CSE 105:
Computation

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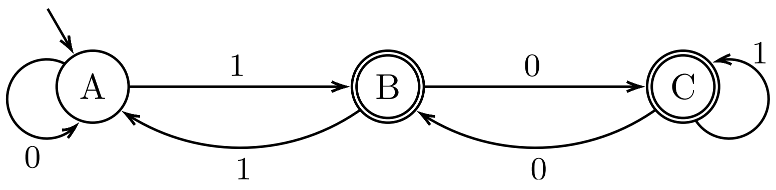
1 Deterministic Finite Automaton (DFA)

A machine consists different drawn in circles with names. Often a state drawn as a double circle is an “acceptive state,” and a plain circle indicates a rejective state. A machine receives a string consisted of ‘1’s and ‘0’s as input and the states change as the machine read through input digits. An arrow is used to indicate which state is to start with. See Example 1.1 for detailed information.

Example 1.1: A DFA

Let’s first look at the DFA below which starts at state A.



If the string “010110” is input to the machine, will it result in true or false? Will the State be acceptive or rejective?

$$\xrightarrow{010110}\boxed{\text{M}}\xrightarrow{1/0\text{(True/False, Accept / Reject)}}$$

There are two arrows leaving state A: one with a label reading ‘1’ which points to state B and one reading ‘0’ which goes back to state A itself. That means, if an input digit reads ‘1,’ the state changes to B, and if ‘0’ the state stays in A

Now step through the procedure:

1. The machine starts off at state A with input ‘0,’ which, as explained above, changes the state to A itself.
2. Next, the second digit ‘1’ is read so the state is changed to B.
3. The next difit ‘0’ makse the state B to switch to state C
4. Then state C reads ‘1’ so no state change occurs.
5. The next digit is ‘1’ again so the state remains still on C.
6. Last, the digit ‘0’ switches the state from C to B.

Thus the input string “010110” changes the machine to state B, which is an acceptive state.

Definition 1.1 DFA. *A DFA is a 5-turple*

$$M = (Q, \Sigma, \delta, s, F)$$

where

Q *is a finite set, for states*

Σ *is a finite set, for input alphabet*

s $\in Q$*, for start states*

F $\subseteq Q$*, for accepting states*

δ $Q \times \Sigma \mapsto Q$*, a function that specifies the transition between states*

Example 1.2: DFA table

According to definition 1.1, the machine in Example 1.1 can be denoted by

$$M = (Q, \Sigma, \delta, s, F)$$

where

- $Q = \{A, B, C\}$
- $\Sigma = \{0, 1\}$
- $s = \{A\}$
- $F = \{B, C\}$

And function δ can be described by the table below.

δ	0	1
A	A	B
B	C	A
C	B	C

Definition 1.2 f_M. *For an DFA $M = (Q, \Sigma, \delta, s, F)$, let*

$$f_M : \Sigma^* \mapsto \{\mathit{True}, \mathit{False}\}$$

where Σ^* is a set of string over Σ.

$$f_M(w) = \begin{cases} \mathit{True}, & \delta^*(s, w) \in F \\ \mathit{False}, & else \end{cases}$$

Definition 1.3 δ^*.

$$\delta^* : Q \times \Sigma^* \mapsto Q$$

which is an inductive function defined as

$$\begin{cases} \delta^*(q, \varepsilon) & = q \\ \delta^*(q, aw) & = \delta^*\left(\delta(q, a), w\right) \end{cases}$$

where $(q \in Q, a \in \Sigma, w \in \Sigma^*)$

Definition 1.4 Configuration.

$$Conf = Q \times \Sigma^*$$

Definition 1.5 Initial Configurations. *The initial configuration of a machine $I_M(w) \in Conf$*

$$I_M(w) = (s, w)$$

Definition 1.6 Final Configurations. *The final configuration of a machine $H_M(w) \subseteq Conf$*

$$H_M = \{(q, u) \mid q \in Q, u = \varepsilon\}$$

Definition 1.7 Machine Output. *The output of a machine is a function that either "True" or "False."*

$$O_M : H_M \mapsto \{\mathit{True}, \mathit{False}\}$$

defined as

$$O_M(q, \varepsilon) = \begin{cases} \mathit{True}, & if \quad q \in F \\ \mathit{False}, & if \quad q \notin F \end{cases}$$

