CHAPTER 14
Related Rates

14.1 The top of a 25-foot ladder, leaning against a vertical wall is slipping down the wall at the rate of 1 foot per second. How fast is the bottom of the ladder slipping along the ground when the bottom of the ladder is 7 feet away from the base of the wall?

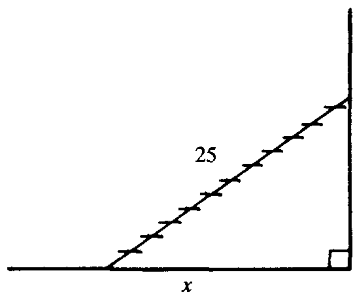

Fig. 14-1

▌ Let y be the distance of the top of the ladder from the ground, and let x be the distance of the bottom of the ladder from the base of the wall (Fig. 14-1). By the Pythagorean theorem, $x^2 + y^2 = (25)^2$. Differentiating with respect to time t, $2x \cdot D_t x + 2y \cdot D_t y = 0$; so, $x \cdot D_t x + y \cdot D_t y = 0$. The given information tells us that $D_t y = -1$ foot per second. (Since the ladder is sliding down the wall, y is decreasing, and, therefore, its derivative is negative.) When $x = 7$, substitution in $x^2 + y^2 = (25)^2$ yields $y^2 = 576$, $y = 24$. Substitution in $x \cdot D_t x + y \cdot D_t y = 0$ yields: $7 \cdot D_t x + 24 \cdot (-1) = 0$, $D_t x = \frac{24}{7}$ feet per second.

14.2 A cylindrical tank of radius 10 feet is being filled with wheat at the rate of 314 cubic feet per minute. How fast is the depth of the wheat increasing? (The volume of a cylinder is $\pi r^2 h$, where r is its radius and h is its height.)

▌ Let V be the volume of wheat at time t, and let h be the depth of the wheat in the tank. Then $V = \pi(10)^2 h$. So, $D_t V = 100\pi \cdot D_t h$. But we are given that $D_t V = 314$ cubic feet per minute. Hence, $314 = 100\pi \cdot D_t h$, $D_t h = 314/(100\pi)$. If we approximate π by 3.14, then $D_t h = 1$. Thus, the depth of the wheat is increasing at the rate of 1 cubic foot per minute.

14.3 A 5-foot girl is walking toward a 20-foot lamppost at the rate of 6 feet per second. How fast is the tip of her shadow (cast by the lamp) moving?

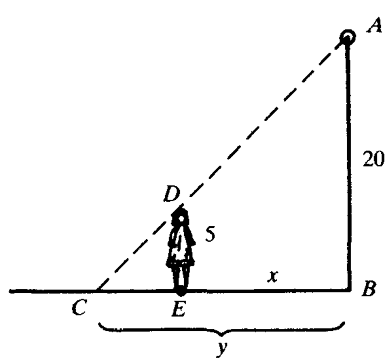

Fig. 14-2

▌ Let x be the distance of the girl from the base of the post, and let y be the distance of the tip of her shadow from the base of the post (Fig. 14-2). $\triangle ABC$ is similar to $\triangle DEC$. Hence, $AB/DE = y/(y-x)$, $\frac{20}{5} = y/(y-x)$, $4 = y/(y-x)$, $4y - 4x = y$, $3y = 4x$. Hence, $3 \cdot D_t y = 4 \cdot D_t x$. But, we are told that $D_t x = -6$ feet per second. (Since she is walking toward the base, x is decreasing, and $D_t x$ is negative.) So $3 \cdot D_t y = 4 \cdot (-6)$, $D_t y = -8$. Thus the tip of the shadow is moving at the rate of 8 feet per second toward the base of the post.

14.4 Under the same conditions as in Problem 14.3, how fast is the length of the girl's shadow changing?

▌ Use the same notation as in Problem 14.3. Let ℓ be the length of her shadow. Then $\ell = y - x$. **Hence,**

$D_t\ell = D_ty - D_tx = (-8) - (-6) = -2$. Thus, the length of the shadow is decreasing at the rate of 2 feet per second.

14.5 A rocket is shot vertically upward with an initial velocity of 400 feet per second. Its height s after t seconds is $s = 400t - 16t^2$. How fast is the distance changing from the rocket to an observer on the ground 1800 feet away from the launching site, when the rocket is still rising and is 2400 feet above the ground?

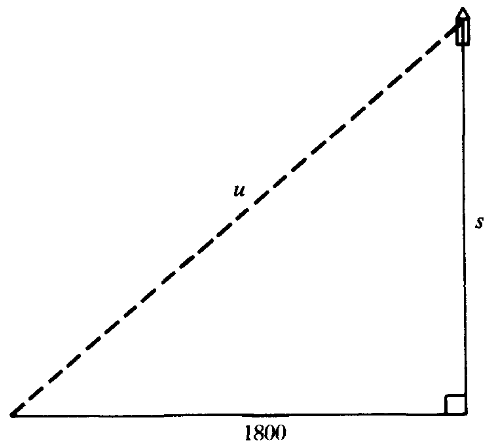

Fig. 14-3

Let u be the distance from the rocket to the observer, as shown in Fig. 14-3. By the Pythagorean theorem, $u^2 = s^2 + (1800)^2$. Hence, $2u \cdot D_tu = 2s \cdot D_ts$, $u \cdot D_tu = s \cdot D_ts$. When $s = 2400$, $u^2 = (100)^2 \cdot (900)$, $u = 100 \cdot 30 = 3000$. Since $s = 400t - 16t^2$, when $s = 2400$, $2400 = 400t - 16t^2$, $t^2 - 25t + 150 = 0$, $(t-10)(t-15) = 0$. So, on the way up, the rocket is at 2400 feet when $t = 10$. But, $D_ts = 400 - 32t$. So, when $t = 10$, $D_ts = 400 - 32 \cdot 10 = 80$. Substituting in $u \cdot D_tu = s \cdot D_ts$, we obtain $3000 \cdot D_tu = 2400 \cdot 80$, $D_tu = 64$. So the distance from the rocket to the observer is increasing at the rate of 64 feet per second when $t = 10$.

14.6 A small funnel in the shape of a cone is being emptied of fluid at the rate of 12 cubic centimeters per second. The height of the funnel is 20 centimeters and the radius of the top is 4 centimeters. How fast is the fluid level dropping when the level stands 5 centimeters above the vertex of the cone? (Remember that the volume of a cone is $\frac{1}{3}\pi r^2 h$.)

