CHAPTER 16
Applied Maximum and Minimum Problems

16.1 A rectangular field is to be fenced in so that the resulting area is c square units. Find the dimensions of that field (if any) for which the perimeter is a minimum, and the dimensions (if any) for which the perimeter is a maximum.

Let ℓ be the length and w the width. Then $\ell w = c$. The perimeter $p = 2\ell + 2w = 2\ell + 2c/\ell$. ℓ can be any positive number. $D_\ell p = 2 - 2c/\ell^2$, and $D_\ell^2 p = 4c/\ell^3$. Hence, solving $2 - 2c/\ell^2 = 0$, we see that $\ell = \sqrt{c}$ is a critical number. The second derivative is positive, and, therefore, there is a relative minimum at $\ell = \sqrt{c}$. Since that is the only critical number and the function $2\ell + 2c/\ell$ is continuous for all positive ℓ, there is an absolute minimum at $\ell = \sqrt{c}$. (If p achieved a still smaller value at some other point ℓ_0, there would have to be a relative maximum at some point between $\sqrt{c}$ and ℓ_0, yielding another critical number.) When $\ell = \sqrt{c}$, $w = \sqrt{c}$. Thus, for a fixed area, the square is the rectangle with the smallest perimeter. Notice that the perimeter does not achieve a maximum, since $p = 2\ell + 2c/\ell \to +\infty$ as $\ell \to +\infty$.

16.2 Find the point(s) on the parabola $2x = y^2$ closest to the point $(1, 0)$.

Refer to Fig. 16-1. Let u be the distance between $(1,0)$ and a point (x, y) on the parabola. Then $u = \sqrt{(x-1)^2 + y^2}$. To minimize u it suffices to minimize $u^2 = (x-1)^2 + y^2$. Now, $u^2 = (x-1)^2 + 2x$. Since (x, y) is a point on $2x = y^2$, x can be any nonnegative number. Now, $D_x(u^2) = 2(x-1) + 2 = 2x > 0$ for $x > 0$. Hence, u^2 is an increasing function, and, therefore, its minimum value is attained at $x = 0$, $y = 0$.

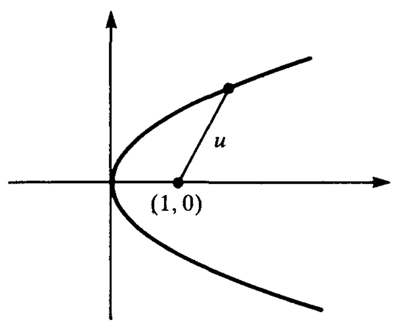

Fig. 16-1

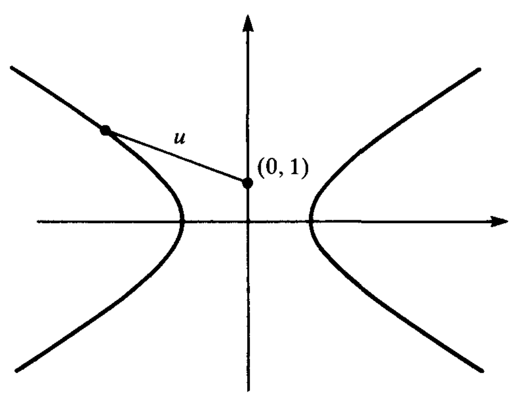

Fig. 16-2

16.3 Find the point(s) on the hyperbola $x^2 - y^2 = 2$ closest to the point $(0, 1)$.

Refer to Fig. 16-2. Let u be the distance between $(0, 1)$ and a point (x, y) on the hyperbola. Then $u = \sqrt{x^2 + (y-1)^2}$. To minimize u, it suffices to minimize $u^2 = x^2 + (y-1)^2 = 2 + y^2 + (y-1)^2$. Since $x^2 = y^2 + 2$, y can be any real number. $D_y(u^2) = 2y + 2(y-1) = 4y - 2$. Also, $D_y^2(u^2) = 4$. The only critical number is $\frac{1}{2}$, and, since the second derivative is positive, there is a relative minimum at $y = \frac{1}{2}$, $x = \pm\frac{3}{2}$. Since there is only one critical number, this point yields the absolute minimum.

16.4 A closed box with a square base is to contain 252 cubic feet. The bottom costs \$5 per square foot, the top costs \$2 per square foot, and the sides cost \$3 per square foot. Find the dimensions that will minimize the cost.

Let s be the side of the square base and let h be the height. Then $s^2h = 252$. The cost of the bottom is $5s^2$, the cost of the top is $2s^2$, and the cost of each of the four sides is $3sh$. Hence, the total cost $C = 5s^2 + 2s^2 + 4(3sh) = 7s^2 + 12sh = 7s^2 + 12s(252/s^2) = 7s^2 + 3024/s$. s can be any positive number. Now, $D_sC = 14s - 3024/s^2$, and $D_s^2C = 14 + 6048/s^3$. Solving $14s - 3024/s^2 = 0$, $14s^3 = 3024$, $s^3 = 216$, $s = 6$. Thus, $s = 6$ is the only critical number. Since the second derivative is always positive for $s > 0$, there is a relative minimum at $s = 6$. Since $s = 6$ is the *only* critical number, it yields an absolute minimum. When $s = 6$, $h = 7$.

16.5 A printed page must contain 60 cm^2 of printed material. There are to be margins of 5 cm on either side and margins of 3 cm on the top and bottom (Fig. 16-3). How long should the printed lines be in order to minimize the amount of paper used?

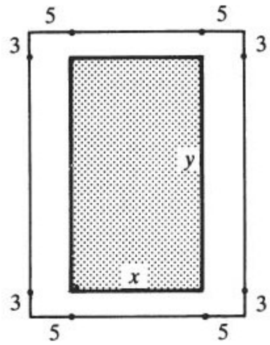

Fig. 16-3

Let x be the length of the line and let y be the height of the printed material. Then $xy = 60$. The amount of paper $A = (x + 10)(y + 6) = (x + 10)(60/x + 6) = 6(10 + x + 100/x + 10) = 6(20 + x + 100/x)$. x can be any positive number. Then $D_xA = 6(1 - 100/x^2)$ and $D_x^2A = 1200/x^3$. Solving $1 - 100/x^2 = 0$, we see that the only critical number is 10. Since the second derivative is positive, there is a relative minimum at $x = 10$, and, since this is the *only* critical number, there is an absolute minimum at $x = 10$.

16.6 A farmer wishes to fence in a rectangular field of 10,000 ft^2. The north-south fences will cost \$1.50 per foot, while the east-west fences will cost \$6.00 per foot. Find the dimensions of the field that will minimize the cost.

Let x be the east-west dimension, and let y be the north-south dimension. Then $xy = 10{,}000$. The cost $C = 6(2x) + 1.5(2y) = 12x + 3y = 12x + 3(10{,}000/x) = 12x + 30{,}000/x$. x can be any positive number. $D_xC = 12 - 30{,}000/x^2$. $D_x^2C = 60{,}000/x^3$. Solving $12 - 30{,}000/x^2 = 0$, $2500 = x^2$, $x = 50$. Thus, 50 is the only critical number. Since the second derivative is positive, there is a relative minimum at $x = 50$. Since this is the only critical number, this is an absolute minimum. When $x = 50$, $y = 200$.

16.7 Find the dimensions of the closed cylindrical can that will have a capacity of k units of volume and will use the minimum amount of material. Find the ratio of the height h to the radius r of the top and bottom.

The volume $k = \pi r^2 h$. The amount of material $M = 2\pi r^2 + 2\pi rh$. (This is the area of the top and bottom, plus the lateral area.) So $M = 2\pi r^2 + 2\pi r(k/\pi r^2) = 2\pi r^2 + 2k/r$. Then $D_rM = 4\pi r - 2k/r^2$, $D_r^2M = 4\pi + 4k/r^3$. Solving $4\pi r - 2k/r^2 = 0$, we find that the only critical number is $r = \sqrt[3]{k/2\pi}$. Since the second derivative is positive, this yields a relative minimum, which, by the uniqueness of the critical number, is an absolute minimum. Note that $k = \pi r^2 h = \pi r^3(h/r) = \pi(k/2\pi)(h/r)$. Hence, $h/r = 2$.

16.8 In Problem 16.7, find the ratio h/r that will minimize the amount of material used if the bottom and top of the can have to be cut from square pieces of metal and the rest of these squares are wasted. Also find the resulting ratio of height to radius.

$k = \pi r^2 h$. Now $M = 8r^2 + 2\pi rh = 8r^2 + 2\pi r(k/\pi r^2) = 8r^2 + 2k/r$. $D_rM = 16r - 2k/r^2$. $D_r^2M = 16 + 4k/r^3$. Solving for the critical number, $r^3 = k/8$, $r = \sqrt[3]{k}/2$. As before, this yields an absolute minimum. Again, $k = \pi r^2 h = \pi r^3(h/r) = \pi(k/8)(h/r)$. So, $h/r = 8/\pi$.