In summary :

- F $\subseteq$ Q
- s $\in$ Q
- $\varepsilon : Q \times \Sigma \mapsto Q$

Example 1.3: Example 1.1 as configurations

With input "10010" write in mathematical language, the confiuration of machine in Example 1.1:

$$I_M(10010) = (A, 10010) \to (B, 0010) \to (C, 010) \to (B, 10) \to (A, 0) \to (A, \varepsilon) \in H_M$$

And thus the out put

$$O_F(A, \varepsilon) = \text{False}$$

.

The machine in fact will only accept integers that are ***not*** multiples of 3.

Definition 1.8 .

$$R_M \subseteq Conf \to_M = \{(q, aw), (\delta(q, a), w) \mid q \in Q, a \in \Sigma, w \in \Sigma^*\}$$

Definition 1.9 n's State's Configuration.

$$f'_n(w) = O_F(C_n)$$

e.g.

$$I_M(w) \to_M C_1 \to_M C_2 \to_M \cdots \to_M \in H_M$$

for example

$$\begin{aligned} L(M) &= \{w \in \Sigma^* \mid f_M(w) = \text{True}\} \\ L(M_1) &\neq \Sigma^* \\ 1001 &\notin 3 \times \mathbb{Z} \end{aligned}$$

Definition 1.10 Language of Machine M. *A subset of Σ^* of a DFA that contains all inputs to which the output of the machine is* True *is called the* ***language*** *of the machine.*

In other word, If A is the set of all strings that machine M accepts, we say that A is the ***language of machine*** M and write $L(M) = A$. (M ***recognizes*** A)

Definition 1.11 Regularity of Language. *$L \subseteq \Sigma^*$ is regular if*

$$\exists \text{DFA} M \mid L(M) = L$$

Which means, a DFA could ***recognize*** L. In short, given a regular language, there always exist a DFA could be draw.

Notice that

- ε(small epsilon) = ***empty string***
- Σ(big Sigma) = ***alphabet set***
- $\varepsilon^* = \{\,\varepsilon\,\}$
- Σ^* = $\{\,\varepsilon, 1, 0, 10, 101, \cdots\} = \{\,0, 1\,\}^*$,

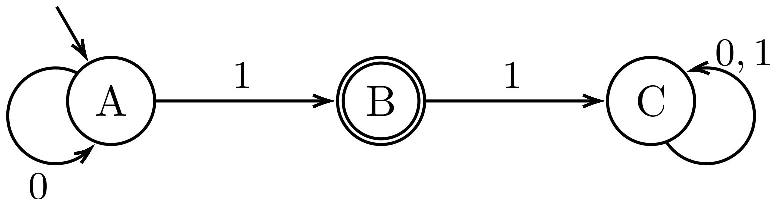
Definition 1.12 Operations on Languages.

Example 1.4: Which of the follwing languages are regular?

Given that and which of the following languages are regular?

- $L_1 = \{ w \in \{0,1\}^* \mid w \text{ is a power of } 2 \}$, and
- $L_2 = \{ w \in \{0,1\}^* \mid w \text{ is a power of } 3 \}$.

L_1 is regular while L_2 is not. A binary number that is a power of 2 consists of only one 1 and all other digits should be 0s. A DFA that recognizes the language would be



Complement $L^C = \{ w \in \Sigma^* \mid w \notin L \}$

Union $L_1 \cup L_2 = \{ w \in \Sigma^* \mid w \in L_1 \vee w \in L_2 \}$

Intersection $L_1 \cap L_2 = \{ w \mid w \in L_1 \wedge \in L_2 \}$

Concatenation $L_1 \cdot L_2 = \{ w_1 \cdot w_2 \mid w \in L_1, w_2 \in L_2 \}$

Theorem 1.1 . *$\mathbb{R}$ is closed under complement.*

Example 1.5: If L is regular, is L^C also regular?

Yes.