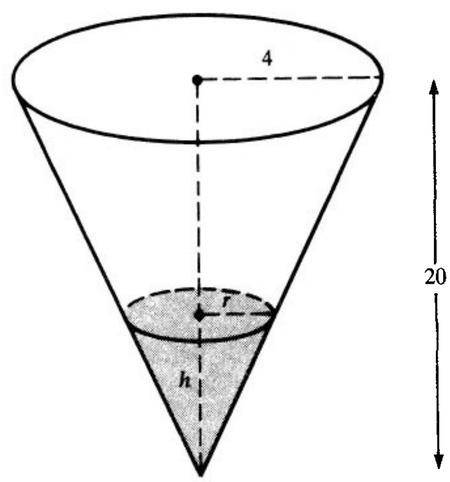

Fig. 14-4

The fluid in the funnel forms a cone with radius r, height h, and volume V. By similar triangles, $r/4 = h/20$, $r = h/5$. So, $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(h/5)^2 h = \frac{1}{75}\pi h^3$. Hence, $D_tV = \frac{1}{25}\pi h^2 \cdot D_th$. We are given that $D_tV = -12$, since the fluid is leaving at the rate of 12 cubic centimeters per second. Hence, $-12 = \frac{1}{25}\pi h^2 \cdot D_th$, $-300 = \pi h^2 \cdot D_th$. When $h = 5$, $-300 = \pi \cdot 25 \cdot D_th$, $D_th = -12/\pi$, or approximately -3.82 centimeters per second. Hence, the fluid level is dropping at the rate of about 3.82 centimeters per second.

14.7 A balloon is being inflated by pumped air at the rate of 2 cubic inches per second. How fast is the diameter of the balloon increasing when the radius is $\frac{1}{2}$ inch?

$V = \frac{4}{3}\pi r^3$. So, $D_tV = 4\pi r^2 \cdot D_tr$. We are told that $D_tV = 2$. So, $2 = 4\pi r^2 \cdot D_tr$. When $r = \frac{1}{2}$, $2 = 4\pi(\frac{1}{4}) \cdot D_tr$, $D_tr = 2/\pi$. Let d be the diameter. Then, $d = 2r$, $D_td = 2 \cdot D_tr = 2 \cdot (2/\pi) = 4/\pi \approx 1.27$. So, the diameter is increasing at the rate of about 1.27 inches per second.

14.8 Oil from an uncapped well in the ocean is radiating outward in the form of a circular film on the surface of the water. If the radius of the circle is increasing at the rate of 2 meters per minute, how fast is the area of the oil film growing when the radius is 100 meters.

The area $A = \pi r^2$. So, $D_tA = 2\pi r \cdot D_tr$. We are given that $D_tr = 2$. Hence, when $r = 100$, $D_tA = 2\pi \cdot 100 \cdot 2 = 400\pi$, which is about $1256\ \mathrm{m}^2/\mathrm{min}$.

14.9 The length of a rectangle of constant area 800 square millimeters is increasing at the rate of 4 millmeters per second. What is the width of the rectangle at the moment the width is decreasing at the rate of 0.5 millimeter per second?

The area $800 = \ell w$. Differentiating, $0 = \ell \cdot D_tw + w \cdot D_t\ell$. We are given that $D_t\ell = 4$. So, $0 = \ell \cdot D_tw + 4w$. When $D_tw = -0.5$, $0 = -0.5\ell + 4w$, $4w = 0.5\ell$. But $\ell = 800/w$. So, $4w = 0.5(800/w) = 400/w$, $w^2 = 100$, $w = 10\ \mathrm{mm}$.

14.10 Under the same conditions as in Problem 14.9, how fast is the diagonal of the rectangle changing when the width is 20 mm?

As in the solution of Problem 14.9. $0 = \ell \cdot D_tw + 4w$. Let u be the diagonal. Then $u^2 = w^2 + \ell^2$, $2u \cdot D_tu = 2w \cdot D_tw + 2\ell \cdot D_t\ell$, $u \cdot D_tu = w \cdot D_tw + \ell \cdot D_t\ell$. When $w = 20$, $\ell = 800/w = 40$. Substitute in $0 = \ell \cdot D_tw + 4w$: $0 = 40 \cdot D_tw + 80$, $D_tw = -2$. When $w = 20$, $u^2 = (20)^2 + (40)^2 = 2000$, $u = 20\sqrt{5}$. Substituting in $u \cdot D_tu = w \cdot D_tw + \ell \cdot D_t\ell$, $20\sqrt{5} \cdot D_tu = 20 \cdot (-2) + 40 \cdot 4 = 120$, $D_tu = 6\sqrt{5}/5 \approx 2.69\ \mathrm{mm/s}$.

14.11 A particle moves on the hyperbola $x^2 - 18y^2 = 9$ in such a way that its y-coordinate increases at a constant rate of 9 units per second. How fast is its x-coordinate changing when $x = 9$?

$2x \cdot D_tx - 36y \cdot D_ty = 0$, $x \cdot D_tx = 18y \cdot D_ty$. We are given that $D_ty = 9$. Hence, $x \cdot D_tx = 18y \cdot 9 = 162y$. When $x = 9$, $(9)^2 - 18y^2 = 9$, $18y^2 = 72$, $y^2 = 4$, $y = \pm 2$. Substituting in $x \cdot D_tx = 162y$, $9 \cdot D_tx = \pm 324$, $D_tx = \pm 36$ units per second.

14.12 An object moves along the graph of $y = f(x)$. At a certain point, the slope of the curve is $\frac{1}{2}$ and the x-coordinate of the object is decreasing at the rate of 3 units per second. At that point, how fast is the y-coordinate of the object changing?

$y = f(x)$. By the chain rule, $D_ty = f'(x) \cdot D_tx$. Since $f'(x)$ is the slope $\frac{1}{2}$ and $D_tx = -3$, $D_ty = \frac{1}{2} \cdot (-3) = -\frac{3}{2}$ units per second.

14.13 If the radius of a sphere is increasing at the constant rate of 3 millimeters per second, how fast is the volume changing when the surface area $4\pi r^2$ is 10 square millimeters?

$V = \frac{4}{3}\pi r^3$. Hence, $D_tV = 4\pi r^2 \cdot D_tr$. We are given that $D_tr = 3$. So, $D_tV = 4\pi r^2 \cdot 3$ When $4\pi r^2 = 10$, $D_tV = 30\ \mathrm{mm}^3/\mathrm{s}$.

14.14 What is the radius of an expanding circle at a moment when the rate of change of its area is numerically twice as large as the rate of change of its radius?

$A = \pi r^2$. Hence, $D_tA = 2\pi r \cdot D_tr$. When $D_tA = 2 \cdot D_tr$, $2 \cdot D_tr = 2\pi r \cdot D_tr$, $1 = \pi r$, $r = 1/\pi$.

14.15 A particle moves along the curve $y = 2x^3 - 3x^2 + 4$. At a certain moment, when $x = 2$, the particle's x-coordinate is increasing at the rate of 0.5 unit per second. How fast is its y-coordinate changing at that moment?

$D_ty = 6x^2 \cdot D_tx - 6x \cdot D_tx = 6x \cdot D_tx(x - 1)$. When $x = 2$, $D_tx = 0.5$. So, at that moment, $D_ty = 12(0.5)(1) = 6$ units per second.

14.16 A plane flying parallel to the ground at a height of 4 kilometers passes over a radar station R (Fig. 14-5). A short time later, the radar equipment reveals that the distance between the plane and the station is 5 kilometers and that the distance between the plane and the station is increasing at a rate of 300 kilometers per hour. At that moment, how fast is the plane moving horizontally?

