac{1}{\sqrt{p_b}} \right) + (\sqrt{p_0} - \sqrt{p_a}) \right) \right] \\
&= L \left[p_t \left(\frac{1}{\sqrt{p_a}} - \frac{1}{\sqrt{p_b}} \right) - \left(p_0 \left(\frac{1}{\sqrt{p_0}} - \frac{1}{\sqrt{p_b}} \right) + (\sqrt{p_0} - \sqrt{p_a}) \right) \right] \\
&= L \left[p_t \left(\frac{1}{\sqrt{p_a}} - \frac{1}{\sqrt{p_b}} \right) + \frac{p_0}{\sqrt{p_b}} - 2\sqrt{p_0} + \sqrt{p_a} \right]
\end{aligned} \tag{119}$$

For $p_t \geq p_b$, using Eq (35):

$$\begin{aligned}
P\&L^{funded} &= (p_t x_t + y_t) - (p_0 x_0 + y_0) \\
&= L \left[(0 + (\sqrt{p_b} - \sqrt{p_a})) - \left(p_0 \left(\frac{1}{\sqrt{p_0}} - \frac{1}{\sqrt{p_b}} \right) + (\sqrt{p_0} - \sqrt{p_a}) \right) \right] \\
&= L \left[(\sqrt{p_b} - \sqrt{p_a}) - \left(p_0 \left(\frac{1}{\sqrt{p_0}} - \frac{1}{\sqrt{p_b}} \right) + (\sqrt{p_0} - \sqrt{p_a}) \right) \right] \\
&= L \left[(\sqrt{p_b} - \sqrt{p_a}) + \frac{p_0}{\sqrt{p_b}} - 2\sqrt{p_0} + \sqrt{p_a} \right] \\
&= L \left[\sqrt{p_b} + \frac{p_0}{\sqrt{p_b}} - 2\sqrt{p_0} \right]
\end{aligned} \tag{120}$$

□

6.2 Proof to Proposition 3.4

Proposition 3.4. Using Eq(7) with Eq(33) for $p \in (p_a, p_b)$:

$$\begin{aligned}
P\&L^{borrowed} &= (p_t x_t + y_t) - (p_t x_0 + y_0) \\
&= L \left[\left(p_t \left(\frac{1}{\sqrt{p_t}} - \frac{1}{\sqrt{p_b}} \right) + (\sqrt{p_t} - \sqrt{p_a}) \right) - \left(p_t \left(\frac{1}{\sqrt{p_0}} - \frac{1}{\sqrt{p_b}} \right) + (\sqrt{p_0} - \sqrt{p_a}) \right) \right] \\
&= L \left[p_t \left(\frac{1}{\sqrt{p_t}} - \frac{1}{\sqrt{p_0}} \right) + (\sqrt{p_t} - \sqrt{p_0}) \right] \\
&= L \left[p_t \left(\frac{\sqrt{p_0} - \sqrt{p_t}}{\sqrt{p_0} \sqrt{p_t}} \right) + (\sqrt{p_t} - \sqrt{p_0}) \right] \\
&= L \left[\frac{\sqrt{p_0 p_t} - p_t}{\sqrt{p_0}} + (\sqrt{p_t} - \sqrt{p_0}) \right] \\
&= L \left[\frac{2\sqrt{p_0 p_t} - p_t - p_0}{\sqrt{p_0}} \right] \\
&= -\frac{L}{\sqrt{p_0}} (\sqrt{p_t} - \sqrt{p_0})^2 = -L\sqrt{p_0} \left(\sqrt{\frac{p_t}{p_0}} - 1 \right)^2
\end{aligned} \tag{121}$$

For $p \leq p_a$, using (34):

$$\begin{aligned}
P\&L^{borrowed} &= (p_t x_t + y_t) - (p_t x_0 + y_0) \\
&= L \left[\left(p_t \left(\frac{1}{\sqrt{p_a}} - \frac{1}{\sqrt{p_b}} \right) + 0 \right) - \left(p_t \left(\frac{1}{\sqrt{p_0}} - \frac{1}{\sqrt{p_b}} \right) + (\sqrt{p_0} - \sqrt{p_a}) \right) \right] \\
&= L \left[p_t \left(\frac{1}{\sqrt{p_a}} - \frac{1}{\sqrt{p_0}} \right) - (\sqrt{p_0} - \sqrt{p_a}) \right]
\end{aligned} \tag{122}$$

