Introduction to Commutative Algebra:
with a brief introduction to the
algebraic geometry of affine
algebraic varieties

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January 4, 2021

Abstract

The motivation for the study of algebraic geometry is how algebraic objects (rings of rational functions) are associated with varieties (zeros of polynomials). This subject florished during the second half of the twentieth century. Algebraic geometry allows us to study the geometry arising from algebraic objects. Core to the deeper understanding of this subject is an understanding of the subject of commutative algebra which studies commutative rings and their ideals and modules. The purpose of the present project is to gain an understanding of commutative algebra through solving exercises from Atiyah-MacDonald's book, Introduction to Commutative Algebra. The reading project comprised of the study of the theory of rings and modules, their tensor product and exact sequences of rings and modules. The project concluded with a proof of the Going-Up Theorem.

Contents

Chapter 1

Introduction

The purpose of the current project is to give an understanding to the reader of how commutative algebra is used to define the basic objects of algebraic geometry. Algebraic Geometry is the study of the geometric properties of the solution set of polynomial equations in arbitary variables. For this purpose it is useful to confine the discussion to algebraically closed fields k and the polynomial ring $k[x_1, \cdots, x_n]$. We define *affine n-space* over the algebraically closed field k as k^n and denote it by A_k^n.

Definition 1.0.1. *The algebraic variety [1] in the affine space $\mathbb{A}_k^n$ of a set S of polynomials $f \in k[x_1, \cdots, x_n]$ is a subset $V(S) \subseteq \mathbb{A}_k^n$*

$$V(S) = \{x = (x_1, \cdots, x_n) \in \mathbb{A}_k^n : f(x) = 0 \textit{ for all } f \in S\}.$$

The affine variety is not the only example of a variety. The projective variety defined over projective spaces $\mathbb{P}_k^n$ is another example of an algebraic variety.

The solution set or *locus* of solutions of polynomials of degree one are discrete points in k. But over k^2 the *locus* are curves. For example, the solution of the elliptic curve given by the polynomial equation in $\mathbb{R}[x, y]$ in $\mathbb{R}^2$

$$y^2 = x^3 - 3x + 5$$

is a curve (Fig 1.). The question then arises: *what does the solution set of a set of polynomials in k^3 look like? What about k^n?* This is same asking the question: *what does an affine algebraic variety in A_k^n look like?* We may go further by demanding: *what is the topology of the affine algebraic variety?* (The answer is called the *Zariski Topology*, which we shall get to in due time.)

Once we have an affine algebraic variety we may define functions on them and we may define functions from one algebraic variety to another. Thus we start to do the usual mathematical *shtick* on affine varieties.

To answer the proposed questions we need to know the properties of an affine algebraic varieties. For example: *what is so algebraic about an affine algebraic variety?* The answer is that we may consider every *affine algebraic variety* to be some *finitely generated nilpotent-free k-algebra A*. To know what each of these terms means we first need some commutative algebra. The purpose of this article is to explain these terms and give a brief introduction to Algebraic Geometry focusing only on affine algebraic varieties.

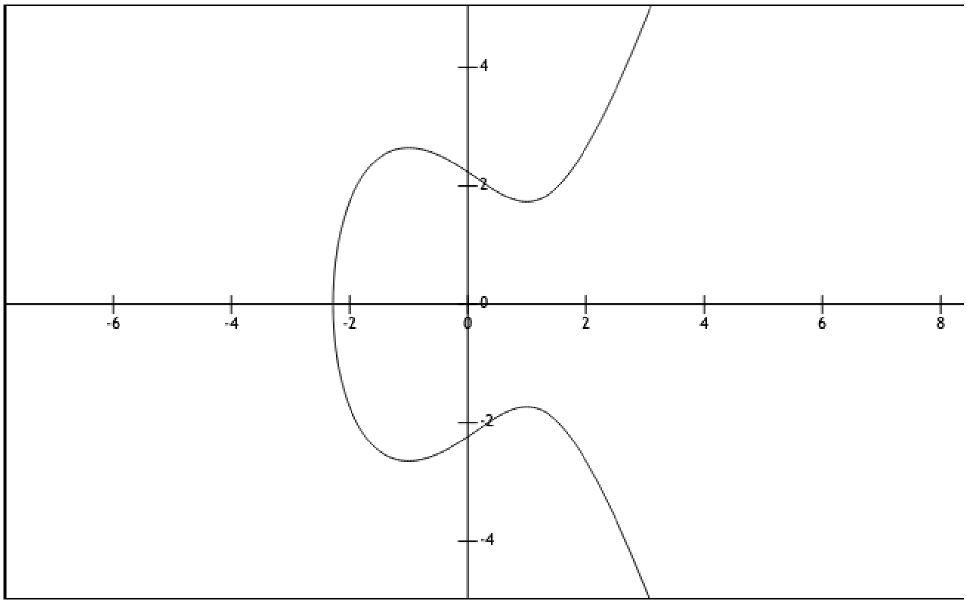

Figure 1.1: Fig 1.

The exposition is based on the knowledge gained after spending many hours working out the exercises contained in [2]. I would advise anyone thinking about learning algebraic geometry to start there and work out those exercises themselves.

We divide the article into three sections. In the first section we introduce the *prime spectrum* of a ring A, $X = \text{Spec}(A)$ by which we denote the set of all prime ideals of A. We endow this space X with a topology called *the Zariski Topology*.

In the next section, we introduce *modules*, which are a sort of generalised vector spaces and *algebras*, which are modules on rings, and we define the tensor product on a collection of such *modules*. Then we discuss a particular sort of structure that can be given to a collection of modules called *the direct limit* – a term loaned from Category Theory.

In the last section we introduce the *rings of fractions* of modules and the concept of *localization* which we use in the formulation of a sort of structure called *presheaf* and *sheaf*.

Chapter 2

The Zariski Topology

2.1 Prime Spectrum

2.2 Zariski Topology

2.3 Irreducible Spaces

2.4 Affine Algebraic Varieties

Chapter 3

Tensor Product and Direct Limits

3.1 Tensor Product

3.2 Flatness

3.3 Direct Limit

Chapter 4

Sheaf and Presheaf

4.1 Support

4.2 Presheaf

4.3 Sheaf

4.4 Constructible Topology

4.5 Absolute Flatness

Bibliography

[1] Ravi Vakil, *Math 145: Algebraic Geometry*

[2] Atiyah-Macdonald, *Introduction to Commutative Algebra*