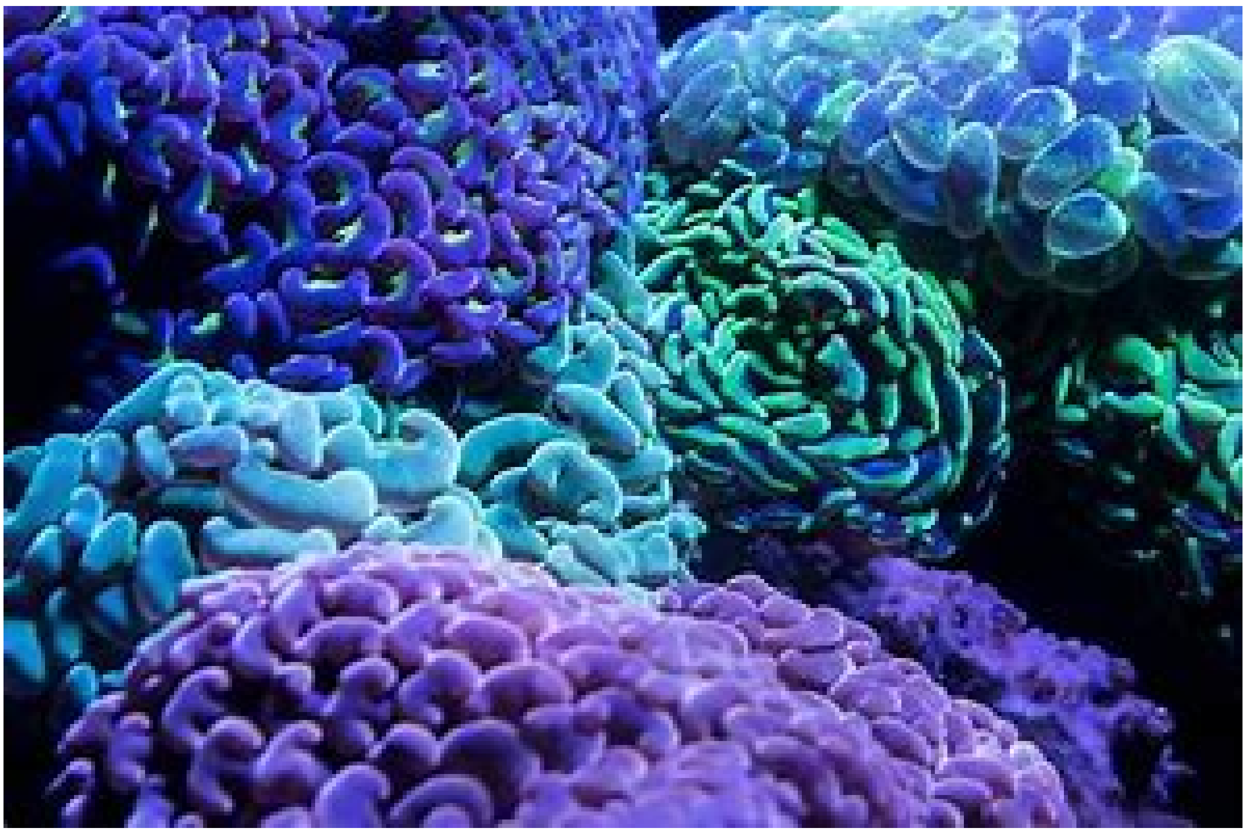
Interim Report

Predicting Algae Bloom spread in the Marlborough Sounds using Bayesian Inference

Luke Pearson



Abstract

Toxic Algae blooms cause shellfish in the affected area to be infected, causing them to carry the toxins that can cause severe illness to humans when they are consumed. When these blooms occur, it causes the shellfish industry in the Marlborough Sounds to shut down collection region-wide out of an abundance of caution, which increases costs. Therefore, they have an interest in predicting Algae blooms, and in particular the size and the spread in an effort to narrow the extent of the shutdown.

For the purposes of this project, the Marlborough sounds have been divided into a set of 386 polygonal approximations of areas of the sounds. The volumes of these polygons have been calculated, and a transport matrix depicting the proportion of particles that move from site i to site j in a particular time interval. (Ross Vennell, 2022)

Our data is recording concentrations of samples retrieved from these sites at discrete time intervals, observed with an error. This data is also observed sparsely.

Due to timing, the actual data is not currently available, so the scope of this project will involve simulating data with randomly generated parameters, and then trying to predict the data by deducing the values of the parameters.

This will be done by constructing a forward in time statistical model of the mass and then using bayesian inference, via a Metropolis-Hastings algorithm to determine the model parameters.

This model will be based upon an exponential growth component during active blooms, a decay component and a transfer component based upon the transport matrix.

Background

Harmful Algal Blooms, or HABs, occur when there is a large buildup of algae that release materials which are toxic to other species. Particularly, *cyanobacteria* variety algae blooms, when ingested by shellfish become toxic to humans and can cause serious illness if consumed[1].

As a result of this danger, when HABs occur in the Marlborough Sounds, the response is to temporarily halt harvesting in the region until the bloom ends. Considering the Marlborough Sounds is responsible for 80%[2] of domestic shellfish production, this causes an immense cost to the industry; approximately $X million annually[3].

There are a number of conditions that could cause a HAB to begin, with the main factors being a warmer temperature[4] and increase in nutrients such as phosphorus, nitrogen, and carbon[5] in the water, however it is hard to predict when exactly one will occur. Climate Change has also had an adverse impact on these conditions causing HABs with more frequency[6].

While the length of the HAB is also variable and hard to predict, there are conditions that will determine the end; primarily, a sudden decrease in temperature that will cause the Algae to go inactive and sink below the surface, stopping the bloom. For the purposes of the Marlborough sounds region, these events happen globally.

The aim of this project is to simulate data of algal blooms over a set time period, and then use Bayesian Inference via a Metropolis-Hastings Algorithm to approximate the parameters of our model. We will also explore other scenarios such as varying amounts of sparseness in the data, and predicting a different number of events than the true amount.

Methods

Bayesian Inference

Bayesian inference is a branch of statistical modelling based upon Bayes' theorem

$$P(\theta|Y) = \frac{P(Y|\theta) \times P(\theta)}{P(Y)}$$

Where:

- $P(\theta)$ is the $prior$ distribution
- $P(Y|\theta)$ is the $likelihood$ distribution
- $P(\theta|Y)$ is the $posterior$ distribution

Where instead of the usual approach where the parameters X are fixed and the data Y is variable to determine the likelihood, the data is used as the fixed variable to determine the distribution of the unknown parameters. Since the data is fixed, $P(Y)$ will be constant.

For the purposes of this project, we can use the likelihood for the Concentrations and an educated construction of prior distributions to determine the distribution of the parameters, or $P(Parameters|Concentration) \alpha P(Concentration|Parameters) \times P(Parameters)$.

However, due to the complex nature of the likelihood in this case $\left(Where\ C_{it} \sim N\left(log\left(\delta + \frac{M_{it}}{V_i}\right), \frac{1}{\sqrt{\tau}}^2\right)\right)$, deriving the posterior distribution mathematically provides a result that is not useful. Therefore, an analytical method of approximating these distributions is needed.

Metropolis-Hastings

The Metropolis-Hastings algorithm is a Monte Carlo Markov Chain sampling method that can be used to approximate a distribution that is otherwise hard to sample from. This method utilises the bayesian inference ideas above, but also uses a $proposal$ distribution to control the chain.

General Method

For each parameter θ, the algorithm generates a candidate value θ^* from a proposal distribution $q\left(\theta^* | \theta_{s-1}\right)$ that is dependent on the previous value for θ. It then calculates a ratio of

$$r = \frac{P\left(Y|\theta^*\right) \times P\left(\theta^*\right) \times q\left(\theta_{s-1}|\theta^*\right)}{P\left(Y|\theta_{s-1}\right) \times P\left(\theta_{s-1}\right) \times q\left(\theta^*|\theta_{s-1}\right)}$$ with:

- $P(Y|\theta)$ being calculated from the $likelihood$ function
- $P(Y)$ being calculated from the $prior$ function
- $P\left(\theta_a|\theta_b\right)$ being calculated from the $likelihood$ function

And then the probability of accepting θ^* as θ_s is r, with the probability of accepting θ_{s-1} as θ_s is $1 - r$

Application

Because our likelihood is different for each site i and time t, and that we have the assumption of independence amongst Y_{it}, the total likelihood $P(Y|\theta)$ is actually $\prod_{i=1}^{n}\left(\prod_{t=1}^{T}\left(P\left(Y_{it}|\theta\right)\right)\right)$. Since our simulation will involve 10 sites and 100 times, this will be the product of 10000 probabilities; to avoid our code rounding the value to zero, $\sum_{i=1}^{n}\left(\sum_{t=1}^{T}\left(log\left(P\left(Y_{it}|\theta\right)\right)\right)\right)$ will be more useful. Therefore, our total ratio is

$$ratio = log\left(P\left(Y|\theta^*\right)\right) + log\left(P\left(\theta^*\right)\right) + log\left(q\left(\theta_{s-1}|\theta^*\right)\right) - \left(log\left(P\left(Y|\theta_{s-1}\right)\right) + log\left(P\left(\theta_{s-1}\right)\right) + log\left(q\left(\theta^*|\theta_{s-1}\right)\right)\right)$$

And then the probability of accepting θ^* as θ_s is e^{ratio}, with the probability of accepting θ_{s-1} as θ_s is $1 - e^{ratio}$.

Transportation Matrix

In preparation for this project, Ross Vennell has constructed a transportation matrix, P, to represent the proportion of mass that moves between sites across the Marlborough Sounds.

This matrix was constructed using OceanTracker**,** an application designed to efficiently simulate hydrodynamic models via Lagrangian Particle Tracking[7].

He then constructed polygons approximating around the 386 different source sites (Labelled in $Figure\ X$), and simulated 10000 particles being released in that polygon for a week. The fraction of particles in each polygon after this time period constitute the values for P.

P is an $n \times n$ matrix where P_{ij} is equal to the proportion of particles in site i that travels to site j, with two main properties:

- $0 \leq P_{ij} \leq 1 \ for\ all\ i,\ j$
- $\sum_{j=1}^{n} P_{ij} = 1 \ for\ all\ i$

$Figure\ X + 1$ is a heatmap of P. P is a very sparse matrix, with 97% of values being 0, and P is also close to a diagonal matrix with the average value of P_{ii} being 0.935. As a result, we shouldn't expect algae to spread amongst a lot of sites such that a bloom should stay relatively contained to the sites in the near vicinity.

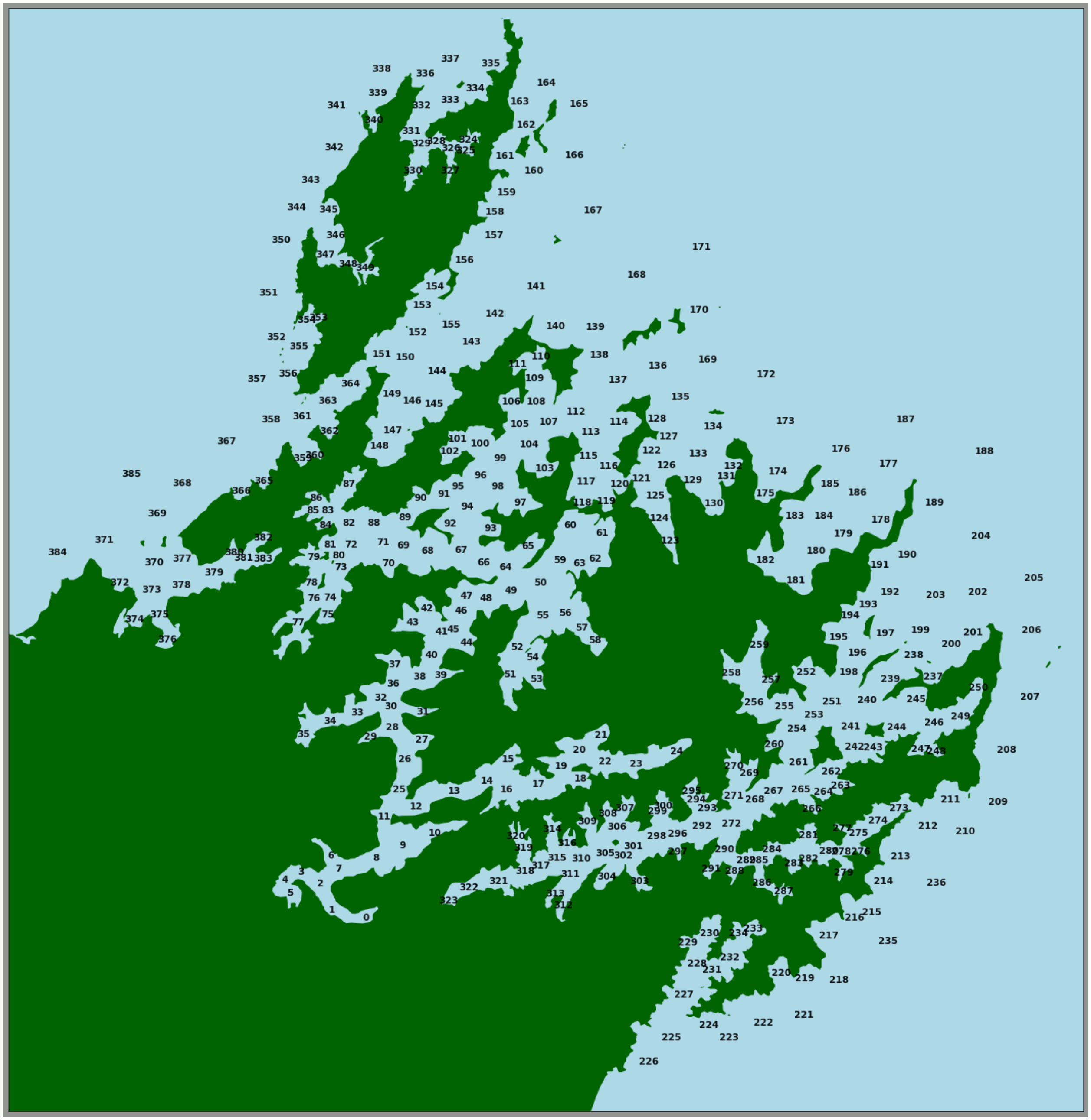
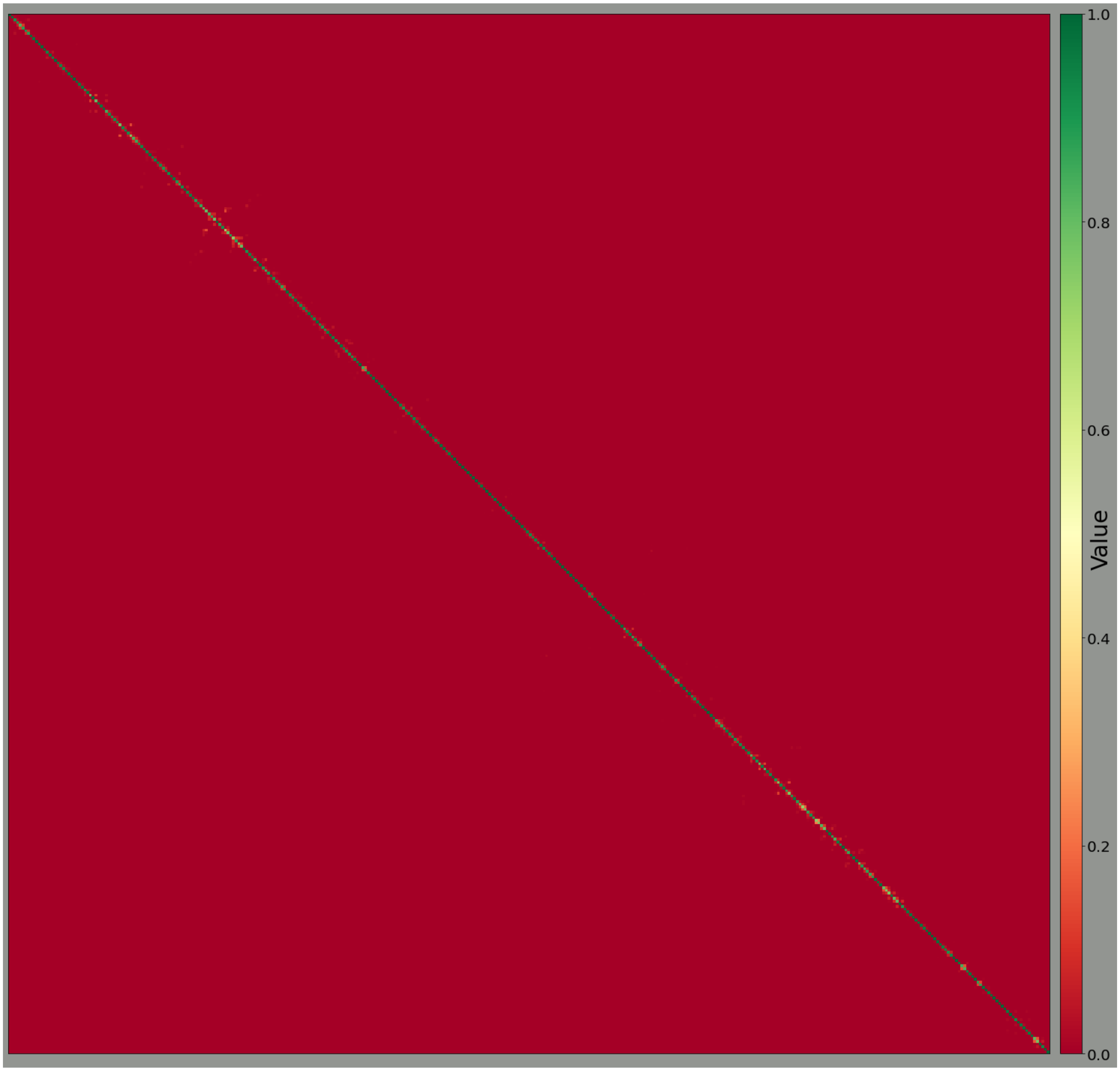


Figure X: Map of Marlborough Sounds with locations of sites labelled



$Figure\ X + 1{:}\ Heatmap\ of\ P$

Metrics

We will be using two metrics to evaluate the comparative performances of our estimation: DIC and the Likelihood. We will be using them over both the full set of data, as well as the observed subset. For each parameter, we will

DIC

DIC, or the Deviance information Criteria, is an extension of Akaike information criterion (AIC) that is particularly useful for performing model selection when they've been approximated via Markov Chain Monte Carlo algorithms like the Metropolis-Hastings algorithm we're using here. This is calculated by using the deviance for a set of parameters $D(\theta)$ as;

$D(\theta) = - 2log(P(Y|\theta)) + C$, where C is a constant that will cancel out when comparing models.

Since our likelihood function $log(P(Y|\theta))$ is dependent on i, t and we are assuming independence amongst$Y_{i,t}$ for all i, t, the deviance can be calculated as

$$D(\theta) = - 2\sum_{i=1}^{n}\left(\sum_{t=1}^{T}\left(log\left(P\left(Y_{it}|\theta\right)\right)\right)\right)$$

Since $D(\theta)$ fits more easily to models with large sets of parameters, there is a penalty p_D calculated as

$p_D = \overline{D(\theta)} - D\left(\hat{\theta}\right)$, where $\overline{D(\theta)}$ is the expected value of all Deviances and $D\left(\hat{\theta}\right)$ is the Deviance of the expected parameter values. So the total DIC is

$$DIC = D(\theta) + p_D$$

Our Model

Mass

Our equation for Mass is:

$$M_{i,t} = \alpha A_{i,t} M_{i,t-1} + \beta\left(\sum_{j=1}^{n} p_{i,j} M_{j,t-1}\right) + B_{i,t}, \text{ where}$$

$$A_{i,t} = Indicator\left(\sum_{k=1}^{N} Indicator\left(I_k = i\right) Indicator\left(T_k \le t \le T_k + W_k\right) > 0\right)$$

and

$$B_{i,t} = \sum_{k=1}^{N} I\left(I_k = i\right) I\left(T_k = t\right) S_k$$

This model is comprised of three main components; start, growth, and transport, where blooms are represented as a fixed number of events N, spanning n sites and over the course of T time periods. For our subscripts, i refers to the site and t refers to the time period.

Start component

The 'Start' component is meant to simulate the starting size of a bloom when it begins at a particular site and time.

If event k starts at site I_k at time T_k, then this component $B_{i,t}$ will be equal to size S_k when $I_k = i$ and $T_k = t$, and all other values will be equal to 0.

In terms of mathematical notation, it is represented as:

$$B_{i,t} = \sum_{k=1}^{n} I\left(I_k = i\right) I\left(T_k = t\right) S_k.$$

Growth Component

The 'Growth' component is meant to simulate how a bloom grows at a particular site over the course of the bloom's length during a single time period.

If event k starts at site I_k and at time T_k with a length of W_k, then this component $A_{i,t}$ will be equal to 1 when $I_k = i$ and $T_k \leq t \leq T_k + W_k$, and all other values will be equal to 0.

In terms of mathematical notation, it is represented as:

$$A_{i,t} = I\left(\sum_{k=1}^{n} I\left(I_k = i\right) I\left(T_k \leq t \leq T_k + W_k\right) > 0\right)$$

If $A_{i,t} = 1$, then the mass at site i will be growing exponentially at a rate of α where α is our growth rate.

Transport Component

The 'transport' component is meant to simulate the mass that transfers from one site to another during a single time period.

For each site i, they will receive $p_{i,j} M_{j,t-1}$ from each site, where $p_{i,j}$ is the proportion of mass that transports from site i to site j, and $M_{j,t-1}$ is the mass at site j at time $t - 1$. All of this mass will be scaled by β, the rate of decay.

In terms of mathematical notation, it is represented as

$$\beta\left(\sum_{j=1}^{n} p_{i,j} M_{j,t-1}\right)$$

Concentration

The equation for $Y_{i,t}$ is

$$log\left(Y_{i,t}\right) \sim N\left(log\left(\delta + \frac{M_{i,t}}{V_i}\right), \frac{1}{\sqrt{\tau}}^2\right)$$

Concentration $Y_{i,t}$ for a site i at time t is on a logarithmic scale of $\frac{M_{i,t}}{V_i}$, where $M_{i,t}$ is the mass for site i at time t, and V_i is the volume of site i. Due to it being logarithmic and $\frac{M_{i,t}}{V_i}$ having a range of $\left\{0, \infty^+\right\}$, an offset δ is added to ensure that $\delta + \frac{M_{i,t}}{V_i} > 0$. This concentration is also presumed to be measured with an error, which is represented as sampling from a normal distribution with mean $log\left(\delta + \frac{M_{i,t}}{V_i}\right)$ and variance $\frac{1}{\tau}^2$, where τ is our precision parameter.

Assumptions

- The growth rate, α. is equal across all sites and times
- The decay rate, β, is equal across all sites and times
- I_k, T_k, W_k, S_k are independent from all other events
- $log\left(Y_{i,t}\right)$ is independently sampled from $N\left(log\left(\delta + \frac{M_{i,t}}{V_i}\right), \frac{1}{\sqrt{\tau}}^2\right)$
- There are no seasonal components

Estimation

Parameters

The parameters that are known are:

- Number of sites, n
  - n will be set to 10
- Time period, T
  - T will be set to 100
- Dispersion Matrix, P
  - P is derived from the transportation matrix above
- Number of events, N
  - N will be set to 5 for our sample
- Volumes, V_i
  - V_i will be samples from $NegBinom(1,\ 1)\ +\ 1$
- Concentration offset, δ
  - δ will be set to 1^{-10}

And the parameters that need to be estimated are:

- Growth Parameter, α
- Decay Parameter, β
- Event site, $\left[I_1, ..., I_N\right]$
- Event Starting time, $\left[T_1, ..., T_N\right]$
- Event time length, $\left[W_1, ..., W_N\right]$
- Event size, $\left[S_1, ..., S_N\right]$
- Precision, τ

Where the initial values will be derived from the priors below.

Priors

$\alpha \sim Gamma(2, 1)$

Such that $E(\alpha) = 2$, since an $\alpha \leq 1$ would cause it to not grow.

$\beta \sim Beta(1, 1)$

Which is equivalent to $Uniform(1, 1)$, such that it is an uninformative prior but any values outside the domain of $\{0, 1\}$ will not be considered.

$I_k \sim DiscreteUniform(1, n)$

Such that there is an uninformative prior that will only consider the sites as possible values and all sites are equally likely.

$T_k \sim DiscreteUniform(1, T)$

Such that there is an uninformative prior that will only consider the range of time periods as possible values and all time periods are equally likely.

$W_k \sim NegBinomial(6, 0.75)$

A prior such that $E(W_k) = 2$, so that event lengths are relatively small but also should give a high enough event length such that α can be approximated.

$S_k \sim Gamma(1, 1)$

An uninformative prior where $E(S_k) = 1$ and prefers a small cluster of values.

$\tau \sim Gamma(1, 1)$

An uninformative prior where $E(\tau) = 1$ and prefers a small cluster of values.