For $p \geq p_b$, using Eq (35):

$$\begin{aligned}
P\&L^{borrowed} &= (p_t x_t + y_t) - (p_t x_0 + y_0) \\
&= L \left[(0 + (\sqrt{p_b} - \sqrt{p_a})) - \left(p_t \left(\frac{1}{\sqrt{p_0}} - \frac{1}{\sqrt{p_b}} \right) + (\sqrt{p_0} - \sqrt{p_a}) \right) \right] \\
&= L \left[(\sqrt{p_b} - \sqrt{p_0}) - p_t \left(\frac{1}{\sqrt{p_0}} - \frac{1}{\sqrt{p_b}} \right) \right]
\end{aligned} \tag{123}$$

□

6.3 Proof to Proposition 3.5

Proof. We apply P&L of Funded LP in Eq (41) as follows.

$$P\&L_{funded}^{(y)}(p_t) = \begin{cases} 2(\sqrt{p_t} - \sqrt{p_0}) + \frac{p_0 - p_t}{\sqrt{p_b}} & p_t \in (p_a, p_b) \\ p_t \left(\frac{1}{\sqrt{p_a}} - \frac{1}{\sqrt{p_b}} \right) + \frac{p_0}{\sqrt{p_b}} - 2\sqrt{p_0} + \sqrt{p_a} & p_t \leq p_a \\ \sqrt{p_b} + \frac{p_0}{\sqrt{p_b}} - 2\sqrt{p_0} & p_t \geq p_b \end{cases} \tag{124}$$

We consider the range part in the interval $p_t \in (p_a, p_b)$ as follows.

$$\begin{aligned}
Range(p_t) &= 2(\sqrt{p_t} - \sqrt{p_0}) + \frac{p_0 - p_t}{\sqrt{p_b}} \\
&= -\frac{p_t}{\sqrt{p_b}} + 2\sqrt{p_t} + \left(\frac{p_0}{\sqrt{p_b}} - 2\sqrt{p_0} \right)
\end{aligned} \tag{125}$$

We extend the range part to the interval $p_t \in (0, +\infty)$ as follows

$$Range(p_t) \equiv -\frac{1}{\sqrt{p_b}} u_1(p_t) + 2u_{1/2}(p_t) + \left(\frac{p_0}{\sqrt{p_b}} - 2\sqrt{p_0} \right) u_0(p_t) \quad (126)$$

where

$$\begin{aligned} u_1(p_t) &= p_t \mathbb{1} \{p_a < p_t < p_b\} \\ u_{1/2}(p_t) &= \sqrt{p_t} \mathbb{1} \{p_a < p_t < p_b\} \\ u_0(p_t) &= \mathbb{1} \{p_a < p_t < p_b\} \end{aligned} \quad (127)$$

are extended on $p_t \in (0, +\infty)$ with zero values outside $p_t \in (p_a, p_b)$.

It is clear that $u_1(p_t)$ can be decomposed on $p_t \in (0, \infty)$ as a collar option position along with put and call digital:

$$u_1(p_t) = p_t + \max \{p_a - p_t, 0\} - \max \{p_t - p_b, 0\} - p_a \mathbb{1} \{p_t \leq p_a\} - p_b \mathbb{1} \{p_t \geq p_b\} \quad (128)$$

We evaluate $u_0(p_t)$ term as follows

$$u_0(p_t) = 1 - \mathbb{1} \{p_t \geq p_b\} - \mathbb{1} \{p_t \leq p_a\} \quad (129)$$