▮ At time t, let x be the horizontal distance of the plane from the point directly over R, and let u be the distance between the plane and the station. Then $u^2 = x^2 + (4)^2$. So, $2u \cdot D_t u = 2x \cdot D_t x$, $u \cdot D_t u = x \cdot D_t x$. When $u = 5$, $(5)^2 = x^2 + (4)^2$, $x = 3$, and we are also told that $D_t u$ is 300. Substituting in $u \cdot D_t u = x \cdot D_t x$, $5 \cdot 300 = 3 \cdot D_t x$, $D_t x = 500$ kilometers per hour.

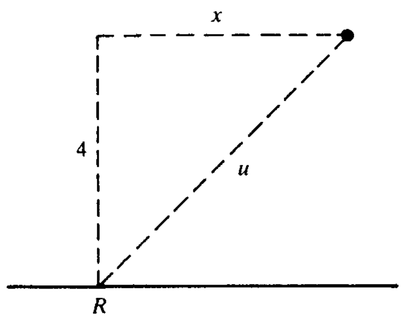

Fig. 14-5

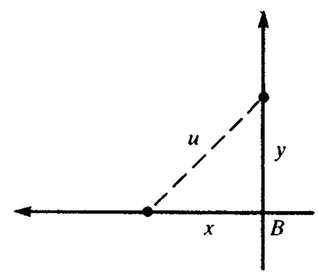

Fig. 14-6

14.17 A boat passes a fixed buoy at 9 a.m. heading due west at 3 miles per hour. Another boat passes the same buoy at 10 a.m. heading due north at 5 miles per hour. How fast is the distance between the boats changing at 11:30 a.m.?

▮ Refer to Fig. 14-6. Let the time t be measured in hours after 9 a.m. Let x be the number of miles that the first boat is west of the buoy at time t, and let y be the number of miles that the second boat is north of the buoy at time t. Let u be the distance between the boats at time t. For any time $t \ge 1$, $u^2 = x^2 + y^2$. Then $2u \cdot D_t u = 2x \cdot D_t x + 2y \cdot D_t y$, $u \cdot D_t u = x \cdot D_t x + y \cdot D_t y$. We are given that $D_t x = 3$ and $D_t y = 5$. So, $u \cdot D_t u = 3x + 5y$. At 11:30 a.m. the first boat has travelled $2\frac{1}{2}$ hours at 3 miles per hour; so, $x = \frac{15}{2}$. Similarly, the second boat has travelled at 5 miles per hour for $1\frac{1}{2}$ hours since passing the buoy; so, $y = \frac{15}{2}$. Also, $u^2 = (\frac{15}{2})^2 + (\frac{15}{2})^2 = \frac{225}{2}$, $u = 15/\sqrt{2}$. Substituting in $u \cdot D_t u = 3x + 5y$, $(15/\sqrt{2}) \cdot D_t u = 3 \cdot \frac{15}{2} + 5 \cdot \frac{15}{2} = 60$, $D_t u = 4\sqrt{2} \approx 5.64$ miles per hour.

14.18 Water is pouring into an inverted cone at the rate of 3.14 cubic meters per minute. The height of the cone is 10 meters, and the radius of its base is 5 meters. How fast is the water level rising when the water stands 7.5 meters above the base?

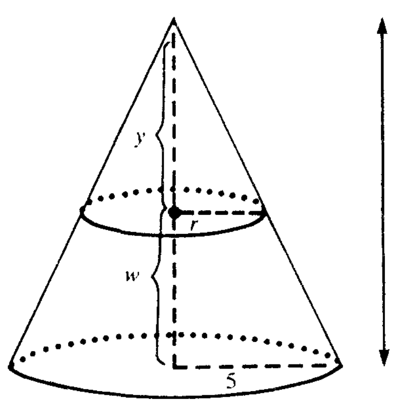

Fig. 14-7

▮ Let w be the level of the water above the base, and let r be the radius of the circle that forms the surface of the water. Let $y = 10 - w$. Then y is the height of the cone-shaped region above the water (see Fig. 14-7). So, the volume of that cone is $V_1 = \frac{1}{3}\pi r^2 y$. The total volume of the conical container is $V_2 = \frac{1}{3}\pi(5)^2 \cdot 10 = 250\pi/3$. Thus, the total volume of the water is $V = V_2 - V_1 = 250\pi/3 - \pi r^2 y/3$. By similar triangles, $10/5 = y/r$, $y = 2r$. So, $V = 250\pi/3 - \pi r^2(2r)/3 = 250\pi/3 - 2\pi r^3/3$. Hence, $D_t V = -2\pi r^2 \cdot D_t r$. We are given that $D_t V = 3.14$. So, $3.14 = -2\pi r^2 \cdot D_t r$. Thus, $D_t r = -3.14/2\pi r^2$. When the water stands 7.5 meters in the cone, $w = 7.5$, $y = 10 - 7.5 = 2.5$ $r = \frac{1}{2}y = 1.25$. So $D_t r = -3.14/2\pi(1.25)^2$. $D_t y = 2 \cdot D_t r = -3.14/\pi(1.25)^2$, $D_t w = -D_t y = 3.14/\pi(1.25)^2 \approx \frac{16}{25} = 0.64$ m/min.

14.19 A particle moves along the curve $y = x^2 + 2x$. At what point(s) on the curve are the x- and y-coordinates of the particle changing at the same rate?

▮ $D_t y = 2x \cdot D_t x + 2 \cdot D_t x = D_t x(2x + 2)$. When $D_t y = D_t x$, $2x + 2 = 1$, $2x = -1$, $x = -\frac{1}{2}$, $y = -\frac{3}{4}$.

14.20 A boat is being pulled into a dock by a rope that passes through a ring on the bow of the boat. The dock is 8 feet higher than the bow ring. How fast is the boat approaching the dock when the length of rope between the dock and the boat is 10 feet, if the rope is being pulled in at the rate of 3 feet per second?

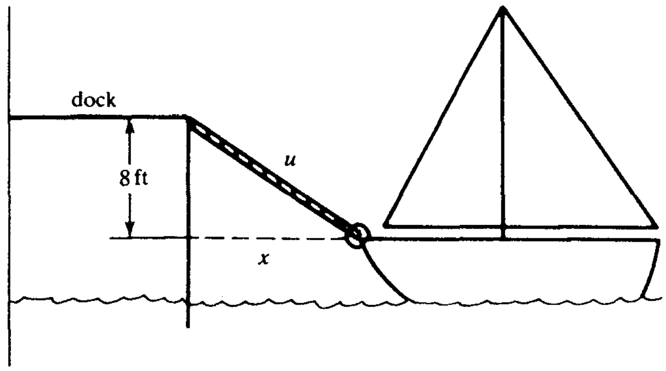

Fig. 14-8

Let x be the horizontal distance from the bow ring to the dock, and let u be the length of the rope between the dock and the boat. Then, $u^2 = x^2 + (8)^2$. So $2u \cdot D_t u = 2x \cdot D_t x$, $u \cdot D_t u = x \cdot D_t x$. We are told that $D_t u = -3$. So $-3u = x \cdot D_t x$. When $u = 10$, $x^2 = 36$, $x = 6$. Hence, $-3 \cdot 10 = 6 \cdot D_t x$, $D_t x = -5$. So the boat is approaching the dock at the rate of 5 ft/s.

14.21 A girl is flying a kite, which is at a height of 120 feet. The wind is carrying the kite horizontally away from the girl at a speed of 10 feet per second. How fast must the kite string be let out when the string is 150 feet long?