Simulation

Since we have no real data, we will need to simulate it. We will do this using the RunSimulation function outlined below, where we randomly generate the true values for our parameters and then use the Mass equation outlined above to calculate $M_{i,t}$ for all $i,\ t$.

However, to generate a sample we will use a variation of RunSimulation with a fixed $N\ =\ 5$ such that we can have the most even spread of varying predicted numbers of events, with both undercounting and overcounting blooms.

Due to the fact that we want to compare the effectiveness of the inference amongst different scenarios, we will generate a singular sample and save it along with the true parameters for that sample for repeated use.

Assumptions

- The growth rate, α. is equal across all sites and times
- The decay rate, β, is equal across all sites and times
- I_k, T_k, W_k, S_k are independent from all other events
- There is an equal probability for $I_k = i$ amongst all sites
- There is an equal probability for $T_k = t$ amongst all time periods
- All parameters are independent
- $log\left(Y_{i,t}\right)$ is independently sampled from $N\left(log\left(\delta + \frac{M_{i,t}}{V_i}\right), \frac{1}{\sqrt{\tau}}^2\right)$ for all $i,\ t$
- There are no seasonal components

Sample

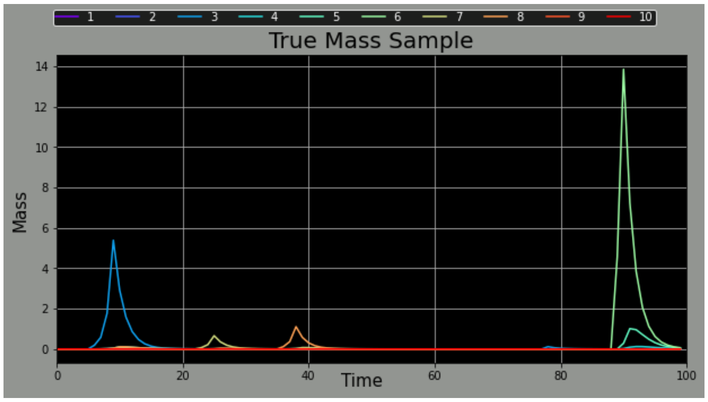
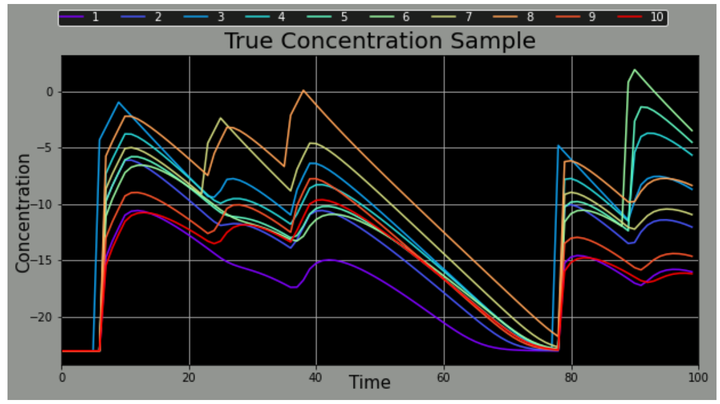
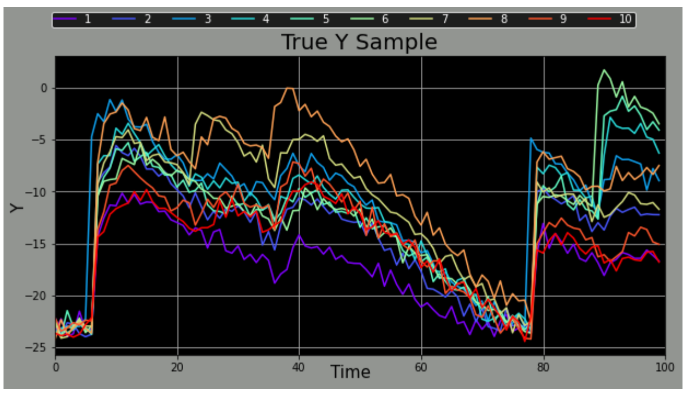
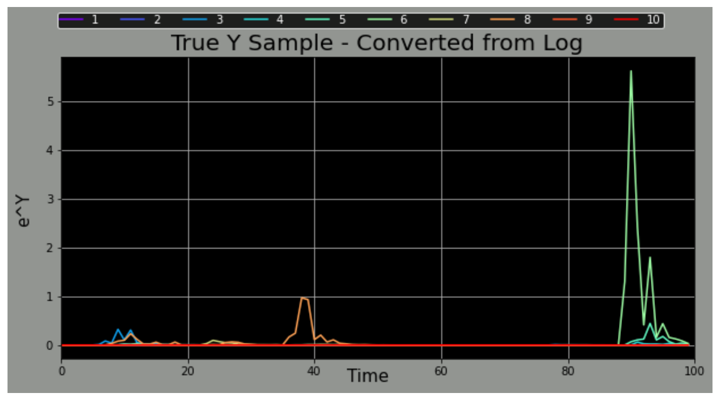
Mass	Concentration
	
	
Y	e^Y

Figure X + 2: Plots of the Mass, Concentration, Y, and e^Y for the true sample

Parameter	Value(s)
α	2.5086
β	0.5927
I	[7, 2, 2, 5, 6]
T	[36, 78, 6, 89, 23]
W	[3, 1, 4, 2, 3]
S	[0.1164, 0.1162, 0.1900, 4.5746, 00689]
τ	2.2986

Figure X + 3: Table of the true parameters

Pseudocode

FindM$_{New}$

The purpose of this function is to calculate and output $M_{i,t}$ from the mass equation outlined above, using $M_{i,t-1}$ and the given parameters. It does this by taking the sum of the separate components which have been calculated. It then returns $M_{i,t}$

Inputs:

- M
- α
- β
- P
- A
- B

$$growth = \alpha \times (A \cdot M)$$

$$transport = \beta \times (P \cdot M)$$

$$M_{New} = B + growth + transport$$

Outputs:

- M_{New}

RunSimulation

The purpose of this function is to generate a random set of parameters and to create the mass data from these parameters. For this, it randomly generates these sets of parameters by taking a sample from predetermined mock distributions, and then uses a for loop over T time periods and continuously feeds $M_{i,t-1}$ and the parameters into **FindM$_{New}$**. It then returns M and all the randomly generated parameters.

Inputs:

- n
- T
- P

$\alpha \sim Gamma(2,\ 1)$

$\beta \sim Beta(4,\ 3)$

$N \sim DiscreteUniform\left(1,\ \frac{T}{10}\right)$

$For\ x\ in\ 1{:}\ N$:

$i \sim DiscreteUniform(1,\ n)$

$t \sim DiscreteUniform(1,\ T)$

$w \sim NegBinom(6,\ 0.75)\ +\ 1$

$s \sim Gamma(1,\ 1)$

$A_{t:min(t+w, T), i} = 1$

$B_{t, i} = s$

$\left[M_{0,0}, \ldots, M_{n,0}\right] = 0$

$For\ t\ in\ 1{:}\ T$:

$M_{New} = FindM_{New}\left(\left[M_{0,t-1}, \ldots, M_{n,t-1}\right], \alpha, \beta, P_{i:n, i:n}, \left[A_{t,1}, \ldots, A_{t,n}\right], \left[B_{t,1}, \ldots, B_{t,n}\right]\right)$

$\left[M_{0,t}, \ldots, M_{n,t}\right] = M_{New}$

Outputs:

- $\left[\left[M_{0,1}, \ldots, M_{n,1}\right], \ldots, \left[M_{0,T}, \ldots, M_{n,T}\right]\right]$
- α
- β
- N
- $\left[I_0, \ldots, I_N\right]$
- $\left[T_0, \ldots, T_N\right]$
- $\left[W_0, \ldots, W_N\right]$
- $\left[S_0, \ldots, S_N\right]$

GetSample

The purpose of this function is to produce the mass data given a predetermined set of parameters. It does this in a similar fashion to **RunSimulation** but takes the parameters as inputs as opposed to randomly generating themselves within the function. It then returns M.

Inputs:

- n
- T
- P
- α
- β
- N
- $\left[I_1, \ldots, I_N\right]$
- $\left[t_1, \ldots, t_N\right]$
- $\left[W_1, \ldots, W_N\right]$
- $\left[s_1, \ldots, s_N\right]$

$For\ x\ in\ 1{:}\ N{:}$

$$A_{t_x : min(t_x + W_x, t), I_x} = 1$$

$$B_{t, i} = s_x$$

$$\left[M_{0,0}, \ldots, M_{n,0}\right] = 0$$

$For\ t\ in\ 1{:}\ T{:}$

$$M_{New} = FindM_{New}\left(\left[M_{0,t-1}, \ldots, M_{n,t-1}\right], \alpha, \beta, P_{i:n, i:n}, \left[A_{t,1}, \ldots, A_{t,n}\right], \left[B_{t,1}, \ldots, B_{t,n}\right]\right)$$

$$\left[M_{0,t}, \ldots, M_{n,t}\right] = M_{New}$$

Outputs:

- $\left[\left[M_{0,1},\ldots, M_{n,1}\right],\ldots,\left[M_{0,T},\ldots, M_{n,T}\right]\right]$

ConvertConcentration

The purpose of this function is to take the mass data returned from either **RunSimulation** or **GetSample**, along with the volumes V and δ, and convert it into $log\left(\delta + \frac{M}{V}\right)$ format that is the mean of $log\left(Y_{i,t}\right)$'s normal distribution. It then returns this array.

Inputs:

- n
- T
- M
- V
- δ

$For\ i\ in\ 1{:}\,n$:

$$C_{i,\,1:T} = \frac{M_{i,\,1:T}}{V_{i}}$$

$$Concentration = log(C + \delta)$$

Outputs:

- $Concentration$

ConvertY

The purpose of this function is to take the array of $log\left(\delta + \frac{M}{V}\right)$ generated from **ConvertConcentration** and sample from the distribution for concentration. It does this by taking this concentration array, along with the precision parameter τ, and sampling from the $N\left(log\left(\delta + \frac{M}{V}\right), \frac{1}{\sqrt{\tau}}^2\right)$ to generate $T \times n$ samples of Y. It then returns the Y_{log} array.

Inputs:

- C
- τ

$$Y_{log} \sim N\left(C, \frac{1}{\sqrt{\tau}}^2\right)$$

Outputs:

- Y_{log}

GenerateObservation

Due to the practical context of taking samples from sites out in the Marlborough sounds, one of the scenarios is dealing with sparse data. Therefore, the purpose of this function is to create an observation array O where $O_{i,t} = 1$ if the data is present and $O_{i,t} = 0$ otherwise.

There are two methods to do this: Remove data randomly based upon a probability $P\left(O_{i,t} = 0\right) = p$ for some value p, where it is independent amongst all i, t; or remove specific sites $\left\{I_1,..., I_X\right\}$ where $O_{i,t} = 0 \; if \; i \in \left\{I_1,..., I_X\right\}$ and $O_{i,t} = 1$ otherwise.

Random

Inputs:

- n
- T
- p

$O_{1:n,\,1:T} = 1$

$For\ i\ in\ 1:n$:

$For\ t\ in\ 1:T$:

$u \sim Uniform(0,\ 1)$

$if\ u\ \geq\ p$:

$O_{i,\,t} = 0$

Outputs:

- O

Sites

Inputs:

- n
- T
- $\left[I_{1}, \ldots I_{X}\right]$

$O_{1:n,\,1:T} = 1$

$For\ I\ in \left[I_{1}, \ldots I_{X}\right]$:

$O_{I,\,1:T} = 0$

Outputs:

- O

Metropolis-Hastings Algorithm

The purpose of this code is to execute the Bayesian inference via the Metropolis-Hastings Method previously outlined and approximate the parameters. It does this by first generating a sample with random parameters using **RunSimulation**, and using **ConvertConcentration** and **ConvertY** to create the Y sample. It also generates initial values for all parameters θ_0 by sampling from the prior distributions, an Observation Array via **GenerateObservation** and volumes from $NegBinom(1, 1) + 1$. It also generates initial parameter values θ_0 from the priors outlined above, and calculates the initial $M_{Current}$ and $C_{Current}$ arrays. Then, looping over the 10000 iterations, for each parameter to be estimated, it:

- Randomly generates $\theta_{Candidate}$ from the proposal distributions $q\left(\theta_S | \theta_{S-1}\right)$
- Generate $M_{Candidate}/C_{Candidate}$ using $\theta_{Candidate}$ with **GetSample** and **ConvertConcentration**
- Calculate $ratio = log\left(P\left(Y|\theta^*\right)\right) + log\left(P\left(\theta^*\right)\right) + log\left(q\left(\theta_{s-1}|\theta^*\right)\right) - \left(log\left(P\left(Y|\theta_{s-1}\right)\right) + log\left(P\left(\theta_{s-1}\right)\right) + log\left(q\left(\theta^*|\theta_{s-1}\right)\right)\right)$
- Accept or reject $\theta_{Candidate}$ with probability e^{ratio}
    - If Accepted, $M_{Current} = M_{Candidate}$, $C_{Current} = C_{Candidate}$
- After all parameters, calculate DIC and $P(Y|\theta)$

Inputs:

- P

$n = 10$

$T = 100$

$\delta = 1^{-10}$

$\tau_{True} \sim Gamma(1, 1)$

$\left[V_1, ..., V_n\right] \sim NegBinom(1, 1) + 1$

$\left[M_{True}, \alpha_{True}, \beta_{True}, N_{True}, I_{True}, T_{True}, W_{True}, S_{True}\right] = runSimulation(n, T, P)$

$C_{True} = ConvertCalculation\left(n, T, M_{True}, V, \delta\right)$

$Y_{True} = ConvertY\left(C_{True}, \tau_{True}\right)$

$O = generateObservationArray$

$\alpha_0 \sim Gamma(2,\ 1)$

$\beta_0 \sim Beta(1,\ 1)$

$N\ =\ N_{True}$

$\left[I_{0,1},...,\ I_{0,N}\right] \sim DiscreteUniform(1,\ n)$

$\left[t_{1,1},...,\ t_{0,N}\right] \sim DiscreteUniform(1,\ T)$

$\left[w_{0,1},...,\ w_{0,N}\right] \sim NegBinom(6,\ 0.75)$

$\left[s_{0,1},...,\ s_{0,N}\right] \sim Gamma(1,\ 1)$

$\tau_0 \sim Gamma(1,\ 1)$

$\phi\ =\ \left[\alpha_0,\ \beta_0,\ N,\ \left[I_{0,1},...,\ I_{0,N}\right],\ \left[t_{0,1},...,\ t_{0,N}\right],\ \left[w_{0,1},...,\ w_{0,N}\right],\ \left[s_{0,1},...,\ s_{0,N}\right],\ \tau_0\right]$

$For\ each\ parameter_0\ in\ \phi$:

$Array_{parameter}\ =\ parameter_0$

$Accept_{parameter}\ =\ 0$

$M_{Current}\ =\ GetSample\left(n,\ T, \phi_\alpha,\ \phi_\beta,\ \phi_N,\ \phi_I,\ \phi_T,\ \phi_W,\ \phi_S\right)$

$C_{Current}\ =\ ConvertConcentration\left(n,\ T,\ M_{Current},\ V,\ \delta\right)$

$For\ s\ in\ 1: 10000$:

Alpha

$\alpha_{Candidate} \sim N\left(\alpha_{Current},\ 8^2\right)$

$M_{Candidate}\ =\ GetSample\left(n,\ T,\ \alpha_{Candidate},\ \phi_\beta,\ \phi_N,\ \phi_I,\ \phi_T,\ \phi_W,\ \phi_S\right)$

$C_{Candidate}\ =\ ConvertConcentration\left(n,\ T,\ M_{Candidate},\ V,\ \delta\right)$

$$p_{prior_{Candidate}} = Gamma(2,\ 1).\, logPDF\left(\alpha_{Candidate}\right)$$

$$p_{prior_{Current}} = Gamma(2,\ 1).\, logPDF\left(\phi_{\alpha}\right)$$

$$p_{Likelihood_{Candidate}} = \sum_{t=1}^{T}\left(\sum_{i=1}^{n} N\left(C_{Candidate_{it}} \cdot O,\ \frac{1}{\sqrt{\phi_{\tau}}}^{2}\right).\, logPDF\left(Y_{True_{it}} \cdot O\right)\right)$$

$$p_{Likelihood_{Current}} = \sum_{t=1}^{T}\left(\sum_{i=1}^{n} N\left(C_{Current_{it}} \cdot O,\ \frac{1}{\sqrt{\phi_{\tau}}}^{2}\right).\, logPDF\left(Y_{True_{it}} \cdot O\right)\right)$$

$$p_{Q_{Candidate}} = N\left(\phi_{\alpha},\ 8^{2}\right).\, logPDF\left(\alpha_{Candidate}\right)$$

$$p_{Q_{Current}} = N\left(\alpha_{Candidate},\ 8^{2}\right).\, logPDF\left(\phi_{\alpha}\right)$$

$$ratio_{\alpha} = p_{prior_{Candidate}} + p_{Likelihood_{Candidate}} - p_{prior_{Current}} - p_{Likelihood_{Current}} + p_{Q_{Current}} - p_{Q_{Candidate}}$$

$$u \sim U(0,\ 1)$$

$if\ ratio > log(u)$:

$$\phi_{\alpha} = \alpha_{Candidate}$$

$$Accept_{\alpha} = Accept_{\alpha} + 1$$

$$M_{Candidate} = M_{Current}$$

$$C_{Candidate} = C_{Current}$$

$$Likelihood_{\alpha_{Observed_{s}}} = \sum_{t=1}^{T}\left(\sum_{i=1}^{n} N\left(C_{Candidate_{it}} \cdot O,\ \frac{1}{\sqrt{\phi_{\tau}}}^{2}\right).\, logPDF\left(Y_{True_{it}} \cdot O\right)\right)$$

$$Likelihood_{\alpha_{Full_{s}}} = \sum_{t=1}^{T}\left(\sum_{i=1}^{n} N\left(C_{Candidate_{it}},\ \frac{1}{\sqrt{\phi_{\tau}}}^{2}\right).\, logPDF\left(Y_{True_{it}}\right)\right)$$

$else$:

$$Likelihood_{\alpha_{Observed_s}} = \sum_{t=1}^{T}\left(\sum_{i=1}^{n} N\left(C_{Current_{it}} \cdot O, \frac{1}{\sqrt{\phi_{\tau}}}^{2}\right).logPDF\left(Y_{True_{it}} \cdot O\right)\right)$$

$$Likelihood_{\alpha_{Full_s}} = \sum_{t=1}^{T}\left(\sum_{i=1}^{n} N\left(C_{Current_{it}} \cdot O, \frac{1}{\sqrt{\phi_{\tau}}}^{2}\right).logPDF\left(Y_{True_{it}}\right)\right)$$

Beta

$\beta_{Candidate} \sim Beta(1, 1)$

$M_{Candidate} = GetSample\left(n, T, \phi_{\alpha}, \beta_{Candidate}, \phi_{N}, \phi_{I}, \phi_{T}, \phi_{W}, \phi_{S}\right)$

$C_{Candidate} = ConvertConcentration\left(n, T, M_{Candidate}, V, \delta\right)$

$p_{prior_{Candidate}} = Beta(1, 1).logPDF\left(\beta_{Candidate}\right)$

$p_{prior_{Current}} = Beta(1, 1).logPDF\left(\phi_{\beta}\right)$

$$p_{Likelihood_{Candidate}} = \sum_{t=1}^{T}\left(\sum_{i=1}^{n} N\left(C_{Candidate_{it}}, \frac{1}{\sqrt{\phi_{\tau}}}^{2}\right).logPDF\left(Y_{True_{it}}\right)\right)$$

$$p_{Likelihood_{Current}} = \sum_{t=1}^{T}\left(\sum_{i=1}^{n} N\left(C_{Current_{it}}, \frac{1}{\sqrt{\phi_{\tau}}}^{2}\right).logPDF\left(Y_{True_{it}}\right)\right)$$

$p_{Q_{Candidate}} = Beta(1, 1).logPDF\left(\beta_{Candidate}\right)$

$p_{Q_{Current}} = Beta(1, 1).logPDF\left(\phi_{\beta}\right)$

$ratio_{\beta} = p_{prior_{Candidate}} + p_{Likelihood_{Candidate}} - p_{prior_{Current}} - p_{Likelihood_{Current}} + p_{Q_{Current}} - p_{Q_{Candidate}}$

$u \sim U(0, 1)$

$if\ ratio\ >\ u:$

$$\phi_{\beta} = \beta_{Candidate}$$

$$Accept_{\beta} = Accept_{\beta} + 1$$

$$M_{Candidate} = M_{Current}$$

$$C_{Candidate} = C_{Current}$$

$$Likelihood_{\beta_{Observed_s}} = \sum_{t=1}^{T}\left(\sum_{i=1}^{n} N\left(C_{Candidate_{it}} \cdot O,\ \frac{1}{\sqrt{\phi_{\tau}}}^{2}\right).logPDF\left(Y_{True_{it}} \cdot O\right)\right)$$

$$Likelihood_{\beta_{Full_s}} = \sum_{t=1}^{T}\left(\sum_{i=1}^{n} N\left(C_{Candidate_{it}},\ \frac{1}{\sqrt{\phi_{\tau}}}^{2}\right).logPDF\left(Y_{True_{it}}\right)\right)$$

$else:$

$$Likelihood_{\beta_{Observed_s}} = \sum_{t=1}^{T}\left(\sum_{i=1}^{n} N\left(C_{Current_{it}} \cdot O,\ \frac{1}{\sqrt{\phi_{\tau}}}^{2}\right).logPDF\left(Y_{True_{it}} \cdot O\right)\right)$$

$$Likelihood_{\beta_{Full_s}} = \sum_{t=1}^{T}\left(\sum_{i=1}^{n} N\left(C_{Current_{it}} \cdot O,\ \frac{1}{\sqrt{\phi_{\tau}}}^{2}\right).logPDF\left(Y_{True_{it}}\right)\right)$$

$for\ k\ in\ 1:n:$

$$I_{k_{Candidate}} \sim DiscreteUniform(1,\ n)$$

$$I_{Candidate} = \phi_{I}$$

$$I_{Candidate_k} = I_{k_{Candidate}}$$

$$M_{Candidate} = GetSample\left(n,\ T,\ \phi_{\alpha},\ \phi_{\beta},\ \phi_{N},\ I_{Candidate},\ \phi_{T},\ \phi_{W},\ \phi_{S}\right)$$

$$C_{Candidate} = ConvertConcentration\left(n,\ T,\ M_{Candidate},\ V,\ \delta\right)$$

$$p_{prior_{Candidate}} = DiscreteUniform(1,\ n).logPMF\left(I_{Candidate_k}\right)$$

$$p_{prior_{Current}} = DiscreteUniform(1,\ n).logPMF\left(\phi_{I_k}\right)$$

$$p_{Likelihood_{Candidate}} = \sum_{t=1}^{T}\left(\sum_{i=1}^{n} N\left(C_{Candidate_{it}},\ \frac{1}{\sqrt{\phi_\tau}}^{2}\right).logPDF\left(Y_{True_{it}}\right)\right)$$

$$p_{Likelihood_{Current}} = \sum_{t=1}^{T}\left(\sum_{i=1}^{n} N\left(C_{Current_{it}},\ \frac{1}{\sqrt{\phi_\tau}}^{2}\right).logPDF\left(Y_{True_{it}}\right)\right)$$

$$p_{Q_{Candidate}} = DiscreteUniform(1,\ n).logPMF\left(I_{Candidate_k}\right)$$

$$p_{Q_{Current}} DiscreteUniform(1,\ n).logPMF\left(\phi_{I_k}\right)$$

$$ratio_{I_k} = p_{prior_{Candidate}} + p_{Likelihood_{Candidate}} - p_{prior_{Current}} - p_{Likelihood_{Current}} + p_{Q_{Current}} - p_{Q_{Candidate}}$$

$$u \sim U(0,\ 1)$$

$$if\ ratio > log(u):$$

$$\phi_I = I_{Candidate}$$

$$Accept_I = Accept_I + \frac{1}{N}$$

$$M_{Candidate} = M_{Current}$$

$$C_{Candidate} = C_{Current}$$

$$Likelihood_{I_{Observed_{s,k}}} = \sum_{t=1}^{T}\left(\sum_{i=1}^{n} N\left(C_{Candidate_{it}} \cdot O,\ \frac{1}{\sqrt{\phi_\tau}}^{2}\right).logPDF\left(Y_{True_{it}} \cdot O\right)\right)$$

$$Likelihood_{I_{Full_{s,k}}} = \sum_{t=1}^{T}\left(\sum_{i=1}^{n} N\left(C_{Candidate_{it}},\ \frac{1}{\sqrt{\phi_\tau}}^{2}\right).logPDF\left(Y_{True_{it}}\right)\right)$$

$else$:

$$Likelihood_{I_{Observed_{s,k}}} = \sum_{t=1}^{T}\left(\sum_{i=1}^{n} N\left(C_{Current_{it}} \cdot O,\ \frac{1}{\sqrt{\phi_{\tau}}}^{2}\right).logPDF\left(Y_{True_{it}} \cdot O\right)\right)$$

$$Likelihood_{I_{Full_{s,k}}} = \sum_{t=1}^{T}\left(\sum_{i=1}^{n} N\left(C_{Current_{it}} \cdot O,\ \frac{1}{\sqrt{\phi_{\tau}}}^{2}\right).logPDF\left(Y_{True_{it}}\right)\right)$$