Then $u_{1/2}(p_t)$ is the only non-linear claim which needs model valuation

$$u_{1/2}(p_t) = \sqrt{p_t} \mathbb{1} \{p_a < p_t < p_b\} \quad (130)$$

Thus the range part on $p_t \in (0, \infty)$ becomes

$$\begin{aligned} Range(p_t) &\equiv -\frac{1}{\sqrt{p_b}} u_1(p_t) + 2u_{1/2}(p_t) + \left(\frac{p_0}{\sqrt{p_b}} - 2\sqrt{p_0} \right) u_0(p_t) \\ &= -\frac{1}{\sqrt{p_b}} [p_t + \max \{p_a - p_t, 0\} - \max \{p_t - p_b, 0\} - p_a \mathbb{1} \{p_t \leq p_a\} - p_b \mathbb{1} \{p_t \geq p_b\}] + 2u_{1/2}(p_t) \\ &\quad + \left(\frac{p_0}{\sqrt{p_b}} - 2\sqrt{p_0} \right) [1 - \mathbb{1} \{p_t \geq p_b\} - \mathbb{1} \{p_t \leq p_a\}] \\ &= -\frac{1}{\sqrt{p_b}} p_t + 2u_{1/2}(p_t) + \left(\frac{p_0}{\sqrt{p_b}} - 2\sqrt{p_0} \right) \\ &\quad - \frac{1}{\sqrt{p_b}} \max \{p_a - p_t, 0\} + \frac{1}{\sqrt{p_b}} \max \{p_t - p_b, 0\} + \frac{1}{\sqrt{p_b}} p_a \mathbb{1} \{p_t \leq p_a\} + \sqrt{p_b} \mathbb{1} \{p_t \geq p_b\} \\ &\quad - \left(\frac{p_0}{\sqrt{p_b}} - 2\sqrt{p_0} \right) [\mathbb{1} \{p_t \geq p_b\} + \mathbb{1} \{p_t \leq p_a\}] \\ &= -\frac{1}{\sqrt{p_b}} p_t + 2u_{1/2}(p_t) + \left(\frac{p_0}{\sqrt{p_b}} - 2\sqrt{p_0} \right) - \frac{1}{\sqrt{p_b}} \max \{p_a - p_t, 0\} + \frac{1}{\sqrt{p_b}} \max \{p_t - p_b, 0\} \\ &\quad + \left[\frac{1}{\sqrt{p_b}} p_a - \left(\frac{p_0}{\sqrt{p_b}} - 2\sqrt{p_0} \right) \right] \mathbb{1} \{p_t \leq p_a\} + \left[\sqrt{p_b} - \left(\frac{p_0}{\sqrt{p_b}} - 2\sqrt{p_0} \right) \right] \mathbb{1} \{p_t \geq p_b\} \\ &= -\frac{1}{\sqrt{p_b}} p_t + 2u_{1/2}(p_t) + \left(\frac{p_0}{\sqrt{p_b}} - 2\sqrt{p_0} \right) - \frac{1}{\sqrt{p_b}} \max \{p_a - p_t, 0\} + \frac{1}{\sqrt{p_b}} \max \{p_t - p_b, 0\} \\ &\quad + \left[\frac{p_a - p_0}{\sqrt{p_b}} + 2\sqrt{p_0} \right] \mathbb{1} \{p_t \leq p_a\} + \left[\frac{p_b - p_0}{\sqrt{p_b}} + 2\sqrt{p_0} \right] \mathbb{1} \{p_t \geq p_b\} \end{aligned} \quad (131)$$

Next we evaluate the put side for $p_t \leq p_a$:

$$\begin{aligned}
Put(p_t) &= p_t \left(\frac{1}{\sqrt{p_a}} - \frac{1}{\sqrt{p_b}} \right) + \frac{p_0}{\sqrt{p_b}} - 2\sqrt{p_0} + \sqrt{p_a} \\
&= \left[\frac{\sqrt{p_b} - \sqrt{p_a}}{\sqrt{p_a}\sqrt{p_b}} \right] (p_t \pm p_a) + \left(\frac{p_0}{\sqrt{p_b}} - 2\sqrt{p_0} + \sqrt{p_a} \right) \\
&= \left[\frac{\sqrt{p_b} - \sqrt{p_a}}{\sqrt{p_a}\sqrt{p_b}} \right] (-\{p_a - p_t\} + p_a) + \left(\frac{p_0}{\sqrt{p_b}} - 2\sqrt{p_0} + \sqrt{p_a} \right) \\
&= - \left[\frac{\sqrt{p_b} - \sqrt{p_a}}{\sqrt{p_a}\sqrt{p_b}} \right] \{p_a - p_t\} + \left[\frac{\sqrt{p_b} - \sqrt{p_a}}{\sqrt{p_a}\sqrt{p_b}} \right] p_a + \left(\frac{p_0}{\sqrt{p_b}} - 2\sqrt{p_0} + \sqrt{p_a} \right) \\
&= - \left[\frac{\sqrt{p_b} - \sqrt{p_a}}{\sqrt{p_a}\sqrt{p_b}} \right] \{p_a - p_t\} + \frac{\sqrt{p_b}\sqrt{p_a} - p_a}{\sqrt{p_b}} + \frac{p_0 - 2\sqrt{p_0}\sqrt{p_b} + \sqrt{p_b}\sqrt{p_a}}{\sqrt{p_b}} \\
&= - \left[\frac{\sqrt{p_b} - \sqrt{p_a}}{\sqrt{p_a}\sqrt{p_b}} \right] \{p_a - p_t\} + \frac{(p_0 - p_a) - 2\sqrt{p_0}\sqrt{p_b} + 2\sqrt{p_b}\sqrt{p_a}}{\sqrt{p_b}} \\
&= - \left[\frac{\sqrt{p_b} - \sqrt{p_a}}{\sqrt{p_a}\sqrt{p_b}} \right] \{p_a - p_t\} + \frac{p_0 - p_a}{\sqrt{p_b}} - 2(\sqrt{p_0} - \sqrt{p_a})
\end{aligned} \tag{132}$$