Let x be the horizontal distance of the kite from the point directly over the girl's head at 120 feet. Let u be the length of the kite string from the girl to the kite. Then $u^2 = x^2 + (120)^2$. So, $2u \cdot D_t u = 2x \cdot D_t x$, $u \cdot D_t u = x \cdot D_t x$. We are told that $D_t x = 10$. Hence, $u \cdot D_t u = 10x$. When $u = 150$, $x^2 = 8100$, $x = 90$. So, $150 \cdot D_t u = 900$, $D_t u = 6$ ft/s.

14.22 A rectangular trough is 8 feet long, 2 feet across the top, and 4 feet deep. If water flows in at a rate of 2 ft^3/min, how fast is the surface rising when the water is 1 ft deep?

Let x be the depth of the water. Then the water is a rectangular slab of dimensions x, 2, and 8. Hence, the volume $V = 16x$. So $D_t V = 16 \cdot D_t x$. We are told that $D_t V = 2$. So, $2 = 16 \cdot D_t x$. Hence, $D_t x = \frac{1}{8}$ ft/min.

14.23 A ladder 20 feet long leans against a house. Find the rate at which the top of the ladder is moving downward if the foot of the ladder is 12 feet away from the house and sliding along the ground away from the house at the rate of 2 feet per second?

Let x be the distance of the foot of the ladder from the base of the house, and let y be the distance of the top of the ladder from the ground. Then $x^2 + y^2 = (20)^2$. So, $2x \cdot D_t x + 2y \cdot D_t y = 0$, $x \cdot D_t x + y \cdot D_t y = 0$. We are told that $x = 12$ and $D_t x = 2$. When $x = 12$, $y^2 = 256$, $y = 16$. Substituting in $x \cdot D_t x + y \cdot D_t y = 0$, $12 \cdot 2 + 16 \cdot D_t y = 0$, $D_t y = -\frac{3}{2}$. So the ladder is sliding down the wall at the rate of 1.5 ft/s.

14.24 In Problem 14.23, how fast is the angle α between the ladder and the ground changing at the given moment?

$\tan \alpha = y/x$. So, by the chain rule, $\sec^2 \alpha \cdot D_t \alpha = \dfrac{x \cdot D_t y - y \cdot D_t x}{x^2} = \dfrac{12 \cdot (-\frac{3}{2}) - 16 \cdot 2}{144} = -\dfrac{50}{144}$. Also, $\tan \alpha = y/x = \frac{16}{12} = \frac{4}{3}$. So, $\sec^2 \alpha = 1 + \tan^2 \alpha = 1 + \frac{16}{9} = \frac{25}{9}$. Thus, $\frac{25}{9} \cdot D_t \alpha = -\frac{50}{144}$, $D_t \alpha = -\frac{1}{8}$. Hence, the angle is decreasing at the rate of $\frac{1}{8}$ radian per second.

14.25 A train, starting at 11 a.m., travels east at 45 miles per hour, while another starting at noon from the same point travels south at 60 miles per hour. How fast is the distance between them increasing at 3 p.m.?

Let the time t be measured in hours, starting at 11 a.m. Let x be the distance that the first train is east of the starting point, and let y be the distance that the second train is south of the starting point. Let u be the distance between the trains. Then $u^2 = x^2 + y^2$, $2u \cdot D_t u = 2x \cdot D_t x + 2y \cdot D_t y$, $u \cdot D_t u = x \cdot D_t x + y \cdot D_t y$. We are told that $D_t x = 45$ and $D_t y = 60$. So $u \cdot D_t u = 45x + 60y$. At 3 p.m., the first train has been travelling for 4 hours at 45 mi/h, and, therefore, $x = 180$; the second train has been travelling for 3 hours at 60 mi/h, and, therefore, $y = 180$. Then, $u^2 = (180)^2 + (180)^2$, $u = 180\sqrt{2}$. Thus, $180\sqrt{2} \cdot D_t u = 45 \cdot 180 + 60 \cdot 180$, $D_t u = 105\sqrt{2}/2$ mi/h.

14.26 A light is at the top of a pole 80 feet high. A ball is dropped from the same height (80 ft) from a point 20 feet from the light. Assuming that the ball falls according to the law $s = 16t^2$, how fast is the shadow of the ball moving along the ground one second later?

I See Fig. 14-9. Let x be the distance of the shadow of the ball from the base of the lightpole. Let y be the height of the ball above the ground. By similar triangles, $y/80 = (x - 20)/x$. But, $y = 80 - 16t^2$. So, $1 - \frac{1}{5}t^2 = 1 - (20/x)$. Differentiating, $-\frac{2}{5}t = (20/x^2) \cdot D_t x$. When $t = 1$, $1 - \frac{1}{5}(1)^2 = 1 - 20/x$, $x = 100$. Substituting in $-\frac{2}{5}t = (20/x^2) \cdot D_t x$, $D_t x = -200$. Hence, the shadow is moving at 200 ft/s.

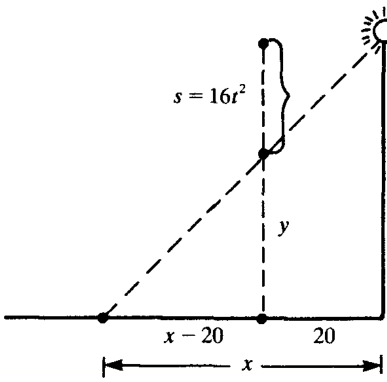

Fig. 14-9

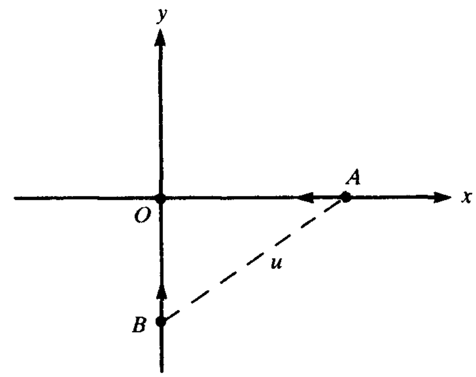

Fig. 14-10

14.27 Ship A is 15 miles east of point O and moving west at 20 miles per hour. Ship B is 60 miles south of O and moving north at 15 miles per hour. Are they approaching or separating after 1 hour, and at what rate?

I Let the point O be the origin of a coordinate system, with A moving on the x-axis and B moving on the y-axis (Fig. 14-10). Since A begins at $x = 15$ and is moving to the left at 20 mi/h, its position is $x = 15 - 20t$. Likewise, the position of B is $y = -60 + 15t$. Let u be the distance between A and B. Then $u^2 = x^2 + y^2$, $2u \cdot D_t u = 2x \cdot D_t x + 2y \cdot D_t y$, $u \cdot D_t u = x \cdot D_t x + y \cdot D_t y$. Since $D_t x = -20$ and $D_t y = 15$, $u \cdot D_t u = -20x + 15y$. When $t = 1$, $x = 15 - 20 = -5$, $y = -60 + 15 = -45$, $u^2 = (-5)^2 + (-45)^2 = (25)(82)$, $u = 5\sqrt{82}$. Substituting in $u \cdot D_t u = -20x + 15y$, $5\sqrt{82} D_t u = -575$, $D_t u = -115/\sqrt{82} \approx -13$. Since the derivative of u is negative, the distance between the ships is getting smaller, at roughly 13 mi/h.