T

$for\ k\ in\ 1\colon n$:

$$T_{k_{Candidate}} \sim DiscreteUniform(1,\ T)$$

$$T_{Candidate} = \phi_{T}$$

$$T_{Candidate_{k}} = T_{k_{Candidate}}$$

$$M_{Candidate} = GetSample\left(n,\ T,\ \phi_{\alpha},\ \phi_{\beta},\ \phi_{N},\ \phi_{I},\ T_{Candidate},\ \phi_{W},\ \phi_{S}\right)$$

$$C_{Candidate} = ConvertConcentration\left(n,\ T,\ M_{Candidate},\ V,\ \delta\right)$$

$$p_{prior_{Candidate}} = DiscreteUniform(1,\ T).logPMF\left(T_{Candidate_{k}}\right)$$

$$p_{prior_{Current}} = DiscreteUniform(1,\ T).logPMF\left(\phi_{T_{k}}\right)$$

$$p_{Likelihood_{Candidate}} = \sum_{t=1}^{T}\left(\sum_{i=1}^{n} N\left(C_{Candidate_{it}},\ \frac{1}{\sqrt{\phi_{\tau}}}^{2}\right).logPDF\left(Y_{True_{it}}\right)\right)$$

$$p_{Likelihood_{Current}} = \sum_{t=1}^{T}\left(\sum_{i=1}^{n} N\left(C_{Current_{it}},\ \frac{1}{\sqrt{\phi_{\tau}}}^{2}\right).logPDF\left(Y_{True_{it}}\right)\right)$$

$$p_{Q_{Candidate}} = DiscreteUniform(1,\ T).logPMF\left(T_{Candidate_k}\right)$$

$$p_{Q_{Current}} = DiscreteUniform(1,\ T).logPMF\left(\phi_{4_k}\right)$$

$$ratio_{T_k} = min\left(p_{prior_{Candidate}} + p_{Likelihood_{Candidate}} - p_{prior_{Current}} - p_{Likelihood_{Current}} + p_{Q_{Current}} - p_{Q_{Candidate}},\ 1\right)$$

$$u \sim U(0,\ 1)$$

$if\ ratio > log(u)$:

$$\phi_T = T_{Candidate}$$

$$Accept_T = Accept_T + \frac{1}{N}$$

$$M_{Candidate} = M_{Current}$$

$$C_{Candidate} = C_{Current}$$

$$Likelihood_{T_{Observed_{s,k}}} = \sum_{t=1}^{T}\left(\sum_{i=1}^{n} N\left(C_{Candidate_{it}} \cdot O,\ \frac{1}{\sqrt{\phi_\tau}}^2\right).logPDF\left(Y_{True_{it}} \cdot O\right)\right)$$

$$Likelihood_{T_{Full_{s,k}}} = \sum_{t=1}^{T}\left(\sum_{i=1}^{n} N\left(C_{Candidate_{it}},\ \frac{1}{\sqrt{\phi_\tau}}^2\right).logPDF\left(Y_{True_{it}}\right)\right)$$

$else$:

$$Likelihood_{T_{Observed_{s,k}}} = \sum_{t=1}^{T}\left(\sum_{i=1}^{n} N\left(C_{Current_{it}} \cdot O,\ \frac{1}{\sqrt{\phi_\tau}}^2\right).logPDF\left(Y_{True_{it}} \cdot O\right)\right)$$

$$Likelihood_{T_{Full_{s,k}}} = \sum_{t=1}^{T}\left(\sum_{i=1}^{n} N\left(C_{Current_{it}} \cdot O,\ \frac{1}{\sqrt{\phi_\tau}}^2\right).logPDF\left(Y_{True_{it}}\right)\right)$$

W

$for\ k\ in\ 1{:}\ n{:}$

$$W_{k_{Candidate}} \sim Binom\left(T,\ \frac{1+\phi_{W_k}}{2+T}\right)$$

$$W_{Candidate} = \phi_W$$

$$W_{Candidate_k} = W_{k_{Candidate}}$$

$$M_{Candidate} = GetSample\left(n,\ T,\ \phi_{\alpha},\ \phi_{\beta},\ \phi_{N},\ \phi_{I},\ \phi_{T},\ W_{Candidate},\ \phi_{S}\right)$$

$$C_{Candidate} = ConvertConcentration\left(n,\ T,\ M_{Candidate},\ V,\ \delta\right)$$

$$p_{prior_{Candidate}} = DiscreteUniform(1,\ T).logPMF\left(W_{Candidate_k}\right)$$

$$p_{prior_{Current}} = DiscreteUniform(1,\ T).logPMF\left(\phi_{W_k}\right)$$

$$p_{Likelihood_{Candidate}} = \sum_{t=1}^{T}\left(\sum_{i=1}^{n} N\left(C_{Candidate_{it}},\ \frac{1}{\sqrt{\phi_{\tau}}}^{2}\right).logPDF\left(Y_{True_{it}}\right)\right)$$

$$p_{Likelihood_{Current}} = \sum_{t=1}^{T}\left(\sum_{i=1}^{n} N\left(C_{Current_{it}},\ \frac{1}{\sqrt{\phi_{\tau}}}^{2}\right).logPDF\left(Y_{True_{it}}\right)\right)$$

$$p_{Q_{Candidate}} = Binom\left(T,\ \frac{1+\phi_{W_k}}{2+T}\right).logPMF\left(W_{Candidate_k}\right)$$

$$p_{Q_{Current}}\ Binom\left(T,\ \frac{1+W_{Candidate_k}}{2+T}\right).logPMF\left(\phi_{W_k}\right)$$

$$ratio_{W_k} = p_{prior_{Candidate}} + p_{Likelihood_{Candidate}} - p_{prior_{Current}} - p_{Likelihood_{Current}} + p_{Q_{Current}} - p_{Q_{Candidate}}$$

$$u \sim Uniform(0,\ 1)$$

$$if\ ratio > log(u):$$

$$\phi_W = W_{Candidate}$$

$$Accept_W = Accept_W + \frac{1}{N}$$

$$M_{Candidate} = M_{Current}$$

$$C_{Candidate} = C_{Current}$$

$$Likelihood_{W_{Observed_{s,k}}} = \sum_{t=1}^{T}\left(\sum_{i=1}^{n} N\left(C_{Candidate_{it}} \cdot O,\ \frac{1}{\sqrt{\phi_\tau}}^2\right).logPDF\left(Y_{True_{it}} \cdot O\right)\right)$$

$$Likelihood_{W_{Full_{s,k}}} = \sum_{t=1}^{T}\left(\sum_{i=1}^{n} N\left(C_{Candidate_{it}},\ \frac{1}{\sqrt{\phi_\tau}}^2\right).logPDF\left(Y_{True_{it}}\right)\right)$$

$$else:$$

$$Likelihood_{W_{Observed_{s,k}}} = \sum_{t=1}^{T}\left(\sum_{i=1}^{n} N\left(C_{Current_{it}} \cdot O,\ \frac{1}{\sqrt{\phi_\tau}}^2\right).logPDF\left(Y_{True_{it}} \cdot O\right)\right)$$

$$Likelihood_{W_{Full_{s,k}}} = \sum_{t=1}^{T}\left(\sum_{i=1}^{n} N\left(C_{Current_{it}} \cdot O,\ \frac{1}{\sqrt{\phi_\tau}}^2\right).logPDF\left(Y_{True_{it}}\right)\right)$$

S

$$for\ k\ in\ 1:n:$$

$$S_{k_{Candidate}} \sim N\left(\phi_{S_k},\ 0.1^2\right)$$

$$S_{Candidate} = \phi_S$$

$$S_{Candidate_k} = S_{k_{Candidate}}$$

$$M_{Candidate} = GetSample\left(n, T, \phi_{\alpha}, \phi_{\beta}, \phi_{N}, \phi_{I}, \phi_{T}, \phi_{W}, S_{Candidate}\right)$$

$$C_{Candidate} = ConvertConcentration\left(n, T, M_{Candidate}, V, \delta\right)$$

$$p_{prior_{Candidate}} = Gamma(1, 1).logPDF\left(S_{Candidate_k}\right)$$

$$p_{prior_{Current}} = Gamma(1, 1).logPDF\left(\phi_{S_k}\right)$$

$$p_{Likelihood_{Candidate}} = \sum_{t=1}^{T}\left(\sum_{i=1}^{n} N\left(C_{Candidate_{it}}, \frac{1}{\sqrt{\phi_{\tau}}}^{2}\right).logPDF\left(Y_{True_{it}}\right)\right)$$

$$p_{Likelihood_{Current}} = \sum_{t=1}^{T}\left(\sum_{i=1}^{n} N\left(C_{Current_{it}}, \frac{1}{\sqrt{\phi_{\tau}}}^{2}\right).logPDF\left(Y_{True_{it}}\right)\right)$$

$$p_{Q_{Candidate}} = N\left(\phi_{S_k}, 0.1^{2}\right).logPDF\left(S_{Candidate_k}\right)$$

$$p_{Q_{Current}} N\left(S_{Candidate_k}, 0.1^{2}\right).logPDF\left(\phi_{S_k}\right)$$

$$ratio_{S_k} = p_{prior_{Candidate}} + p_{Likelihood_{Candidate}} - p_{prior_{Current}} - p_{Likelihood_{Current}} + p_{Q_{Current}} - p_{Q_{Candidate}}$$

$$u \sim U(0, 1)$$

$$if\ ratio > log(u):$$

$$\phi_{S} = S_{Candidate}$$

$$Accept_{S} = Accept_{S} + \frac{1}{N}$$

$$M_{Candidate} = M_{Current}$$

$$C_{Candidate} = C_{Current}$$

$$Likelihood_{S_{Observed_{s,k}}} = \sum_{t=1}^{T}\left(\sum_{i=1}^{n} N\left(C_{Candidate_{it}} \cdot O, \frac{1}{\sqrt{\phi_{\tau}}}^{2}\right).logPDF\left(Y_{True_{it}} \cdot O\right)\right)$$

$$Likelihood_{S_{Full_{s,k}}} = \sum_{t=1}^{T}\left(\sum_{i=1}^{n} N\left(C_{Candidate_{it}}, \frac{1}{\sqrt{\phi_{\tau}}}^{2}\right).logPDF\left(Y_{True_{it}}\right)\right)$$

$else$:

$$Likelihood_{S_{Observed_{s,k}}} = \sum_{t=1}^{T}\left(\sum_{i=1}^{n} N\left(C_{Current_{it}} \cdot O, \frac{1}{\sqrt{\phi_{\tau}}}^{2}\right).logPDF\left(Y_{True_{it}} \cdot O\right)\right)$$

$$Likelihood_{S_{Full_{s,k}}} = \sum_{t=1}^{T}\left(\sum_{i=1}^{n} N\left(C_{Current_{it}} \cdot O, \frac{1}{\sqrt{\phi_{\tau}}}^{2}\right).logPDF\left(Y_{True_{it}}\right)\right)$$

Tau

$$\tau_{Candidate} \sim N\left(\phi_{\tau}, \frac{1}{2}^{2}\right)$$

$$p_{prior_{Candidate}} = Gamma(1, 1).logPDF\left(\tau_{Candidate}\right)$$

$$p_{prior_{Current}} = Gamma(1, 1).logPDF\left(\phi_{\tau}\right)$$

$$p_{Likelihood_{Candidate}} = \sum_{t=1}^{T}\left(\sum_{i=1}^{n} N\left(C_{Current_{it}} \cdot O, \frac{1}{\sqrt{\tau_{Candidate}}}^{2}\right).logPDF\left(Y_{True_{it}} \cdot O\right)\right)$$

$$p_{Likelihood_{Current}} = \sum_{t=1}^{T}\left(\sum_{i=1}^{n} N\left(C_{Current_{it}} \cdot O, \frac{1}{\sqrt{\phi_{\tau}}}^{2}\right).logPDF\left(Y_{True_{it}} \cdot O\right)\right)$$

$$p_{Q_{Candidate}} = N\left(\phi_{\tau}, \frac{1}{2}^{2}\right).logPDF\left(\tau_{Candidate}\right)$$

$$p_{Q_{Current}} N\left(\tau_{Candidate}, \frac{1}{2}^{2}\right).logPDF\left(\phi_{\tau}\right)$$

$$ratio_{\tau} = p_{prior_{Candidate}} + p_{Likelihood_{Candidate}} - p_{prior_{Current}} - p_{Likelihood_{Current}} + p_{Q_{Current}} - p_{Q_{Candidate}}$$

$$u \sim U(0,\ 1)$$

$if\ ratio\ >\ log(u)$:

$$\phi_{\tau} = \tau_{Candidate}$$

$$Accept_{\tau} = Accept_{\tau} + 1$$

$$Likelihood_{\tau_{Observed_s}} = \sum_{t=1}^{T}\left(\sum_{i=1}^{n} N\left(C_{Candidate_{it}} \cdot O,\ \frac{1}{\sqrt{\phi_{\tau}}}^{2}\right).logPDF\left(Y_{True_{it}} \cdot O\right)\right)$$

$$Likelihood_{\tau_{Full_s}} = \sum_{t=1}^{T}\left(\sum_{i=1}^{n} N\left(C_{Candidate_{it}},\ \frac{1}{\sqrt{\phi_{\tau}}}^{2}\right).logPDF\left(Y_{True_{it}}\right)\right)$$

$else$:

$$Likelihood_{\tau_{Observed_s}} = \sum_{t=1}^{T}\left(\sum_{i=1}^{n} N\left(C_{Current_{it}} \cdot O,\ \frac{1}{\sqrt{\phi_{\tau}}}^{2}\right).logPDF\left(Y_{True_{it}} \cdot O\right)\right)$$

$$Likelihood_{\tau_{Full_s}} = \sum_{t=1}^{T}\left(\sum_{i=1}^{n} N\left(C_{Current_{it}} \cdot O,\ \frac{1}{\sqrt{\phi_{\tau}}}^{2}\right).logPDF\left(Y_{True_{it}}\right)\right)$$

Proposal Distributions

$\alpha^* \sim N\left(\phi_{\alpha}, 2^2\right)$

Using a random walk proposal with $E\left(\alpha^*\right) = \phi_{\alpha}$. Using a standard deviation of $\sigma = 2$ because through testing, α has had a higher variance, especially compared to other parameters with normally distributed proposals like S_k and τ.

$\beta^* \sim Beta(1, 1)$

An independence proposal in order to sample only from β's possible range of values of $[0, 1]$

$I_k^* \sim DiscreteUniform(1, n)$

An independence proposal in order to sample only from I_k's possible range of values of $[0, n]$

$T_k^* \sim DiscreteUniform(1, T)$

An independence proposal in order to sample only from T_k's possible range of values of $[0, T]$

$W_k^* \sim Binom\left(T, \frac{1+\phi_{W_k}}{2+T}\right)$

A random walk sampler over a discrete domain where $E\left(W_k^*\right) \approx \phi_{W_k}$ as possible, with the added requirements that $p\left(W_k^* | \phi_{W_k} = 0\right) \neq 0$ (Hence the $+ 1$), and $p\left(W_k^* | \phi_{W_k} = T\right) \neq 1$ (Hence the $+ 2$)

$S_k^* \sim N\left(\phi_{S_k}, 1^2\right)$

Using a random walk proposal with $E\left(S_k^*\right) = \phi_{S_k}$. Using a standard deviation of $\sigma = 1$ because through testing, the default value of 1 allowed an appropriate amount of variance for S_k.

$\tau^* \sim N\left(\phi_{\tau}, \frac{1}{2}^2\right)$

Using a random walk proposal with $E\left(\tau^*\right) = \phi_{\tau}$. Using a standard deviation of $\sigma = \frac{1}{2}$ because through testing, τ had a fairly tight distribution compared to α and S_k.

Acceptance Probabilities

Alpha

$$p_{prior_{Candidate}} = log(\alpha_{Candidate}) - \alpha_{Candidate}$$

$$p_{prior_{Current}} = log(\alpha_{Current}) - \alpha_{Current}$$

$$p_{Likelihood_{Candidate}} = \frac{-\tau}{32}\sum_{t=1}^{T}\left(\sum_{i=1}^{n}\left(Y_{True_{it}} - C_{Candidate_{it}}\right)^2\right)$$

$$p_{Likelihood_{Current}} = \frac{-\tau}{2}\sum_{t=1}^{T}\left(\sum_{i=1}^{n}\left(Y_{True_{it}} - C_{Current_{it}}\right)^2\right)$$

$$p_{Q_{Candidate}} = -\frac{1}{8}\left(\alpha_{Candidate} - \alpha_{Current}\right)^2$$

$$p_{Q_{Candidate}} = -\frac{1}{8}\left(\alpha_{Current} - \alpha_{Candidate}\right)^2$$

$$ratio_{log} = log\left(\frac{\alpha_{Candidate}}{\alpha_{Current}}\right) - \alpha_{Candidate} + \alpha_{Current} + \frac{\tau}{2}\sum_{t=1}^{T}\left(\sum_{i=1}^{n}\left(Y_{True_{it}} - C_{Current_{it}}\right)^2 - \left(Y_{True_{it}} - C_{Candidate_{it}}\right)^2\right)$$

$$P_{Acceptance}(\alpha) = e^{ratio_{log}}$$

Beta

$$p_{prior_{Candidate}} = 0$$

$$p_{prior_{Current}} = 0$$

$$p_{Likelihood_{Candidate}} = \frac{-\tau}{2}\sum_{t=1}^{T}\left(\sum_{i=1}^{n}\left(Y_{True_{it}} - C_{Candidate_{it}}\right)^{2}\right)$$

$$p_{Likelihood_{Current}} = \frac{-\tau}{2}\sum_{t=1}^{T}\left(\sum_{i=1}^{n}\left(Y_{True_{it}} - C_{Current_{it}}\right)^{2}\right)$$

$$p_{Q_{Candidate}} = 0$$

$$p_{Q_{Candidate}} = 0$$

$$ratio_{log} = \frac{\tau}{2}\sum_{t=1}^{T}\left(\sum_{i=1}^{n}\left(Y_{True_{it}} - C_{Current_{it}}\right)^{2} - \left(Y_{True_{it}} - C_{Candidate_{it}}\right)^{2}\right)$$

$$P_{Acceptance}(\beta) = e^{ratio_{log}}$$

I_k

$$p_{prior_{Candidate}} = -\log(n)$$

$$p_{prior_{Current}} = -\log(n)$$

$$p_{Likelihood_{Candidate}} = \frac{-\tau}{2}\sum_{t=1}^{T}\left(\sum_{i=1}^{n}\left(Y_{True_{it}} - C_{Candidate_{it}}\right)^2\right)$$

$$p_{Likelihood_{Current}} = \frac{-\tau}{2}\sum_{t=1}^{T}\left(\sum_{i=1}^{n}\left(Y_{True_{it}} - C_{Current_{it}}\right)^2\right)$$

$$p_{Q_{Candidate}} = -\log(n)$$

$$p_{Q_{Candidate}} = -\log(n)$$

$$ratio_{log} = \frac{\tau}{2}\sum_{t=1}^{T}\left(\sum_{i=1}^{n}\left(Y_{True_{it}} - C_{Current_{it}}\right)^2 - \left(Y_{True_{it}} - C_{Candidate_{it}}\right)^2\right)$$

$$P_{Acceptance}\left(I_k\right) = e^{ratio_{log}}$$

T_k

$$p_{prior_{Candidate}} = -\log(T)$$

$$p_{prior_{Current}} = -\log(T)$$

$$p_{Likelihood_{Candidate}} = \frac{-\tau}{2}\sum_{t=1}^{T}\left(\sum_{i=1}^{n}\left(Y_{True_{it}} - C_{Candidate_{it}}\right)^2\right)$$

$$p_{Likelihood_{Current}} = \frac{-\tau}{2}\sum_{t=1}^{T}\left(\sum_{i=1}^{n}\left(Y_{True_{it}} - C_{Current_{it}}\right)^2\right)$$

$$p_{Q_{Candidate}} = -\log(T)$$

$$p_{Q_{Candidate}} = -\log(T)$$

$$ratio_{log} = \frac{\tau}{2}\sum_{t=1}^{T}\left(\sum_{i=1}^{n}\left(Y_{True_{it}} - C_{Current_{it}}\right)^2 - \left(Y_{True_{it}} - C_{Candidate_{it}}\right)^2\right)$$

$$P_{Acceptance}\left(T_k\right) = e^{ratio_{log}}$$

W_k

$$p_{prior_{Candidate}} = log\left(\left(W_{Candidate_k} + 5\right)C\left(W_{Candidate_k}\right)\right) + 6log\left(\frac{3}{4}\right) - W_{Candidate_k}log(4)$$

$$p_{prior_{Current}} = log\left(\left(W_{Current_k} + 5\right)C\left(W_{Current_k}\right)\right) + 6log\left(\frac{3}{4}\right) - W_{Current_k}log(4)$$

$$p_{Likelihood_{Candidate}} = \frac{-\tau}{2}\sum_{t=1}^{T}\left(\sum_{i=1}^{n}\left(Y_{True_{it}} - C_{Candidate_{it}}\right)^2\right)$$

$$p_{Likelihood_{Current}} = \frac{-\tau}{2}\sum_{t=1}^{T}\left(\sum_{i=1}^{n}\left(Y_{True_{it}} - C_{Current_{it}}\right)^2\right)$$

$$p_{Q_{Candidate}} = log\left((T)C\left(W_{Candidate_k}\right)\right) + W_{Candidate_k}log\left(W_{Current_k} + 1\right) + \left(T - W_{Candidate_k}\right)log\left(T + 1 - W_{Current_k}\right) + Tlog(T + 2)$$

$$p_{Q_{Candidate}} = log\left((T)C\left(W_{Current_k}\right)\right) + W_{Current_k}log\left(W_{Candidate_k} + 1\right) + \left(T - W_{Current_k}\right)log\left(T + 1 - W_{Candidate_k}\right) + Tlog(T + 2)$$

$$\begin{aligned} ratio_{log} = {} & log\left(\left(W_{Candidate_k} + 5\right)C\left(W_{Candidate_k}\right)\right) + \left(W_{Current_k} - W_{Candidate_k}\right)log(4) - log\left(\left(W_{Current_k} + 5\right)C\left(W_{Current_k}\right)\right) \\ & + \frac{\tau}{2}\sum_{t=1}^{T}\left(\sum_{i=1}^{n}\left(Y_{True_{it}} - C_{Current_{it}}\right)^2 - \left(Y_{True_{it}} - C_{Candidate_{it}}\right)^2\right) + log\left((T)C\left(W_{Candidate_k}\right)\right) + W_{Candidate_k}log\left(W_{Current_k} + 1\right) \\ & + \left(T - W_{Candidate_k}\right)log\left(T + 1 - W_{Current_k}\right) - log\left((T)C\left(W_{Current_k}\right)\right) - W_{Current_k}log\left(W_{Candidate_k} + 1\right) - \left(T - W_{Current_k}\right)log\left(T + 1 - W_{Candidate_k}\right) \end{aligned}$$

$$P_{Acceptance}\left(W_k\right) = e^{ratio_{log}}$$

S_k

$$p_{prior_{Candidate}} = -S_{Candidate_k}$$

$$p_{prior_{Current}} = -S_{Current_k}$$

$$p_{Likelihood_{Candidate}} = \frac{-\tau}{2}\sum_{t=1}^{T}\left(\sum_{i=1}^{n}\left(Y_{True_{it}} - C_{Candidate_{it}}\right)^2\right)$$

$$p_{Likelihood_{Current}} = \frac{-\tau}{2}\sum_{t=1}^{T}\left(\sum_{i=1}^{n}\left(Y_{True_{it}} - C_{Current_{it}}\right)^2\right)$$

$$p_{Q_{Candidate}} = -50\left(S_{Candidate_k} - S_{Current_k}\right)^2$$

$$p_{Q_{Candidate}} = -50\left(S_{Current_k} - S_{Candidate_k}\right)^2$$

$$ratio_{log} = -S_{Candidate_k} + S_{Current_k} + \frac{\tau}{2}\sum_{t=1}^{T}\left(\sum_{i=1}^{n}\left(Y_{True_{it}} - C_{Current_{it}}\right)^2 - \left(Y_{True_{it}} - C_{Candidate_{it}}\right)^2\right)$$

$$P_{Acceptance}\left(S_k\right) = e^{ratio_{log}}$$

Tau

$$p_{prior_{Candidate}} = log\left(\tau_{Candidate}\right) - \tau_{Candidate}$$

$$p_{prior_{Current}} = log\left(\tau_{Current}\right) - \tau_{Current}$$

$$p_{Likelihood_{Candidate}} = \frac{-\tau_{Candidate}}{2}\sum_{t=1}^{T}\left(\sum_{i=1}^{n}\left(Y_{True_{it}} - C_{Candidate_{it}}\right)\right)^{2}$$