We further extend the last expression on $p_t \in (0, +\infty)$

$$Put(p_t) = - \left[\frac{\sqrt{p_b} - \sqrt{p_a}}{\sqrt{p_a}\sqrt{p_b}} \right] \max \{p_a - p_t, 0\} + \left[\frac{p_0 - p_a}{\sqrt{p_b}} - 2(\sqrt{p_0} - \sqrt{p_a}) \right] \mathbb{1} \{p_t \leq p_a\} \tag{133}$$

Next we extend the call side for $p_t \in (0, +\infty)$ as follows

$$Call(p_t) = \left[\sqrt{p_b} + \frac{p_0}{\sqrt{p_b}} - 2\sqrt{p_0} \right] \mathbb{1} \{p_t \geq p_b\} \tag{134}$$

Finally we sum up the three parts on $p_t \in (0, +\infty)$ as

$$\begin{aligned}
&Range(p_t) + Put(p_t) + Call(p_t) \\
&= - \frac{1}{\sqrt{p_b}} p_t + 2u_{1/2}(p_t) + \left(\frac{p_0}{\sqrt{p_b}} - 2\sqrt{p_0} \right) - \frac{1}{\sqrt{p_b}} \max \{p_a - p_t, 0\} + \frac{1}{\sqrt{p_b}} \max \{p_t - p_b, 0\} \\
&\quad + \left[\frac{p_a - p_0}{\sqrt{p_b}} + 2\sqrt{p_0} \right] \mathbb{1} \{p_t \leq p_a\} + \left[\frac{p_b - p_0}{\sqrt{p_b}} + 2\sqrt{p_0} \right] \mathbb{1} \{p_t \geq p_b\} \\
&\quad - \left[\frac{\sqrt{p_b} - \sqrt{p_a}}{\sqrt{p_a}\sqrt{p_b}} \right] \max \{p_a - p_t, 0\} + \left[\frac{p_0 - p_a}{\sqrt{p_b}} - 2(\sqrt{p_0} - \sqrt{p_a}) \right] \mathbb{1} \{p_t \leq p_a\} \\
&\quad + \left[\sqrt{p_b} + \frac{p_0}{\sqrt{p_b}} - 2\sqrt{p_0} \right] \mathbb{1} \{p_t \geq p_b\} \\
&= - \frac{1}{\sqrt{p_b}} p_t + 2u_{1/2}(p_t) + \left(\frac{p_0}{\sqrt{p_b}} - 2\sqrt{p_0} \right) \\
&\quad - \left[\frac{1}{\sqrt{p_b}} + \frac{\sqrt{p_b} - \sqrt{p_a}}{\sqrt{p_a}\sqrt{p_b}} \right] \max \{p_a - p_t, 0\} + \frac{1}{\sqrt{p_b}} \max \{p_t - p_b, 0\} \\
&\quad + \left[\frac{p_a - p_0}{\sqrt{p_b}} + 2\sqrt{p_0} + \frac{p_0 - p_a}{\sqrt{p_b}} - 2(\sqrt{p_0} - \sqrt{p_a}) \right] \mathbb{1} \{p_t \leq p_a\} \\
&\quad + \left[\frac{p_b - p_0}{\sqrt{p_b}} + 2\sqrt{p_0} + \sqrt{p_b} + \frac{p_0}{\sqrt{p_b}} - 2\sqrt{p_0} \right] \mathbb{1} \{p_t \geq p_b\} \\
&= - \frac{1}{\sqrt{p_b}} p_t + 2u_{1/2}(p_t) + \left(\frac{p_0}{\sqrt{p_b}} - 2\sqrt{p_0} \right) \\
&\quad - \frac{1}{\sqrt{p_a}} \max \{p_a - p_t, 0\} + \frac{1}{\sqrt{p_b}} \max \{p_t - p_b, 0\} + 2\sqrt{p_a} \mathbb{1} \{p_t \leq p_a\} + 2\sqrt{p_b} \mathbb{1} \{p_t \geq p_b\}
\end{aligned} \tag{135}$$

