Interim Report

Predicting Algae Bloom spread in the Marlborough Sounds using Bayesian Inference

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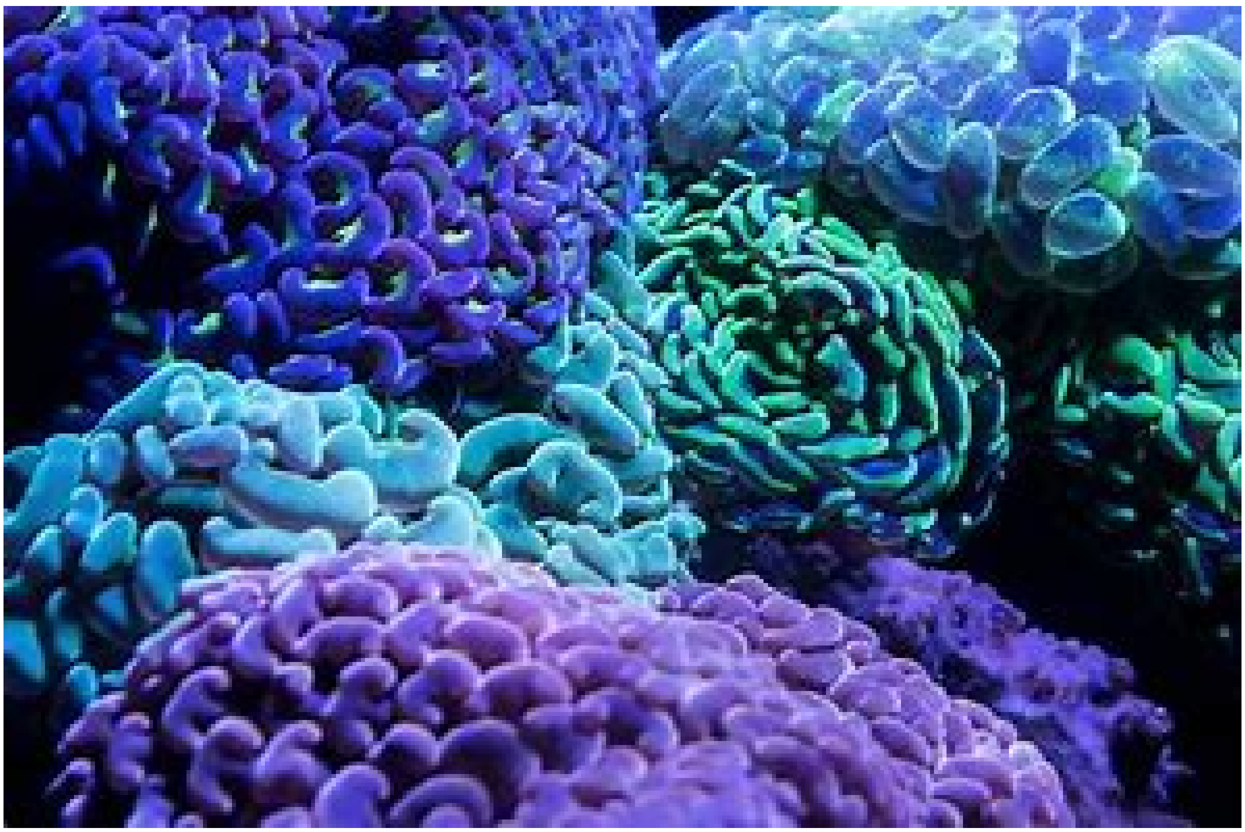


Introduction

This is an interim technical report describing the current state of my Metropolis-Hastings algorithm for Predicting Algae Bloom spread, using a simplified model involving a singular event.

Our Model

Mass

Mass is observed as:

- n Sites and T discrete Times
- Growth parameter α
- Decay parameter β
- Dispersion Matrix P where p_{ij} is the proportion of Mass at site i that travels to site j (Such that $\sum_{j=1}^{n} p_{ij} = 1$ for all $i \in \{1,..., n\}$)
- N events
- Each of $k \in \{1,..., N\}$ events has:
  - Starting site I_k with uniform probability $\frac{1}{n}$
  - Starting time $T_k \in \{1,..., T\}$
  - Length of time W_k
  - Starting size S_k

Where the equation for $M_{i,t}$ is:

$$M_{i,t} = \alpha A_{i,t} M_{i,t-1} + \beta\left(\sum_{j=1}^{n} p_{i,j} M_{j,t-1}\right) + B_{i,t}$$

Where

$$A_{i,t} = I\left(\sum_{k=1}^{n} I\left(I_k = i\right) I\left(T_k \le t \le T_k + W_k\right) > 0\right)$$

$$B_{i,t} = \sum_{k=1}^{n} I\left(I_k = i\right) I\left(T_k = t\right) S_k$$

Concentration

Concentration is calculated from Mass where:

- Each site $i \in \{1,..., n\}$ has volume V_i
- δ is a concentration offset of approximately 0^+
- Precision parameter τ

Where the equation for $C_{i,t}$ is

$$log(Y_{i,t}) \sim N\left(log\left(\delta + \frac{M_{i,t}}{V_i}\right), \frac{1}{\tau}^2\right)$$

Parameters

Known Parameters:

- Number of sites, n
- Time period, T
- Dispersion Matrix, P
- Number of events, N
- Volumes, V_i
- Concentration offset, δ

Parameters to be estimated

- Growth Parameter, α
- Decay Parameter, β
- Event site, I_k
- Event Starting time, T_k
- Event time length, W_k
- Event size, S_k
- Precision τ

Priors

$\alpha \sim Gamma\left(a_{\alpha}, b_{\alpha}\right)$

$\beta \sim Beta\left(a_{\beta}, b_{\beta}\right)$

$I_{k} \sim DiscreteUniform(1, n)$

$T_{k} \sim DiscreteUniform(1, T)$

$W_{k} \sim NegBinomial\left(r_{W}, p_{W}\right)$

$S_{k} \sim Gamma\left(a_{S}, b_{S}\right)$

$\tau \sim Gamma\left(a_{\tau}, b_{\tau}\right)$

Where:

$a_{\beta}, a_{S}, a_{\tau}, b = 1$

$a_{\alpha} = 2$

$r_{W} = 6$

$p_{W} = 0.75$

Pseudocode

FindM$_{New}$

Inputs:

- M
- α
- β
- P
- A
- B

$growth = \alpha \times (AM)$

$transport = \beta \times (P \cdot M)$

$M_{New} = B + growth + transport$

Outputs:

- M_{New}

RunSimulation

Inputs:

- n
- T
- P

$\alpha \sim Gamma(2,\ 1)$

$\beta \sim Beta(4,\ 3)$

$N \sim DiscreteUniform\left(1,\ \frac{T}{10}\right)$

$For\ x\ in\ 1{:}\ N{:}$

$i \sim DiscreteUniform(1,\ n)$

$t \sim DiscreteUniform(1,\ T)$

$w \sim NegBinom(6,\ 0.75)\ +\ 1$

$s \sim Gamma(1,\ 1)$

$A_{t:min(t+w,T),\, i} = 1$

$B_{t,\, i} = s$

$\left[M_{0,\, 0},\ldots,\ M_{n,\, 0}\right] = 0$

$For\ t\ in\ 1{:}\ T{:}$

$M_{New} = FindM_{New}\left(\left[M_{0,\, t-1},\ldots,\ M_{n,\, t-1}\right],\ \alpha,\ \beta,\ P_{i:n,\, i:n},\ \left[A_{t,\, 1},\ldots, A_{t,\, n}\right],\ \left[B_{t,\, 1},\ldots, B_{t,\, n}\right]\right)$

$\left[M_{0,\, t},\ldots,\ M_{n,\, t}\right] = M_{New}$

Outputs:

- $\left[\left[M_{0,\, 1},\ldots,\ M_{n,\, 1}\right],\ldots, \left[M_{0,\, T},\ldots,\ M_{n,\, T}\right]\right]$
- α
- β
- N
- $\left[I_{0},\ldots,\ I_{N}\right]$
- $\left[T_{0},\ldots,\ T_{N}\right]$
- $\left[W_{0},\ldots,\ W_{N}\right]$
- $\left[S_{0},\ldots,\ S_{N}\right]$

GetSample

Inputs:

- n
- T
- P
- α
- β
- N
- $\left[I_1, ..., I_N\right]$
- $\left[t_1, ..., t_N\right]$
- $\left[W_1, ..., W_N\right]$
- $\left[s_1, ..., s_N\right]$

$For\ x\ in\ 1: N:$

$$A_{t_x : min(t_x + W_x, t), I_x} = 1$$

$$B_{t, i} = s_x$$

$$\left[M_{0,0}, ..., M_{n,0}\right] = 0$$

$For\ t\ in\ 1: T:$

$$M_{New} = FindM_{New}\left(\left[M_{0, t-1}, ..., M_{n, t-1}\right], \alpha, \beta, P_{i:n, i:n}, \left[A_{t,1}, ..., A_{t,n}\right], \left[B_{t,1}, ..., B_{t,n}\right]\right)$$

$$\left[M_{0,t}, ..., M_{n,t}\right] = M_{New}$$

Outputs:

- $\left[\left[M_{0,1}, ..., M_{n,1}\right], ..., \left[M_{0,T}, ..., M_{n,T}\right]\right]$

ConvertConcentration

Inputs:

- n
- T
- M
- V
- δ

$For\ i\ in\ 1:n:$

$$C_{i,\,1:T} = \frac{M_{i,\,1:T}}{V_i}$$

$$C_{log} = log(C + \delta)$$

Outputs:

- C_{log}

ConvertY

Inputs:

- C
- τ

$$Y_{log} \sim N\left(C, \frac{1}{\sqrt{\tau}}\right)$$

Outputs:

- Y_{log}

Metropolis-Hastings Algorithm

Inputs:

- P

$n = 10$

$T = 100$

$\delta = 1^{-10}$

$\tau_{True} \sim Gamma(1, 1)$

$[V_1, ..., V_n] \sim NegBinom(1, 1) + 1$

$[M_{True}, \alpha_{True}, \beta_{True}, N_{True}, I_{True}, T_{True}, W_{True}, S_{True}] = runSimulation(n, T, P)$

$C_{True} = ConvertCalculation(n, T, M_{True}, V, \delta)$

$Y_{True} = ConvertY(C_{True}, \tau_{True})$

$\alpha_0 \sim Gamma(2, 1)$

$\beta_0 \sim Beta(1, 1)$

$N = N_{True}$

$[I_{0,1}, ..., I_{0,N}] \sim DiscreteUniform(1, n)$

$[t_{1,1}, ..., t_{0,N}] \sim DiscreteUniform(1, T)$

$[w_{0,1}, ..., w_{0,N}] \sim NegBinom(6, 0.75)$

$[s_{0,1}, ..., s_{0,N}] \sim Gamma(1, 1)$

$\tau_0 \sim Gamma(1, 1)$

$\phi = [\alpha_0, \beta_0, N, [I_{0,1}, ..., I_{0,N}], [t_{0,1}, ..., t_{0,N}], [w_{0,1}, ..., w_{0,N}], [s_{0,1}, ..., s_{0,N}], \tau_0]$

$For\ each\ parameter_0\ in\ \phi$:

$Array_{parameter} = parameter_0$

$Accept_{parameter} = 0$

$For\ s\ in\ 1{:}\ 10000{:}$

Alpha

$$\alpha_{Candidate} \sim N\left(\alpha_{Candidate}, \frac{1}{2}^{2}\right)$$

$$M_{Candidate} = GetSample\left(n, T, \alpha_{Candidate}, \phi_{\beta}, \phi_{N}, \phi_{I}, \phi_{T}, \phi_{W}, \phi_{S}\right)$$

$$C_{Candidate} = ConvertConcentration\left(n, T, M_{Candidate}, V, \delta\right)$$

$$M_{Current} = GetSample\left(n, T, \phi_{\alpha}, \phi_{\beta}, \phi_{N}, \phi_{I}, \phi_{T}, \phi_{W}, \phi_{S}\right)$$

$$C_{Current} = ConvertConcentration\left(n, T, M_{Current}, V, \delta\right)$$

$$p_{prior_{Candidate}} = Gamma(2, 1).logPDF\left(\alpha_{Candidate}\right)$$

$$p_{prior_{Current}} = Gamma(2, 1).logPDF\left(\phi_{\alpha}\right)$$

$$p_{Likelihood_{Candidate}} = \sum_{t=1}^{T}\left(\sum_{i=1}^{n} N\left(C_{Candidate_{it}}, \frac{1}{\sqrt{\phi_{\tau}}}^{2}\right).logPDF\left(Y_{True_{it}}\right)\right)$$

$$p_{Likelihood_{Current}} = \sum_{t=1}^{T}\left(\sum_{i=1}^{n} N\left(C_{Current_{it}}, \frac{1}{\sqrt{\phi_{\tau}}}^{2}\right).logPDF\left(Y_{True_{it}}\right)\right)$$

$$p_{Q_{Candidate}} = N\left(\phi_{1}, \frac{1}{2}^{2}\right).logPDF\left(\alpha_{Candidate}\right)$$

$$p_{Q_{Current}} = N\left(\alpha_{Candidate}, \frac{1}{2}^{2}\right).logPDF\left(\phi_{\alpha}\right)$$

$$ratio_{\alpha} = min\left(p_{prior_{Candidate}} + p_{Likelihood_{Candidate}} - p_{prior_{Current}} - p_{Likelihood_{Current}} + p_{Q_{Current}} - p_{Q_{Candidate}}, 1\right)$$

$$u \sim U(0, 1)$$

$$if\ ratio > log(u){:}$$

$$\phi_{\alpha} = \alpha_{Candidate}$$

$$Accept_{\alpha} = Accept_{\alpha} + 1$$

$$Likelihood_{\alpha_s} = \sum_{t=1}^{T}\left(\sum_{i=1}^{n} N\left(C_{Candidate_{it}}, \frac{1}{\sqrt{\phi_{\tau}}}^{2}\right).logPDF\left(Y_{True_{it}}\right)\right)$$

else:

$$Likelihood_{\alpha_s} = \sum_{t=1}^{T}\left(\sum_{i=1}^{n} N\left(C_{Current_{it}}, \frac{1}{\sqrt{\phi_{\tau}}}^{2}\right).logPDF\left(Y_{True_{it}}\right)\right)$$