$$p_{Likelihood_{Current}} = \frac{-\tau_{Current}}{2}\sum_{t=1}^{T}\left(\sum_{i=1}^{n}\left(Y_{True_{it}} - C_{Current_{it}}\right)\right)^{2}$$

$$p_{Q_{Candidate}} = -2\left(\tau_{Candidate} - \tau_{Current}\right)^{2}$$

$$p_{Q_{Candidate}} = -2\left(\tau_{Current} - \tau_{Candidate}\right)^{2}$$

Since for τ, $C_{Candidate_{it}} = C_{Current_{it}}$ for all $i,\ t$

$$ratio_{log} = log\left(\frac{\tau_{Candidate}}{\tau_{Current}}\right) - \tau_{Candidate} + \tau_{Current}$$

$$P_{Acceptance}(\tau) = e^{ratio_{log}}$$

Results

Changing Sparseness

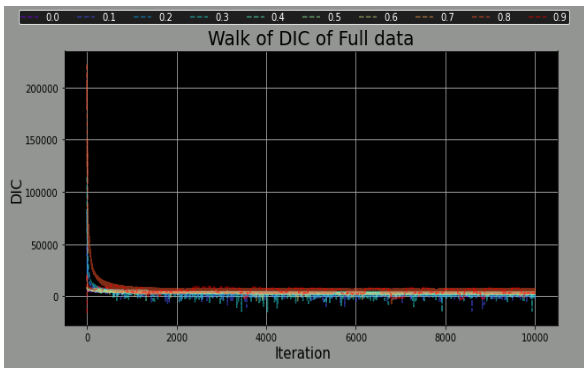
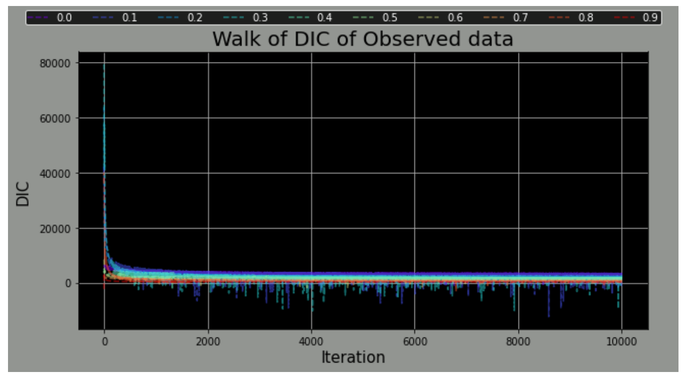
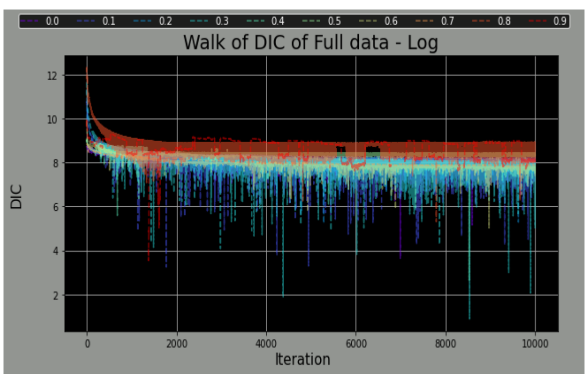
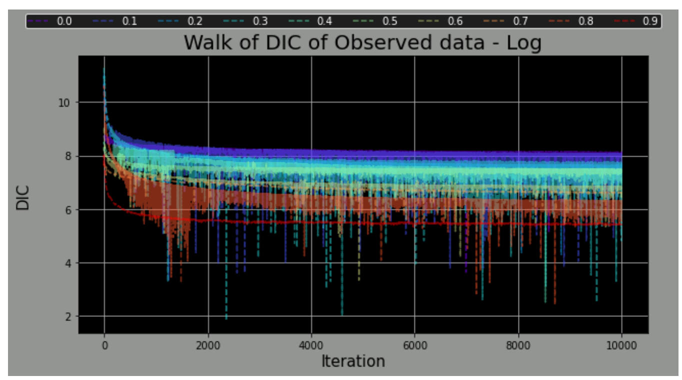
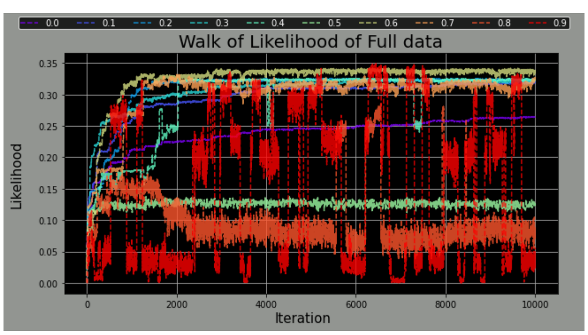
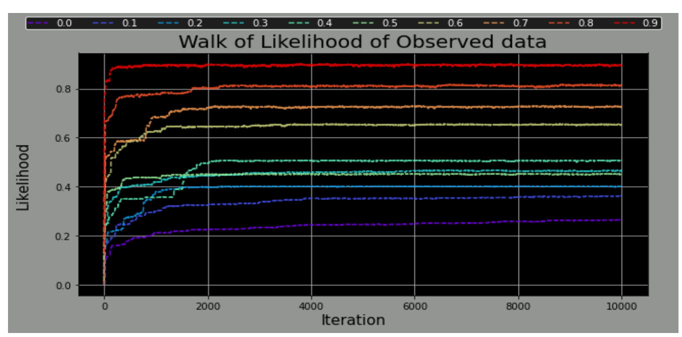
Model Comparison Metrics	Full Data	Observed Data
DIC		
DIC - Log		
Likelihood		

Figure X + 4: Table of Model Comparison metrics for models with varying sparseness of data

p = 0.0

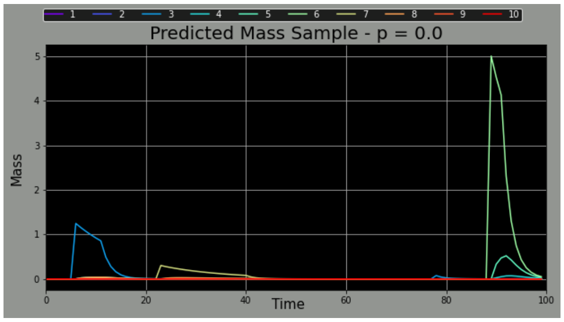
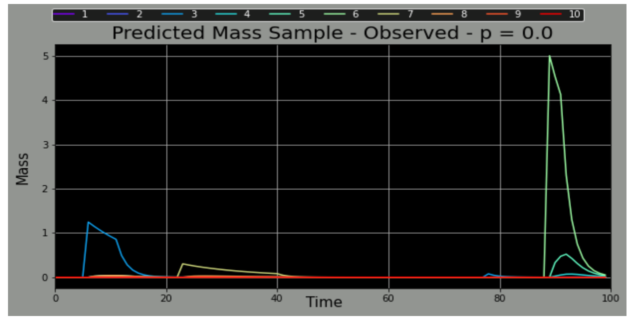
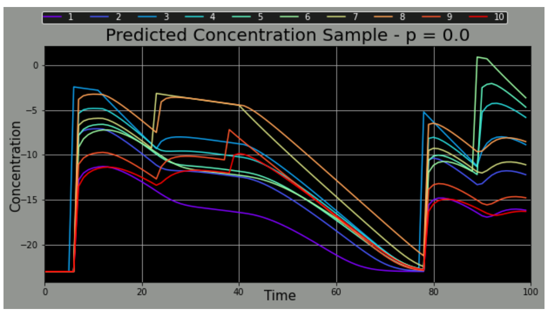
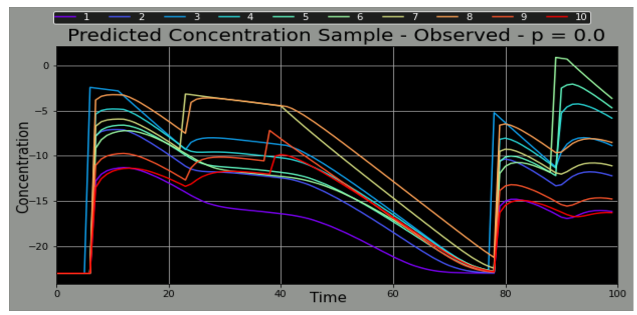
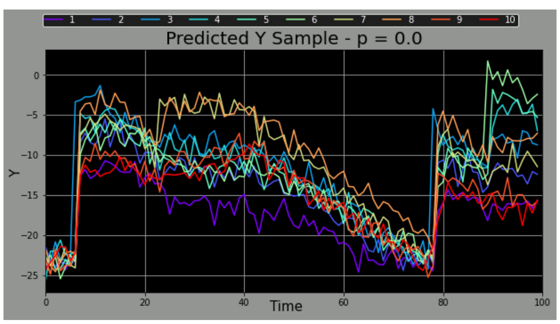
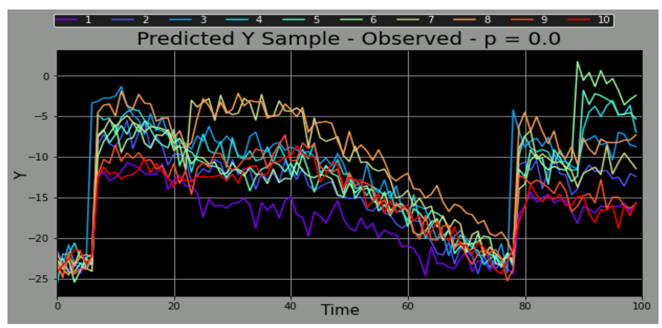
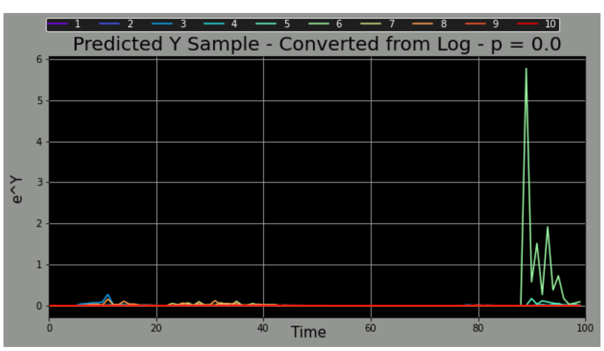
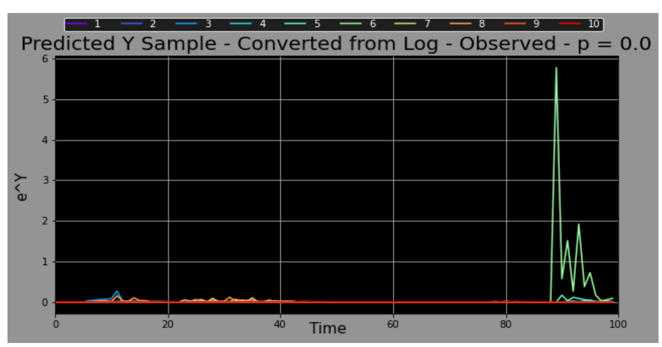
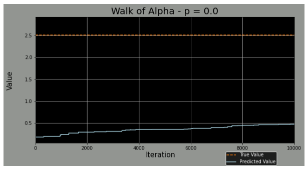
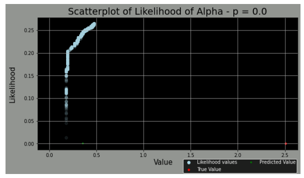
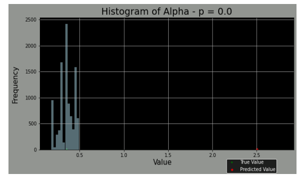
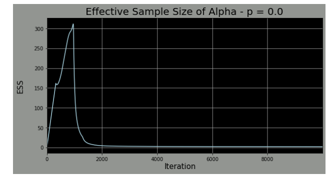
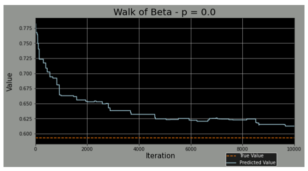
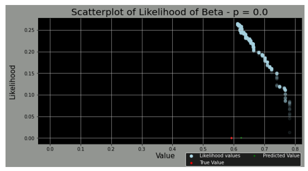
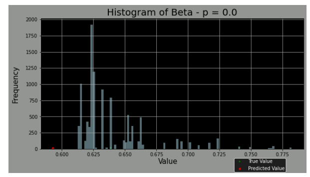
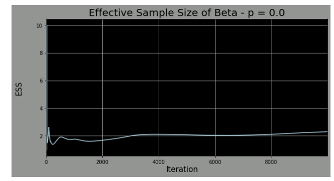
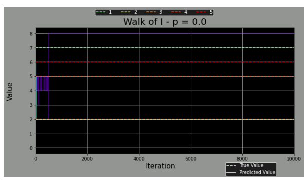
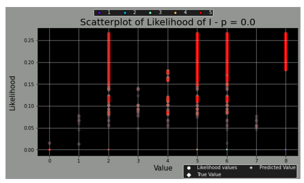
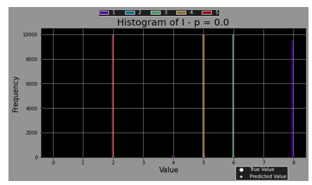
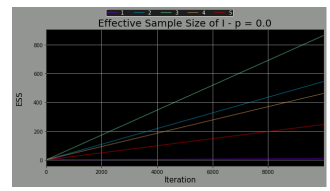
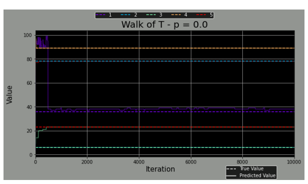
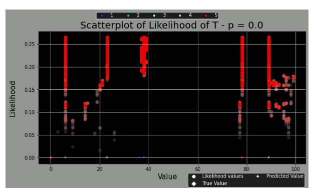
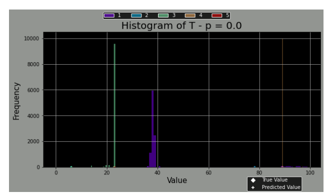
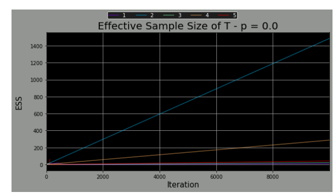
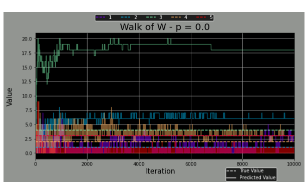
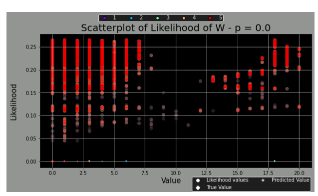
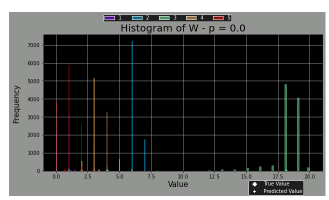
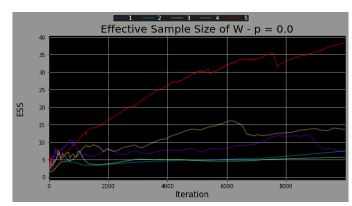
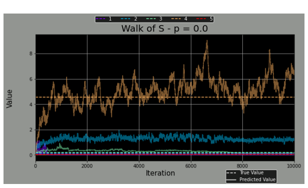
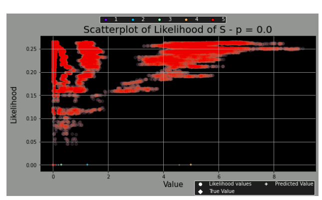
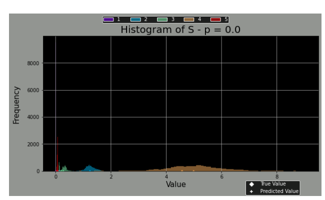
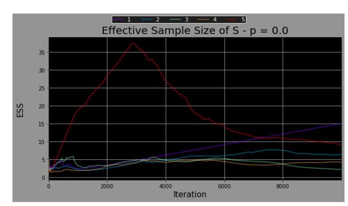
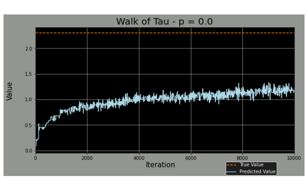
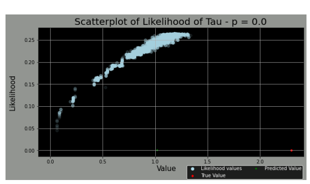
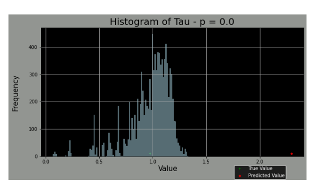
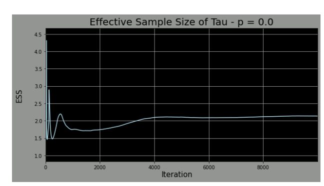
Results	Full	Observed
Mass		
Concentration		
Y		
e^Y		

Figure X + 5: Table of results for sparse data (p = 0.0)

Parameter	Median	95% CI - Lower Bound	95% CI - Upper Bound	P(Acceptance)
α	0.3557	0.3501	0.3535	0.0037
β	0.6252	0.6883	0.6394	0.0059
I	8, 2, 6, 5, 2	7.8188, 2.0016, 5.9958, 4.9960, 2.0027	7.8482, 2.0076, 5.9998, 4.9990, 2.0067	0.1022
T	36, 6, 23, 89, 78	40.5308, 5.9999, 22.7976, 89.0103, 77.9859	40.9932, 6.0033, 22.8432, 89.0287, 77.9901	0.0122
W	1, 6, 18, 3, 1	1.4617, 5.9910, 18.1646, 3.3658, 0.6595	1.5065, 6.0218, 18.2184, 3.4002, 0.6861	0.3117
S	0.0043, 1.2469, 0.3017, 5.0037, 0.0764	0.0228, 1.2527, 0.2907, 4.9664, 0.0730	0.0271, 1.2600, 0.2951, 5.0101, 0.0742	0.3875
τ	1.0197	0.9717	0.9795	0.1138

$Figure\ X + 6\colon\ Table\ of\ summary\ statistics\ of\ parameters\ for\ sparse\ data\ (p = 0.0)$

Parameter	Walk of value	Scatter Plot of Likelihood	Histogram	Walk of ESS
α				
β				
I				
T				
W				
S				
τ				

$Figure\ X + 7\text{: } Table\ of\ graphs\ of\ parameter\ metrics\ for\ sparse\ data\ (p = 0.0)$

p = 0.1

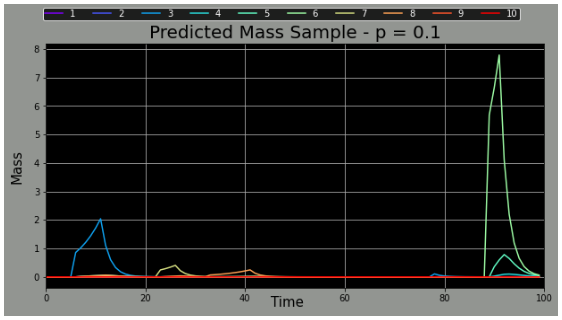
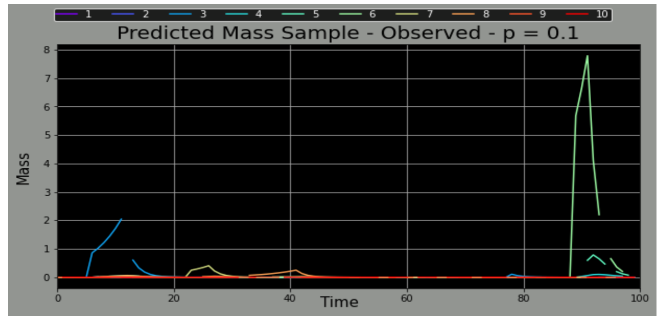
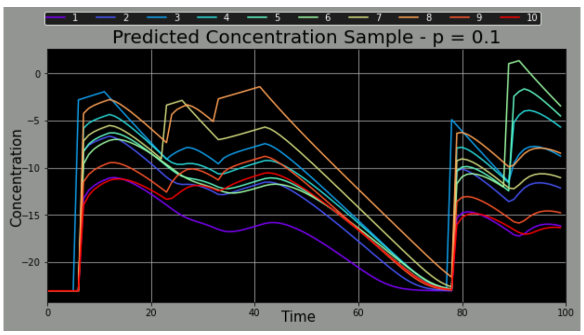
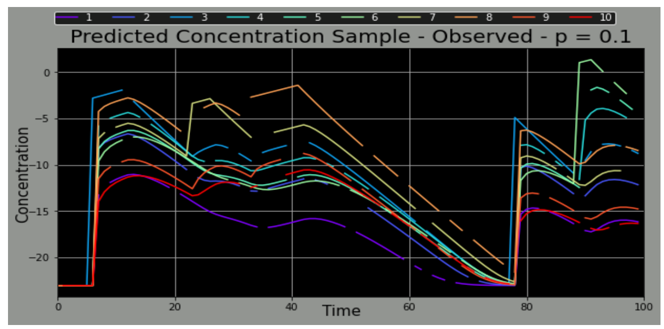
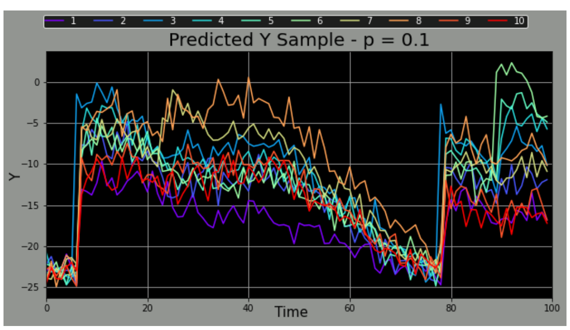
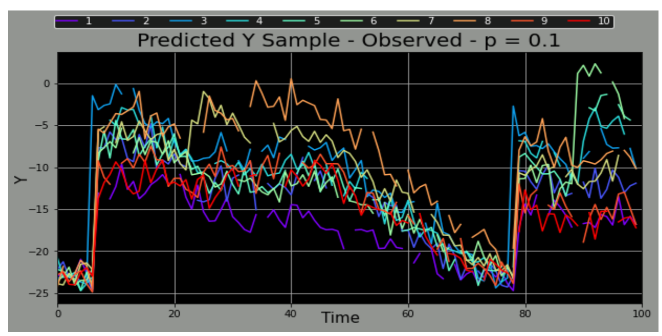
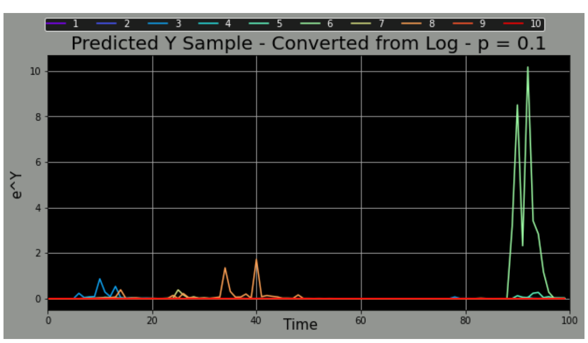
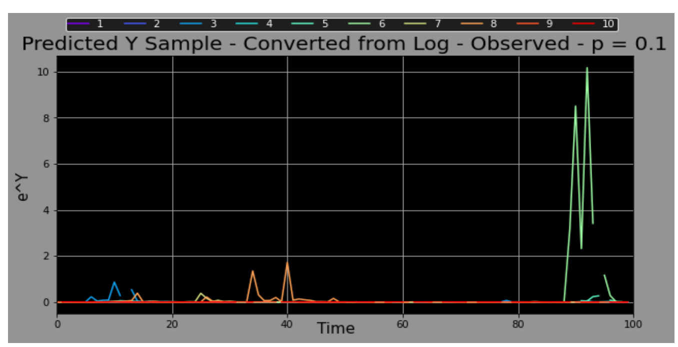
Results	Full	Observed
Mass		
Concentration		
Y		
e^Y		

Figure X + 8: Table of results for sparse data (p = 0.1)

Parameter	Median	95% CI - Lower Bound	95% CI - Upper Bound	P(Acceptance)
α	0.6470	0.6088	0.6141	0.0049
β	0.5937	0.6043	0.6055	0.0047
I	5, 2, 6, 2, 7	4.9953, 2.0014, 5.9756, 1.9999, 6.9741	4.9985, 2.0040, 5.9836, 2.0003, 6.9805	0.0982
T	89, 78, 23, 6, 33	88.9857, 77.9957, 23.9584, 5.9976, 32.5263	89.0249, 78.0011, 24.3292, 6.0150, 32.5927	0.0104
W	3, 1, 4, 6, 9	3.0190, 0.5942, 3.6240, 5.9021, 8.8898	3.0370, 0.6148, 3.6514, 5.9231, 8.9298	0.2500
S	5.6794, 0.1061, 0.2440, 0.8569, 0.0604	5.5443, 0.1014, 0.2742, 0.8896, 0.0709	5.5721, 0.1023, 0.2808, 0.8972, 0.0726	0.3807
τ	1.6465	1.5506	1.5623	0.1799

$Figure\ X + 9:\ Table\ of\ summary\ statistics\ of\ parameters\ for\ sparse\ data\ (p = 0.1)$