□

6.4 Proof of Proposition 6.4

Proof. Using P&L of borrowed LP in Eq (45) we obtain the following.

$$P\&L \text{ borrowed}^{(y)}(p_t) = \begin{cases} -\sqrt{p_0} \left(\sqrt{\frac{p_t}{p_0}} - 1 \right)^2 & p_t \in (p_a, p_b) \\ p_t \left(\frac{1}{\sqrt{p_a}} - \frac{1}{\sqrt{p_0}} \right) - (\sqrt{p_0} - \sqrt{p_a}) & p_t \leq p_a \\ (\sqrt{p_b} - \sqrt{p_0}) - p_t \left(\frac{1}{\sqrt{p_0}} - \frac{1}{\sqrt{p_b}} \right) & p_t \geq p_b \end{cases} \quad (136)$$

where $L = 1$

We split the payoff in the three parts.

The range part we evaluate for $p_t \in (p_a, p_b)$ as follows

$$\begin{aligned} Range(p_t) &= -\sqrt{p_0} \left(\frac{p_t}{p_0} - 2\sqrt{\frac{p_t}{p_0}} + 1 \right) \\ &\equiv \left(\frac{1}{\sqrt{p_0}} u_1(p_t) - 2u_{1/2}(p_t) + \sqrt{p_0} u_0(p_t) \right) \end{aligned} \quad (137)$$

It is clear that the first term can be decomposed for $p_t \in (0, \infty)$ as a collar option position :

$$u_1(p_t) = p_t + \max \{p_a - p_t, 0\} - \max \{p_t - p_b, 0\} - p_a \mathbb{1} \{p_t \leq p_a\} - p_b \mathbb{1} \{p_t \geq p_b\} \quad (138)$$

where $\mathbb{1}\{x\}$ is the indicator function.

We evaluate $u_0(p_t)$ term is as follows:

$$u_0(p_t) = 1 - \mathbb{1} \{p_t \geq p_b\} - \mathbb{1} \{p_t \leq p_a\} \quad (139)$$

Then $u_{1/2}(p_t)$ is the only non-linear claim which needs model valuation

$$u_{1/2}(p_t) = \sqrt{p_t} \mathbb{1} \{p_a < p_t < p_b\} \quad (140)$$

We evaluate the put side for $p_t \leq p_a$ as follows

$$\begin{aligned} Put(p_t) &= (\sqrt{p_0} - \sqrt{p_a}) - p_t \left(\frac{\sqrt{p_0} - \sqrt{p_a}}{\sqrt{p_0 p_a}} \right) \\ &= \left(\frac{\sqrt{p_0} - \sqrt{p_a}}{\sqrt{p_0 p_a}} \right) [\pm p_a + \sqrt{p_0 p_a} - p_t] \\ &= \left(\frac{\sqrt{p_0} - \sqrt{p_a}}{\sqrt{p_0 p_a}} \right) [(p_a - p_t) + (\sqrt{p_0 p_a} - p_a)] \\ &= \frac{\sqrt{p_0} - \sqrt{p_a}}{\sqrt{p_0 p_a}} [p_a - p_t] + \frac{\sqrt{p_0} - \sqrt{p_a}}{\sqrt{p_0 p_a}} (\sqrt{p_0 p_a} - p_a) \\ &= \frac{\sqrt{p_0} - \sqrt{p_a}}{\sqrt{p_0 p_a}} [p_a - p_t] + \frac{(\sqrt{p_0} - \sqrt{p_a})^2}{\sqrt{p_0}} \end{aligned} \quad (141)$$