14.28 Under the same hypotheses as in Problem 14.27, when are the ships nearest each other?

I When the ships are nearest each other, their distance u assumes a relative minimum, and, therefore, $D_t u = 0$. Substituting in $u \cdot D_t u = -20x + 15y$, $0 = -20x + 15y$. But $x = 15 - 20t$ and $y = -60 + 15t$. So, $0 = -20(15 - 20t) + 15(-60 + 15t)$, $t = \frac{48}{25}$ hours, or approximately, 1 hour and 55 minutes.

14.29 Water, at the rate of 10 cubic feet per minute, is pouring into a leaky cistern whose shape is a cone 16 feet deep and

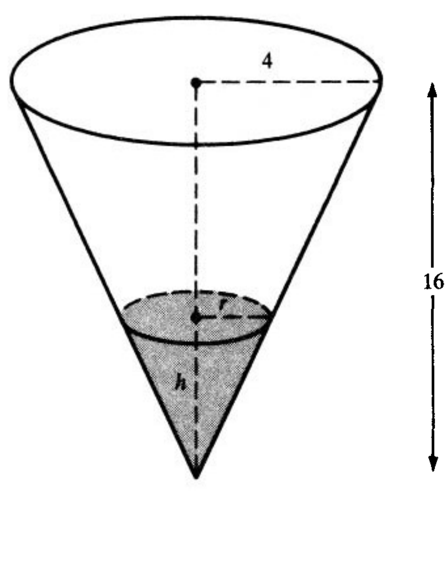

Fig. 14-11

8 feet in diameter at the top. At the time the water is 12 feet deep, the water level is observed to be rising 4 inches per minute. How fast is the water leaking out?

Let h be the depth of the water, and let r be the radius of the water surface (Fig. 14-11). The water's volume $V=\frac{1}{3}\pi r^2h$. By similar triangles, $r/4=h/16$, $r=\frac{1}{4}h$, $V=\frac{1}{3}\pi(h/4)^2h=\frac{1}{48}\pi h^3$. So $D_tV=\frac{1}{16}\pi h^2\cdot D_th$. We are told that when $h=12$, $D_th=\frac{1}{3}$. Hence, at that moment, $D_tV=\frac{1}{16}\pi(144)(\frac{1}{3})=3\pi$. Since the rate at which the water is pouring in is 10, the rate of leakage is $(10-3\pi)\ \mathrm{ft}^3/\mathrm{min}$.

14.30 An airplane is ascending at a speed of 400 kilometers per hour along a line making an angle of 60° with the ground. How fast is the altitude of the plane changing?

Let h be the altitude of the plane, and let u be the distance of the plane from the ground along its flight path (Fig. 14-12). Then $h/u=\sin 60°=\sqrt{3}/2$, $2h=\sqrt{3}u$, $2\cdot D_th=\sqrt{3}D_tu=\sqrt{3}\cdot 400$. Hence, $D_th=200\sqrt{3}$ kilometers per hour.

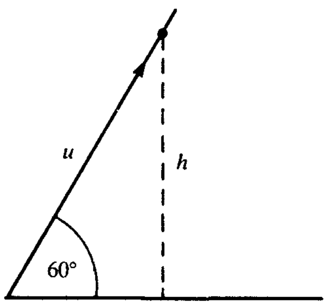

Fig. 14-12

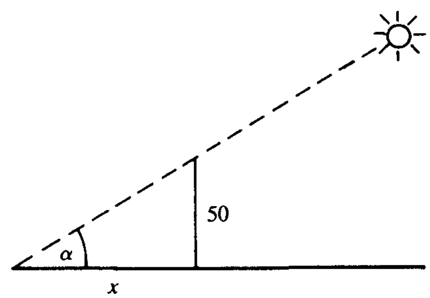

Fig. 14-13

14.31 How fast is the shadow cast on level ground by a pole 50 feet tall lengthening when the angle α of elevation of the sun is 45° and is decreasing by $\frac{1}{4}$ radian per hour? (See Fig. 14.13.)

Let x be the length of the shadow. $\tan\alpha=50/x$. By the chain rule, $\sec^2\alpha\cdot D_t\alpha=(-50/x^2)\cdot D_tx$. When $\alpha=45°$, $\tan\alpha=1$, $\sec^2\alpha=1+\tan^2\alpha=2$, $x=50$. So, $2(-\frac{1}{4})=-\frac{1}{50}\cdot D_tx$. Hence, $D_tx=25\ \mathrm{ft/h}$.

14.32 A revolving beacon is situated 3600 feet off a straight shore. If the beacon turns at 4π radians per minute, how fast does its beam sweep along the shore at its nearest point A?

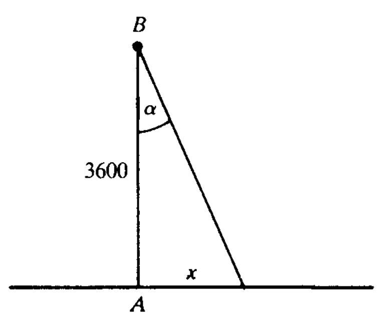

Fig. 14-14

Let x be the distance from A to the point on the shore hit by the beacon, and let α be the angle between the line from the lighthouse B to A, and the beacon (Fig. 14-14). Then $\tan\alpha=x/3600$, so $\sec^2\alpha\cdot D_t\alpha=\frac{1}{3600}\cdot D_tx$. We are told that $D_t\alpha=4\pi$. When the beacon hits point A, $\alpha=0$, $\sec\alpha=1$, so $4\pi=\frac{1}{3600}D_tx$, $D_tx=14{,}400\pi\ \mathrm{ft/min}=240\pi\ \mathrm{ft/s}$.

14.33 Two sides of a triangle are 15 and 20 feet long, respectively. How fast is the third side increasing when the angle α between the given sides is 60° and is increasing at the rate of 2° per second?

Let x be the third side. By the law of cosines, $x^2=(15)^2+(20)^2-2(15)(20)\cdot\cos\alpha$. Hence, $2x\cdot D_tx=600\sin\alpha\cdot D_t\alpha$. $x\cdot D_tx=300\sin\alpha\cdot D_t\alpha$. We are told that $D_t\alpha=2\cdot(\pi/180)=\pi/90\ \mathrm{rad/s}$. When $\alpha=60°$, $\sin\alpha=\sqrt{3}/2$, $\cos\alpha=\frac{1}{2}$, $x^2=225+400-600\cdot\frac{1}{2}=325$, $x=5\sqrt{13}$. Hence, $5\sqrt{13}\cdot D_tx=300\cdot(\sqrt{3}/2)\cdot(\pi/90)$, $D_tx=(\pi/\sqrt{39})\ \mathrm{ft/s}$.

14.34 The area of an expanding rectangle is increasing at the rate of 48 square centimeters per second. The length of the rectangle is always equal to the square of its width (in centimeters). At what rate is the length increasing at the instant when the width is 2 cm?