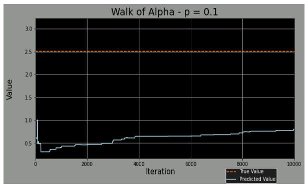
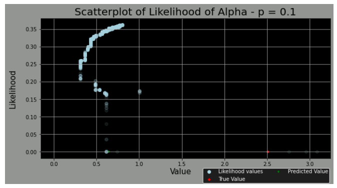
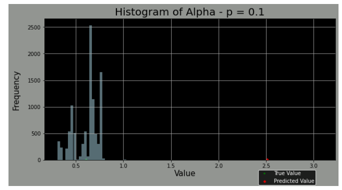
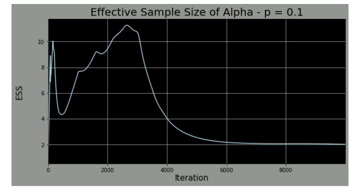
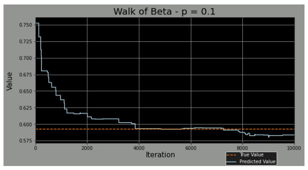
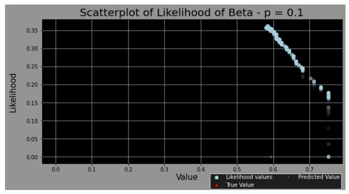
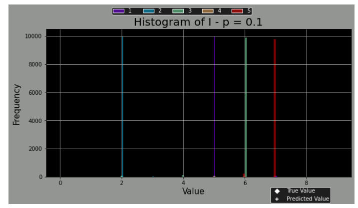
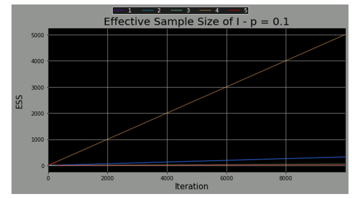
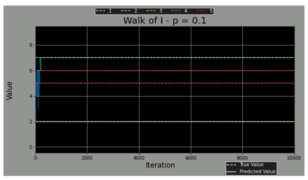
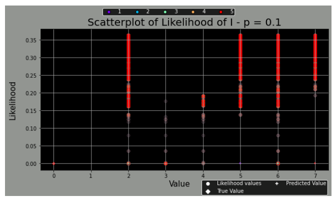
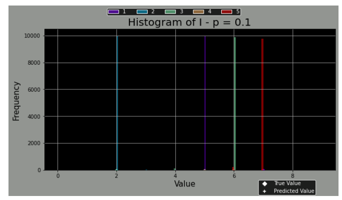
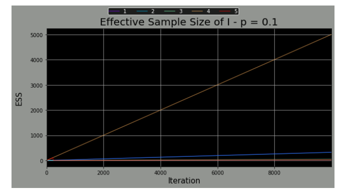
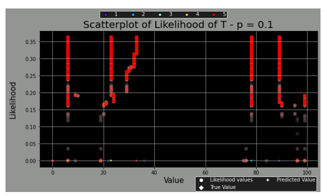
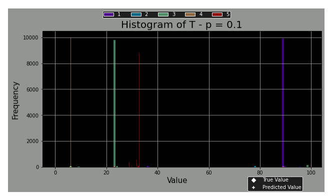
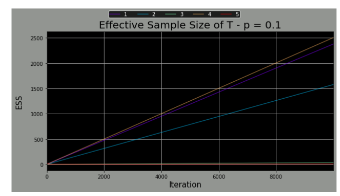
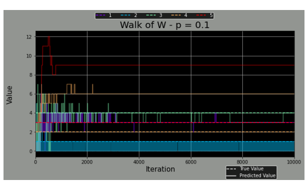
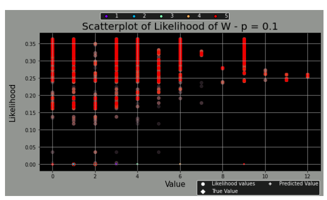
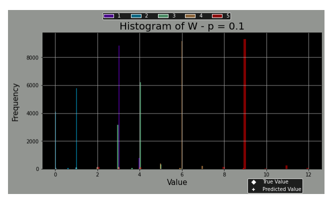
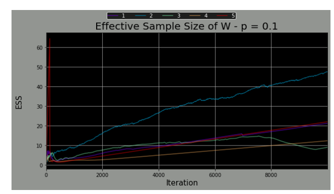
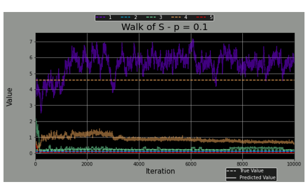
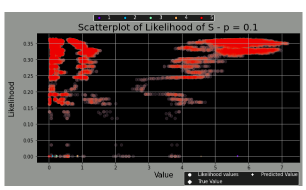
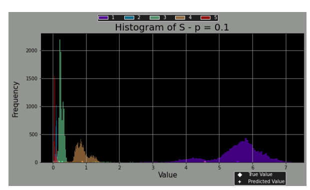
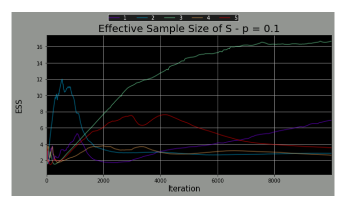
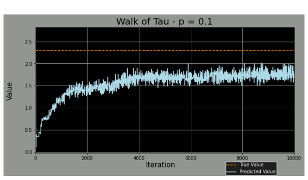
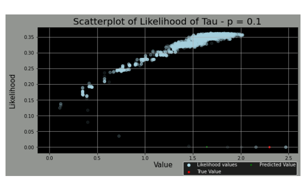
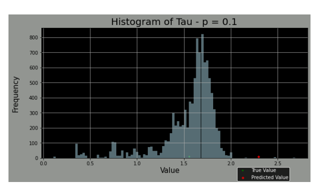
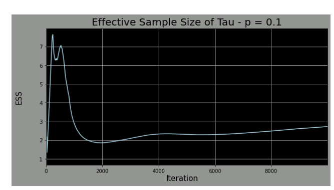
Parameter	Walk of value	Scatter Plot of Likelihood	Histogram	Walk of ESS
α				
β				
I				
T				
W				
S				
τ				

Figure X + 10: Table of graphs of parameter metrics for sparse data $(p = 0.1)$

p = 0.2

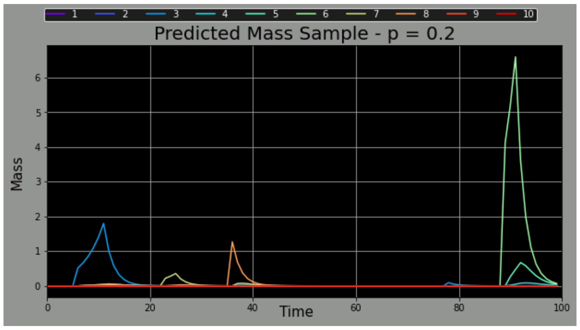
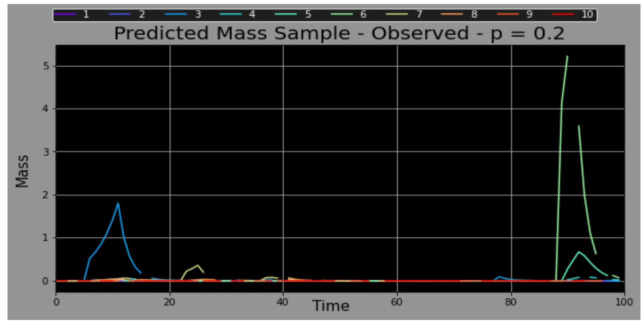
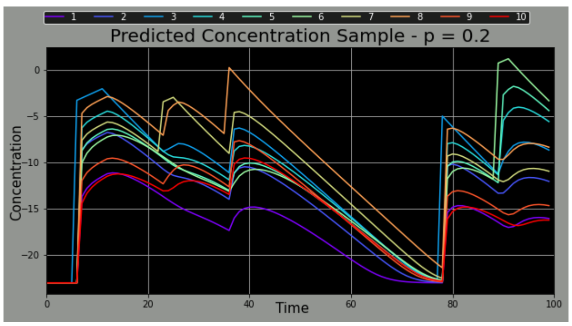
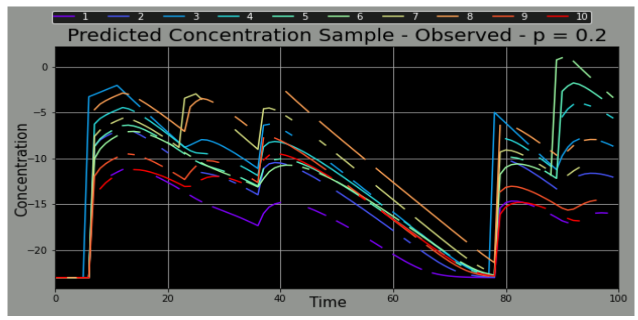
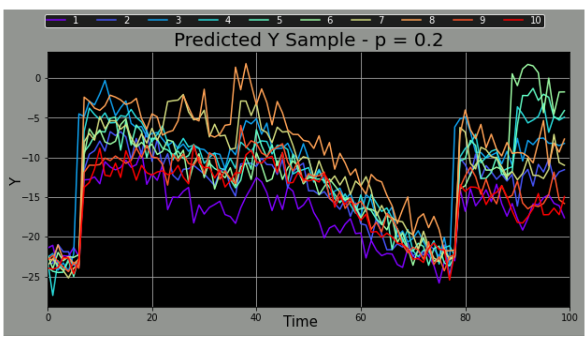
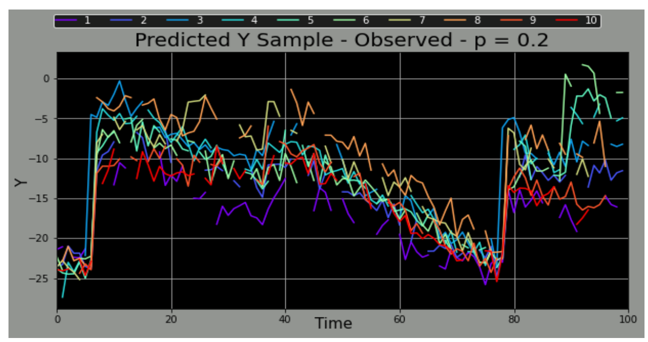
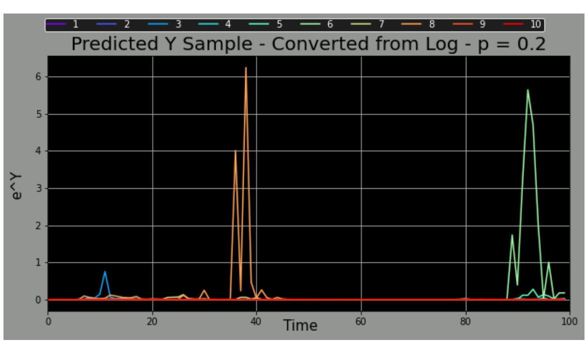
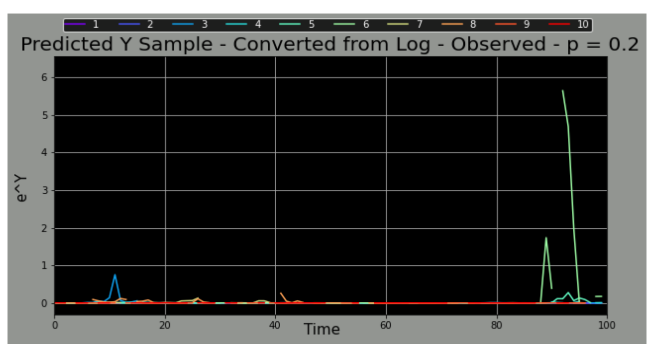
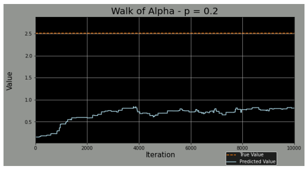
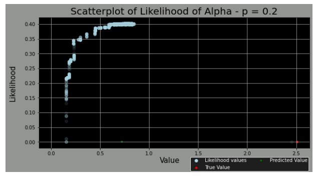
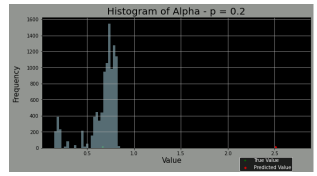
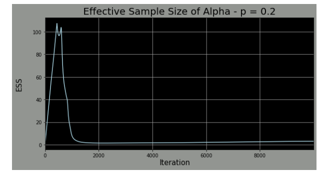
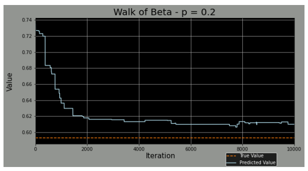
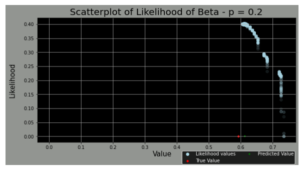
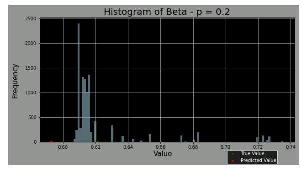
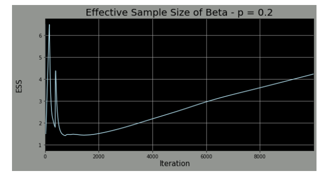
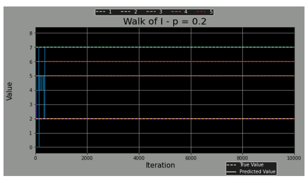
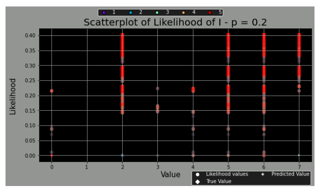
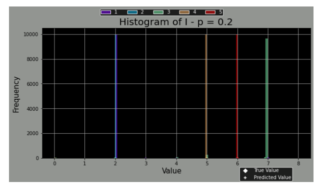
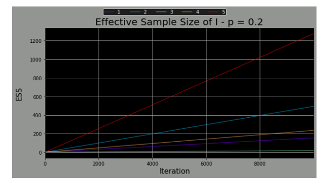
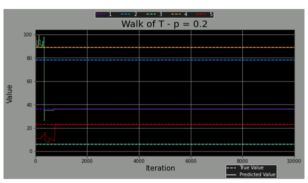
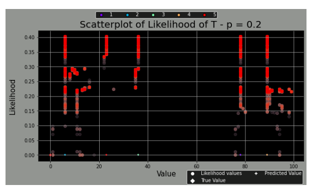
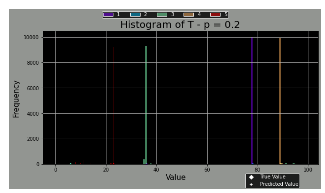
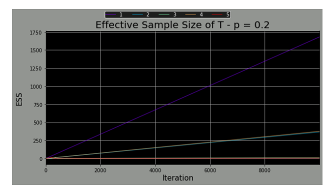
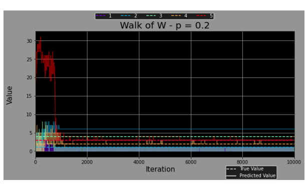
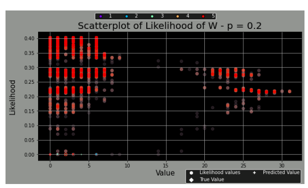
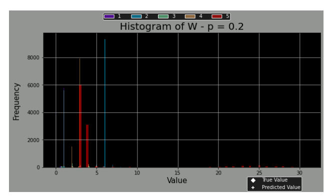
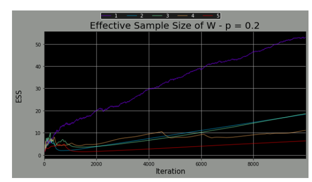
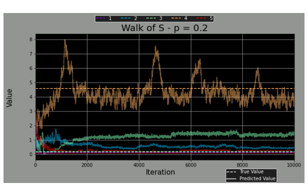
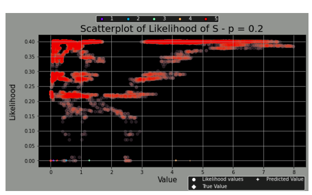
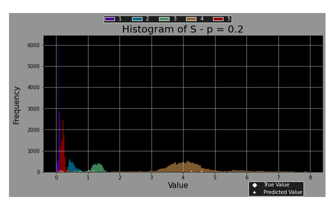
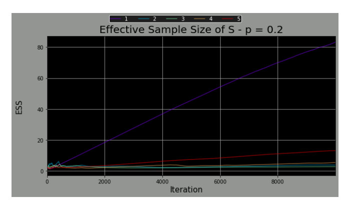
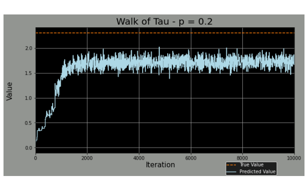
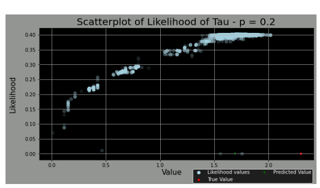
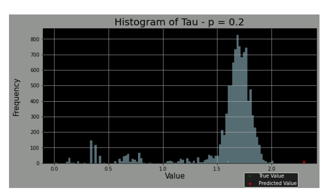
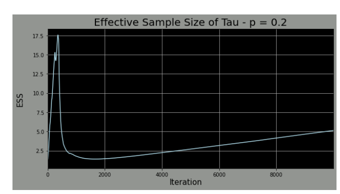
Results	Full	Observed
Mass		
Concentration		
Y		
e^Y		

Figure X + 11: Table of results for sparse data ($p = 0.2$)

Parameter	Median	95% CI - Lower Bound	95% CI - Upper Bound	P(Acceptance)
α				
β				
I				
T				
W				
S				
τ				

$Figure\ X\ +\ 12\!:\ Table\ of\ summary\ statistics\ of\ parameters\ for\ sparse\ data\ (p\ =\ 0.2)$

Parameter	Walk of value	Scatter Plot of Likelihood	Histogram	Walk of ESS
α				
β				
I				
T				
W				
S				
τ				

$Figure\ X + 13\!: Table\ of\ graphs\ of\ parameter\ metrics\ for\ sparse\ data\ (p = 0.2)$

p = 0.3

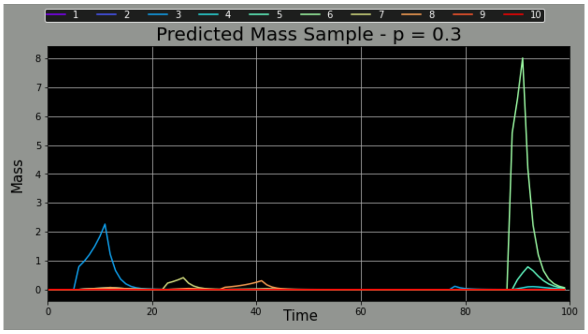
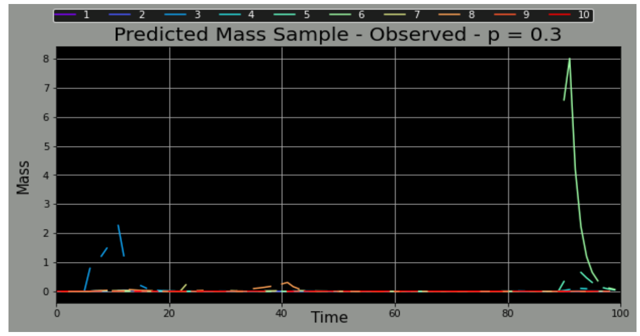
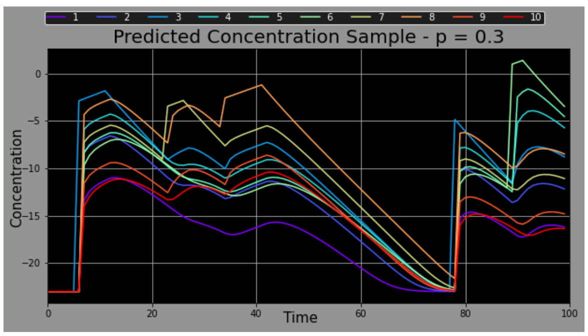
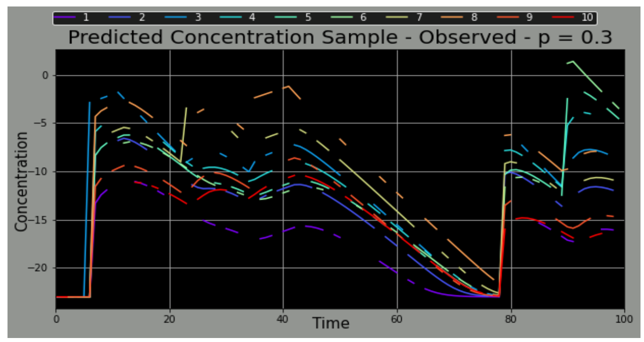
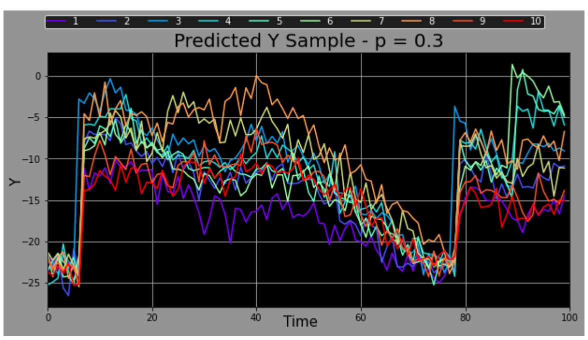
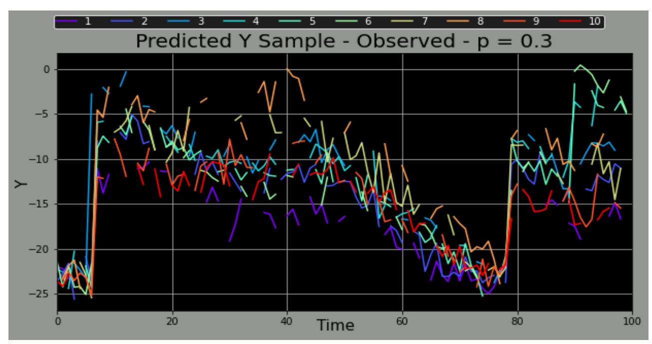
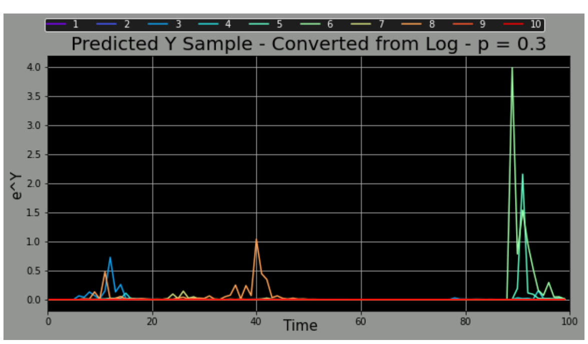
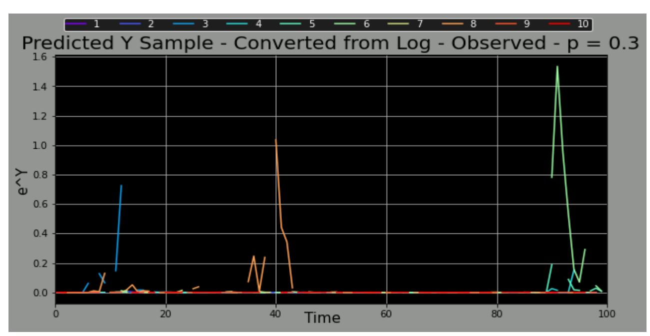
Results	Full	Observed
Mass		
Concentration		
Y		
e^Y		

Figure X + 14: Table of results for sparse data (p = 0.3)

Parameter	Median	95% CI - Lower Bound	95% CI - Upper Bound	P(Acceptance)
α				
β				
I				
T				
W				
S				
τ				

$Figure\ X + 15{:}\ Table\ of\ summary\ statistics\ of\ parameters\ for\ sparse\ data\ (p = 0.3)$