Proof of Theorem 1.1. Let $L \in \mathbb{R}$, prove $L^C \in \mathbb{R}$:

By definition,

$$\exists M = (Q, \Sigma, \delta, s, F) \text{ s.t. } L(M) = L.$$

Let $M' = (Q, \Sigma, \delta, s, F^C)$,
then $L(M') = L(M)^C = L^C$.
$L^C \in \mathbb{R}$ because $L^C = L(M')$. □

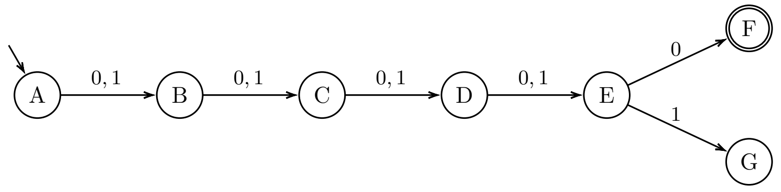
Example 1.6: $\forall L_1, L_2$
$L_1 \in \mathbb{R} \vee L_2 \in \mathbb{R} \implies L_1 \cup L_2 \in \mathbb{R}$

Yes, $\mathbb{R}$ is closed under union.

Example 2.1: How many states needed?

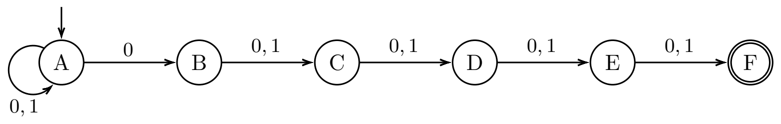
$$L = \{\, w \in \{\,0,1\,\}^* \mid \text{ the } 5^{th} \text{ (from the left) of } w \text{ is } 0 \,\}$$

We will need 7 states



What if

$$L = \{\, w \in \{\,0,1\,\}^* \mid \text{ the } 5^{th} \text{ from the } \textbf{\textit{r}}\text{ight) of } w \text{ is } 0 \,\}$$



Notice this is a **DFA**

2 Nondeterministic Finite Automaton (NFA)

Definition 2.1 NFA. *An NFA is a* 5*-tuple*

$$N = (Q, \Sigma, \delta, s, F)$$

where

- Q *and* Σ *are finite sets*
- $s \in Q$
- $F \subseteq Q$
- $\delta \colon Q \times \Sigma_\varepsilon$[1] $\mapsto \mathbb{P}(Q)$
  $\delta(A, \varepsilon) = \{\, H \,\}$ $\delta(D, \varepsilon) = \emptyset$

In DFA, only one output; In NFA, many possible ways from the same input.

$C_0, C_1, \ldots, C_n$ Computation is a sequence of Configurations, such that $C_0 = I(w)$ $\forall i, (C_i, C_{i+1} \in \mathbb{R})$ $[C_i \mapsto C_{i+1}], C_n \in H$

[1] $\Sigma_\varepsilon = \Sigma \cup \{\, \varepsilon \,\}$

$\boxed{\text{Accepting if } O(C_n) = \text{ True Rejecting if } O(C_n) = \text{ False}}$

Definition 2.2 Language of NFA.

$$L(N) = \{\, w \mid \exists \textit{accepting computation on input } w \,\}$$

Theorem 2.1 .

$$\forall N = (\, Q, \Sigma, \delta, s, F\,),\ \exists \text{DFA} \quad M = (\, Q', \Sigma', \delta', s', F'\,) \textit{s.t.} L(N) = L(M)$$

Proof of Theorem 2.1.

$$\begin{aligned}
Q' &= \mathbb{P}(Q) \\
F' &= \{\, A \subseteq Q \mid A \cap F \neq \varnothing \,\} \\
s' &= E(\{\, s \,\}) = \left\{ q \in Q \mid \exists s \xrightarrow{\varepsilon} q_1 \xrightarrow{\varepsilon} q_2 \xrightarrow{\varepsilon} q_3 \cdots \xrightarrow{\varepsilon} q_n \right\} \\
\delta'(A, a) &= E(\bigcup_{q \in A} \delta(q, a))
\end{aligned}$$

□