16.9 A thin-walled cone-shaped cup (Fig. 16-4) is to hold 36π in^3 of water when full. What dimensions will minimize the amount of material needed for the cup?

Let r be the radius and h be the height. Then the volume $36\pi = \frac{1}{3}\pi r^2 h$. The lateral surface area $A = \pi rs$, where s is the slant height of the cone. $s^2 = r^2 + h^2$ and $h = 108/r^2$. Hence, $A = \pi r\sqrt{r^2 + (108/r^2)^2} = (\pi/r)\sqrt{r^6 + (108)^2}$. Then,

$$D_rA = \frac{\pi[r \cdot \frac{1}{2} \cdot 6r^5/\sqrt{r^6 + (108)^2} - \sqrt{r^6 + (108)^2}]}{r^2}$$

$$= \frac{\pi\{3r^6 - [r^6 + (108)^2]\}}{r^2\sqrt{r^6 + (108)^2}} = \frac{\pi[2r^6 - (108)^2]}{r^2\sqrt{r^6 + (108)^2}}$$

Solving $2r^6 - (108)^2 = 0$ for the critical number, $r = 3\sqrt{2}$. The first-derivative test yields the case $\{-, +\}$, showing that $r = 3\sqrt{2}$ gives a relative minimum, which, by the uniqueness of the critical number, must be an absolute minimum. When $r = 3\sqrt{2}$, $h = 6$.

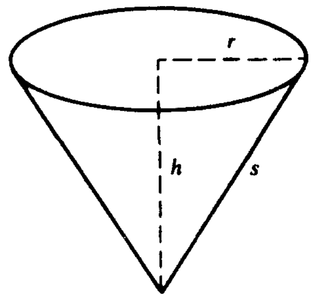

Fig. 16-4

16.10 A rectangular bin, open at the top, is required to contain 128 cubic meters. If the bottom is to be a square, at a cost of \$2 per square meter, while the sides cost \$0.50 per square meter, what dimensions will minimize the cost?

Let s be the side of the bottom square and let h be the height. Then $128 = s^2h$. The cost (in dollars) $C = 2s^2 + \frac{1}{2}(4sh) = 2s^2 + 2s(128/s^2) = 2s^2 + 256/s$, so $D_sC = 4s - 256/s^2$, $D_s^2C = 4 + 512/s^3$. Solving $4s - 256/s^2 = 0$, $s^3 = 64$, $s = 4$. Since the second derivative is positive, the critical number $s = 4$ yields a relative minimum, which, by the uniqueness of the critical number, is an absolute minimum. When $s = 4$, $h = 8$.

16.11 The selling price P of an item is $100 - 0.02x$ dollars, where x is the number of items produced per day. If the cost C of producing and selling x items is $40x + 15{,}000$ dollars per day, how many items should be produced and sold every day in order to maximize the profit?

The total income per day is $x(100 - 0.02x)$. Hence the profit $G = x(100 - 0.02x) - (40x + 15{,}000) = 60x - 0.02x^2 - 15{,}000$, and $D_xG = 60 - 0.04x$ and $D_x^2G = -0.04$. Hence, the unique critical number is the solution of $60 - 0.04x = 0$, $x = 1500$. Since the second derivative is negative, this yields a relative maximum, which, by the uniqueness of the critical number, is an absolute maximum.

16.12 Find the point(s) on the graph of $3x^2 + 10xy + 3y^2 = 9$ closest to the origin.

It suffices to minimize $u = x^2 + y^2$, the square of the distance from the origin. By implicit differentiation, $D_xu = 2x + 2yD_xy$ and $6x + 10(xD_xy + y) + 6yD_xy = 0$. From the second equation, $D_xy = -(3x + 5y)/(5x + 3y)$, and, then, substituting in the first equation, $D_xu = 2x + 2y[-(3x + 5y)/(5x + 3y)]$. Setting $D_xu = 0$, $x(5x + 3y) - y(3x + 5y) = 0$, $5(x^2 - y^2) = 0$, $x^2 = y^2$, $x = \pm y$. Substituting in the equation of the graph, $6x^2 \pm 10x^2 = 9$. Hence, we have the + sign, and $16x^2 = 9$, $x = \pm\frac{3}{4}$ and $y = \pm\frac{3}{4}$. Thus, the two points closest to the origin are $(\frac{3}{4}, \frac{3}{4})$ and $(-\frac{3}{4}, -\frac{3}{4})$.

16.13 A man at a point P on the shore of a circular lake of radius 1 mile wants to reach the point Q on the shore diametrically opposite P (Fig. 16-5). He can row 1.5 miles per hour and walk 3 miles per hour. At what angle θ $(0 \le \theta \le \pi/2)$ to the diameter PQ should he row in order to minimize the time required to reach Q?

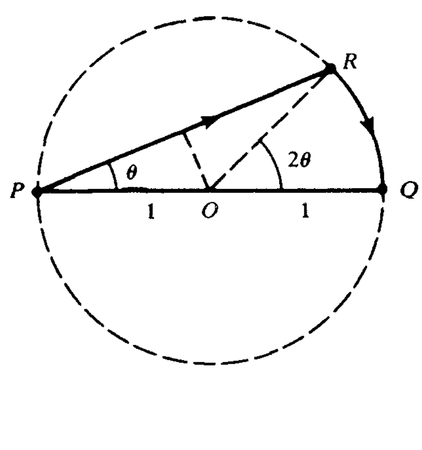

Fig. 16-5

Let R be the point where the boat lands, and let O be the center of the circle. Since $\triangle OPR$ is isosceles, $PR = 2\cos\theta$. The arc length $\widehat{RQ} = 2\theta$. Hence, the time $T = PR/1.5 + \widehat{RQ}/3 = \frac{4}{3}\cos\theta + \frac{2}{3}\theta$. So, $D_\theta T = -\frac{4}{3}\sin\theta + \frac{2}{3}$. Setting $D_\theta T = 0$, we find $\sin\theta = \frac{1}{2}$, $\theta = \pi/6$. Since T is a continuous function on the closed interval $[0, \pi/2]$, we can use the tabular method. List the critical number $\pi/6$ and the endpoints 0 and $\pi/2$, and compute the corresponding values of T. The smallest of these values is the absolute minimum. Clearly, $\pi/3 < \frac{4}{3}$, and it is easy to check that $\pi/3 < (6\sqrt{3} + \pi)/9$. (Assume the contrary and obtain the false consequence that $\pi > 3\sqrt{3}$.) Thus, the absolute minimum is attained when $\theta = \pi/2$. That means that the man walks all the way.

θ	$\pi/6$	0	$\pi/2$
T	$(6\sqrt{3} + \pi)/9$	$\frac{4}{3}$	$\pi/3$

16.14 Find the answer to Problem 16.13 when, instead of rowing, the man can paddle a canoe at 4 miles per hour.

Using the same notation as in Problem 16.13, we find $T = \frac{1}{2}\cos\theta + \frac{2}{3}\theta$, $D_\theta T = -\frac{1}{2}\sin\theta + \frac{2}{3}$. Setting $D_\theta T = 0$, We obtain $\sin\theta = \frac{4}{3}$, which is impossible. Hence, we use the tabular method for just the endpoints 0 and $\pi/2$. Then, since $\frac{1}{2} < \pi/3$, the absolute minimum is $\frac{1}{2}$, attained when $\theta = 0$. Hence, in this case, the man paddles all the way.

θ	0	$\pi/2$
T	$\frac{1}{2}$	$\pi/3$

16.15 A wire 16 feet long has to be formed into a rectangle. What dimensions should the rectangle have to maximize the area?

Let x and y be the dimensions. Then $16 = 2x + 2y$, $8 = x + y$. Thus, $0 \le x \le 8$. The area $A = xy = x(8 - x) = 8x - x^2$, so $D_x A = 8 - 2x$, $D_x^2 A = -2$. Hence, the only critical number is $x = 4$. We can use the tabular method. Then the maximum value 16 is attained when $x = 4$. When $x = 4$, $y = 4$. Thus, the rectangle is a square.

x	4	0	8
A	16	0	0

16.16 Find the height h and radius r of a cylinder of greatest volume that can be cut within a sphere of radius b.

The axis of the cylinder must lie on a diameter of the sphere. From Fig. 16-6, $b^2 = r^2 + (h/2)^2$, so the volume of the cylinder $V = \pi r^2 h = \pi(b^2 - h^2/4)h = \pi(b^2 h - h^3/4)$. Then $D_h V = \pi(b^2 - 3h^2/4)$ and $D_h^2 V = -(3\pi/2)h$, so the critical number is $h = 2b/\sqrt{3}$. Since the second derivative is negative, there is a relative maximum at $h = 2b/\sqrt{3}$, which, by virtue of the uniqueness of the critical number, is an absolute maximum. When $h = 2b/\sqrt{3}$, $r = b\sqrt{\frac{2}{3}}$.