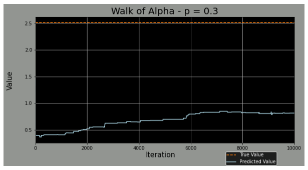
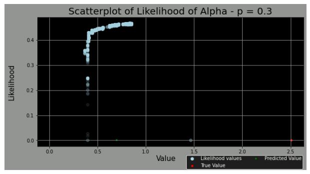
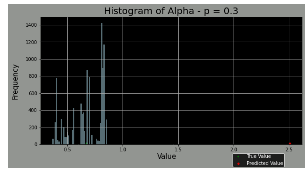
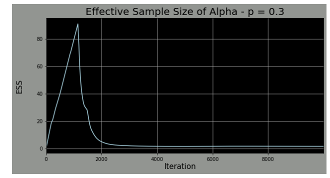
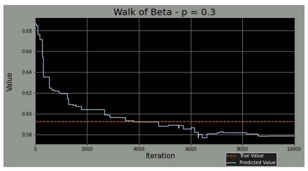
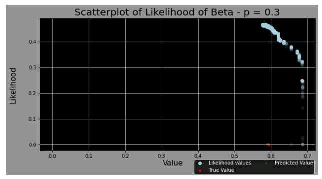
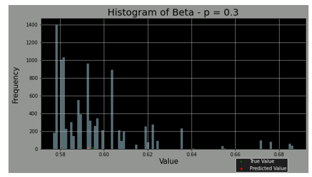
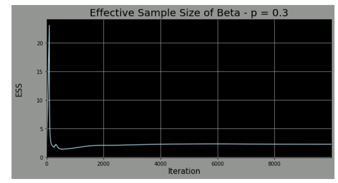
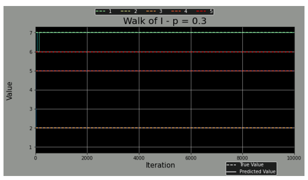
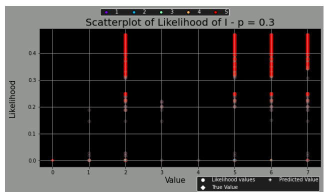
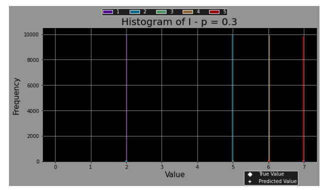
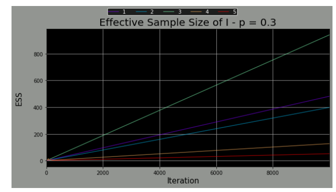
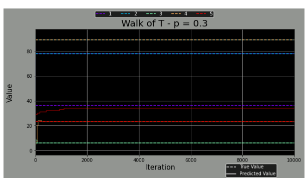
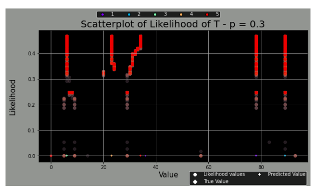
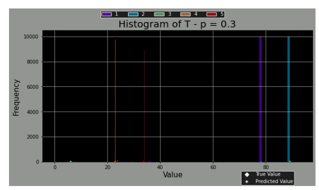
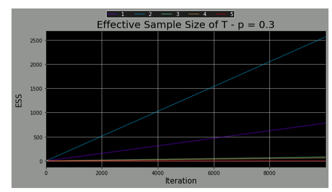
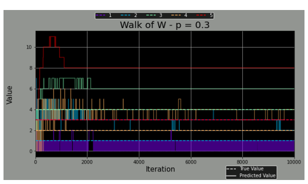
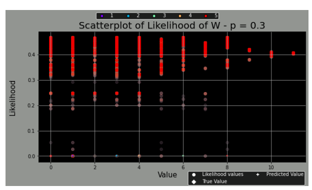
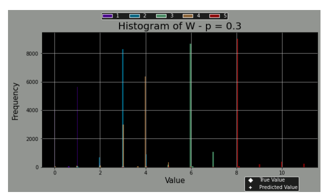
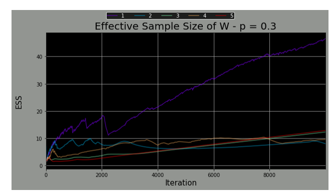
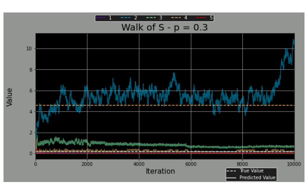
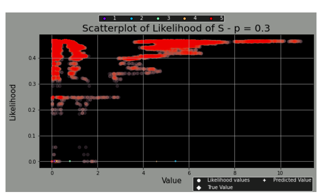
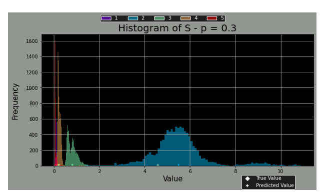
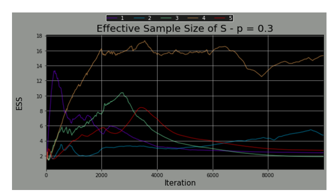
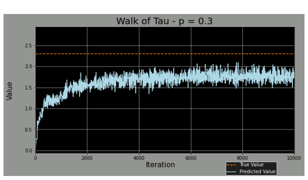
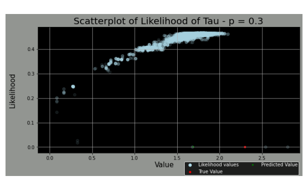
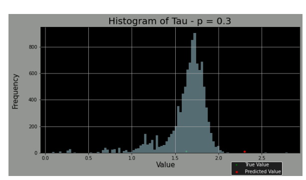
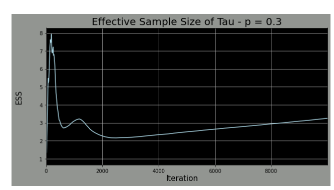
Parameter	Walk of value	Scatter Plot of Likelihood	Histogram	Walk of ESS
α				
β				
I				
T				
W				
S				
τ				

Figure X + 16: Table of graphs of parameter metrics for sparse data ($p = 0.3$)

p = 0.4

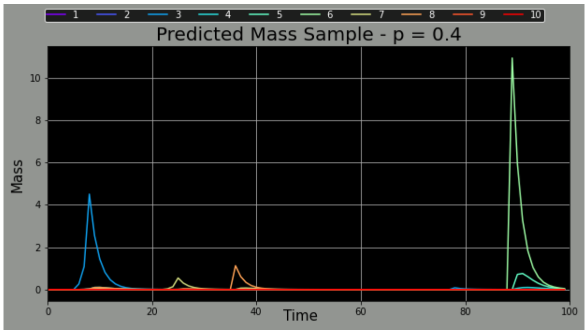
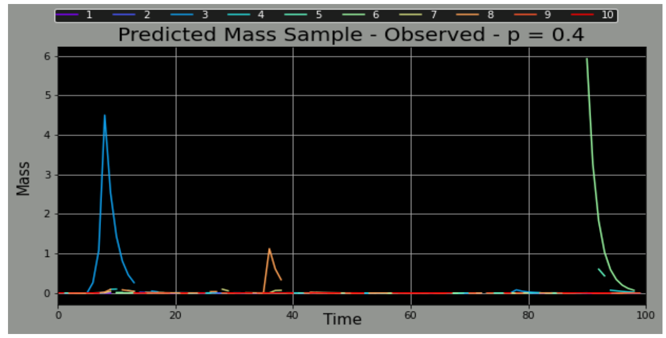
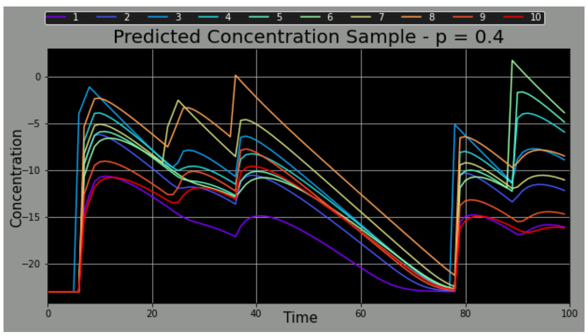
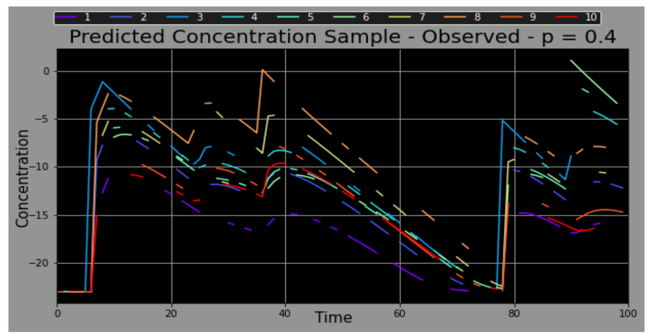
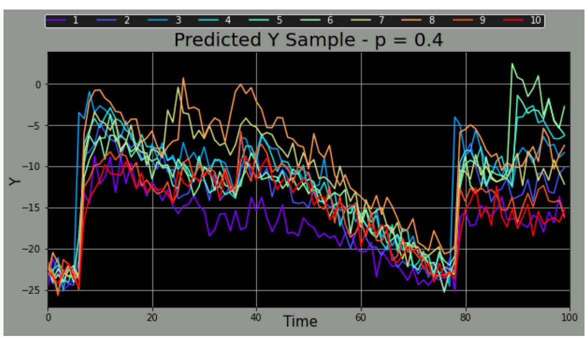
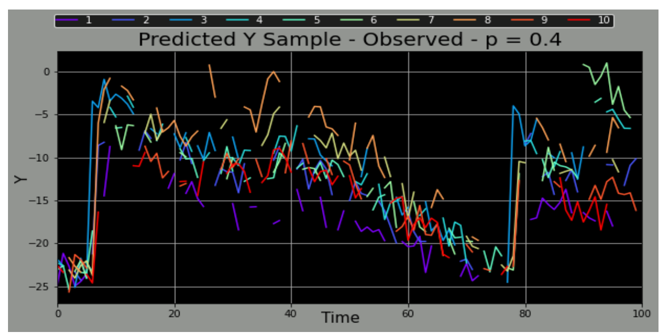
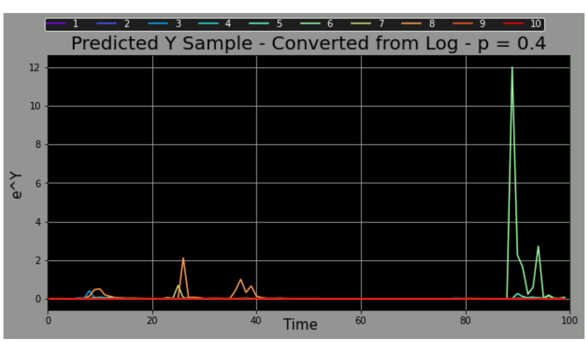
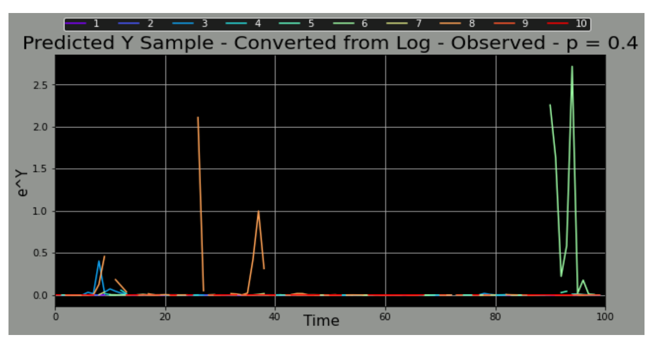
<table>
<tr><th>Results</th><th>Full</th><th>Observed</th></tr>
<tr><td>Mass</td><td>
</td><td>
</td></tr>
<tr><td>Concentration</td><td>
</td><td>
</td></tr>
<tr><td>Y</td><td>
</td><td>
</td></tr>
<tr><td>e^Y</td><td>
</td><td>
</td></tr>
</table>

Figure X + 17: Table of results for sparse data (p = 0.4)

Parameter	Median	95% CI - Lower Bound	95% CI - Upper Bound	P(Acceptance)
α				
β				
I				
T				
W				
S				
τ				

$Figure\ X\ +\ 18\!:\ Table\ of\ summary\ statistics\ of\ parameters\ for\ sparse\ data\ (p\ =\ 0.4)$

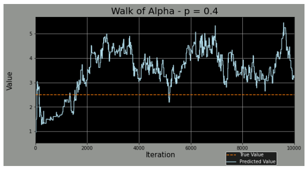
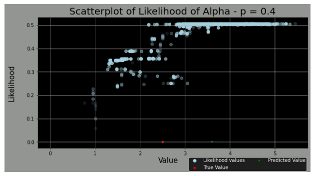
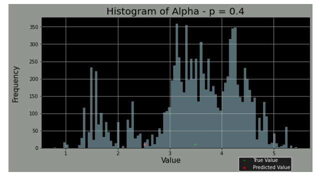
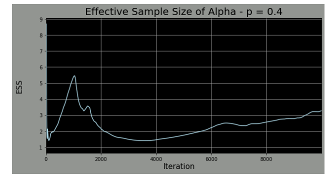
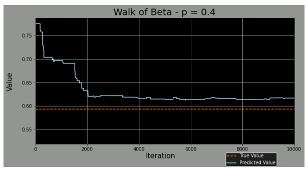
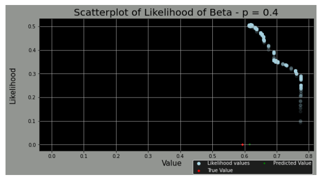
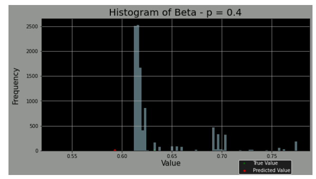
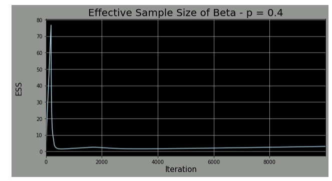
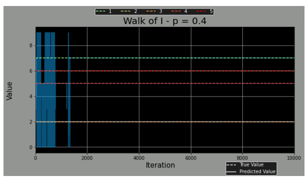
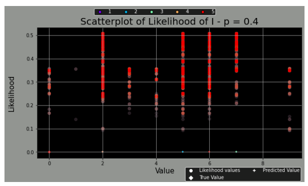
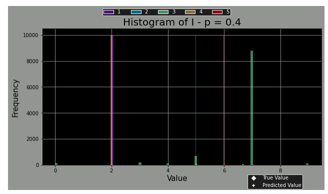
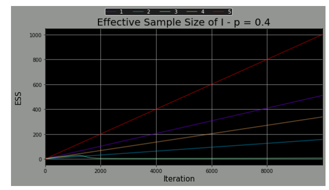
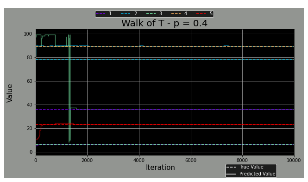
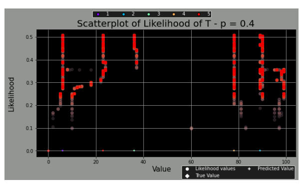
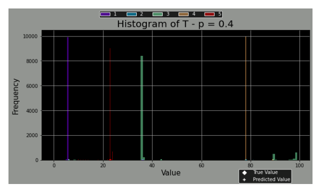
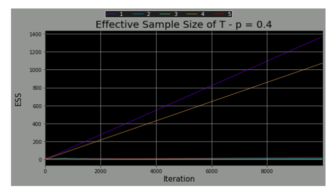
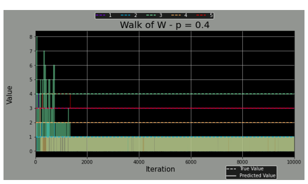
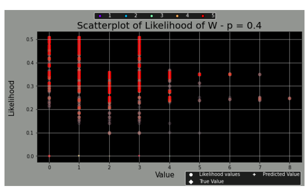
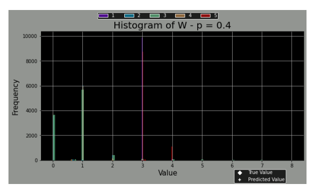
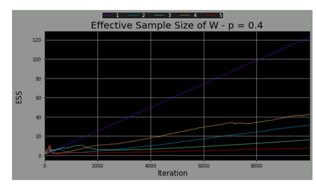
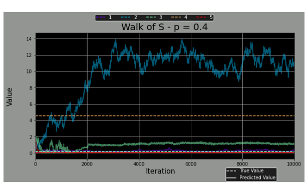
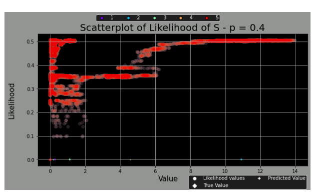
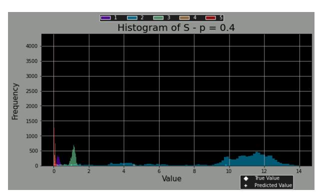
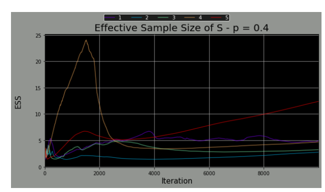
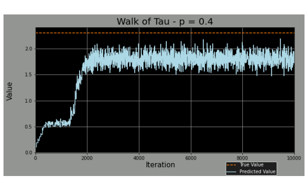
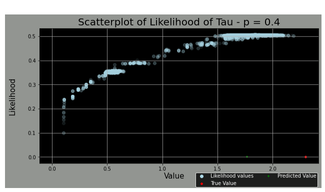
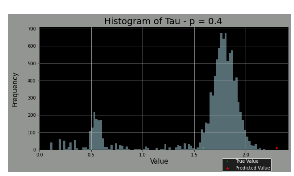
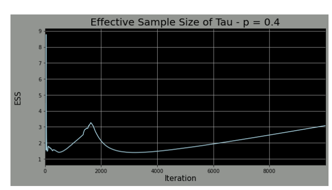
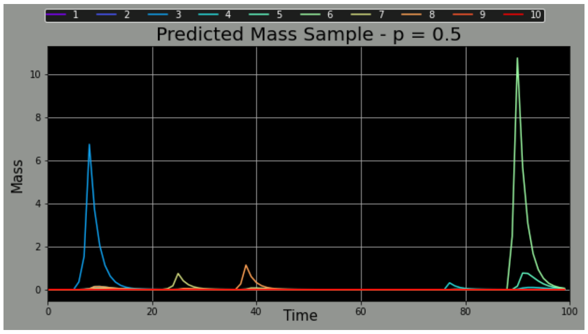
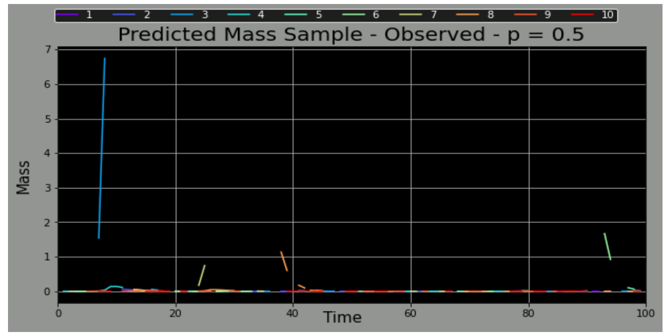
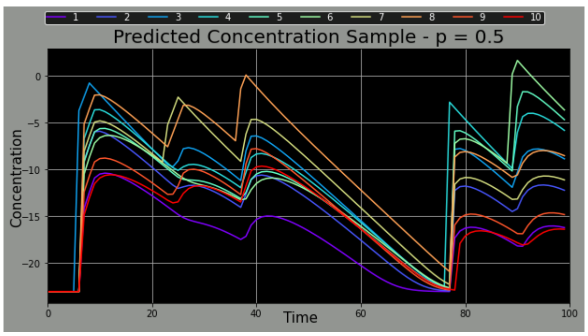
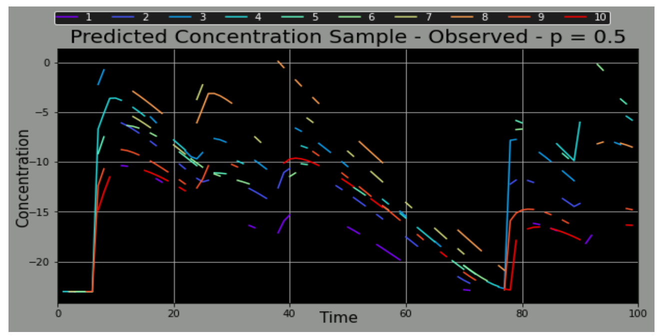
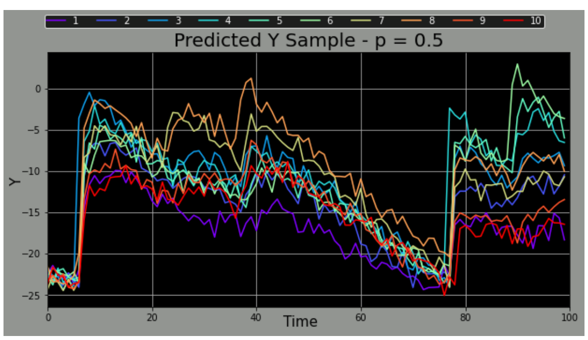
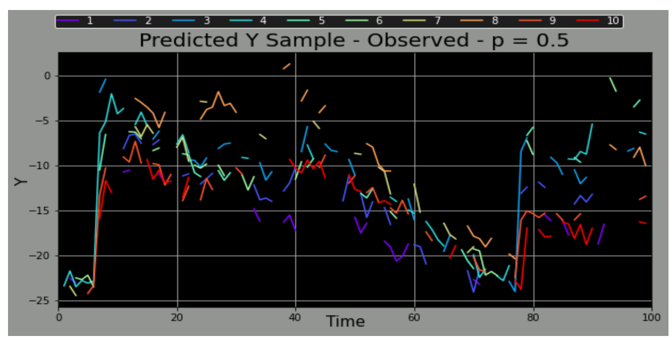
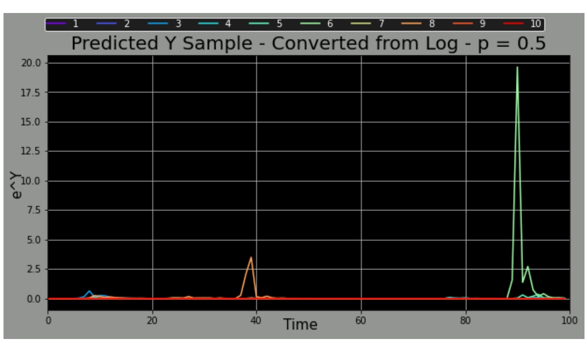
Parameter	Walk of value	Scatter Plot of Likelihood	Histogram	Walk of ESS
α				
β				
I				
T				
W				
S				
τ				

Figure X + 19: Table of graphs of parameter metrics for sparse data ($p = 0.4$)

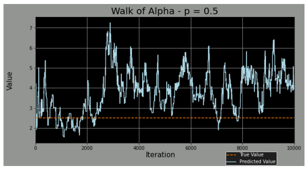
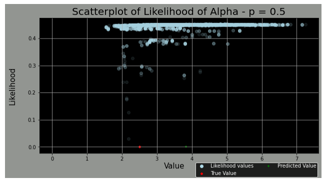
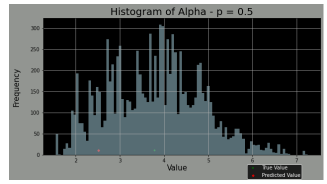
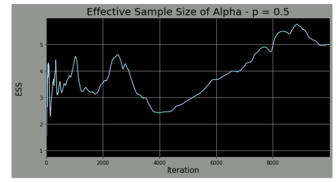
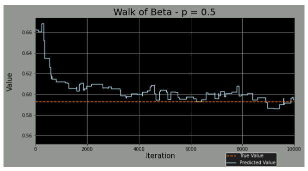
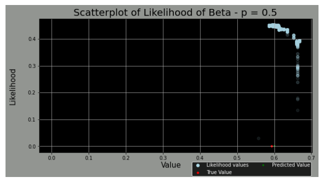
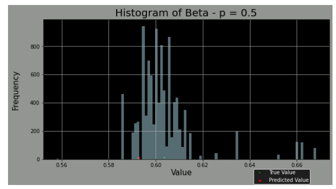
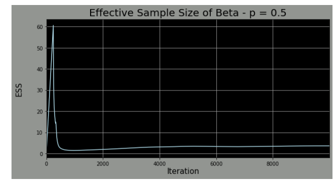
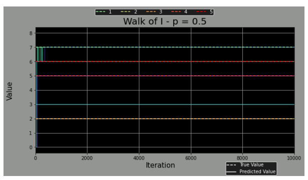
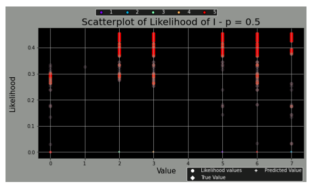
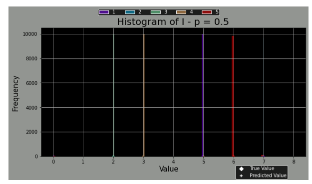
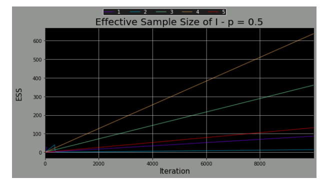
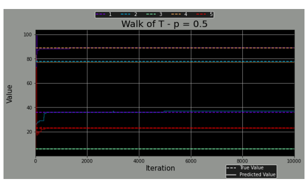
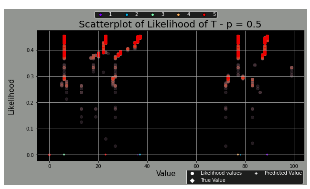
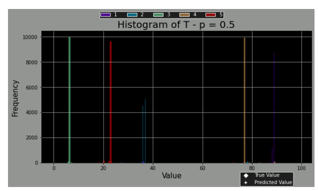
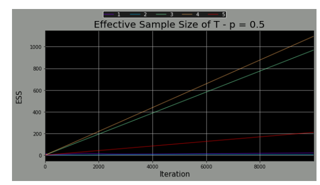
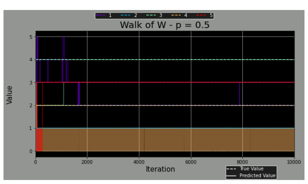
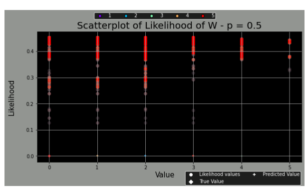
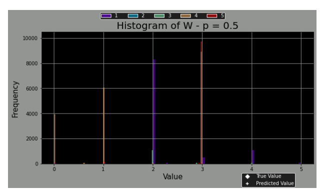
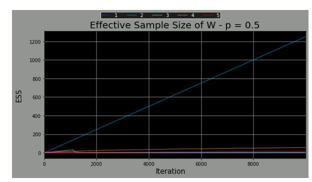
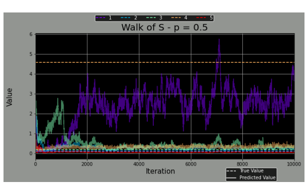
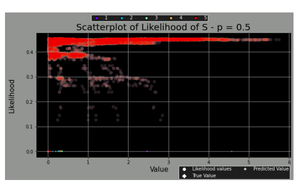
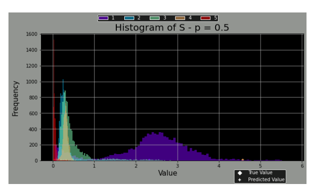
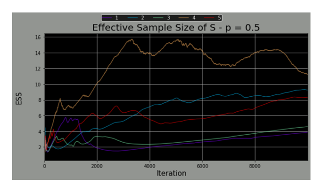
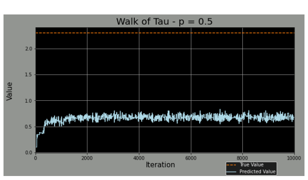
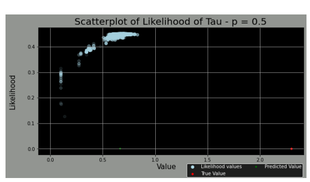
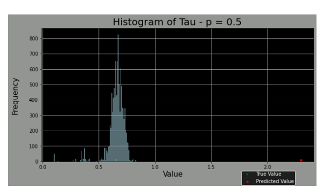
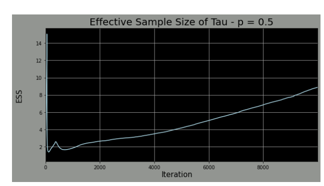
p = 0.5