Beta

$$\beta_{Candidate} \sim Beta(1,\ 1)$$

$$M_{Candidate} = GetSample\left(n,\ T,\ \phi_{\alpha},\ \beta_{Candidate},\ \phi_{N},\ \phi_{I},\ \phi_{T},\ \phi_{W},\ \phi_{S}\right)$$

$$C_{Candidate} = ConvertConcentration\left(n,\ T,\ M_{Candidate},\ V,\ \delta\right)$$

$$M_{Current} = GetSample\left(n,\ T, \phi_{\alpha},\ \phi_{\beta},\ \phi_{N},\ \phi_{I},\ \phi_{T},\ \phi_{W},\ \phi_{S}\right)$$

$$C_{Current} = ConvertConcentration\left(n,\ T,\ M_{Current},\ V,\ \delta\right)$$

$$p_{prior_{Candidate}} = Beta(1,\ 1).logPDF\left(\beta_{Candidate}\right)$$

$$p_{prior_{Current}} = Beta(1,\ 1).logPDF\left(\phi_{2}\right)$$

$$p_{Likelihood_{Candidate}} = \sum_{t=1}^{T}\left(\sum_{i=1}^{n} N\left(C_{Candidate_{it}}, \frac{1}{\sqrt{\phi_{\tau}}}^{2}\right).logPDF\left(Y_{True_{it}}\right)\right)$$

$$p_{Likelihood_{Current}} = \sum_{t=1}^{T}\left(\sum_{i=1}^{n} N\left(C_{Current_{it}}, \frac{1}{\sqrt{\phi_{\tau}}}^{2}\right).logPDF\left(Y_{True_{it}}\right)\right)$$

$$p_{Q_{Candidate}} = Beta(1,\ 1).logPDF\left(\beta_{Candidate}\right)$$

$$p_{Q_{Current}} = Beta(1,\ 1).logPDF\left(\phi_{2}\right)$$

$$ratio_{\beta} = min\left(p_{prior_{Candidate}} + p_{Likelihood_{Candidate}} - p_{prior_{Current}} - p_{Likelihood_{Current}} + p_{Q_{Current}} - p_{Q_{Candidate}},\ 1\right)$$

$$u \sim U(0,\ 1)$$

$if\ ratio > u$:

$$\phi_{\beta} = \beta_{Candidate}$$

$$Accept_{\beta} = Accept_{\beta} + 1$$

$$Likelihood_{\beta_s} = \sum_{t=1}^{T}\left(\sum_{i=1}^{n} N\left(C_{Candidate_{it}},\ \frac{1}{\sqrt{\phi_{\tau}}}^{2}\right).logPDF\left(Y_{True_{it}}\right)\right)$$

$else$:

$$Likelihood_{\beta_s} = \sum_{t=1}^{T}\left(\sum_{i=1}^{n} N\left(C_{Current_{it}},\ \frac{1}{\sqrt{\phi_{\tau}}}^{2}\right).logPDF\left(Y_{True_{it}}\right)\right)$$

I

$for\ k\ in\ 1:n$:

$$I_{k_{Candidate}} \sim DiscreteUniform(1,\ n)$$

$$I_{Candidate} = \phi_{3}$$

$$I_{Candidate_k} = I_{k_{Candidate}}$$

$$M_{Candidate} = GetSample\left(n,\ T,\ \phi_{\alpha},\ \phi_{\beta},\ \phi_{N},\ I_{Candidate},\ \phi_{T},\ \phi_{W},\ \phi_{S}\right)$$

$$C_{Candidate} = ConvertConcentration\left(n,\ T,\ M_{Candidate},\ V,\ \delta\right)$$

$$M_{Current} = GetSample\left(n,\ T, \phi_{\alpha},\ \phi_{\beta},\ \phi_{N},\ \phi_{I},\ \phi_{T},\ \phi_{W},\ \phi_{S}\right)$$

$$C_{Current} = ConvertConcentration\left(n,\ T,\ M_{Current},\ V,\ \delta\right)$$

$$p_{prior_{Candidate}} = DiscreteUniform(1,\ n).\ logPMF\left(I_{Candidate_k}\right)$$

$$p_{prior_{Current}} = DiscreteUniform(1,\ n).\ logPMF\left(\phi_{I_k}\right)$$

$$p_{Likelihood_{Candidate}} = \sum_{t=1}^{T}\left(\sum_{i=1}^{n} N\left(C_{Candidate_{it}},\ \frac{1}{\sqrt{\phi_{\tau}}}^{2}\right).\ logPDF\left(Y_{True_{it}}\right)\right)$$

$$p_{Likelihood_{Current}} = \sum_{t=1}^{T}\left(\sum_{i=1}^{n} N\left(C_{Current_{it}},\ \frac{1}{\sqrt{\phi_{\tau}}}^{2}\right).\ logPDF\left(Y_{True_{it}}\right)\right)$$

$$p_{Q_{Candidate}} = DiscreteUniform(1,\ n).\ logPMF\left(I_{Candidate_k}\right)$$

$$p_{Q_{Current}}\ DiscreteUniform(1,\ n).\ logPMF\left(\phi_{I_k}\right)$$

$$ratio_{I_k} = min\left(p_{prior_{Candidate}} + p_{Likelihood_{Candidate}} - p_{prior_{Current}} - p_{Likelihood_{Current}} + p_{Q_{Current}} - p_{Q_{Candidate}},\ 1\right)$$

$$u \sim U(0,\ 1)$$

$if\ ratio > log(u)$:

$$\phi_{I} = I_{Candidate}$$

$$Accept_{I} = Accept_{I} + \frac{1}{N}$$

$$Likelihood_{I_{s,k}} = \sum_{t=1}^{T}\left(\sum_{i=1}^{n} N\left(C_{Candidate_{it}},\ \frac{1}{\sqrt{\phi_{\tau}}}^{2}\right).\ logPDF\left(Y_{True_{it}}\right)\right)$$

$else$:

$$Likelihood_{I_{s,k}} = \sum_{t=1}^{T}\left(\sum_{i=1}^{n} N\left(C_{Current_{it}},\ \frac{1}{\sqrt{\phi_{\tau}}}^{2}\right).\ logPDF\left(Y_{True_{it}}\right)\right)$$