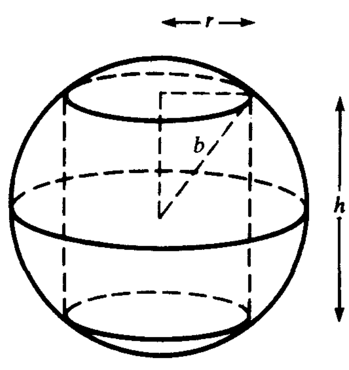

Fig. 16-6

16.17 Among all pairs of nonnegative numbers that add up to 5, find the pair that maximizes the product of the square of the first number and the cube of the second number.

Let x and y be the numbers. Then $x + y = 5$. We wish to maximize $P = x^2y^3 = (5 - y)^2y^3$. Clearly, $0 \le y \le 5$. $D_y P = (5 - y)^2(3y^2) + y^3[2(5 - y)(-1)] = (5 - y)y^2[3(5 - y) - 2y] = (5 - y)y^2(15 - 5y)$. Hence, the critical numbers are 0, 3, and 5. By the tabular method, the absolute maximum for P is 108, corresponding to $y = 3$. When $y = 3$, $x = 2$.

y	0	3	5
P	0	108	0

16.18 A solid steel cylinder is to be produced so that the sum of its height h and diameter $2r$ is to be at most 3 units. Find the dimensions that will maximize its volume.

We may assume that $h+2r=3$. So, $0 \le r \le \frac{3}{2}$. Then $V=\pi r^2 h = \pi r^2(3-2r) = 3\pi r^2 - 2\pi r^3$, so $D_r V = 6\pi r - 6\pi r^2$. Hence, the critical numbers are 0 and 1. We use the tabular method. The maximum value π is attained when $r=1$. When $r=1$, $h=1$.

r	0	1	2
V	0	π	0

16.19 Among all right triangles with fixed perimeter p, find the one with maximum area.

Let the triangle $\triangle ABC$ have a right angle at C, and let the two sides have lengths x and y (Fig. 16-7). Then the hypotenuse $AB = p - x - y$. Therefore, $(p-x-y)^2 = x^2 + y^2$, so $2(p-x-y)(-1-D_x y) = 2x + 2yD_x y$, $D_x y[(p-x-y)+y] = (-p+x-y)+x$, $D_x y(p-x) = y-p$, $D_x y = (y-p)/(p-x)$. Now, the area $A = \frac{1}{2}xy$, $D_x A = \frac{1}{2}(xD_x y + y) = \frac{1}{2}[x(y-p)/(p-x)+y] = \frac{1}{2}\{[x(y-p)+y(p-x)]/(p-x)\} = \frac{1}{2}[p(y-x)/(p-x)]$. Then, when $D_x A = 0$, $y = x$, and the triangle is isosceles. Then, $(p-2x)^2 = 2x^2$, $p - 2x = x\sqrt{2}$, $x = p/(2+\sqrt{2})$. Thus, the only critical number is $x = p/(2+\sqrt{2})$. Since $0 \le x \le p$, we can use the tabular method. When $x = p/(2+\sqrt{2})$, $y = p/(2+\sqrt{2})$, and $A = \frac{1}{2}[p(2+\sqrt{2})]^2$. When $x=0$, $A=0$. When $x = p/2$, $y=0$ and $A=0$. Hence, the maximum is attained when $x = y = p/(2+\sqrt{2})$.

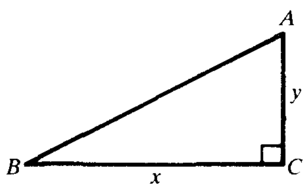

Fig. 16-7

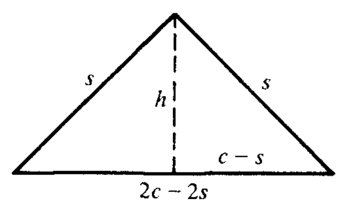

Fig. 16-8

16.20 Of all isosceles triangles with a fixed perimeter, which one has the maximum area?

Let s be the length of the equal sides, and let h be the altitude to the base. Let the fixed perimeter be $2c$. Then the base is $2c-2s$, and (Fig. 16-8) $h^2 = s^2 - (c-s)^2 = 2cs - c^2$. Hence, $2hD_s h = 2c$, $hD_s h = c$. The area $A = \frac{1}{2}h(2c-2s) = h(c-s)$. So, $D_s A = (c-s)D_s h - h = (c-s)c/h - h = \dfrac{c(c-s)-h^2}{h} = \dfrac{c^2 - cs - 2cs + c^2}{h} = \dfrac{2c^2 - 3cs}{h}$. Solving $2c^2 - 3cs = 0$, we find the critical number $s = \frac{2}{3}c$. Now, $D_s^2 A = \dfrac{hc(-3) - c(2c-3s)D_s h}{h^2}$, which eventually evaluates to $-(c^2/h^3)(3s-c)$. When $s = \frac{2}{3}c$, the second derivative becomes $-(c^3/h^3) < 0$. Hence, $s = \frac{2}{3}c$ yields a relative maximum, which by virtue of the uniqueness of the critical number, must be an absolute maximum. When $s = \frac{2}{3}c$, the base $2c - 2s = \frac{2}{3}c$. Hence, the triangle that maximizes the area is equilateral. [Can you see from the ellipse of Fig. 16-9 that of *all* triangles with a fixed perimeter, the equilateral has the greatest area?]

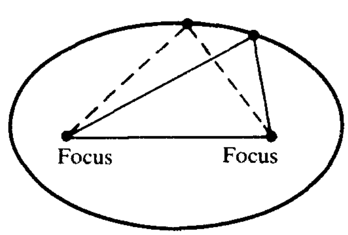

Fig. 16-9

16.21 A rectangular yard is to be built which encloses 400 ft^2. Two opposite sides are to be made from fencing which costs \$1 per foot, while the other two opposite sides are to be made from fencing which costs \$2 per foot. Find the least possible cost.

Let x be the length of the each side costing \$1 per foot, and let y be the length of each side costing \$2 per foot. Then $400 = xy$. The cost $C = 4y + 2x = 1600/x + 2x$. Hence, $D_xC = -1600/x^2 + 2$ and $D_x^2C = 1600/x^3$. Solving $-1600/x^2 + 2 = 0$, $x^2 = 800$, $x = 20\sqrt{2}$. Since the second derivative is positive, this yields a relative minimum, which, by virtue of the uniqueness of the critical number, must be an absolute minimum. Then the least cost is $1600/20\sqrt{2} + 40\sqrt{2} = 40\sqrt{2} + 40\sqrt{2} = 80\sqrt{2}$, which is approximately \$112.

16.22 A closed, right cylindrical container is to have a volume of 5000 in^3. The material for the top and bottom of the container will cost \$2.50 per in^2, while the material for the rest of the container will cost \$4 per in^2. How should you choose the height h and the radius r in order to minimize the cost?

$5000 = \pi r^2 h$. The lateral surface area is $2\pi rh$. Hence, the cost $C = 2(2.50)\pi r^2 + 4(2\pi rh) = 5\pi r^2 + 8\pi rh = 5\pi r^2 + 8\pi r(5000/\pi r^2) = 5\pi r^2 + 40{,}000/r$. Hence, $D_rC = 10\pi r - 40{,}000/r^2$ and $D_r^2C = 10\pi + (80{,}000/r^3)$. Solving $10\pi r - 40{,}000/r^2 = 0$, we find the unique critical number $r = 10\sqrt[3]{4/\pi}$. Since the second derivative is positive, this yields a relative minimum, which, by virtue of the uniqueness of the critical number, is an absolute minimum. The height $h = 5000/\pi r^2 = 25/\sqrt[3]{2\pi}$.

16.23 The sum of the squares of two nonnegative numbers is to be 4. How should they be chosen so that the product of their cubes is a maximum?

Let x and y be the numbers. Then $x^2 + y^2 = 4$, so $0 \le x \le 2$. Also, $2x + 2yD_xy = 0$, $D_xy = -x/y$. The product of their cubes $P = x^3y^3$, so $D_xP = x^3(3y^2D_xy) + 3x^2y^3 = x^3[3y^2(-x/y)] + 3x^2y^3 = -3x^4y + 3x^2y^3 = 3x^2y(-x^2 + y^2)$. Hence, when $D_xP = 0$, either $x = 0$ or $y = 0$ (and $x = 2$), or $y = x$ (and then, by $x^2 + y^2 = 4$, $x = \sqrt{2}$). Thus, we have three critical numbers $x = 0$, $x = 2$, and $x = \sqrt{2}$. Using the tabular method, with the endpoints 0 and 2, we find that the maximum value of P is achieved when $x = \sqrt{2}$, $y = \sqrt{2}$.

x	0	$\sqrt{2}$	2
P	0	8	0

16.24 Two nonnegative numbers are such that the first plus the square of the second is 10. Find the numbers if their sum is as large as possible.