Results	Full	Observed
Mass		
Concentration		
Y		
e^Y		

$Figure\ X + 20\!: Table\ of\ results\ for\ sparse\ data\ (p = 0.5)$

Parameter	Median	95% CI - Lower Bound	95% CI - Upper Bound	P(Acceptance)
α				
β				
I				
T				
W				
S				
τ				

$Figure\ X + 21\text{:}\ Table\ of\ summary\ statistics\ of\ parameters\ for\ sparse\ data\ (p = 0.5)$

Parameter	Walk of value	Scatter Plot of Likelihood	Histogram	Walk of ESS
α				
β				
I				
T				
W				
S				
τ				

$Figure\ X + 22{:}\ Table\ of\ graphs\ of\ parameter\ metrics\ for\ sparse\ data\ (p = 0.5)$

p = 0.6

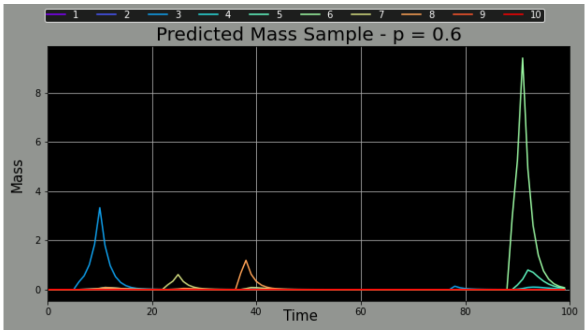
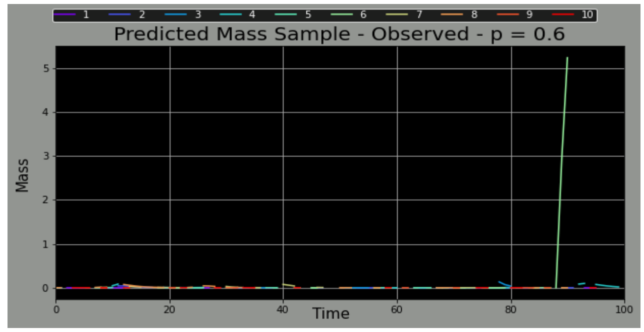
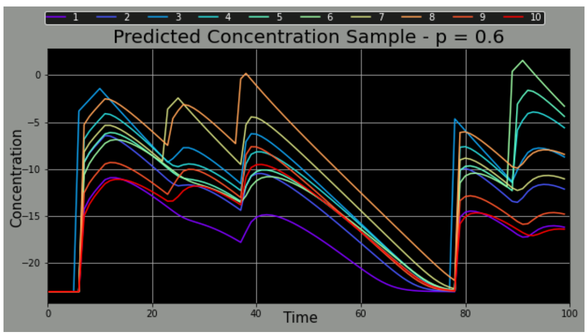
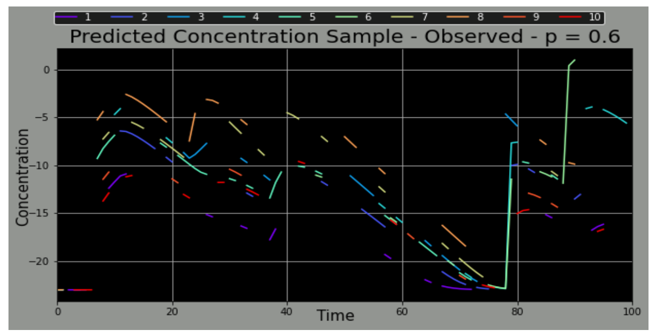
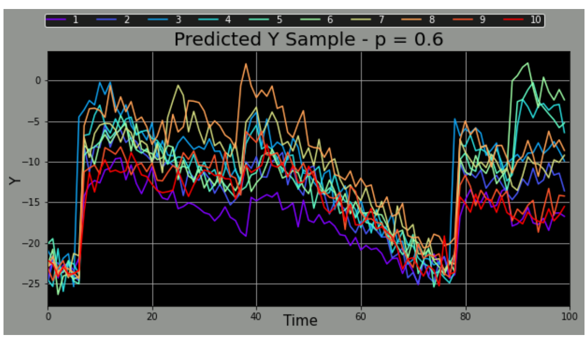
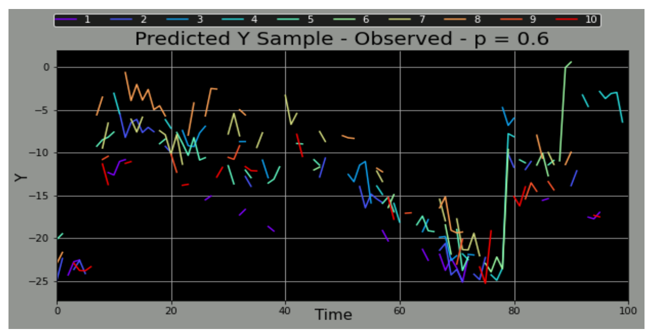
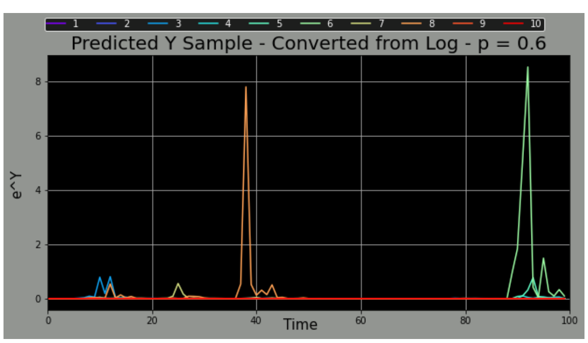
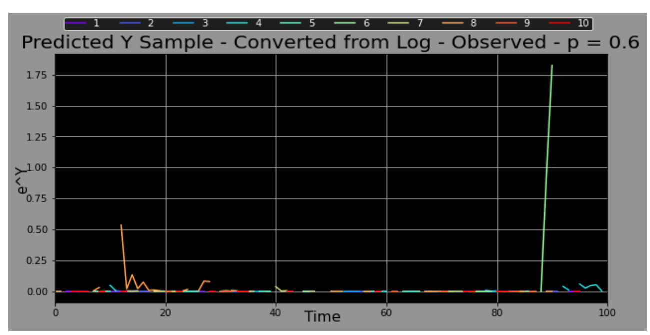
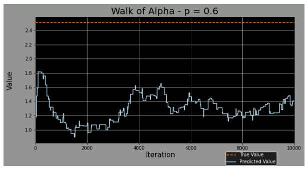
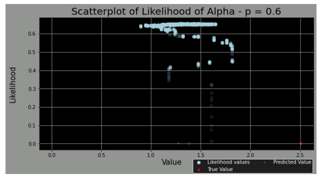
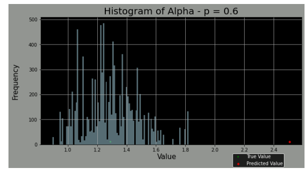
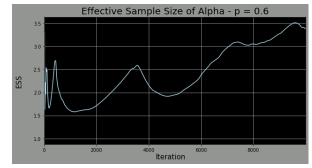
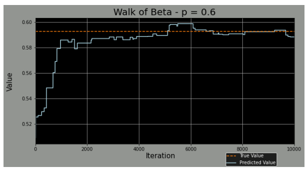
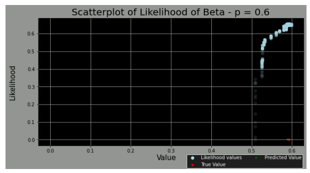
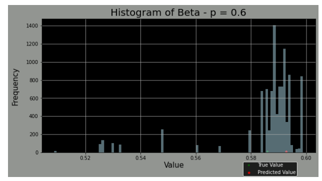
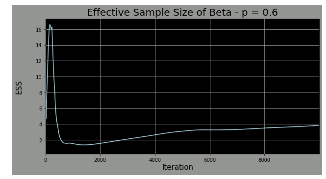
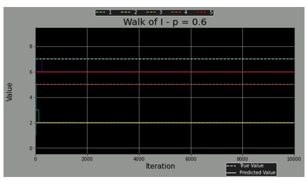
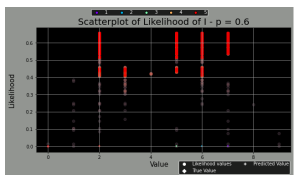
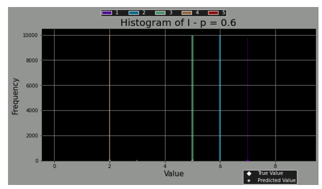
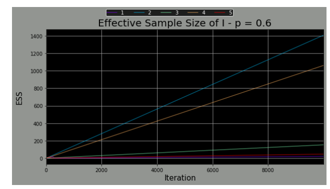
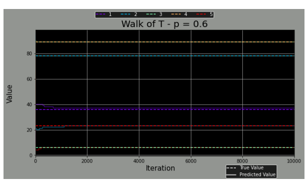
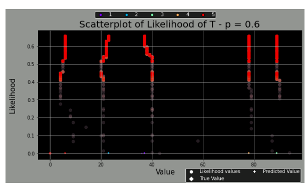
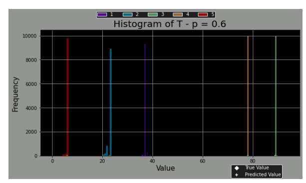
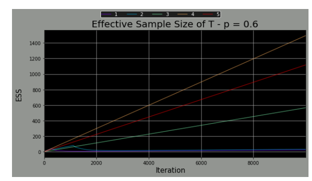
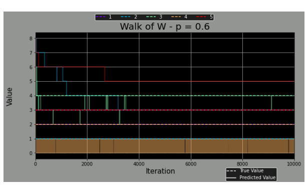
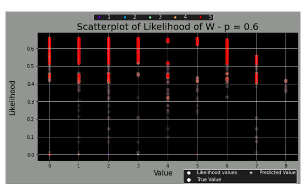
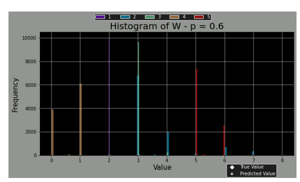
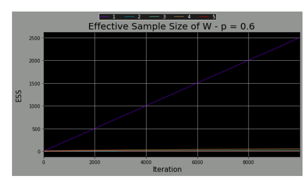
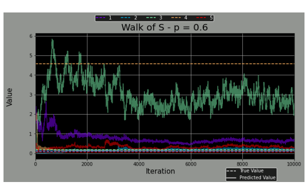
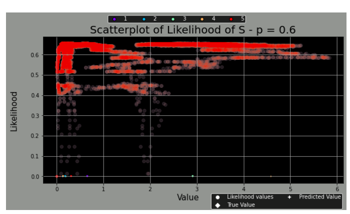
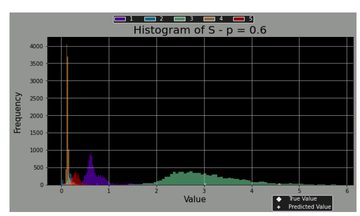
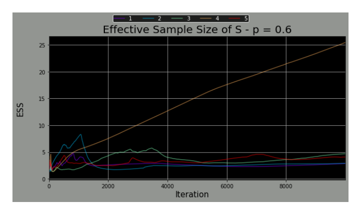
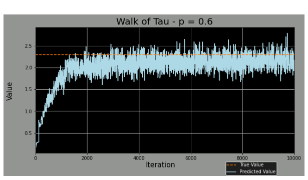
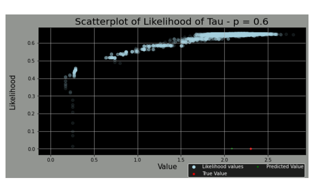
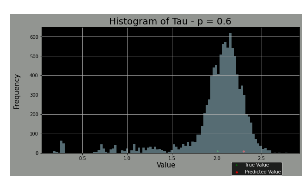
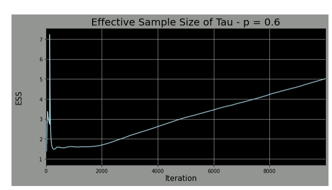
Results	Full	Observed
Mass		
Concentration		
Y		
e^Y		

Figure X + 23: Table of results for sparse data (p = 0.6)

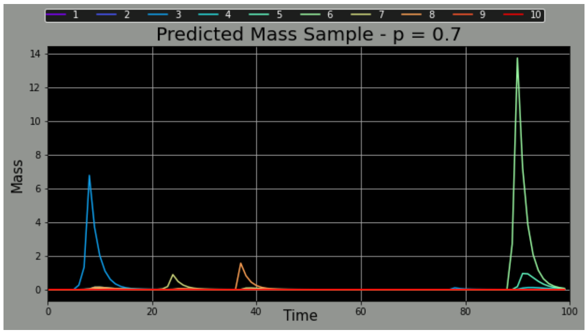
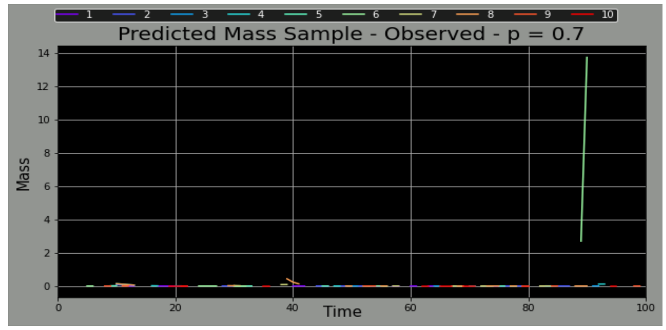
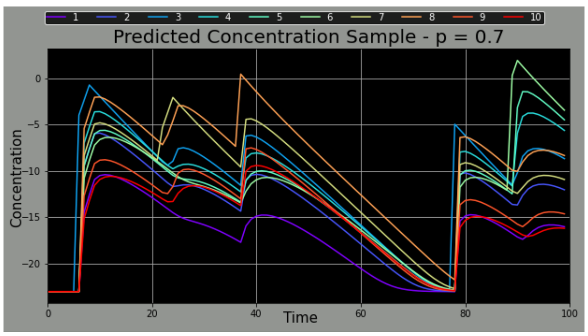
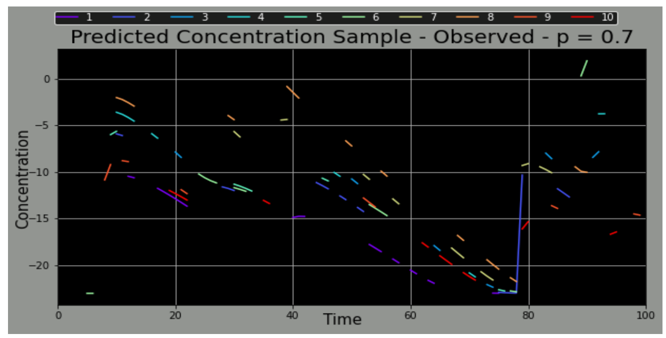
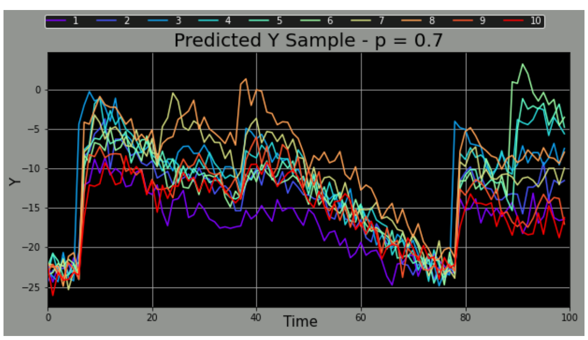
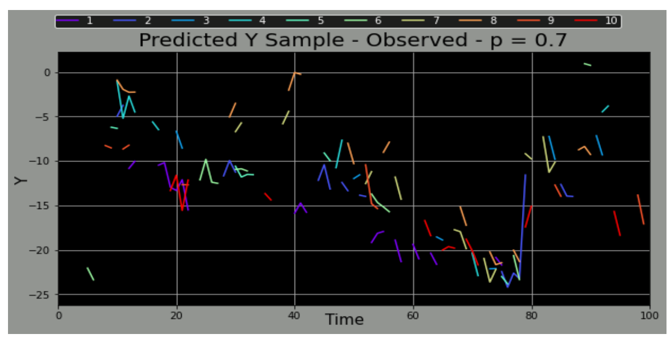
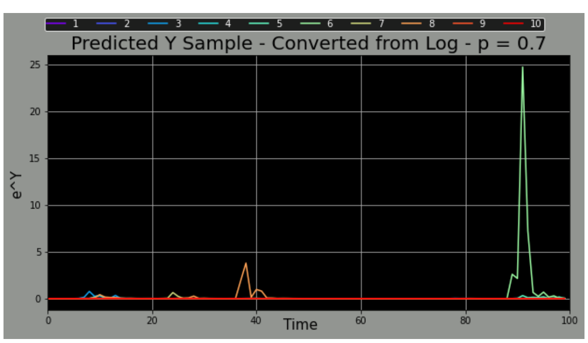
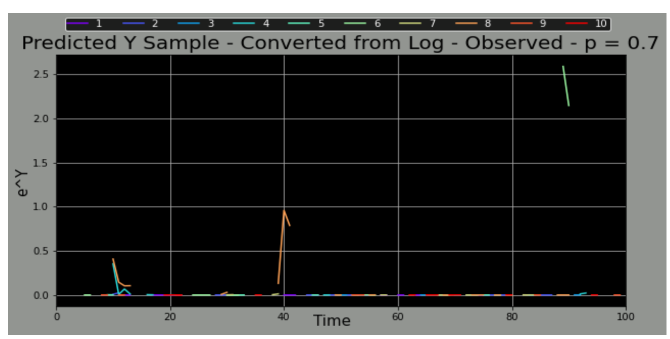
Parameter	Median	95% CI - Lower Bound	95% CI - Upper Bound	P(Acceptance)
α				
β				
I				
T				
W				
S				
τ				

$Figure\ X\ +\ 24\colon\ Table\ of\ summary\ statistics\ of\ parameters\ for\ sparse\ data\ (p\ =\ 0.0.6)$

Parameter	Walk of value	Scatter Plot of Likelihood	Histogram	Walk of ESS
α				
β				
I				
T				
W				
S				
τ				

$Figure\ X + 25$: $Table\ of\ graphs\ of\ parameter\ metrics\ for\ sparse\ data\ (p = 0.6)$

p = 0.7

Results	Full	Observed
Mass		
Concentration		
Y		
e^Y		

$Figure\ X\ +\ 26\colon\ Table\ of\ results\ for\ sparse\ data\ (p\ =\ 0.7)$

Parameter	Median	95% CI - Lower Bound	95% CI - Upper Bound	P(Acceptance)
α				
β				
I				
T				
W				
S				
τ				

$Figure\ X\ +\ 27\colon\ Table\ of\ summary\ statistics\ of\ parameters\ for\ sparse\ data\ (p\ =\ 0.7)$