T

$for\ k\ in\ 1{:}\ n{:}$

$$T_{k_{Candidate}} \sim DiscreteUniform(1,\ T)$$

$$T_{Candidate} = \phi_{T}$$

$$T_{Candidate_{k}} = T_{k_{Candidate}}$$

$$M_{Candidate} = GetSample\left(n,\ T,\ \phi_{\alpha},\ \phi_{\beta},\ \phi_{N},\ \phi_{I},\ T_{Candidate},\ \phi_{W},\ \phi_{S}\right)$$

$$C_{Candidate} = ConvertConcentration\left(n,\ T,\ M_{Candidate},\ V,\ \delta\right)$$

$$M_{Current} = GetSample\left(n,\ T,\ \phi_{\alpha},\ \phi_{\beta},\ \phi_{N},\ \phi_{I},\ \phi_{T},\ \phi_{W},\ \phi_{S}\right)$$

$$C_{Current} = ConvertConcentration\left(n,\ T,\ M_{Current},\ V,\ \delta\right)$$

$$p_{prior_{Candidate}} = DiscreteUniform(1,\ T).\ logPMF\left(T_{Candidate_{k}}\right)$$

$$p_{prior_{Current}} = DiscreteUniform(1,\ T).\ logPMF\left(\phi_{T_{k}}\right)$$

$$p_{Likelihood_{Candidate}} = \sum_{t=1}^{T}\left(\sum_{i=1}^{n} N\left(C_{Candidate_{it}},\ \frac{1}{\sqrt{\phi_{\tau}}}^{2}\right).\ logPDF\left(Y_{True_{it}}\right)\right)$$

$$p_{Likelihood_{Current}} = \sum_{t=1}^{T}\left(\sum_{i=1}^{n} N\left(C_{Current_{it}},\ \frac{1}{\sqrt{\phi_{\tau}}}^{2}\right).\ logPDF\left(Y_{True_{it}}\right)\right)$$

$$p_{Q_{Candidate}} = DiscreteUniform(1,\ T).\ logPMF\left(T_{Candidate_{k}}\right)$$

$$p_{Q_{Current}} = DiscreteUniform(1,\ T).\ logPMF\left(\phi_{4_{k}}\right)$$

$$ratio_{T_k} = min\left(p_{prior_{Candidate}} + p_{Likelihood_{Candidate}} - p_{prior_{Current}} - p_{Likelihood_{Current}} + p_{Q_{Current}} - p_{Q_{Candidate}}, 1\right)$$

$$u \sim U(0,\ 1)$$

$$if\ ratio\ >\ log(u):$$

$$\phi_T = T_{Candidate}$$

$$Accept_T = Accept_T + \frac{1}{N}$$

$$Likelihood_{T_{s,k}} = \sum_{t=1}^{T}\left(\sum_{i=1}^{n} N\left(C_{Candidate_{it}}, \frac{1}{\sqrt{\phi_\tau}}^2\right).logPDF\left(Y_{True_{it}}\right)\right)$$

$$else:$$

$$Likelihood_{T_{s,k}} = \sum_{t=1}^{T}\left(\sum_{i=1}^{n} N\left(C_{Current_{it}}, \frac{1}{\sqrt{\phi_\tau}}^2\right).logPDF\left(Y_{True_{it}}\right)\right)$$

W

$$for\ k\ in\ 1:n:$$

$$W_{k_{Candidate}} \sim Binom\left(T, \frac{1+\phi_{W_k}}{2+T}\right)$$

$$W_{Candidate} = \phi_W$$

$$W_{Candidate_k} = W_{k_{Candidate}}$$

$$M_{Candidate} = GetSample\left(n, T, \phi_\alpha, \phi_\beta, \phi_N, \phi_I, \phi_T, W_{Candidate}, \phi_S\right)$$

$$C_{Candidate} = ConvertConcentration\left(n, T, M_{Candidate}, V, \delta\right)$$

$$M_{Current} = GetSample\left(n, T, \phi_\alpha, \phi_\beta, \phi_N, \phi_I, \phi_T, \phi_W, \phi_S\right)$$

$$C_{Current} = ConvertConcentration\left(n, T, M_{Current}, V, \delta\right)$$

$$p_{prior_{Candidate}} = DiscreteUniform(1,\ T).\,logPMF\left(W_{Candidate_k}\right)$$

$$p_{prior_{Current}} = DiscreteUniform(1,\ T).\,logPMF\left(\phi_{W_k}\right)$$

$$p_{Likelihood_{Candidate}} = \sum_{t=1}^{T}\left(\sum_{i=1}^{n} N\left(C_{Candidate_{it}},\ \frac{1}{\sqrt{\phi_\tau}}^2\right).\,logPDF\left(Y_{True_{it}}\right)\right)$$

$$p_{Likelihood_{Current}} = \sum_{t=1}^{T}\left(\sum_{i=1}^{n} N\left(C_{Current_{it}},\ \frac{1}{\sqrt{\phi_\tau}}^2\right).\,logPDF\left(Y_{True_{it}}\right)\right)$$

$$p_{Q_{Candidate}} = Binom\left(T,\ \frac{1+\phi_{W_k}}{2+T}\right).\,logPMF\left(W_{Candidate_k}\right)$$

$$p_{Q_{Current}} Binom\left(T,\ \frac{1+W_{Candidate_k}}{2+T}\right).\,logPMF\left(\phi_{W_k}\right)$$

$$ratio_{W_k} = min\left(p_{prior_{Candidate}} + p_{Likelihood_{Candidate}} - p_{prior_{Current}} - p_{Likelihood_{Current}} + p_{Q_{Current}} - p_{Q_{Candidate}},\ 1\right)$$

$$u \sim U(0,\ 1)$$

$if\ ratio > log(u)$:

$$\phi_W = W_{Candidate}$$

$$Accept_W = Accept_W + \frac{1}{N}$$

$$Likelihood_{W_{s,k}} = \sum_{t=1}^{T}\left(\sum_{i=1}^{n} N\left(C_{Candidate_{it}},\ \frac{1}{\sqrt{\phi_\tau}}^2\right).\,logPDF\left(Y_{True_{it}}\right)\right)$$

$else$:

$$Likelihood_{W_{s,k}} = \sum_{t=1}^{T}\left(\sum_{i=1}^{n} N\left(C_{Current_{it}},\ \frac{1}{\sqrt{\phi_\tau}}^2\right).\,logPDF\left(Y_{True_{it}}\right)\right)$$