As a result, the payoff on the put side can be written for $p_t \in (0, \infty)$ as follows

$$Put(p_t) = \frac{\sqrt{p_0} - \sqrt{p_a}}{\sqrt{p_0 p_a}} \max \{p_a - p_t, 0\} + \frac{(\sqrt{p_0} - \sqrt{p_a})^2}{\sqrt{p_0}} \mathbb{1} \{p_t \leq p_a\} \quad (142)$$

Second, we evaluate the call side for $p_t \geq p_b$ as follows

$$\begin{aligned}
Call(p_t) &= p_t \left(\frac{\sqrt{p_b} - \sqrt{p_0}}{\sqrt{p_0 p_b}} \right) - (\sqrt{p_b} - \sqrt{p_0}) \\
&= \left(\frac{\sqrt{p_b} - \sqrt{p_0}}{\sqrt{p_0 p_b}} \right) [p_t - \sqrt{p_0 p_b} \pm p_b] \\
&= \left(\frac{\sqrt{p_b} - \sqrt{p_0}}{\sqrt{p_0 p_b}} \right) [p_t - p_b] + \left(\frac{\sqrt{p_b} - \sqrt{p_0}}{\sqrt{p_0 p_b}} \right) [p_b - \sqrt{p_0 p_b}] \\
&= \left(\frac{\sqrt{p_b} - \sqrt{p_0}}{\sqrt{p_0 p_b}} \right) [p_t - p_b] + \frac{(\sqrt{p_b} - \sqrt{p_0})^2}{\sqrt{p_0}}
\end{aligned} \tag{143}$$

As a result, the payoff on the call side can be written for $p_t \in (0, \infty)$ as follows

$$Call(p_t) = \left(\frac{\sqrt{p_b} - \sqrt{p_0}}{\sqrt{p_0 p_b}} \right) \max \{p_t - p_b, 0\} + \frac{(\sqrt{p_b} - \sqrt{p_0})^2}{\sqrt{p_0}} \mathbb{1} \{p_t \geq p_b\} \tag{144}$$

Summing all together:

$$\begin{aligned}
&Range(p_t) + Put(p_t) + Call(p_t) \\
&= -2u_{1/2}(p_t) + \frac{1}{\sqrt{p_0}} [p_t + \max \{p_a - p_t, 0\} - \max \{p_t - p_b, 0\} - p_a \mathbb{1} \{p_t \leq p_a\} - p_b \mathbb{1} \{p_t \geq p_b\}] \\
&\quad + \sqrt{p_0} [1 - \mathbb{1} \{p_t \geq p_b\} - \mathbb{1} \{p_t \leq p_a\}] \\
&\quad + \frac{\sqrt{p_0} - \sqrt{p_a}}{\sqrt{p_0 p_a}} \max \{p_a - p_t, 0\} + \frac{(\sqrt{p_0} - \sqrt{p_a})^2}{\sqrt{p_0}} \mathbb{1} \{p_t \leq p_a\} \\
&\quad + \left(\frac{\sqrt{p_b} - \sqrt{p_0}}{\sqrt{p_0 p_b}} \right) \max \{p_t - p_b, 0\} + \frac{(\sqrt{p_b} - \sqrt{p_0})^2}{\sqrt{p_0}} \mathbb{1} \{p_t \geq p_b\} \\
&= -2u_{1/2}(p_t) + \frac{1}{\sqrt{p_0}} p_t + \sqrt{p_0} \\
&\quad + \left[\frac{\sqrt{p_0} - \sqrt{p_a}}{\sqrt{p_0 p_a}} + \frac{1}{\sqrt{p_0}} \right] \max \{p_a - p_t, 0\} \\
&\quad + \left[\frac{\sqrt{p_b} - \sqrt{p_0}}{\sqrt{p_0 p_b}} - \frac{1}{\sqrt{p_0}} \right] \max \{p_t - p_b, 0\} \\
&\quad + \left[\frac{(\sqrt{p_0} - \sqrt{p_a})^2}{\sqrt{p_0}} - \frac{p_a}{\sqrt{p_0}} - \sqrt{p_0} \right] \mathbb{1} \{p_t \leq p_a\} \\
&\quad + \left[\frac{(\sqrt{p_b} - \sqrt{p_0})^2}{\sqrt{p_0}} - \frac{p_b}{\sqrt{p_0}} - \sqrt{p_0} \right] \mathbb{1} \{p_t \geq p_b\} \\
&= -2r_1(p_t) + \frac{1}{\sqrt{p_0}} p_t + \sqrt{p_0} \\
&\quad + \frac{1}{\sqrt{p_a}} \max \{p_a - p_t, 0\} \\
&\quad - \frac{1}{\sqrt{p_b}} \max \{p_t - p_b, 0\} \\
&\quad - 2\sqrt{p_a} \mathbb{1} \{p_t \leq p_a\} \\
&\quad - 2\sqrt{p_b} \mathbb{1} \{p_t \geq p_b\}
\end{aligned} \tag{145}$$