▮ $A = \ell w$, and $\ell = w^2$. So, $A = w^3$. Hence, $D_t A = 3w^2 \cdot D_t w$. We are told that $D_t A = 48$. Hence, $48 = 3w^2 \cdot D_t w$, $16 = w^2 \cdot D_t w$. When $w = 2$, $16 = 4 \cdot D_t w$, $D_t w = 4$. Since $\ell = w^2$, $D_t \ell = 2w \cdot D_t w$. Hence, $D_t \ell = 2 \cdot 2 \cdot 4 = 16$ cm/s.

14.35 A spherical snowball is melting (symmetrically) at the rate of 4π cubic centimeters per hour. How fast is the diameter changing when it is 20 centimeters?

▮ The volume $V = \frac{4}{3}\pi r^3$. So, $D_t V = 4\pi r^2 \cdot D_t r$. We are told that $D_t V = -4\pi$. Hence, $-4\pi = 4\pi r^2 \cdot D_t r$. Thus, $-1 = r^2 \cdot D_t r$. When the diameter is 20 centimeters, the radius $r = 10$. Hence, $-1 = 100 \cdot D_t r$, $D_t r = -0.01$. Since the diameter $d = 2r$, $D_t d = 2 \cdot D_t r = 2 \cdot (-0.01) = -0.02$. So, the diameter is decreasing at the rate of 0.02 centimeter per hour.

14.36 A trough is 10 feet long and has a cross section in the shape of an equilateral triangle 2 feet on each side (Fig. 14-15). If water is being pumped in at the rate of 20 ft^3/min, how fast is the water level rising when the water is 1 ft deep?

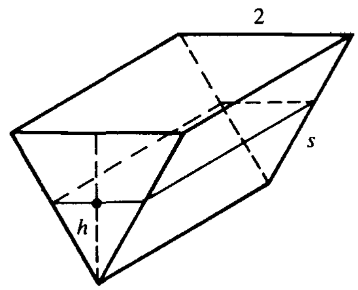

Fig. 14-15

▮ The water in the trough will have a cross section that is an equilateral triangle, say of height h and side s. In an equilateral triangle with side s, $s = 2h/\sqrt{3}$. Hence, the cross-sectional area of the water is $\frac{1}{2} \cdot (2h/\sqrt{3}) \cdot h = h^2/\sqrt{3}$. Therefore, the volume V of water is $10h^2/\sqrt{3}$. So, $D_t V = (20h/\sqrt{3}) \cdot D_t h$. We are told that $D_t V = 20$. So, $20 = (20h/\sqrt{3}) \cdot D_t h$, $\sqrt{3} = h \cdot D_t h$. When $h = 1$ ft, $D_t h = \sqrt{3}$ ft/min.

14.37 If a mothball evaporates at a rate proportional to its surface area $4\pi r^2$, show that its radius decreases at a constant rate.

▮ The volume $V = \frac{4}{3}\pi r^3$. So, $D_t V = 4\pi r^2 \cdot D_t r$. We are told that $D_t V = k \cdot 4\pi r^2$ for some constant k. Hence, $k = D_t r$.

14.38 Sand is being poured onto a conical pile at the constant rate of 50 cubic feet per minute. Frictional forces in the sand are such that the height of the pile is always equal to the radius of its base. How fast is the height of the pile increasing when the sand is 5 feet deep?

▮ The volume $V = \frac{1}{3}\pi r^2 h$. Since $h = r$, $V = \frac{1}{3}\pi h^3$. So, $D_t V = \pi h^2 \cdot D_t h$. We are told that $D_t V = 50$, so $50 = \pi h^2 \cdot D_t h$. When $h = 5$, $50 = \pi \cdot 25 \cdot D_t h$, $D_t h = 2/\pi$ ft/min.

14.39 At a certain moment, a sample of gas obeying Boyle's law, $pV =$ constant, occupies a volume V of 1000 cubic inches at a pressure p of 10 pounds per square inch. If the gas is being compressed at the rate of 12 cubic inches per minute, find the rate at which the pressure is increasing at the instant when the volume is 600 cubic inches.

▮ Since $pV =$ constant, $p \cdot D_t V + V \cdot D_t p = 0$. We are told that $D_t V = -12$, so $-12p + V \cdot D_t p = 0$. When $V = 1000$ and $p = 10$, $D_t p = 0.12$ pound per square inch per minute.

14.40 A ladder 20 feet long is leaning against a wall 12 feet high with its top projecting over the wall (Fig. 14-16). Its bottom is being pulled away from the wall at the constant rate of 5 ft/min. How rapidly is the height of the top of the ladder decreasing when the top of the ladder reaches the top of the wall?

▮ Let y be the height of the top of the ladder, let x be the distance of the bottom of the ladder from the wall, and let u be the distance from the bottom of the ladder to the top of the wall. Now, $u^2 = x^2 + (12)^2$, $2u \cdot D_t u = 2x \cdot D_t x$, $u \cdot D_t u = x \cdot D_t x$. We are told that $D_t x = 5$. So, $u \cdot D_t u = 5x$. When the top of the ladder reaches the top of the wall, $u = 20$, $x^2 = (20)^2 - (12)^2 = 256$, $x = 16$. Hence, $20 \cdot D_t u = 5 \cdot 16$, $D_t u = 4$. By similar triangles, $y/12 = 20/u$, $y = 240/u$, $D_t y = -(240/u^2) \cdot D_t u = -\frac{240}{400} \cdot 4 = -2.4$ ft/min. Thus, the height of the ladder is decreasing at the rate of 2.4 feet per minute.

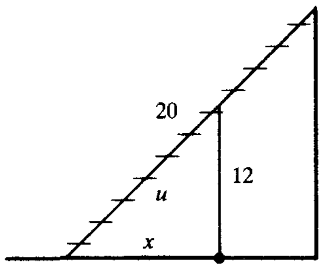

Fig. 14-16

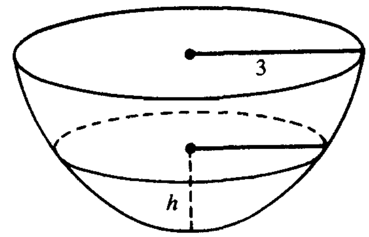

Fig. 14-17

14.41 Water is being poured into a hemispherical bowl of radius 3 inches at the rate of 1 cubic inch per second. How fast is the water level rising when the water is 1 inch deep? [The spherical segment of height h shown in Fig. 14-17 has volume $V = \pi h^2(r - h/3)$, where r is the radius of the sphere.]

I $V = \pi h^2(3 - h/3) = 3\pi h^2 - (\pi/3)h^3$. So, $D_t V = 6\pi h \cdot D_t h - \pi h^2 \cdot D_t h = \pi h D_t h(6 - h)$. We are told that $D_t V = 1$, so $1 = \pi h D_t h(6 - h)$. When $h = 1$, $D_t h = 1/5\pi$ in/s.

14.42 A metal ball of radius 90 centimeters is coated with a uniformly thick layer of ice, which is melting at the rate of 8π cubic centimeters per hour. Find the rate at which the thickness of the ice is decreasing when the ice is 10 centimeters thick?