S

$for\ k\ in\ 1{:}\ n{:}$

$$S_{k_{Candidate}} \sim N\left(\phi_{S_k},\ 1^2\right)$$

$$S_{Candidate} = \phi_S$$

$$S_{Candidate_k} = S_{k_{Candidate}}$$

$$M_{Candidate} = GetSample\left(n,\ T,\ \phi_{\alpha},\ \phi_{\beta},\ \phi_{N},\ \phi_{I},\ \phi_{T},\ \phi_{W},\ S_{Candidate}\right)$$

$$C_{Candidate} = ConvertConcentration\left(n,\ T,\ M_{Candidate},\ V,\ \delta\right)$$

$$M_{Current} = GetSample\left(n,\ T,\ \phi_{\alpha},\ \phi_{\beta},\ \phi_{N},\ \phi_{I},\ \phi_{T},\ \phi_{W},\ \phi_{S}\right)$$

$$C_{Current} = ConvertConcentration\left(n,\ T,\ M_{Current},\ V,\ \delta\right)$$

$$p_{prior_{Candidate}} = Gamma(1,\ 1).\ logPDF\left(S_{Candidate_k}\right)$$

$$p_{prior_{Current}} = Gamma(1,\ 1).\ logPDF\left(\phi_{S_k}\right)$$

$$p_{Likelihood_{Candidate}} = \sum_{t=1}^{T}\left(\sum_{i=1}^{n} N\left(C_{Candidate_{it}},\ \frac{1}{\sqrt{\phi_{\tau}}}^{2}\right).\ logPDF\left(Y_{True_{it}}\right)\right)$$

$$p_{Likelihood_{Current}} = \sum_{t=1}^{T}\left(\sum_{i=1}^{n} N\left(C_{Current_{it}},\ \frac{1}{\sqrt{\phi_{\tau}}}^{2}\right).\ logPDF\left(Y_{True_{it}}\right)\right)$$

$$p_{Q_{Candidate}} = N\left(\phi_{S_k},\ 1^2\right).\ logPDF\left(S_{Candidate_k}\right)$$

$$p_{Q_{Current}} N\left(S_{Candidate_k},\ 1^2\right).\ logPDF\left(\phi_{S_k}\right)$$

$$ratio_{S_k} = min\left(p_{prior_{Candidate}} + p_{Likelihood_{Candidate}} - p_{prior_{Current}} - p_{Likelihood_{Current}} + p_{Q_{Current}} - p_{Q_{Candidate}}, 1\right)$$

$$u \sim U(0,\ 1)$$

$$if\ ratio\ > log(u):$$

$$\phi_S = S_{Candidate}$$

$$Accept_S = Accept_S + \frac{1}{N}$$

$$Likelihood_{S_{s,k}} = \sum_{y=1}^{T}\left(\sum_{x=1}^{N} N\left(C_{Candidate}, \frac{1}{\sqrt{\phi_\tau}}^2\right). logPDF\left(Y_{True}\right)\right)$$

$$else:$$

$$Likelihood_{S_{s,k}} = \sum_{y=1}^{T}\left(\sum_{x=1}^{N} N\left(C_{Current}, \frac{1}{\sqrt{\phi_\tau}}^2\right). logPDF\left(Y_{True}\right)\right)$$

Tau

$$\tau_{Candidate} \sim N\left(\phi_\tau, \frac{1}{2}^2\right)$$

$$M_{Candidate} = GetSample\left(n, T, \phi_\alpha, \phi_\beta, \phi_N, \phi_I, \phi_T, \phi_W, \phi_S\right)$$

$$C_{Candidate} = ConvertConcentration\left(n, T, M_{Candidate}, V, \delta\right)$$

$$M_{Current} = GetSample\left(n, T, \phi_\alpha, \phi_\beta, \phi_N, \phi_I, \phi_T, \phi_W, \phi_S\right)$$

$$C_{Current} = ConvertConcentration\left(n, T, M_{Current}, V, \delta\right)$$

$$p_{prior_{Candidate}} = Gamma(1,\ 1). logPDF\left(\tau_{Candidate}\right)$$

$$p_{prior_{Current}} = Gamma(1,\ 1). logPDF\left(\phi_\tau\right)$$

$$p_{Likelihood_{Candidate}} = \sum_{t=1}^{T}\left(\sum_{i=1}^{n} N\left(C_{Candidate_{it}}, \frac{1}{\sqrt{\tau_{Candidate}}}^{2}\right). logPDF\left(Y_{True_{it}}\right)\right)$$

$$p_{Likelihood_{Current}} = \sum_{t=1}^{T}\left(\sum_{i=1}^{n} N\left(C_{Current_{it}}, \frac{1}{\sqrt{\phi_{\tau}}}^{2}\right). logPDF\left(Y_{True_{it}}\right)\right)$$

$$p_{Q_{Candidate}} = N\left(\phi_{\tau}, \frac{1}{2}^{2}\right). logPDF\left(\tau_{Candidate}\right)$$

$$p_{Q_{Current}} N\left(\tau_{Candidate}, \frac{1}{2}^{2}\right). logPDF\left(\phi_{\tau}\right)$$

$$ratio_{\tau} = min\left(p_{prior_{Candidate}} + p_{Likelihood_{Candidate}} - p_{prior_{Current}} - p_{Likelihood_{Current}} + p_{Q_{Current}} - p_{Q_{Candidate}}, 1\right)$$

$$u \sim U(0, 1)$$

$if\ ratio > log(u)$:

$$\phi_{\tau} = \tau_{Candidate}$$

$$Accept_{\tau} = Accept_{\tau} + 1$$

$$Likelihood_{\tau_s} = \sum_{t=1}^{T}\left(\sum_{i=1}^{n} N\left(C_{Candidate_{it}}, \frac{1}{\sqrt{\phi_{\tau}}}^{2}\right). logPDF\left(Y_{True_{it}}\right)\right)$$

$else$:

$$Likelihood_{\tau_s} = \sum_{t=1}^{T}\left(\sum_{i=1}^{n} N\left(C_{Current_{it}}, \frac{1}{\sqrt{\phi_{\tau}}}^{2}\right). logPDF\left(Y_{True_{it}}\right)\right)$$

Proposal Distributions

$$\alpha^* \sim N\left(\phi_1, \frac{1}{2}^2\right)$$

$$\beta^* \sim Beta(1, 1)$$

$$I_k^* \sim DiscreteUniform(1, n)$$

$$T_k^* \sim DiscreteUniform(1, T)$$

$$W_k^* \sim Binom\left(T, \frac{1+\phi_{W_k}}{2+T}\right)$$

$$S_k^* \sim N\left(\phi_{S_k}, 1^2\right)$$

$$\tau^* \sim N\left(\phi_\tau, \frac{1}{2}^2\right)$$

Acceptance Probabilities

Alpha

$$p_{prior_{Candidate}} = log\left(\alpha_{Candidate}\right) - \alpha_{Candidate}$$

$$p_{prior_{Current}} = log\left(\alpha_{Current}\right) - \alpha_{Current}$$

$$p_{Likelihood_{Candidate}} = \frac{-\tau}{2}\sum_{t=1}^{T}\left(\sum_{i=1}^{n}\left(Y_{True_{it}} - C_{Candidate_{it}}\right)^{2}\right)$$

$$p_{Likelihood_{Current}} = \frac{-\tau}{2}\sum_{t=1}^{T}\left(\sum_{i=1}^{n}\left(Y_{True_{it}} - C_{Current_{it}}\right)^{2}\right)$$