The final result follows by collecting payoffs of vanilla puts and calls and digitals. \square

6.5 Proof to Proposition 4.1

Proof. For the put side, it is clear that

$$\Pi(K) = \sum_{n=1}^N w_n P_n(K) = \sum_{n=1}^N w_n \max(K_n - K, 0) \quad (146)$$

We define the first-order derivatives at discrete strike points as follows:

$$\begin{aligned} \delta IL(K_n) &= \frac{IL(K_n) - IL(K_{n-1})}{K_n - K_{n-1}} \\ \delta \Pi(K_n) &= \frac{\Pi(K_n) - \Pi(K_{n-1})}{K_n - K_{n-1}} \end{aligned} \quad (147)$$

In particular for $K_n \in \mathcal{K}$:

$$\begin{aligned} \Pi(K_n) &= \sum_{n' \geq n}^N w_{n'} P_{n'}(K_n) = \sum_{n' \geq n}^N w_{n'} (K_{n'} - K_n) \\ \Pi(K_{n-1}) &= \sum_{n' \geq n-1}^N w_{n'} (K_{n'} - K_{n-1}) = w_{n-1} (K_{n-1} - K_{n-1}) + \sum_{n' \geq n}^N w_{n'} (K_{n'} - K_{n-1}) = \sum_{n' \geq n}^N w_{n'} (K_{n'} - K_{n-1}) \end{aligned} \quad (148)$$

As a result:

$$\begin{aligned} \delta \Pi(K_n) &= \frac{1}{K_n - K_{n-1}} (\Pi(K_n) - \Pi(K_{n-1})) \\ &= \frac{1}{K_n - K_{n-1}} \left(\sum_{n' \geq n}^N w_{n'} (K_{n'} - K_n) - \sum_{n' \geq n}^N w_{n'} (K_{n'} - K_{n-1}) \right) \\ &= - \sum_{n' \geq n}^N w_{n'} \end{aligned} \quad (149)$$

and

$$\delta \Pi(K_n) - \delta \Pi(K_{n-1}) = -w_{n-1} \quad (150)$$

By piece-wise approximation over the interval $x \in (K_{n-1} - K_n)$:

$$\delta IL(x) = \delta L(K_n) - \frac{K_n - x}{K_n - K_{n-1}} (\delta IL(K_n) - \delta IL(K_{n-1})) \quad (151)$$

and

$$\begin{aligned} \delta \Pi(x) &= \delta \Pi(K_n) - \frac{K_n - x}{K_n - K_{n-1}} (\delta \Pi(K_n) - \delta \Pi(K_{n-1})) \\ &= \delta \Pi(K_n) - \frac{x - K_n}{K_{n-1} - K_n} (-w_{n-1}) \end{aligned} \quad (152)$$

The proof for the call side follows by analogy. \square

6.6 Proof of Proposition 5.2

Proof. We use Eq (71) as follows

$$\begin{aligned} U^{funded}(t, p_t) &= -e^{-r(T-t)} \mathbb{E}^{\mathbb{Q}} \left[u_0^{funded}(x_t + x_{\tau}) + u_{1/2}(x_t + x_{\tau}) + u_1(x_t + x_{\tau}) \right], \\ U^{borrowed}(t, p_t) &= -e^{-r(T-t)} \mathbb{E}^{\mathbb{Q}} \left[u_0^{borrowed}(x_t + x_{\tau}) + u_{1/2}(x_t + x_{\tau}) + u_1(x_t + x_{\tau}) \right], \end{aligned} \quad (153)$$

