Introduction to Commutative Algebra

and affine algebraic varieties

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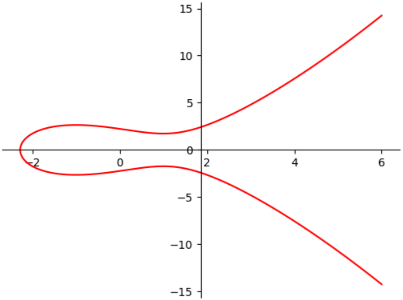
Introduction

The Plan

- Study undergraduate algebraic geometry
- Read and do the exercies from Atiyah-Macdonald, Introduction to Commutative Algebra
- Read first chapter of Hartshorne's Algebraic Geometry

Curves

$y^2 = x^3 - 3x + 5$ a polynomial in $\mathbb{R}[x, y]$. The set of zeros of this polynomial looks like this



Polynomial Ring

Given a algebraicaly closed field k we can form the polynomial ring in n indeterminants

$$k[x_1, \cdots, x_n]$$

Every polynomial $p(x_1, \cdots, x_n) \in k[x_1, \cdots, x_n]$ can be thought of as a mapping from $k^n \to k$. We call k^n the *affine n-space* and denote it by $\mathbb{A}_k^n$.

Affine Algebraic Varieties

S is a set of polynomial in $k[x_1, \cdots, x_n]$. $V(S)$ is points in $\mathbb{A}_k^n$ at which every polynomial in S *vanishes*. $V(S)$ is called the *affine algebraic variety*.

The Coordinate Ring

Given a variety V in $\mathbb{A}_k^n$ the *ideal of a variety* is the ideal $I(V)$ which consists of all polynomials in $k[x_1, \cdots, x_n]$ that vanish on V. The Coordinate ring of a variety is the ring

$$P(X) = k[x_1, \cdots, x_n]/I(X)$$

Hilbert's Nullstellensatz

Nullstellensatz means the theorem of zeros.

Algebra	**Geometry**
$k[x_1, \cdots, x_n]$	$\mathbb{A}_k^n \cong k^n$
$I(V)$	$V(I)$
$(x - a_1, \cdots, x - a_n)$	the point $(a_1, \cdots, a_n)$

Algebraic - Geometry

There is a connection between geometric objects such as curves and the algebraical objects like a ring.

Regular mappings

Given $f_1, \cdots, f_m \in k[x_1, \cdots, x_n]$ we have a polynomial mapping by $\phi(x) = (f_1(x) + \cdots + f_m(x))$ that takes $k^n \to k^m$.
Regular mappings are maps between two varieties $X \subset k^n$ and $Y \subset k^m$ given by the restriction $\phi|_X : X \to Y$.

What sort of Commutative Algebra do we use?

What sort of commutative algebra machinery do we use?

1. Modules
2. Tensor products
3. Exact sequnces
4. Direct Limits

Topology

Given an algebraic variety we can have a topology on it. One such topology is the Zariski Topology.

- $X = \text{Spec}(A)$ is the prime spectrum of the ring A
- $V(E)$ are the closed sets on X
- $V(E)$ satisfies the three axioms for a topological space

Constructible Topology

We can have another topology called the Constructible Topology on $X = \text{Spec}(A)$.

- For each $f : A \to B$ we have $f^* : \text{Spec}(B) \to \text{Spec}(A)$
- The subset $f^*(\text{Spec}(B))$ of $\text{Spec}(A)$ is closed.

When is the Zariski Topology and the Constructible Topology the same?

Presheaf and Sheaf

Definition

Given an open set U on a topological space X, the presheaf of rings, F on X is defined as the following data

1. *To each open set U, a ring $F(U)$ is associated. The elements $f \in F(U)$ are called the sections of F over U. The ring $F(X)$ is called a ring of global section.*
2. *Let U and V be two open subsets of X such that $V \subseteq U$. The ring homomorphim $res_{V,U} : F(U) \to F(V)$ associated with this mapping is called the restriction homomorphim. If $f \in F(U)$ is a section, then $res_{V,U}(f)$ is called the restriction of f and is denoted by $f|_V$.*

Definition

Additionally, the restriction homomorphim must sastisfy the following conditions

1. *For each open set U of X, the restriction homomorphim $res_{U,U} : F(U) \to F(U)$ is the identity homomorphim.*
2. *If we have $W \subseteq V \subseteq U$ for three open sets, then $res_{W,V} \circ res_{V,U} = res_{W,U}$. That is we have the commutative diagram*

Definition

Let U_i be a collection of open sets which cover the open set U. A presheaf F is called a sheaf if given a collection of sections $f_i \in F(U_i)$ which satisfy

$$f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}, \forall i, j$$

there exists a unique $f \in F(U)$ such that $f|_{U_i} = f_i$.

How this any useful?

- Algebraic geometry is used in Arithmetic geometry as there is a deep connection between algebraic geometry and number theoy
- Algebraic geometry is used in String Theory, eg, to study the properties of "curled up dimensions of spacetime"
- A recent brance of mathematical biology called Phylogenetic algebraic geometry

Acknowledgement

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