$$p_{Q_{Candidate}} = -2\left(\alpha_{Candidate} - \alpha_{Current}\right)^{2}$$

$$p_{Q_{Candidate}} = -2\left(\alpha_{Current} - \alpha_{Candidate}\right)^{2}$$

$$ratio_{log} = log\left(\frac{\alpha_{Candidate}}{\alpha_{Current}}\right) - \alpha_{Candidate} + \alpha_{Current} + \frac{\tau}{2}\sum_{t=1}^{T}\left(\sum_{i=1}^{n}\left(Y_{True_{it}} - C_{Current_{it}}\right)^{2} - \left(Y_{True_{it}} - C_{Candidate_{it}}\right.\right.$$

$$P_{Acceptance}(\alpha) = e^{ratio_{log}}$$

Beta

$$p_{prior_{Candidate}} = 0$$

$$p_{prior_{Current}} = 0$$

$$p_{Likelihood_{Candidate}} = \frac{-\tau}{2}\sum_{t=1}^{T}\left(\sum_{i=1}^{n}\left(Y_{True_{it}} - C_{Candidate_{it}}\right)^{2}\right)$$

$$p_{Likelihood_{Current}} = \frac{-\tau}{2}\sum_{t=1}^{T}\left(\sum_{i=1}^{n}\left(Y_{True_{it}} - C_{Current_{it}}\right)^{2}\right)$$

$$p_{Q_{Candidate}} = 0$$

$$p_{Q_{Candidate}} = 0$$

$$ratio_{log} = \frac{\tau}{2}\sum_{t=1}^{T}\left(\sum_{i=1}^{n}\left(Y_{True_{it}} - C_{Current_{it}}\right)^{2} - \left(Y_{True_{it}} - C_{Candidate_{it}}\right)^{2}\right)$$

$$P_{Acceptance}(\beta) = e^{ratio_{log}}$$

I_k

$$p_{prior_{Candidate}} = -\log(n)$$

$$p_{prior_{Current}} = -\log(n)$$

$$p_{Likelihood_{Candidate}} = \frac{-\tau}{2}\sum_{t=1}^{T}\left(\sum_{i=1}^{n}\left(Y_{True_{it}} - C_{Candidate_{it}}\right)^{2}\right)$$

$$p_{Likelihood_{Current}} = \frac{-\tau}{2}\sum_{t=1}^{T}\left(\sum_{i=1}^{n}\left(Y_{True_{it}} - C_{Current_{it}}\right)^{2}\right)$$

$$p_{Q_{Candidate}} = -\log(n)$$

$$p_{Q_{Candidate}} = -\log(n)$$

$$ratio_{log} = \frac{\tau}{2}\sum_{t=1}^{T}\left(\sum_{i=1}^{n}\left(Y_{True_{it}} - C_{Current_{it}}\right)^{2} - \left(Y_{True_{it}} - C_{Candidate_{it}}\right)^{2}\right)$$

$$P_{Acceptance}\left(I_k\right) = e^{ratio_{log}}$$

T_k

$$p_{prior_{Candidate}} = -\log(T)$$

$$p_{prior_{Current}} = -\log(T)$$

$$p_{Likelihood_{Candidate}} = \frac{-\tau}{2}\sum_{t=1}^{T}\left(\sum_{i=1}^{n}\left(Y_{True_{it}} - C_{Candidate_{it}}\right)^2\right)$$

$$p_{Likelihood_{Current}} = \frac{-\tau}{2}\sum_{t=1}^{T}\left(\sum_{i=1}^{n}\left(Y_{True_{it}} - C_{Current_{it}}\right)^2\right)$$

$$p_{Q_{Candidate}} = -\log(T)$$

$$p_{Q_{Candidate}} = -\log(T)$$

$$ratio_{log} = \frac{\tau}{2}\sum_{t=1}^{T}\left(\sum_{i=1}^{n}\left(Y_{True_{it}} - C_{Current_{it}}\right)^2 - \left(Y_{True_{it}} - C_{Candidate_{it}}\right)^2\right)$$

$$P_{Acceptance}\left(T_k\right) = e^{ratio_{log}}$$

W_k

$$p_{prior_{Candidate}} = log\left(\left(W_{Candidate_k} + 5\right)C\left(W_{Candidate_k}\right)\right) + 6log\left(\frac{3}{4}\right) - W_{Candidate_k} log(4)$$

$$p_{prior_{Current}} = log\left(\left(W_{Current_k} + 5\right)C\left(W_{Current_k}\right)\right) + 6log\left(\frac{3}{4}\right) - W_{Current_k} log(4)$$

$$p_{Likelihood_{Candidate}} = \frac{-\tau}{2}\sum_{t=1}^{T}\left(\sum_{i=1}^{n}\left(Y_{True_{it}} - C_{Candidate_{it}}\right)^2\right)$$

$$p_{Likelihood_{Current}} = \frac{-\tau}{2}\sum_{t=1}^{T}\left(\sum_{i=1}^{n}\left(Y_{True_{it}} - C_{Current_{it}}\right)^2\right)$$

$$p_{Q_{Candidate}} = log\left((T)C\left(W_{Candidate_k}\right)\right) + W_{Candidate_k} log\left(W_{Current_k} + 1\right) + \left(T - W_{Candidate_k}\right)log\left(T + 1 - W_{Current_k}\right) + Tlog(T + 2)$$

$$p_{Q_{Candidate}} = log\left((T)C\left(W_{Current_k}\right)\right) + W_{Current_k} log\left(W_{Candidate_k} + 1\right) + \left(T - W_{Current_k}\right)log\left(T + 1 - W_{Candidate_k}\right) + Tlog(T + 2)$$

$$\begin{aligned} ratio_{log} = {} & log\left(\left(W_{Candidate_k} + 5\right)C\left(W_{Candidate_k}\right)\right) + \left(W_{Current_k} - W_{Candidate_k}\right)log(4) - log\left(\left(W_{Current_k} + 5\right)C\left(W_{Current_k}\right)\right) \\ & + \frac{\tau}{2}\sum_{t=1}^{T}\left(\sum_{i=1}^{n}\left(Y_{True_{it}} - C_{Current_{it}}\right)^2 - \left(Y_{True_{it}} - C_{Candidate_{it}}\right)^2\right) + log\left((T)C\left(W_{Candidate_k}\right)\right) + W_{Candidate_k} log\left(W_{Current_k} + 1\right) \\ & + \left(T - W_{Candidate_k}\right)log\left(T + 1 - W_{Current_k}\right) - log\left((T)C\left(W_{Current_k}\right)\right) - W_{Current_k} log\left(W_{Candidate_k} + 1\right) - \left(T - W_{Current_k}\right)log\left(T + 1 - W_{Candidate_k}\right) \end{aligned}$$