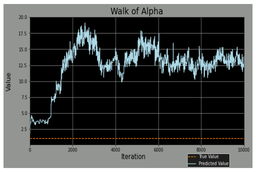
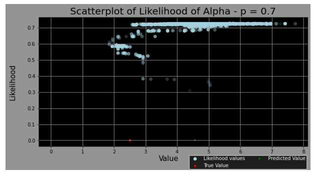
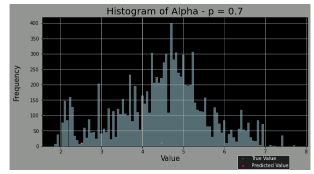
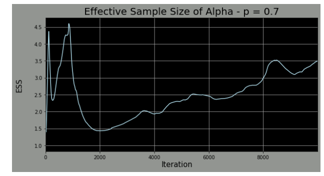
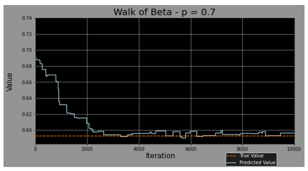
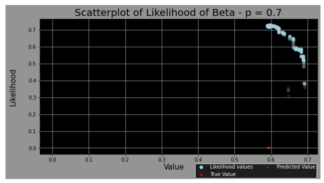
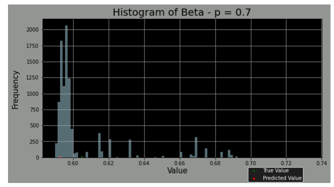
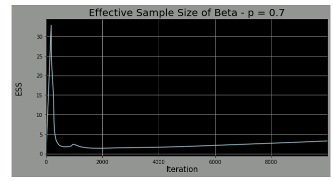
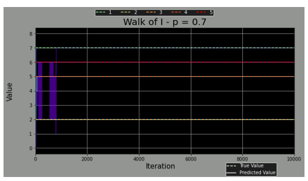
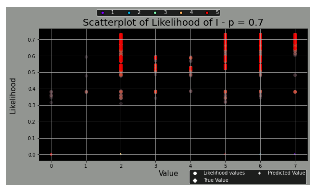
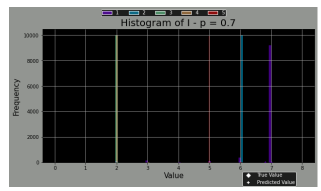
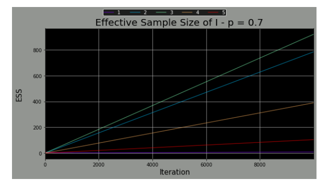
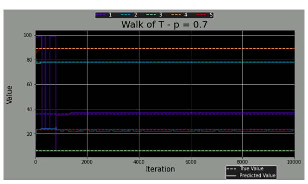
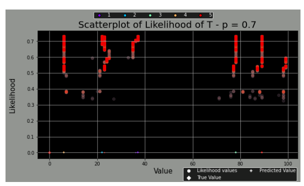
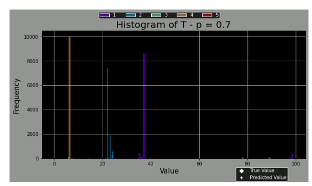
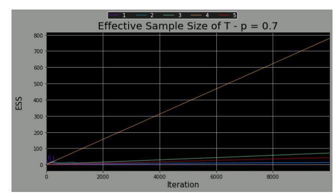
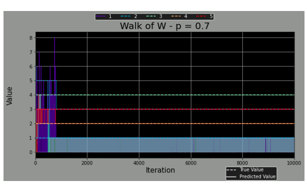
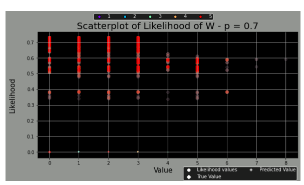
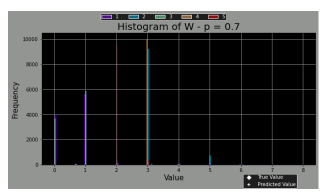
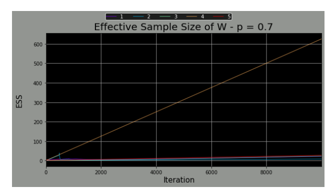
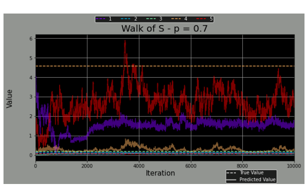
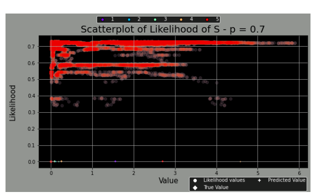
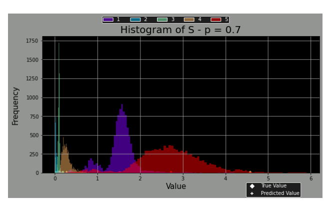
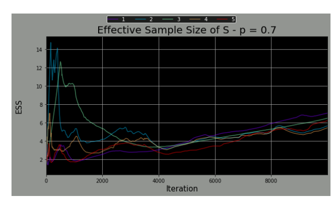
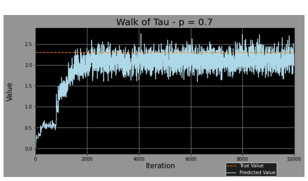
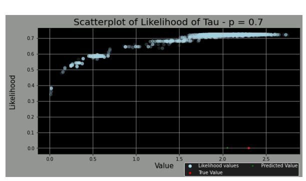
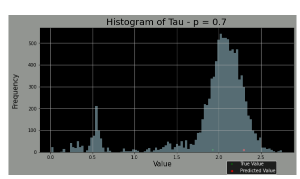
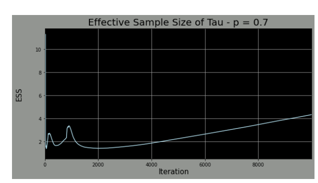
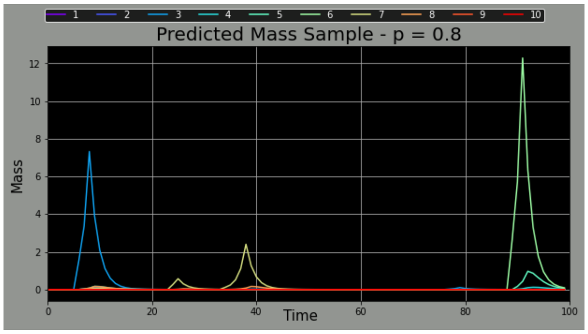
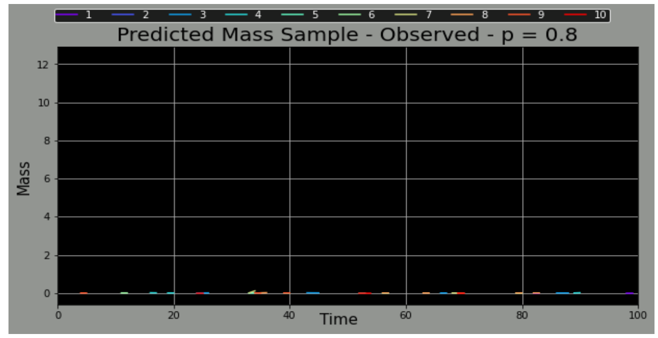
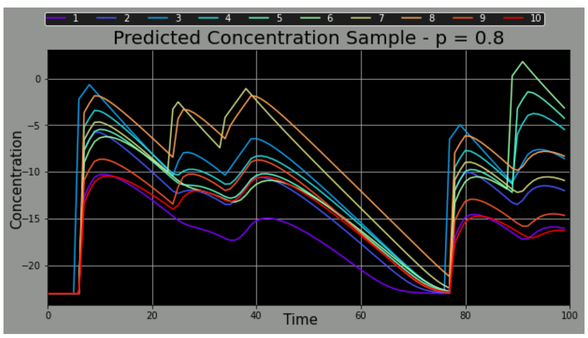
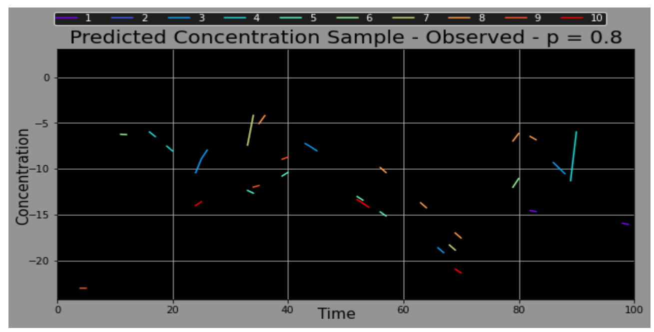
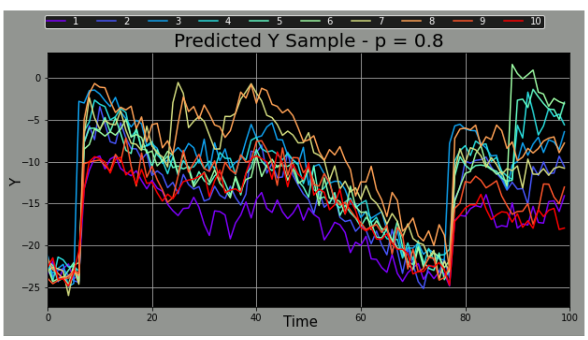
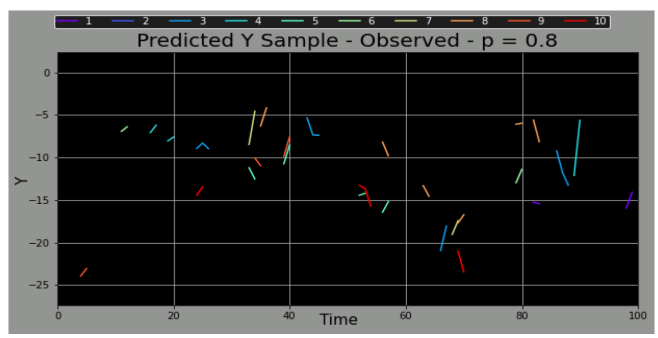
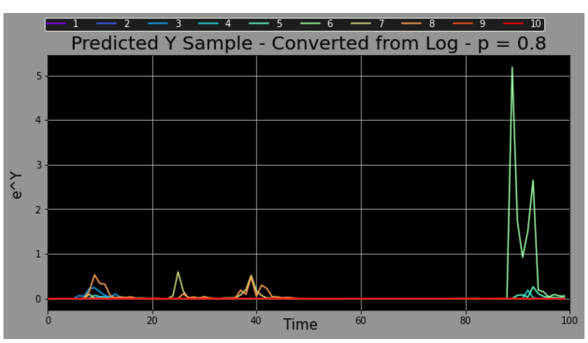
Parameter	Walk of value	Scatter Plot of Likelihood	Histogram	Walk of ESS
α				
β				
I				
T				
W				
S				
τ				

Figure X + 28: Table of graphs of parameter metrics for sparse data ($p = 0.7$)

p = 0.8

Results	Full	Observed
Mass		
Concentration		
Y		
e^Y		

$Figure\ X\ +\ 29\!:\ Table\ of\ results\ for\ sparse\ data\ (p\ =\ 0.8)$

Parameter	Median	95% CI - Lower Bound	95% CI - Upper Bound	P(Acceptance)
α	1.6410	1.7614	1.7817	0.0310
β	0.5828	0.5784	0.5790	0.0080
I	2, 6, 6, 5, 2	2.0812, 5.9319, 6.0000, 4.9992, 2.0029	2.1057, 5.9509, 6.0010, 5.0012, 2.0094	0.1014
T	6, 24, 34, 89, 77	6.0248, 24.1365, 33.9730, 89.0000, 76.9558	6.0502, 24.3975, 33.9782, 89.0000, 77.0054	0.0101
W	3, 2, 5, 3, 3	3.4057, 1.9275, 4.9997, 3.0427, 3.1174	3.4283, 1.9511, 5.0001, 3.0607, 3.1572	0.2100
S	1.5509, 0.2637, 0.0996, 2.6468, 0.0211	1.5357, 0.3833, 0.1161, 2.7619, 0.0276	1.5607, 0.4040, 0.1193, 2.7985, 0.0285	0.4903
τ	1.6578	1.5891	1.6033	0.3684

$Figure\ X + 30\!:\ Table\ of\ summary\ statistics\ of\ parameters\ for\ sparse\ data\ (p = 0.8)$

Parameter	Walk of value	Scatter Plot of Likelihood	Histogram	Walk of ESS

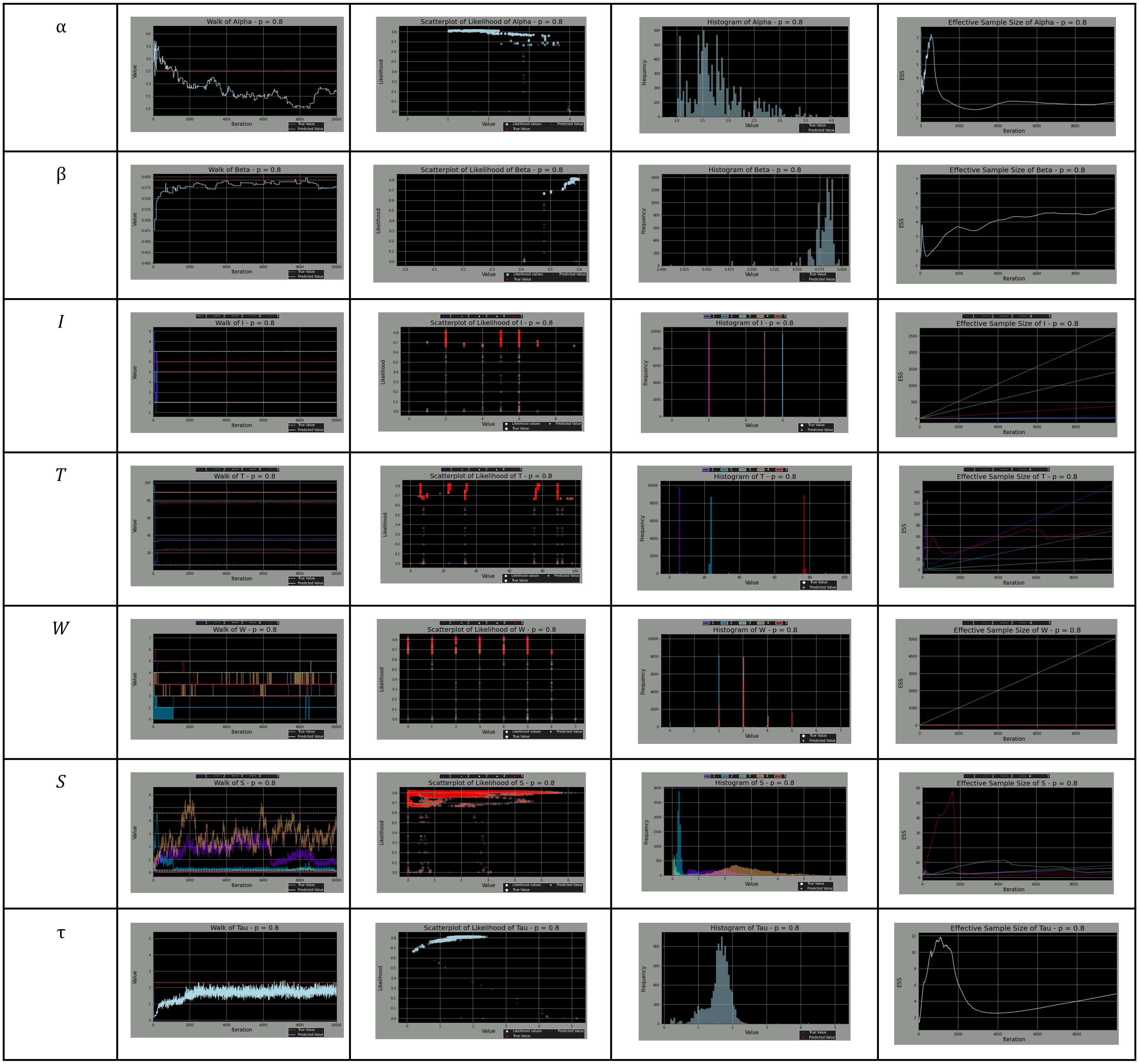


Figure X + 31: Table of graphs of parameter metrics for sparse data (p = 0.8)

p = 0.9

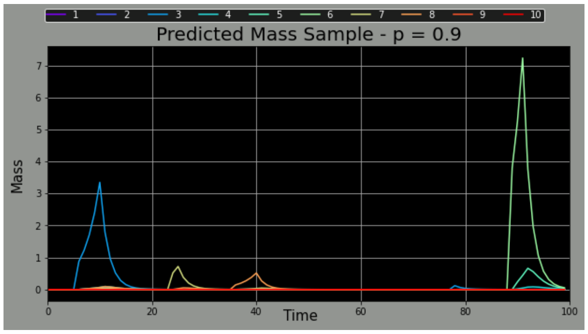
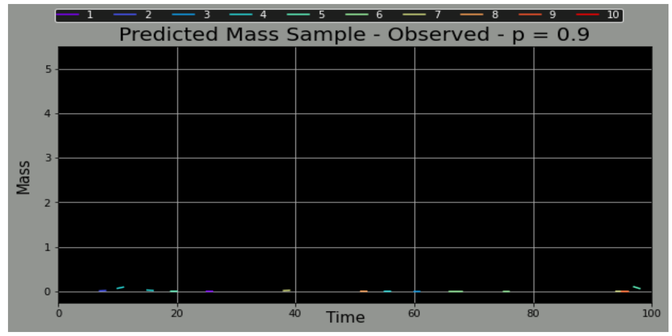
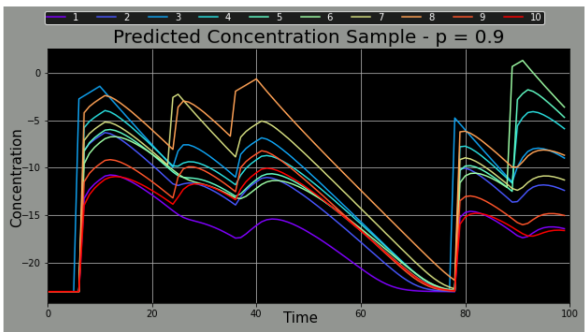
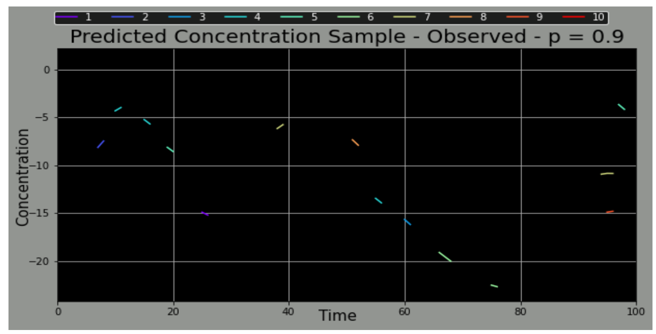
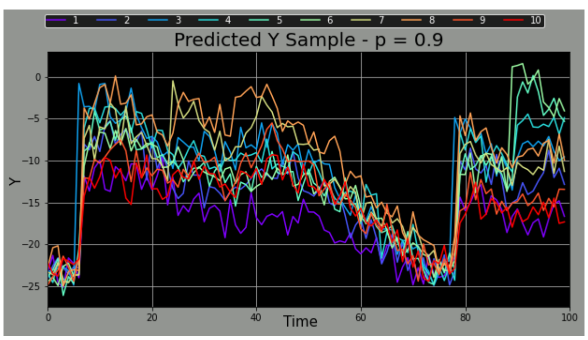
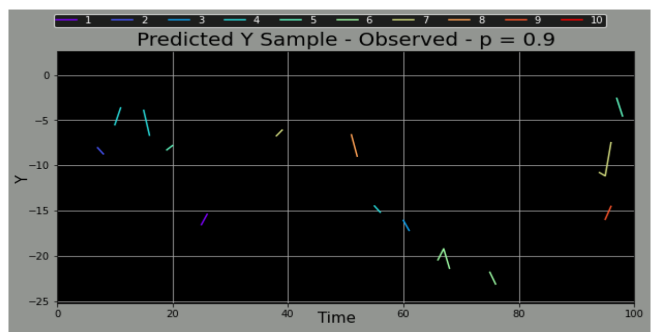
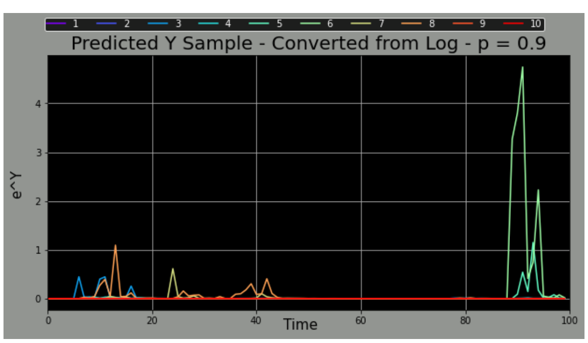
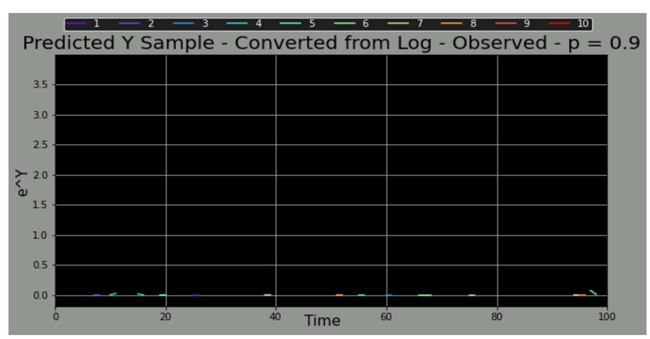
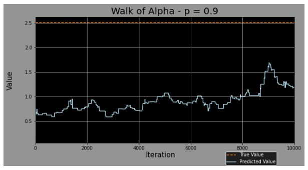
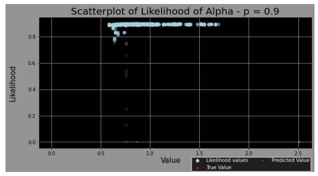
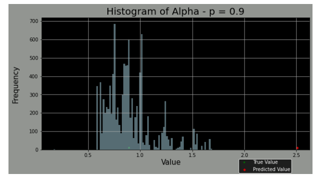
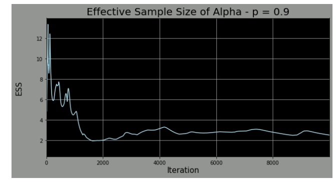
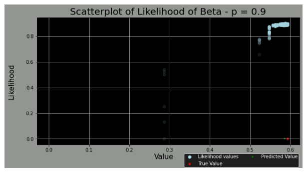
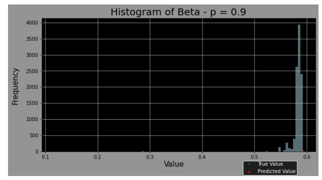
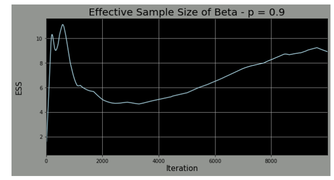
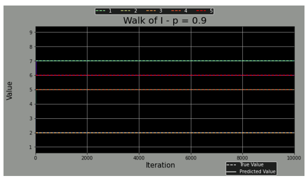
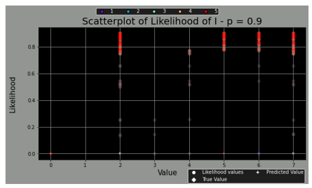
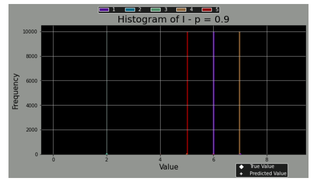
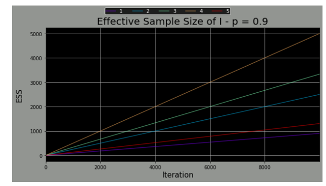
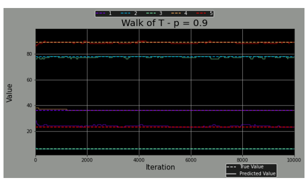
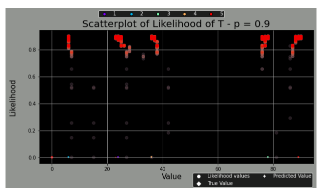
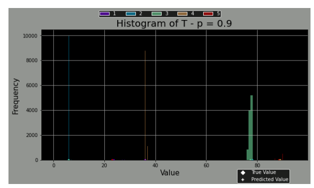
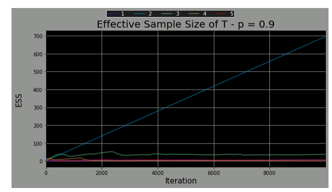
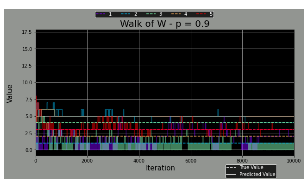
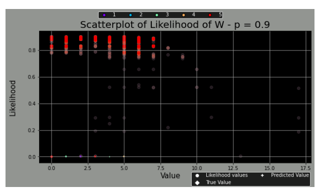
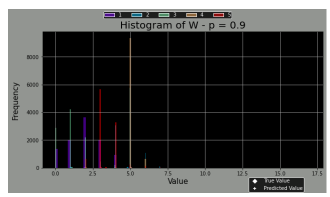
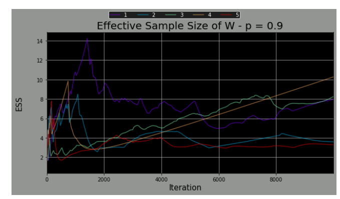
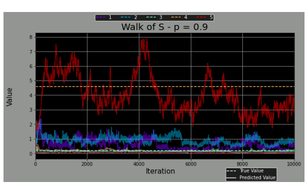
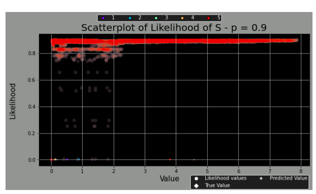
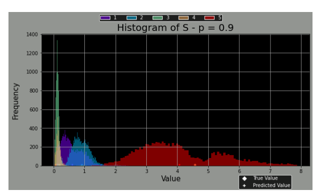
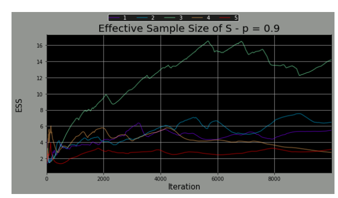
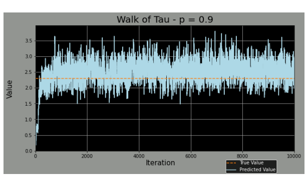
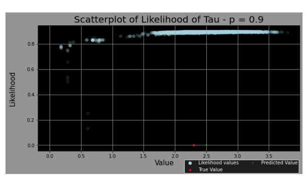
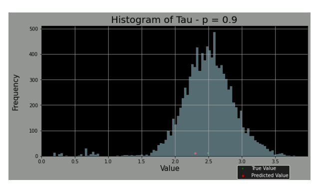
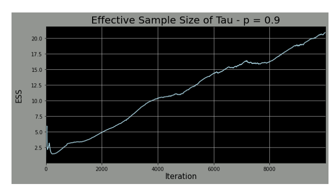
<table>
<tr><th>Results</th><th>Full</th><th>Observed</th></tr>
<tr><td>Mass</td><td>
</td><td>
</td></tr>
<tr><td>Concentration</td><td>
</td><td>
</td></tr>
<tr><td>Y</td><td>
</td><td>
</td></tr>
<tr><td>e^Y</td><td>
</td><td>
</td></tr>
</table>

Figure X + 32: Table of results for sparse data ($p = 0.9$)

Parameter	Median	95% CI - Lower Bound	95% CI - Upper Bound	P(Acceptance)
α	0.8361	0.8909	0.8996	0.0178
β	0.5854	0.5834	0.5839	0.0076
I	6, 2, 2, 7, 5	5.9967, 1.9998, 1.9998, 6.9988, 4.9980	6.0009, 2.0010, 2.0006, 7.0004, 5.0004	0.0991
T	24, 6, 78, 36, 89	23.7761, 6.0013, 77.3759, 36.1253, 88.6104	23.8027, 6.0083, 77.4365, 36.1411, 88.6344	0.0136
W	2, 5, 1, 5, 3	1.9120, 4.8065, 1.0840, 5.0582, 3.3555	1.9596, 4.8321, 1.1224 5.0690, 3.3841	0.2671
S	0.5174, 0.8775, 0.1189, 0.1403, 3.8028	0.5972, 0.9046, 0.1259, 0.1464, 4.0372	0.6112, 0.9155, 0.1290, 0.1489, 4.0881	0.5529
τ	2.5012	2.4859	2.5017	0.5760

$Figure\ X\ +\ 33\text{:}\ Table\ of\ summary\ statistics\ of\ parameters\ for\ sparse\ data\ (p\ =\ 0.9)$

Parameter	Walk of value	Scatter Plot of Likelihood	Histogram	Walk of ESS
α				
β				
I				
T				
W				
S				
τ				

$Figure\ X + 34$: $Table\ of\ graphs\ of\ parameter\ metrics\ for\ sparse\ data\ (p = 0.9)$