Let x be the first and y the second number. Then $x + y^2 = 10$. Their sum $S = x + y = 10 - y^2 + y$. Hence, $D_yS = -2y + 1$, $D_y^2S = -2$, so the critical number is $y = \frac{1}{2}$. Since the second derivative is negative, this yields a relative maximum, which, by virtue of the uniqueness of the critical number, is an absolute maximum. $x = 10 - (\frac{1}{2})^2 = \frac{39}{4}$.

16.25 Find two nonnegative numbers x and y whose sum is 300 and for which x^2y is a maximum.

$x + y = 300$. The product $P = x^2y = x^2(300 - x) = 300x^2 - x^3$. $D_xP = 600x - 3x^2$, so the critical numbers are $x = 0$ and $x = 200$. Clearly, $0 \le x \le 300$, so using the tabular method, we find that the absolute maximum is attained when $x = 200$, $y = 100$.

x	0	200	300
P	0	4×10^6	0

16.26 A publisher decides to print the pages of a large book with $\frac{1}{2}$-inch margins on the top, bottom, and one side, and a 1-inch margin on the other side (to allow for the binding). The area of the entire page is to be 96 square inches. Find the dimensions of the page that will maximize the printed area of the page.

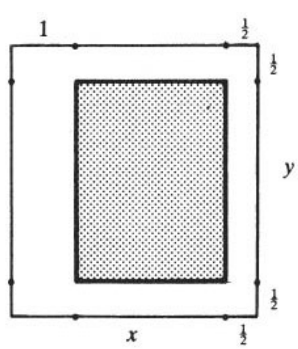

Fig. 16-10

Let x be the width and y be the height of the page. Then $96 = xy$, and (Fig. 16-10) the printed area $A = (x - \frac{3}{2})(y - 1)$. Hence, $0 = xD_xy + y$, $D_xy = -y/x$. Now $D_xA = (x - \frac{3}{2})D_xy + y - 1 = (x - \frac{3}{2})(-y/x) + y - 1 = 3y/2x - 1$. Therefore, $D_x^2A = \frac{3}{2}(xD_xy - y)/x^2 = \frac{3}{2}(-2y/x^2) < 0$. Setting $D_xA = 0$, we obtain $y = \frac{2}{3}x$, $96 = x(\frac{2}{3}x)$, $x^2 = 144$, $x = 12$, $y = 8$. Since the second derivative is negative, the unique critical number $x = 12$ yields an absolute maximum for A.

16.27 A paint manufacturer can produce anywhere from 10 to 30 cubic meters of paint per day. The profit for the day (in hundreds of dollars) is given by $P = (x - 15)^3/1000 - 3(x - 15)/10 + 300$, where x is the volume produced and sold. What value of x maximizes the profit?

$D_xP = \frac{3}{1000}(x - 15)^2 - \frac{3}{10}$. Setting $D_xP = 0$, $(x - 15)^2 = 100$, $x - 15 = \pm 10$, $x = 25$ or $x = 5$. Since $x = 5$ is not within the permissible range, the only critical number is $x = 25$. Using the tabular method, we find that the maximum profit is achieved when $x = 10$.

x	10	25	30
P	301.375	298	298.875

16.28 A printed page is to have a total area of 80 in^2 and margins of 1 inch at the top and on each side and of 1.5 inches at the bottom. What should the dimensions of the page be so that the printed area will be a maximum?

Let x be the width and y be the height of the page. Then $80 = xy$, $0 = xD_xy + y$, $D_xy = -y/x$. The area of the printed page $A = (x - 2)(y - 2.5)$, so $D_xA = (x - 2)\,D_xy + y - 2.5 = (x - 2)(-y/x) + y - 2.5 = -y + 2y/x + y - 2.5 = 2y/x - 2.5$. Also, $D_x^2A = 2(xD_xy - y)/x^2 = 2(-y - y)/x^2 = -4y/x^2 < 0$. Solving $D_xA = 0$, we find $y = 1.25x$, $80 = 1.25x^2$, $64 = x^2$, $x = 8$, $y = 10$. Since the second derivative is negative, this unique critical number yields an absolute maximum for A.

16.29 One side of an open field is bounded by a straight river. Determine how to put a fence around the other sides of a rectangular plot in order to enclose as great an area as possible with 2000 feet of fence.

Let x be the length of the side parallel to the river, and let y be the length of each of the other sides. Then $2y + x = 2000$. The area $A = xy = y(2000 - 2y) = 2000y - 2y^2$, $D_yA = 2000 - 4y$, and $D_y^2A = -4$. Solving $D_yA = 0$, we find the critical number $y = 500$. Since the second derivative is negative, this unique critical number yields an absolute maximum. $x = 2000 - 2(500) = 1000$.

16.30 A box will be built with a square base and an open top. Material for the base costs \$8 per square foot, while material for the sides costs \$2 per square foot. Find the dimensions of the box of maximum volume that can be built for \$2400.

Let s be the side of the base and h be the height. Then $V = s^2h$. We are told that $2400 = 8s^2 + 2(4hs)$, so $300 = s^2 + hs$, $h = 300/s - s$. Hence, $V = s^2(300/s - s) = 300s - s^3$. Then $D_sV = 300 - 3s^2$, $D_s^2V = -6s$. Solving $D_sV = 0$, we find the critical number $s = 10$. Since the second derivative is negative, the unique critical number yields an absolute maximum. $h = \frac{300}{10} - 10 = 20$.

16.31 Find the maximum area of any rectangle which may be inscribed in a circle of radius 1.

Let the center of the circle be the origin. We may assume that the sides of the rectangle are parallel to the coordinate axes. Let $2x$ be the length of the horizontal sides and $2y$ be the length of the vertical sides. Then $x^2 + y^2 = 1$, so $2x + 2yD_xy = 0$, $D_xy = -x/y$. The area $A = (2x)(2y) = 4xy$. So, $D_xA = 4(xD_xy + y) = 4\left(-\frac{x^2}{y} + y\right) = \frac{4}{y}(-x^2 + y^2)$. Also, $D_x^2A = 4\left(-\frac{2xy - x^2D_xy}{y^2} + D_xy\right) = 4\left(-\frac{2xy + x^3/y}{y^2} - \frac{x}{y}\right) = -4\left(\frac{2xy^2 + x^3 - xy^2}{y^3}\right) = -4\left(\frac{x^3 + xy^2}{y^3}\right) = -\frac{4x}{y^3}(x^2 + y^2) = -\frac{4x}{y^3}$. Solving $D_xA = 0$, we find $y = x$, $2x^2 = 1$, $x = 1/\sqrt{2}$, $y = 1/\sqrt{2}$. Since the second derivative is negative, the unique critical number yields an absolute maximum. The maximum area is $(2 \cdot 1/\sqrt{2})(2 \cdot 1/\sqrt{2}) = 2$.

16.32 A factory producing a certain type of electronic component has fixed costs of \$250 per day and variable costs of $90x$, where x is the number of components produced per day. The demand function for these components is $p(x) = 250 - x$, and the feasible production levels satisfy $0 \le x \le 90$. Find the level of production for maximum profit.

The daily income is $x(250-x)$, since $250-x$ is the price at which x units are sold. The profit $G=x(250-x)-(250+90x)=250x-x^2-250-90x=160x-x^2-250$. Hence, $D_xG=160-2x$, $D_x^2G=-2$. Solving $D_xG=0$, we find the critical number $x=80$. Since the second derivative is negative, the unique critical number yields an absolute maximum. Notice that this maximum, taken over a continuous variable x, is assumed for the integral value $x=80$. So it certainly has to remain the maximum when x is restricted to integral values (whole numbers of electronic components).

16.33 A gasoline station selling x gallons of fuel per month has fixed cost of \$2500 and variable costs of $0.90x$. The demand function is $1.50-0.00002x$ and the station's capacity allows no more than 20,000 gallons to be sold per month. Find the maximum profit.

The price that x gallons can be sold at is the value of the demand function. Hence, the total income is $x(1.50-0.00002x)$, and the profit $G=x(1.50-0.00002x)-2500-0.90x=0.60x-0.00002x^2-2500$. Hence, $D_xG=0.60-0.00004x$, and $D_x^2G=-0.00004$. Solving $D_xG=0$, we find $0.60=0.00004x$, $60{,}000=4x$, $x=15{,}000$. Since the second derivative is negative, the unique critical number $x=15{,}000$ yields the maximum profit \$2000.

16.34 Maximize the volume of a box, open at the top, which has a square base and which is composed of 600 square inches of material.

Let s be the side of the base and h be the height. Then $V=s^2h$. We are told that $600=s^2+4hs$. Hence, $h=(600-s^2)/4s$. So $V=s^2[(600-s^2)/4s]=(s/4)(600-s^2)=150s-\frac{1}{4}s^3$. Then $D_sV=150-\frac{3}{4}s^2$, $D_s^2V=-\frac{3}{2}s$. Solving $D_sV=0$, we find $200=s^2$, $10\sqrt{2}=s$. Since the second derivative is negative, this unique critical number yields an absolute maximum. When $s=10\sqrt{2}$, $h=5\sqrt{2}$.