$$P_{Acceptance}\left(W_k\right) = e^{ratio_{log}}$$

S_k

$$p_{prior_{Candidate}} = -S_{Candidate_k}$$

$$p_{prior_{Current}} = -S_{Current_k}$$

$$p_{Likelihood_{Candidate}} = \frac{-\tau}{2}\sum_{t=1}^{T}\left(\sum_{i=1}^{n}\left(Y_{True_{it}} - C_{Candidate_{it}}\right)^2\right)$$

$$p_{Likelihood_{Current}} = \frac{-\tau}{2}\sum_{t=1}^{T}\left(\sum_{i=1}^{n}\left(Y_{True_{it}} - C_{Current_{it}}\right)^2\right)$$

$$p_{Q_{Candidate}} = -\frac{1}{2}\left(S_{Candidate_k} - S_{Current_k}\right)^2$$

$$p_{Q_{Candidate}} = -\frac{1}{2}\left(S_{Current_k} - S_{Candidate_k}\right)^2$$

$$ratio_{log} = -S_{Candidate_k} + S_{Current_k} + \frac{\tau}{2}\sum_{t=1}^{T}\left(\sum_{i=1}^{n}\left(Y_{True_{it}} - C_{Current_{it}}\right)^2 - \left(Y_{True_{it}} - C_{Candidate_{it}}\right)^2\right)$$

$$P_{Acceptance}\left(S_k\right) = e^{ratio_{log}}$$

Tau

$$p_{prior_{Candidate}} = log\left(\tau_{Candidate}\right) - \tau_{Candidate}$$

$$p_{prior_{Current}} = log\left(\tau_{Current}\right) - \tau_{Current}$$

$$p_{Likelihood_{Candidate}} = \frac{-\tau_{Candidate}}{2} \sum_{t=1}^{T}\left(\sum_{i=1}^{n}\left(Y_{True_{it}} - C_{Candidate_{it}}\right)^2\right)$$

$$p_{Likelihood_{Current}} = \frac{-\tau_{Current}}{2} \sum_{t=1}^{T}\left(\sum_{i=1}^{n}\left(Y_{True_{it}} - C_{Current_{it}}\right)^2\right)$$

$$p_{Q_{Candidate}} = -2\left(\tau_{Candidate} - \tau_{Current}\right)^2$$

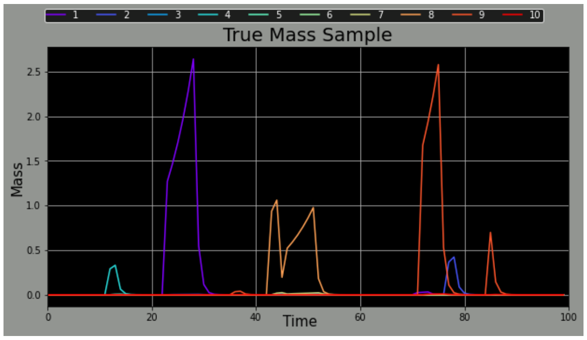
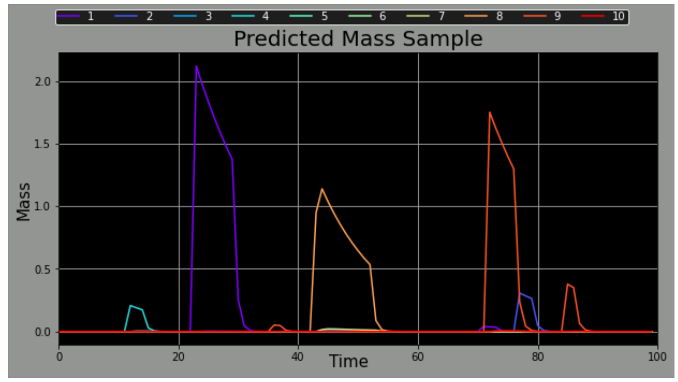
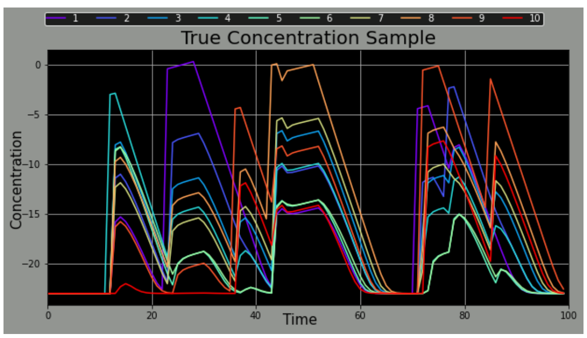
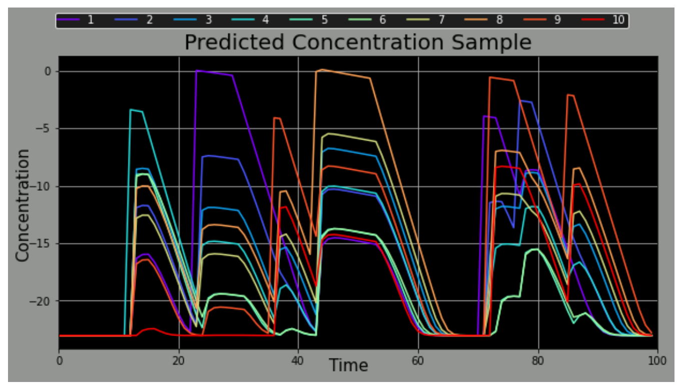
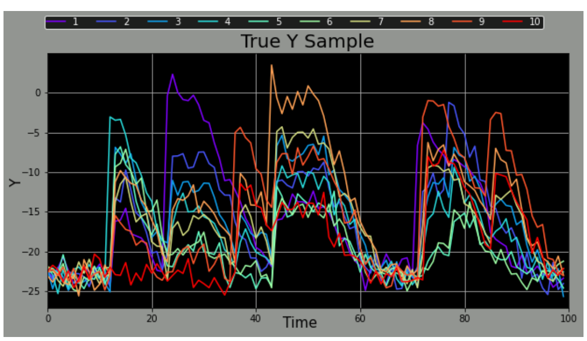
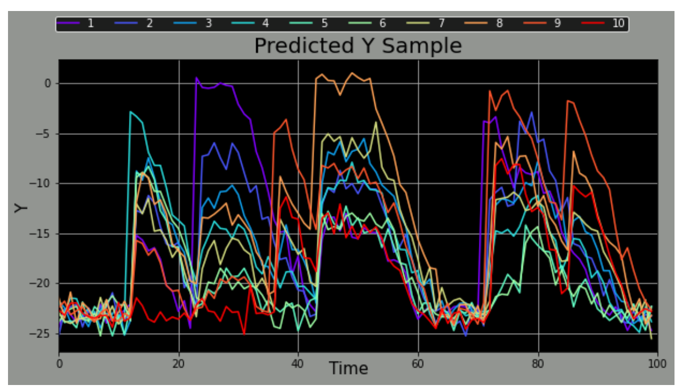
$$p_{Q_{Candidate}} = -2\left(\tau_{Current} - \tau_{Candidate}\right)^2$$