The square payoff in Eq (73) is computed using:

$$\begin{aligned} U_{1/2}(t, p_t) &= e^{-r(T-t)} \mathbb{E}^{\mathbb{Q}} \left[\sqrt{p_0 e^{x_t + x_{\tau}}} \mathbb{1}_{\{x_a < x_t + x_{\tau} < x_b\}} \right] \\ &= e^{-r(T-t)} \sqrt{p_t} \int_{-\infty}^{\infty} \exp \left\{ \frac{1}{2} \left(\mu - \frac{1}{2} \sigma^2 \right) \tau + \frac{1}{2} \sigma \sqrt{\tau} x \right\} \mathbb{1}_{\{x_a - x_t < x < p_b - x_t\}} \mathbf{n}(x) dx \\ &= e^{-r(T-t)} \sqrt{p_t} \int_{-d_{-(p_t, p_b)}}^{-d_{-(p_t, p_a)}} \exp \left\{ \frac{1}{2} \left(\mu - \frac{1}{2} \sigma^2 \right) \tau + \frac{1}{2} \sigma \sqrt{\tau} x \right\} \mathbf{n}(x) dx \\ &= e^{-r(T-t)} \sqrt{p_t} \exp \left\{ \frac{1}{2} \left(\mu - \frac{1}{2} \sigma^2 \right) \tau \right\} \int_{-d_{-(p_t, p_b)}}^{-d_{-(p_t, p_a)}} \exp \left\{ \frac{1}{2} \sigma \sqrt{\tau} x \right\} \mathbf{n}(x) dx \\ &= e^{-r(T-t)} \sqrt{p_t} \exp \left\{ \frac{1}{2} \left(\mu - \frac{1}{2} \sigma^2 \right) \tau \right\} (m(-d_{-(p_t, p_b)}) - m(-d_{-(p_t, p_a)})) \\ &= e^{-r(T-t)} \sqrt{p_t} \exp \left\{ \frac{1}{2} \mu \tau - \frac{1}{8} \sigma^2 \tau \right\} \left(\mathbf{N} \left(\frac{\ln(p_b/p_t) - (r-q)\tau}{\sigma \sqrt{\tau}} \right) - \mathbf{N} \left(\frac{\ln(p_a/p_t) - (r-q)\tau}{\sigma \sqrt{\tau}} \right) \right) \end{aligned} \quad (154)$$

where \mathbf{n} is normal pdf and

$$m(x) = \exp \left\{ \frac{1}{8} \sigma^2 \tau \right\} \mathbf{N} \left(x - \frac{1}{2} \sigma \sqrt{\tau} \right) \quad (155)$$

The option part is computed using the option pricing formulas for BSM model. \square

6.7 Carr-Madan representation

In Carr-Madan (2001) a representation of an arbitrary, twice-differentiable function (payoff) in terms of put and call payoffs was given. Here we derive the same representation relaxing the smoothness assumption, only requiring that its first derivative possesses the generalized derivative everywhere.

To this end, assume that $f : \mathbb{R} \mapsto \mathbb{R}$, is such that f' has generalized derivative at every point, and fix arbitrary $S, F \geq 0$. Then we have

$$\begin{aligned} f(S) &= f(F) + I_{\{S>F\}} \int_F^S f'(u) du - I_{\{S<F\}} \int_S^F f'(u) du \\ &= f(F) + I_{\{S>F\}} \int_F^S \left[f'(F) + \int_F^u f''(v) dv \right] du - I_{\{S<F\}} \int_S^F \left[f'(F) - \int_u^F f''(v) dv \right] du \\ &= f(F) + f'(F)(S-F) + I_{\{S>F\}} \int_F^S \int_F^u f''(v) dv du + I_{\{S<F\}} \int_S^F \int_u^F f''(v) dv du \\ &= f(F) + f'(F)(S-F) + \int_F^S f''^+ dv + \int_0^F f''^+ dv, \end{aligned} \quad (156)$$

where in the last step we used Fubini's theorem. (Note that the upper limit in the first integral in (156) can be replaced by infinity.)