16.35 A rectangular garden is to be completely fenced in, with one side of the garden adjoining a neighbor's yard. The neighbor has agreed to pay for half of the section of the fence that separates the plots. If the garden is to contain 432 ft^2, find the dimensions that minimize the cost of the fence to the garden's owner.

Let y be the length of the side adjoining the neighbor, and let x be the other dimension. Then $432=xy$, $0=x\,D_xy+y$, $D_xy=-y/x$. The cost $C=2x+y+\frac{1}{2}y=2x+\frac{3}{2}y$. Then, $D_xC=2+\frac{3}{2}(D_xy)=2+\frac{3}{2}(-y/x)$ and $D_x^2C=-\frac{3}{2}(xD_xy-y)/x^2=-\frac{3}{2}(-y-y)/x^2=3y/x^2$. Setting $D_xC=0$, We obtain $2=3y/2x$, $4x=3y$, $y=\frac{4}{3}x$, $432=x(4x/3)$, $324=x^2$, $x=18$, $y=24$. Since the second derivative is positive, the unique critical number $x=18$ yields the absolute minimum cost.

16.36 A rectangular box with open top is to be formed from a rectangular piece of cardboard which is 3 inches × 8 inches. What size square should be cut from each corner to form the box with maximum volume? (The cardboard is folded along the dotted lines to form the box.)

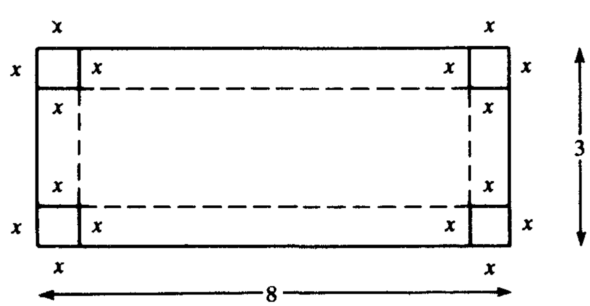

Fig. 16-11

Let x be the side of the square that is cut out. The length will be $8-2x$, the width $3-2x$, and the height x. Hence, the volume $V=x(3-2x)(8-2x)$; so $D_xV=(1)(3-2x)(8-2x)+x(-2)(8-2x)+x(3-2x)(-2)=4(3x-2)(x-3)$, and $D_x^2V=24x-44$. Setting $D_xV=0$, we find $x=\frac{2}{3}$ or $x=3$. Since the width 3 of the cardboard is greater than $2x$, we must have $x<\frac{3}{2}$. Hence, the value $x=3$ is impossible. Thus, we have a unique critical number $x=\frac{2}{3}$, and, for that value, the second derivative turns out to be negative. Hence, that critical number determines an absolute maximum for the volume.

16.37 Refer to Fig. 16-12. At 9 a.m., ship B was 65 miles due east of another ship, A. Ship B was then sailing due west at 10 miles per hour, and A was sailing due south at 15 miles per hour. If they continue their respective courses, when will they be nearest one another?

▮ Let the time t be measured in hours from 9 a.m. Choose a coordinate system with B moving along the x-axis and A moving along the y-axis. Then the x-coordinate of B is $65-10t$, and the y-coordinate of B is $-15t$. Let u be the distance between the ships. Then $u^2=(15t)^2+(65-10t)^2$. It suffices to minimize u^2. $D_t(u^2)=2(15t)(15)+2(65-10t)(-10)=650t-1300$, and $D_t^2(u^2)=650$. Setting $D_t(u^2)=0$, we obtain $t=2$, Since the second derivative is positive, the unique critical number yields an absolute minimum. Hence, the ships will be closest at 11 a.m.

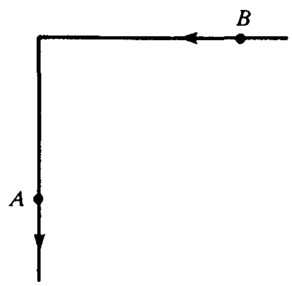

Fig. 16-12

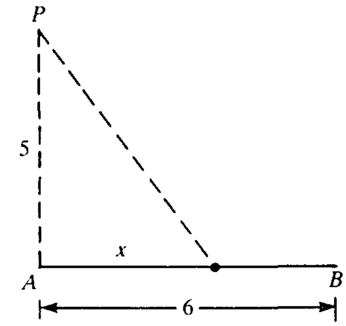

Fig. 16-13

16.38 A woman in a rowboat at P, 5 miles from the nearest point A on a straight shore, wishes to reach a point B, 6 miles from A along the shore (Fig. 16-13). If she wishes to reach B in the shortest time, where should she land if she can row 2 mi/h and walk 4 mi/h?

▮ Let x be the distance between A and the landing point. Then the distance rowed is $\sqrt{25+x^2}$ and the distance walked is $6-x$. Hence the total time $t=(\sqrt{25+x^2})/2+(6-x)/4$. Then $D_xt=\frac{1}{2}\frac{x}{\sqrt{25+x^2}}-\frac{1}{4}$. $D_x^2t=\frac{1}{2}\frac{\sqrt{25+x^2}-x^2/\sqrt{25+x^2}}{25+x^2}=\frac{1}{2}\frac{25}{(25+x^2)^{3/2}}$. Setting $D_xt=0$, we obtain $2x=\sqrt{25+x^2}$, $4x^2=25+x^2$, $3x^2=25$, $x=5\sqrt{3}/3\approx 2.89$. Since the second derivative is positive, the unique critical number yields the absolute minimum time.

16.39 A wall 8 feet high is 3.375 feet from a house. Find the shortest ladder that will reach from the ground to the house when leaning over the wall.

▮ Let x be the distance from the foot of the ladder to the wall. Let y be the height above the ground of the point where the ladder touches the house. Let L be the length of the ladder. Then $L^2=(x+3.375)^2+y^2$. It suffices to minimize L^2. By similar triangles, $y/8=(x+3.375)/x$. Then $D_xy=-27/x^2$. Now, $D_x(L^2)=2(x+3.375)+2yD_xy=2(x+3.375)+2(8/x)(x+3.375)(-27/x^2)=2(x+3.375)(1-216/x^3)$. Solving $D_x(L^2)=0$, we find the unique positive critical number $x=6$. Calculation of $D_x^2(L^2)$ yields $2+(2/x^4)[(27)^2+yx]>0$. Hence, the unique positive critical number yields the minimum length. When $x=6$, $y=\frac{25}{2}$, $L=\frac{125}{8}=15.625$ ft.

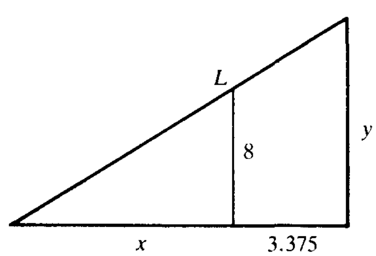

Fig. 16-14

16.40 A company offers the following schedule of charges: \$30 per thousand for orders of 50,000 or less, with the charge per thousand decreased by 37.5 cents for each thousand above 50,000. Find the order that will maximize the company's income.

▮ Let x be the number of orders in thousands. Then the price per thousand is 30 for $x\le 50$ and $30-\frac{3}{8}(x-50)$ for $x>50$. Hence, for $x\le 50$, the income $I=30x$, and, for $x>50$, $I=x[30-\frac{3}{8}(x-50)]=\frac{390}{8}x-\frac{3}{8}x^2$. So, for $x\le 50$, the maximum income is 1500 thousand. For $x>50$, $D_xI=\frac{390}{8}-\frac{3}{4}x$ and $D_x^2I=-\frac{3}{4}$. Solving $D_xI=0$, $x=65$. Since the second derivative is negative, $x=65$ yields the maximum income for $x>50$. That maximum is 3084.375 thousand. Hence, the maximum income is achieved when 65,000 orders are received.

16.41 A rectangle is inscribed in the ellipse $x^2/400 + y^2/225 = 1$ with its sides parallel to the axes of the ellipse (Fig. 16-15). Find the dimensions of the rectangle of maximum perimeter which can be so inscribed.

$x/200 + (2y/225)D_xy = 0$, $D_xy = -(9x/16y)$. The perimeter $P = 4x + 4y$, so $D_xP = 4 + 4D_xy = 4(1 - 9x/16y) = 4(16y - 9x)/16y$ and $D_x^2P = -\frac{9}{4}[y - x(-9x/16y)]/y^2 = -\frac{9}{4}(16y^2 + 9x^2)/16y^3 < 0$. Solving $D_xP = 0$, $16y = 9x$ and, then, substituting in the equation of the ellipse, we find $x^2 = 256$, $x = 16$, $y = 9$. Since the second derivative is negative, this unique critical number yields the maximum perimeter.