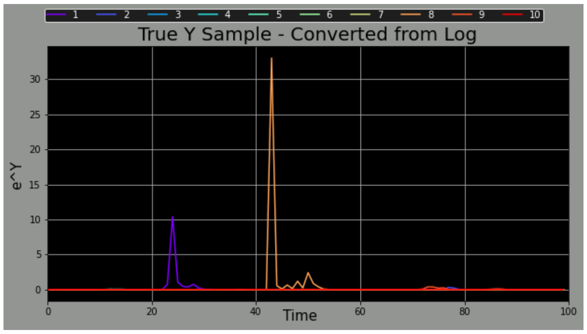
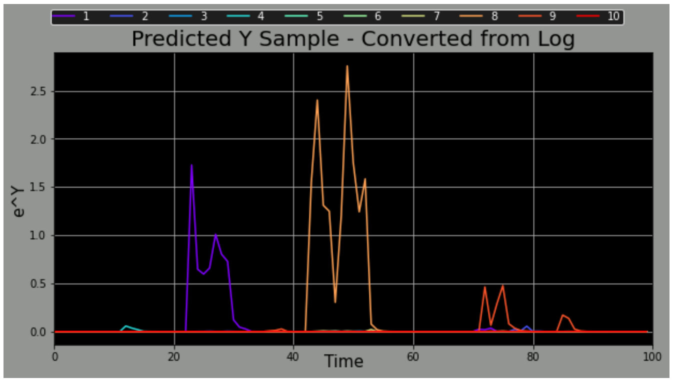
Since for τ, $C_{Candidate_{it}} = C_{Current_{it}}$ for all i, t

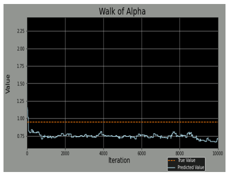
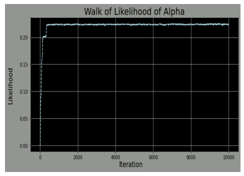
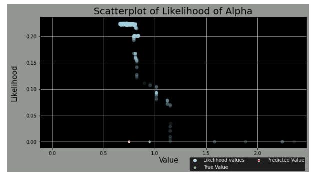
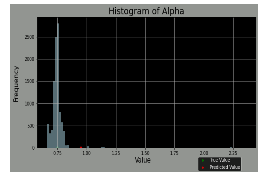
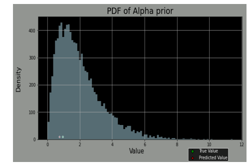
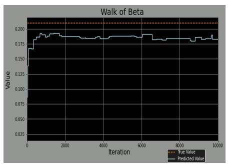
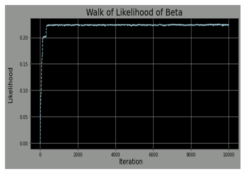
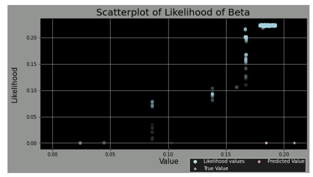
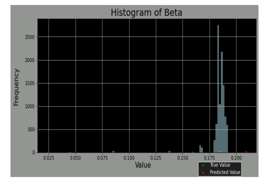
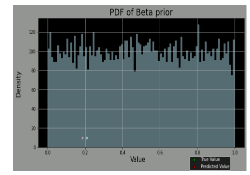
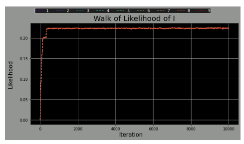
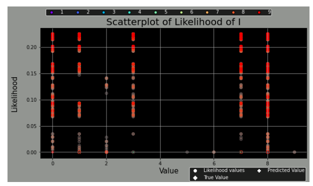
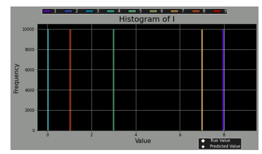
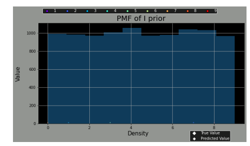
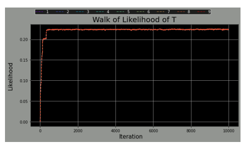
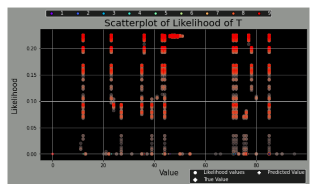
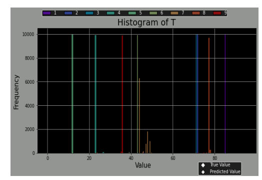
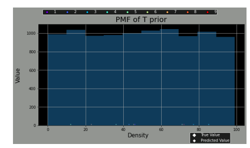
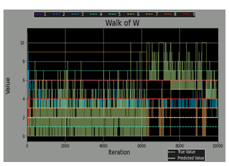
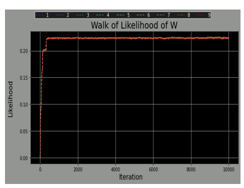
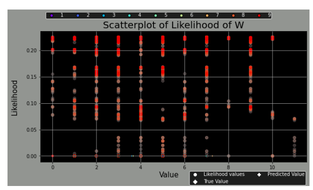
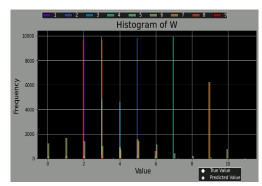
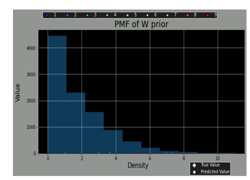
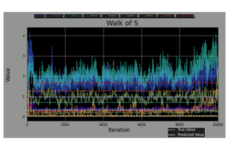
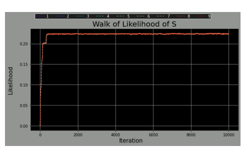
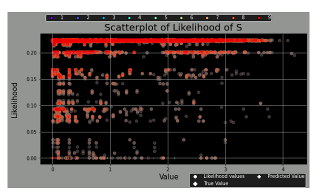
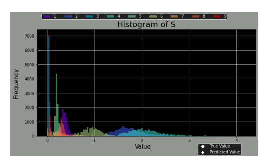
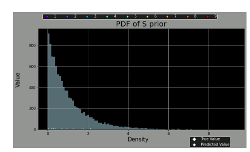
$$ratio_{log} = log\left(\frac{\tau_{Candidate}}{\tau_{Current}}\right) - \tau_{Candidate} + \tau_{Current}$$

$$P_{Acceptance}(\tau) = e^{ratio_{log}}$$

Results

	True	Predicted
Mass		
Concentration		
Y		

e^Y	1 2 3 4 5 6 7 8 9 10 True Y Sample - Converted from Log e^Y Time	1 2 3 4 5 6 7 8 9 10 Predicted Y Sample - Converted from Log e^Y Time

Parameter P(Acceptance)		Walk of value	Walk of Likelihood	Scatter Plot of Likelihood	Histogram	Prior PDF
α	0.0257					
β	0.0056					
I	0.0992					
T	0.0104					
W	0.2327					
S	0.1059					
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