I Let h be the thickness of the ice. The volume of the ice $V = \frac{4}{3}\pi(90 + h)^3 - \frac{4}{3}\pi(90)^3$. So, $D_t V = 4\pi(90 + h)^2 \cdot D_t h$. We are told that $D_t V = -8\pi$. Hence, $-2 = (90 + h)^2 \cdot D_t h$. When $h = 10$, $-2 = (100)^2 \cdot D_t h$, $D_t h = -0.0002$ cm/h.

14.43 A snowball is increasing in volume at the rate of $10\ \text{cm}^3/\text{h}$. How fast is the surface area growing at the moment when the radius of the snowball is 5 cm?

I The surface area $A = 4\pi r^2$. So, $D_t A = 8\pi r \cdot D_t r$. Now, $V = \frac{4}{3}\pi r^3$, $D_t V = 4\pi r^2 \cdot D_t r$. We are told that $D_t V = 10$. So, $10 = 4\pi r^2 \cdot D_t r = \frac{1}{2} r \cdot 8\pi r \cdot D_t r = \frac{1}{2} r \cdot D_t A$. When $r = 5$, $10 = \frac{1}{2} \cdot 5 \cdot D_t A$, $D_t A = 4\ \text{cm}^2/\text{h}$.

14.44 If an object is moving on the curve $y = x^3$, at what point(s) is the y-coordinate of the object changing three times more rapidly than the x-coordinate?

I $D_t y = 3x^2 \cdot D_t x$. When $D_t y = 3 \cdot D_t x$, $x^2 = 1$, $x = \pm 1$. So, the points are $(1, 1)$ and $(-1, -1)$. (Other solutions occur when $D_t x = 0$, $D_t y = 0$. This happens within an interval of time when the object remains fixed at one point on the curve.)

14.45 If the diagonal of a cube is increasing at a rate of 3 cubic inches per minute, how fast is the side of the cube increasing?

I Let u be the length of the diagonal of a cube of side s. Then $u^2 = s^2 + s^2 + s^2 = 3s^2$, $u = s\sqrt{3}$, $D_t u = \sqrt{3} D_t s$. Thus, $3 = \sqrt{3} D_t s$, $D_t s = \sqrt{3}$ in/min.

14.46 The two equal sides of an isosceles triangle with fixed base b are decreasing at the rate of 3 inches per minute. How fast is the area decreasing when the two equal sides are equal to the base?

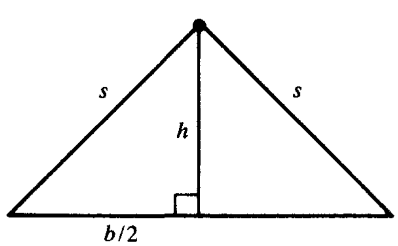

Fig. 14-18

I Let s be the length of the two equal sides, and let h be the height. Then, from Fig. 14-18, $h^2 = s^2 - b^2/4$, $2h \cdot D_th = 2s \cdot D_ts$, $h \cdot D_th = s \cdot D_ts$. When $s = b$, $h^2 = \frac{3}{4}b^2$, $h = (\sqrt{3}/2)b$, $(\sqrt{3}/2)b \cdot D_th = b \cdot D_ts$, $(\sqrt{3}/2) \cdot D_th = D_ts$. We are told that $D_ts = -3$. Hence, $D_th = -2\sqrt{3}$. Now, $A = \frac{1}{2}bh$, $D_tA = \frac{1}{2}b \cdot D_th = \frac{1}{2}b \cdot (-2\sqrt{3}) = -b\sqrt{3}\ \text{in}^2/\text{min}$.

14.47 An object moves on the parabola $3y = x^2$. When $x = 3$, the x-coordinate of the object is increasing at the rate of 1 foot per minute. How fast is the y-coordinate increasing at that moment?

I $3 \cdot D_ty = 2x \cdot D_tx$. When $x = 3$, $D_tx = 1$. So $3 \cdot D_ty = 6$, $D_ty = 2$ ft/min.

14.48 A solid is formed by a cylinder of radius r and altitude h, together with two hemispheres of radius r attached at each end (Fig. 14-19). If the volume V of the solid is constant but r is increasing at the rate of $1/(2\pi)$ meters per minute, how fast must h be changing when r and h are 10 meters?

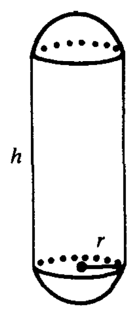

Fig. 14-19

I $V = \pi r^2 h + \frac{4}{3}\pi r^3$. $D_tV = \pi r^2 \cdot D_th + 2\pi rh \cdot D_tr + 4\pi r^2 \cdot D_tr$. But, since V is constant, $D_tV = 0$. We are told that $D_tr = 1/(2\pi)$. Hence, $0 = \pi r^2 \cdot D_th + rh + 2r^2$. When $r = h = 10$, $100\pi \cdot D_th + 300 = 0$, $D_th = -3/\pi$ meters per minute.

14.49. If $y = 7x - x^3$ and x increases at the rate of 4 units per second, how fast is the slope of the graph changing when $x = 3$?

I The slope $D_xy = 7 - 3x^2$. Hence, the rate of change of the slope is $D_t(D_xy) = -6x \cdot D_tx = -6x \cdot 4 = -24x$. When $x = 3$, $D_t(D_xy) = -72$ units per second.

14.50 A segment UV of length 5 meters moves so that its endpoints U and V stay on the x-axis and y-axis, respectively. V is moving away from the origin at the rate of 2 meters per minute. When V is 3 meters from the origin, how fast is U's position changing?

I Let x be the x-coordinate of U and let y be the y-coordinate of V. Then $y^2 + x^2 = 25$, $2y \cdot D_ty + 2x \cdot D_tx = 0$, $y \cdot D_ty + x \cdot D_tx = 0$. We are told that $D_ty = 2$. So, $2y + x \cdot D_tx = 0$. When $y = 3$, $x = 4$, $2 \cdot 3 + 4 \cdot D_tx = 0$, $D_tx = -\frac{3}{2}$ meters per minute.

14.51 A railroad track crosses a highway at an angle of 60°. A train is approaching the intersection at the rate of 40 mi/h, and a car is approaching the intersection from the same side as the train, at the rate of 50 mi/h. If, at a certain moment, the train and car are both 2 miles from the intersection, how fast is the distance between them changing?

I Refer to Fig. 14-20. Let x and y be the distances of the train and car, respectively, from the intersection, and

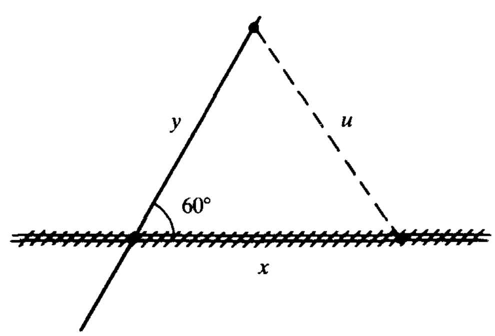

Fig. 14-20

let u be the distance between the train and the car. By the law of cosines, $u^2 = x^2 + y^2 - 2xy \cdot \cos 60° = x^2 + y^2 - xy$ (since $\cos 60° = \frac{1}{2}$). Hence, $2u \cdot D_t u = 2x \cdot D_t x + 2y \cdot D_t y - x \cdot D_t y - y \cdot D_t x$. We are told that $D_t x = -40$ and $D_t y = -50$, so $2u \cdot D_t u = -80x - 100y + 50x + 40y = -30x - 60y$. When $y = 2$ and $x = 2$, $u^2 = 4 + 4 - 4 = 4$, $u = 2$. Hence, $4 \cdot D_t u = -60 - 120 = -180$, $D_t u = -45$ mi/h.