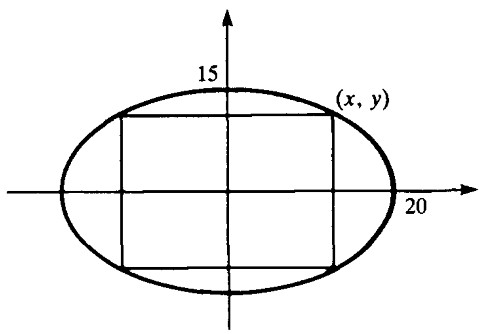

Fig. 16-15

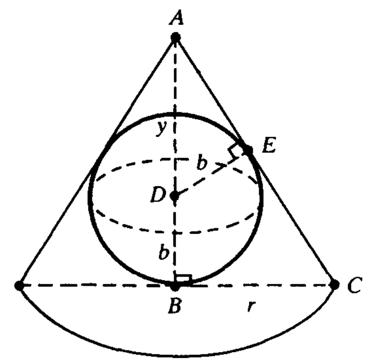

Fig. 16-16

16.42 Find the dimensions of the right circular cone of minimum volume which can be circumscribed about a sphere of radius b.

See Fig. 16-16. Let r be the radius of the base of the cone, and let $y + b$ be the height of the cone. From the similar triangles ABC and AED, $\frac{r}{b} = -\frac{y+b}{\sqrt{y^2 - b^2}}$. Then $r^2 = \frac{b^2(y+b)^2}{y^2 - b^2} = \frac{b^2(y+b)}{y-b}$. The volume of the cone $V = \frac{(\pi r^2)(y+b)}{3} = \frac{\pi b^2(y+b)^2}{3(y-b)}$. Hence, $D_yV = \frac{\pi b^2(y+b)(y-3b)}{3(y-b)^2}$. The only critical number $> b$ is $y = 3b$, and, by the first derivative test, this yields a relative minimum, which, by the uniqueness of the critical number, must be an absolute minimum.

16.43 Find the dimensions of the right circular cylinder of maximum volume that can be inscribed in a right circular cone of radius R and height H (Fig. 16-17).

Let r and h be the radius and height of the cylinder. By similar triangles, $r/(H - h) = R/H$, $r = (R/H)(H - h)$. The volume of the cylinder $V = \pi r^2h = \pi(R^2/H^2)(H - h)^2h$. Then $D_hV = (\pi R^2/H^2)(H - h)(H - 3h)$, so the only critical number for $h < H$ is $h = H/3$. By the first-derivative test, this yields a relative maximum, which, by the uniqueness of the critical number, is an absolute maximum. The radius $r = \frac{2}{3}R$.

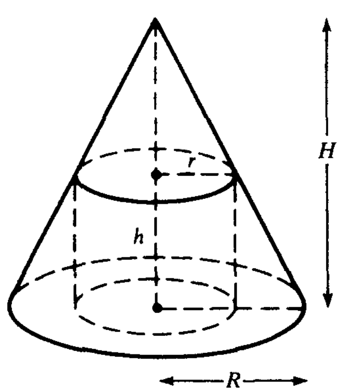

Fig. 16-17

16.44 A rectangular yard must be enclosed by a fence and then divided into two yards by a fence parallel to one of the sides. If the area A is given, find the ratio of the sides that will minimize the total length of the fencing.

Let y be the length of the side with the parallel inside fence, and let x be the length of the other side. Then $A = xy$. The length of fencing is $F = 3y + 2x = 3(A/x) + 2x$. So, $D_xF = -3A/x^2 + 2$, and $D_x^2F = 6A/x^3$. Solving $D_xF = 0$, we obtain $x^2 = \frac{3}{2}A$, $x = \sqrt{\frac{3}{2}A}$. Since the second derivative is positive, this unique critical number yields the absolute minimum for F. When $x = \sqrt{\frac{3}{2}A}$, $y = \sqrt{\frac{2}{3}A}$, and $y/x = \frac{2}{3}$.

16.45 Two vertices of a rectangle are on the positive x-axis. The other two vertices are on the lines $y = 4x$ and $y = -5x + 6$ (Fig. 16-18). What is the maximum possible area of the rectangle?

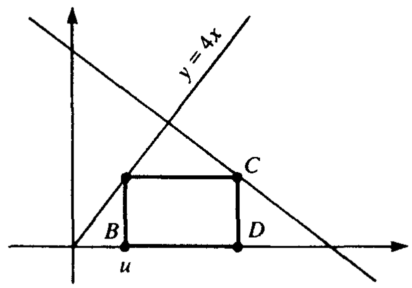

Fig. 16-18

Let u be the x-coordinate of the leftmost vertex B of the rectangle on the x-axis. Then the y-coordinate of the other two vertices is $4u$. The x-coordinate of the vertex C opposite B is obtained by solving the equation $y = -5x + 6$ for x when $y = 4u$. This yields $x = (6 - 4u)/5$. Hence, this is the x-coordinate of the other vertex D on the x-axis. Thus, the base of the rectangle is equal to $(6 - 4u)/5 - u = (6 - 9u)/5$. Therefore, the area of the rectangle $A = 4u(6 - 9u)/5 = \frac{24}{5}u - \frac{36}{5}u^2$. Then $D_uA = \frac{24}{5} - \frac{72}{5}u$ and $D_u^2A = -\frac{72}{5}$. Solving $D_uA = 0$, we find that the only positive critical number is $u = \frac{1}{3}$. Since the second derivative is negative, this yields the maximum area. When $u = \frac{1}{3}$, $A = \frac{4}{5}$.

16.46 A window formed by a rectangle surmounted by a semicircle is to have a fixed perimeter P. Find the dimensions that will admit the most light.

Let $2y$ be the length of the side on which the semicircle rests, and let x be the length of the other side. Then $P = 2x + 2y + \pi y$. Hence, $0 = 2D_yx + 2 + \pi$, $D_yx = -(2 + \pi)/2$. To admit the most light, we must maximize the area $A = 2xy + \pi y^2/2$. $D_yA = 2(x + D_yx \cdot y) + \pi y = 2x - 2y$, and $D_y^2A = 2D_yx - 2 = -\pi - 4 < 0$. Solving $D_yA = 0$, we obtain $x = y$, $P = (4 + \pi)x$, $x = P/(4 + \pi)$. Since the second derivative is negative, this unique critical number yields the maximum area, so the side on which the semicircle rests is twice the other side.

16.47 Find the y-coordinate of the point on the parabola $x^2 = 2py$ that is closest to the point $(0, b)$ on the axis of the parabola (Fig. 16-19).

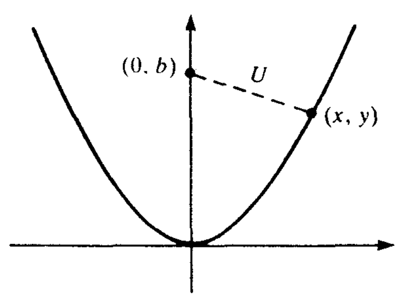

Fig. 16-19

It suffices to find the point (x, y) that minimizes the square of the distance between (x, y) and $(0, b)$. $U = x^2 + (y - b)^2$, and $D_xU = 2x + 2(y - b) \cdot D_xy$. But $2x = 2p\,D_xy$, $D_xy = x/p$. So $D_xU = (2x/p)(p + y - b)$. Also, $D_x^2U = (2/p)(x^2/p + p + y - b)$. Setting $D_xU = 0$, we obtain $x = 0$ or $y = b - p$. **Case 1.** $b \le p$. Then $b - p \le 0$, and, therefore, the only possible critical number is $x = 0$. By the first-derivative test, we see that $x = 0$, $y = 0$ yields the absolute minimum for U. **Case 2.** $b > p$. When $x = 0$, $U = b^2$. When $y = b - p$, $U = p(2b - p) < b^2$. When $y > b - p$, $D_xU > 0$ (for positive x) and, therefore, the value of U is greater than its value when $y = b - p$. Thus, the minimum value occurs when $y = b - p$.

16.48 A wire of length L is cut into two pieces, one is formed into a square and the other into a circle. How should the wire be divided to maximize or minimize the sum of the areas of the pieces?

Let the part used to form the circle be of length x. Then the radius of the circle is $x/2\pi$ and its area is $\pi(x/2\pi)^2 = x^2/4\pi$. The part used to form the square is $L - x$, the side of the square is $(L-x)/4$, and its area is $[(L-x)/4]^2$. So the total area $A = x^2/4\pi + [(L-x)/4]^2$. Then $D_xA = x/2\pi - \frac{1}{8}(L-x)$. Solving $D_xA = 0$, we obtain the critical value $x = \pi L/(4+\pi)$. Notice that $0 \le x \le L$. So, to obtain the minimum and maximum values for A, we need only calculate the values of A at the critical number and at the endpoints 0 and L. Clearly, $L^2/4(4+\pi) < L^2/16 < L^2/4\pi$. Hence, the maximum area is attained when $x = L$, that is, when all the wire is used for the circle. The minimum area is obtained when $x = \pi L/(4+\pi)$.

x	0	$\pi L/(4+\pi)$	L
A	$L^2/16$	$L^2/4(4+\pi)$	$L^2/4\pi$

16.49 Find the positive number x that exceeds its square by the largest amount.

We must maximize $f(x) = x - x^2$ for positive x. Then $f'(x) = 1 - 2x$ and $f''(x) = -2$. Hence, the only critical number is $x = \frac{1}{2}$. Since the second derivative is negative, this unique critical number yields an absolute maximum.

16.50 An east-west and a north-south road intersect at a point O. A diagonal road is to be constructed from a point E east of O to a point N north of O passing through a town C that is a miles east and b miles north of O. Find the distances of E and N from O if the area of $\triangle NOE$ is to be as small as possible.

Let x be the x-coordinate of E. Let y be the y-coordinate of N. By similar triangles, $y/b = x/(x-a)$, $y = bx/(x-a)$. Hence, the area A of $\triangle NOE$ is given by $A = \frac{1}{2}x\, bx/(x-a) = (b/2)x^2/(x-a)$. Using the quotient rule, $D_xA = (b/2)(x^2 - 2ax)/(x-a)^2$, and $D_x^2A = a^2h/(x-a)^3$. Solving $D_xA = 0$, we obtain the critical number $x = 2a$. The second derivative is positive, since $x > a$ is obviously necessary. Hence, $x = 2a$ yields the minimum area A. When $x = 2a$ $y = 2b$.

16.51 A wire of length L is to be cut into two pieces, one to form a square and the other to form an equilateral triangle. How should the wire be divided to maximize or to minimize the sum of the areas of the square and triangle?

Let x be the part used for the triangle. Then the side of the triangle is $x/3$ and its height is $x\sqrt{3}/6$. Hence, the area of the triangle is $\frac{1}{2}(x/3)(x\sqrt{3}/6) = \sqrt{3}x^2/36$. The side of the square is $(L-x)/4$, and its area is $[(L-x)/4]^2$. Hence, the total area $A = \sqrt{3}x^2/36 + [(L-x)/4]^2$. Then $D_xA = x\sqrt{3}/18 - (L-x)/8$. Setting $D_xA = 0$, we obtain the critical number $x = 9L/(9+4\sqrt{3})$. Since $0 \le x \le L$, to find the maximum and minimum values of A we need only compute the values of A at the critical number and the endpoints. It is clear that, since $16 < 12\sqrt{3} < 16 + 12\sqrt{3}$, the maximum area corresponds to $x = 0$, where everything goes into the square, and the minimum value corresponds to the critical number.

x	0	$9L/(9+4\sqrt{3})$	L
A	$L^2/16$	$L^2/(16+12\sqrt{3})$	$L^2/12\sqrt{3}$

16.52 Two towns A and B are, respectively, a miles and b miles from a railroad line (Fig. 16-20). The points C and D on the line nearest to A and B, respectively, are at a distance of c miles from each other. A station S is to be located on the line so that the sum of the distances from A and B to S is minimal. Find the position of S.

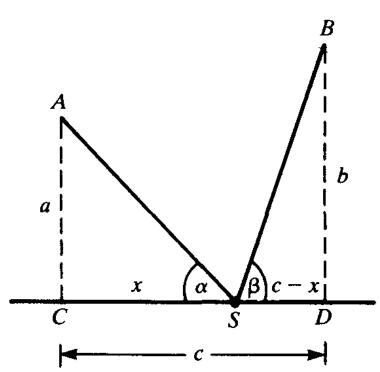

Fig. 16-20

Let x be the distance of S from C. Then the sum of the distances from A and B to S is given by the function $f(x)=\sqrt{a^2+x^2}+\sqrt{b^2+(c-x)^2}$. Hence, $f'(x)=x/\sqrt{a^2+x^2}-(c-x)/\sqrt{b^2+(c-x)^2}$. Setting $f'(x)=0$, we eventually obtain the equation (*) $a/x=b/(c-x)$, $x=ac/(a+b)$. To see that this yields the absolute minimum, computation of $f''(x)$ yields (after extensive simplifications) $a^2/(a^2+x^2)^{3/2}+b^2/[b^2+(c-x)^2]^{3/2}$, which is positive. [Notice that the equation (*) also tells us that the angles α and β are equal. If we reinterpret this problem in terms of a light ray from A being reflected off a mirror to B, we have found that the angle of incidence α is equal to the angle of reflection β.]

16.53 A telephone company has to run a line from a point A on one side of a river to another point B that is on the other side, 5 miles down from the point opposite A (Fig. 16-21). The river is uniformly 12 miles wide. The company can run the line along the shoreline to a point C and then run the line under the river to B. The cost of laying the line along the shore is \$1000 per mile, and the cost of laying it under water is twice as great. Where should the point C be located to minimize the cost?

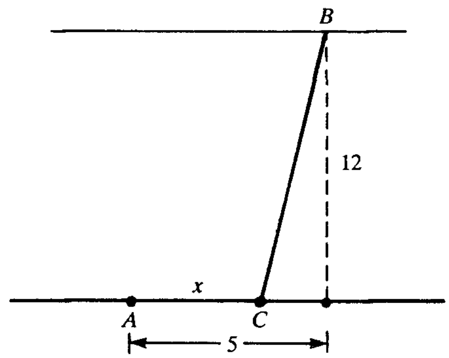

Fig. 16-21

Let x be the distance from A to C. Then the cost of running the line is $f(x)=1000x+2000\sqrt{144+(5-x)^2}$. $f'(x)=1000-\dfrac{2000(5-x)}{\sqrt{144+(5-x)^2}}$, and $f''(x)=2000\dfrac{144+2(5-x)^2}{[144+(5-x)^2]^{3/2}}>0$. Setting $f'(x)=0$ and solving for x, $48=(x-5)^2$, $x=5\pm 4\sqrt{3}$. Since x cannot be negative or greater than 5, neither critical number is feasible. So, the minimum occurs at an endpoint. Since $f(0)=26{,}000$ and $f(5)=29{,}000$, the minimum occurs at $x=0$.

16.54 Let m and n be given positive integers. If x and y are positive numbers such that $x+y$ is a constant S, find the values of x and y that maximize $P=x^m y^n$.

$P=x^m(S-x)^n$. $D_xP=mx^{m-1}(S-x)^n-nx^m(S-x)^{n-1}$. Setting $D_xP=0$, we obtain $x=mS/(m+n)$. The first-derivative test shows that this yields a relative maximum, and, therefore, by the uniqueness of the critical number, an absolute maximum. When $x=mS/(m+n)$, $y=nS/(m+n)$.

16.55 Show that of all triangles with given base and given area, the one with the least perimeter is isosceles. (Compare with Problem 16.20.)

Let the base of length $2c$ lie on the x-axis with the origin as its midpoint, and let the other vertex (x, y) lie in the upper half-plane (Fig. 16-22). By symmetry we may assume $x\ge 0$. To minimize the perimeter, we must minimize $AC+BC$, which is given by the function $f(x)=\sqrt{(c+x)^2+h^2}+\sqrt{(c-x)^2+h^2}$, where h is the altitude. Then $f'(x)=\dfrac{c+x}{\sqrt{(c+x)^2+h^2}}-\dfrac{c-x}{\sqrt{(c-x)^2+h^2}}$. Setting $f'(x)=0$, we obtain $c+x=\pm(c-x)$. The minus sign leads to the contradiction $c=0$. Therefore, $c+x=c-x$, $x=0$. Thus, the third vertex lies on the y-axis and the triangle is isosceles. That the unique critical number $x=0$ yields an absolute minimum follows from a computation of the second derivative, which turns out to be positive.

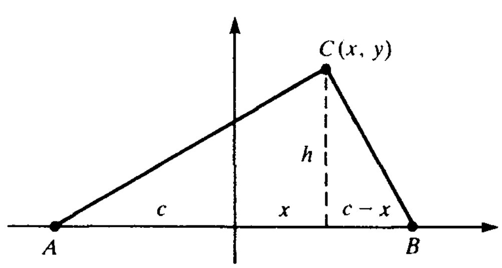

Fig. 16-22

16.56 Let two corridors of widths a and b intersect at a right angle. Find the minimum length of all segments DE that touch the outer walls of the corridors and are tangent to the corner C.

Let θ be the angle between DE and the vertical (see Fig. 16-23). Then $0 < \theta < \pi/2$. Let L be the length of DE. $L = b \sec\theta + a\csc\theta$, and $D_\theta L = b\sec\theta\tan\theta - a\csc\theta\cot\theta$. Setting $D_\theta L = 0$, $b\sec\theta\tan\theta = a\csc\theta\cot\theta$, $b\sin\theta/\cos^2\theta = a\cos\theta/\sin^2\theta$, $b\sin^3\theta = a\cos^3\theta$, $\tan^3\theta = a/b$, $\tan\theta = \sqrt[3]{a/b} = \sqrt[3]{a}/\sqrt[3]{b}$. Consider the hypotenuse u of a right triangle with legs $\sqrt[3]{a}$ and $\sqrt[3]{b}$. Then $u^2 = a^{2/3} + b^{2/3}$, $u = (a^{2/3} + b^{2/3})^{1/2}$. $\sec\theta = (a^{2/3} + b^{2/3})^{1/2}/b^{1/3}$, $\csc\theta = (a^{2/3} + b^{2/3})^{1/2}/a^{1/3}$. So, $L = (a^{2/3} + b^{2/3})^{1/2}(b^{2/3} + a^{2/3}) = (a^{2/3} + b^{2/3})^{3/2}$. Observe that $D_\theta L = (b\cos\theta/\sin^2\theta)(\tan^3\theta - a/b)$. Hence, the first-derivative test yields the case $\{-, +\}$, which, by virtue of the uniqueness of the critical number, shows that $L = (a^{2/3} + b^{2/3})^{3/2}$ is the absolute minimum. Notice that this value of L is the minimal length of all poles that cannot turn the corner from one corridor into the other.