14.52 A trough 20 feet long has a cross section in the shape of an equilateral trapezoid, with a base of 3 feet and whose sides make a 45° angle with the vertical. Water is flowing into it at the rate of 14 cubic feet per hour. How fast is the water level rising when the water is 2 feet deep?

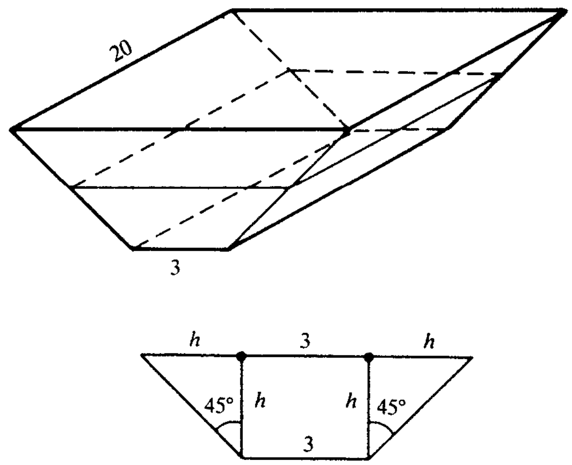

Fig. 14-21

Let h be the depth of the water at time t. The cross-sectional area is $3h + h^2$ (see Fig. 14-21), and, therefore, the volume $V = 20(3h + h^2)$. So $D_t V = 20(3 \cdot D_t h + 2h \cdot D_t h) = 20 \cdot D_t h \cdot (3 + 2h)$. We are told that $D_t V = 14$, so $14 = 20 \cdot D_t h \cdot (3 + 2h)$. When $h = 2$, $14 = 20 \cdot D_t h \cdot 7$, $D_t h = 0.1$ ft/h.

4.53 A lamppost 10 feet tall stands on a walkway that is perpendicular to a wall. The distance from the post to the wall is 15 feet. A 6-foot man moves on the walkway toward the wall at the rate of 5 feet per second. When he is 5 feet from the wall, how fast is the shadow of his head moving up the wall?

See Fig. 14-22. Let x be the distance from the man to the wall. Let u be the distance between the base of the wall and the intersection with the ground of the line from the lamp to the man's head. Let z be the height of the shadow of the man's head on the wall. By similar triangles, $6/(x + u) = 10/(15 + u) = z/u$. From the first equation, we obtain $u = \frac{5}{2}(9 - x)$. Hence, $x + u = \frac{3}{2}(15 - x)$. Since $z/u = 6/(x + u)$, $z = 6u/(x + u) = 10(9 - x)/(15 - x) = 10[1 - 6/(15 - x)]$. Thus, $D_t z = [10 \cdot 6/(15 - x)^2] \cdot (-D_t x)$. We are told that $D_t x = -5$. Hence, $D_t z = 300/(15 - x)^2$. When $x = 5$, $D_t z = 3$ ft/s. Thus, the shadow is moving up the wall at the rate of 3 feet per second.

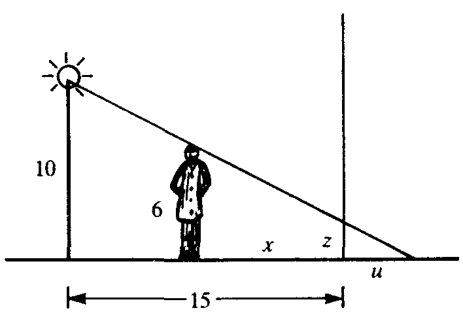

Fig. 14-22

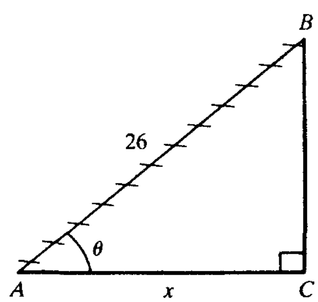

Fig. 14-23

4.54 In Fig. 14-23, a ladder 26 feet long is leaning against a vertical wall. If the bottom of the ladder, A, is slipping away from the base of the wall at the rate of 3 feet per second, how fast is the angle between the ladder and the ground changing when the bottom of the ladder is 10 feet from the base of the wall?

Let x be the distance of A from the base of the wall at C. Then $D_t x = 3$. Since $\cos\theta = x/26$, $(-\sin\theta) \cdot D_t\theta = \frac{1}{26} D_t x = \frac{3}{26}$. When $x = 10$, $CB = \sqrt{(26)^2 - (10)^2} = \sqrt{576} = 24$, and $\sin\theta = \frac{24}{26}$. So, $-\frac{24}{26} D_t\theta = \frac{3}{26}$, $D_t\theta = -\frac{1}{8}$ radian per second.

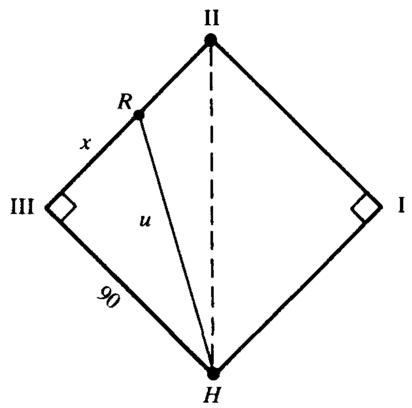

Fig. 14-24

14.55 In Fig. 14-24, a baseball field is a square of side 90 feet. If a runner on second base (II) starts running toward third base (III) at a rate of 20 ft/s. how fast is his distance from home plate (H) changing when he is 60 ft from II?

Let x and u be the distances of the runner from III and H, respectively. Then $u^2 = x^2 + (90)^2$, $2u \dfrac{du}{dt} = 2x \dfrac{dx}{dt} = 2x(-20)$, $\dfrac{du}{dt} = -20 \dfrac{x}{u}$. When $x = 90 - 60 = 30$, $u = \sqrt{(30)^2 + (90)^2} = 30\sqrt{10}$; therefore, $\dfrac{du}{dt} = -20\left(\dfrac{30}{30\sqrt{10}}\right) = -2\sqrt{10}$ ft/s.

14.56 An open pipe with length 3 meters and outer radius of 10 centimeters has an outer layer of ice that is melting at the rate of 2π cm^3/min. How fast is the thickness of ice decreasing when the ice is 2 centimeters thick?

Let x be the thickness of the ice. Then the volume of the ice $V = 300[\pi(10 + x)^2 - 100\pi]$, So $D_tV = 300\pi[2(10 + x) \cdot D_tx]$. Since $D_tV = -2\pi$, we have $-2\pi = 600\pi(10 + x) \cdot D_tx$, $D_tx = -1/[300(10 + x)]$. When $x = 2$, $D_tx = -\frac{1}{3600}$ cm/min. So the thickness is decreasing at the rate of $\frac{1}{3600}$ centimeter per minute (4 millimeters per day).