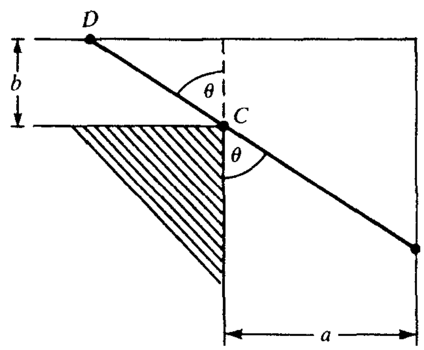

Fig. 16-23

16.57 A rectangular yard is to be laid out and fenced in, and then divided into 10 enclosures by fences parallel to one side of the yard. If a fixed length K of fencing is available, what dimensions will maximize the area?

Let x be the length of the sides of the enclosure fences, and let y be the other side. Then $K = 11x + 2y$. The area $A = xy = x(K - 11x)/2 = (K/2)x - \frac{11}{2}x^2$. Hence, $D_xA = K/2 - 11x$, and $D_x^2A = -11$. Setting $D_xA = 0$, we obtain the critical number $x = K/22$. Since the second derivative is negative, we have a relative maximum, and, since the critical number is unique, the relative maximum is an absolute maximum. When $x = K/22$, $y = K/4$.

16.58 Two runners A and B start at the origin and run along the positive x-axis, with B running 3 times as fast as A. An observer, standing one unit above the origin, keeps A and B in view. What is the maximum angle of sight θ between the observer's view of A and B? (See Fig. 16-24.)

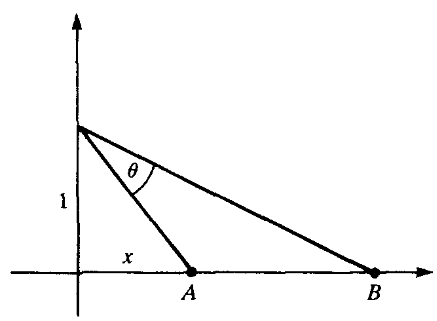

Fig. 16-24

Let x be the distance of A from the origin. Then B is $3x$ units from the origin. Let θ_1 be the angle between the y-axis and the line of sight of A, and let θ_2 be the corresponding angle for B. Then $\theta = \theta_2 - \theta_1$. Note that $\tan\theta_1 = x$ and $\tan\theta_2 = 3x$. So $\tan\theta = \dfrac{\tan\theta_2 - \tan\theta_1}{1 + \tan\theta_1\tan\theta_2} = \dfrac{3x - x}{1 + x(3x)} = \dfrac{2x}{1 + 3x^2}$. Since θ is between 0 and $\pi/2$, maximizing θ is equivalent to maximizing $\tan\theta$. Now, $D_x(\tan\theta) = \dfrac{(1 + 3x^2)(2) - 2x(6x)}{(1 + 3x^2)^2} = \dfrac{2(1 + 3x^2 - 6x^2)}{(1 + 3x^2)^2} = \dfrac{2(1 - 3x^2)}{(1 + 3x^2)^2}$. Setting $D_x(\tan\theta) = 0$, we obtain $1 = 3x^2$, $x = 1/\sqrt{3}$. The first derivative test shows that we have a relative maximum, which, by uniqueness, must be the absolute maximum. When $\tan\theta = 1/\sqrt{3}$, $\theta = 30°$.

16.59 A painting of height 3 feet hangs on the wall of a museum, with the bottom of the painting 6 feet above the floor. If the eyes of an observer are 5 feet above the floor, how far from the base of the wall should the observer stand to maximize his angle of vision θ? See Fig. 16-25.

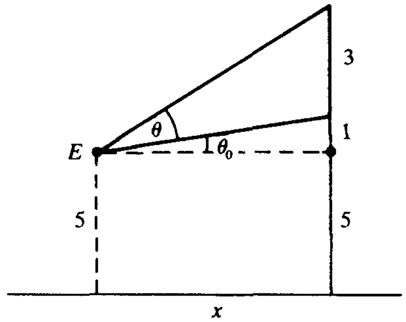

Fig. 16-25

Let x be the distance from the observer to the base of the wall, and let θ_0 be the angle between the line of sight of the bottom of the painting and the horizontal. Then $\tan(\theta+\theta_0)=4/x$ and $\tan\theta_0=1/x$. Hence, $\tan\theta=\dfrac{\tan(\theta+\theta_0)-\tan\theta_0}{1+\tan(\theta+\theta_0)\tan\theta_0}=\dfrac{4/x-1/x}{1+4/x^2}=\dfrac{3x}{x^2+4}$. Since maximizing θ is equivalent to maximizing $\tan\theta$, it suffices to do the latter. Now, $D_x(\tan\theta)=3\left[\dfrac{(x^2+4)-x(2x)}{(x^2+4)^2}\right]=\dfrac{3(4-x^2)}{(x^2+4)^2}$. Hence, the unique positive critical number is $x=2$. The first-derivative test shows this to be a relative maximum, and, by the uniqueness of the positive critical number, this is an absolute maximum.

16.60 A large window consists of a rectangle with an equilateral triangle resting on its top (Fig. 16-26). If the perimeter P of the window is fixed at 33 feet, find the dimensions of the rectangle that will maximize the area of the window.

Let s be the side of the rectangle on which the triangle rests, and let y be the other side. Then $33=2y+3s$. The height of the triangle is $(\sqrt{3}/2)s$. So the area $A=sy+\frac{1}{2}s(\sqrt{3}/2)s=s(33-3s)/2+(\sqrt{3}/4)s^2=\frac{33}{2}s+[(\sqrt{3}-6)/4]s^2$. $D_sA=\frac{33}{2}+[(\sqrt{3}-6)/2]s$. Setting $D_sA=0$, we find the critical number $s=33/(6-\sqrt{3})=6+\sqrt{3}$. The first-derivative test shows that this yields a relative maximum, which, by virtue of the uniqueness of the critical number, must be an absolute maximum. When $s=6+\sqrt{3}$, $y=\frac{3}{2}(5-\sqrt{3})$.

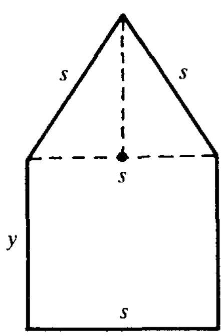

Fig. 16-26

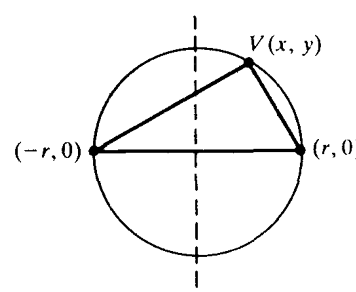

Fig. 16-27

16.61 Consider triangles with one side on a diameter of a circle of radius r and with the third vertex V on the circle (Fig. 16-27). What location of V maximizes the perimeter of the triangle?

Let the origin be the center of the circle, with the diameter along the x-axis, and let (x, y), the third vertex, lie in the upper half-plane. Then the perimeter $P=2r+\sqrt{(x+r)^2+y^2}+\sqrt{(x-r)^2+y^2}$, with $-r<x<r$. $D_xP=\dfrac{(x+r)+y\,D_xy}{\sqrt{(x+r)^2+y^2}}+\dfrac{(x-r)+y\,D_xy}{\sqrt{(x-r)^2+y^2}}$. But, from the equation of the circle $x^2+y^2=r^2$, we find by implicit differentiation that $D_xy=-x/y$. Hence, D_xP becomes $\dfrac{x+r-x}{\sqrt{(x+r)^2+y^2}}+\dfrac{x-r-x}{\sqrt{(x-r)^2+y^2}}=r\left(\dfrac{\sqrt{(x-r)^2+y^2}-\sqrt{(x+r)^2+y^2}}{\sqrt{(x+r)^2+y^2}\sqrt{(x-r)^2+y^2}}\right)$. Then we set $D_xP=0$, and solving for x, find that the critical number is $x=0$. The corresponding value of P is $2r(1+\sqrt{2})$. At the endpoints $x=-r$ and $x=r$, the value of P is $4r$. Since $4r<2r(1+\sqrt{2})$, the maximum perimeter is attained when $x=0$ and $y=r$, that is, V is on the diameter perpendicular to the base of the triangle.