Introduction to Commutative Algebra

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Abstract

The motivation for the study of algebraic geometry is how algebraic objects (rings of rational functions) are associated with varieties (zeros of polynomials). This subject florished during the second half of the twentieth century. Algebraic geometry allows us to study the geometry arising from algebraic objects. Core to the deeper understanding of this subject is an understanding of the subject of commutative algebra which studies commutative rings and their ideals and modules. The purpose of the present project is to gain an understanding of commutative algebra through solving exercises from Atiyah-MacDonald's book, Introduction to Commutative Algebra. The reading project comprised of the study of the theory of rings and modules, their tensor product and exact sequences of rings and modules. The project concluded with a proof of the Going-Up Theorem.

Chapter 1

Introduction: What is Algebraic Geometry?

1.1 Some historical problems

1.1.1 27 lines

1.1.2 Bezout's Theorem

1.1.3 Brief History of Algebraic Geometry

1.2 What is geometry?

1.3 What is algebra?

Chapter 2

Hilbert's Nullstellensatz

2.1 Basics ideas in ring theory

2.2 The Nullstellensatz

Chapter 3

Rings and Ideals

3.1 Basic Definitions

Definition 3.1.1. *A ring A is a set with two binary operations (addition and multiplication) such that*

1. *A is an abelian group with respect to addition*
2. *Multipicatio is associative ($(xy)z = x(yz)$) and distributive over addition ($x(y+z) = xy + xz, (y+z)x = yx + zx$).*
3. *$xy = yx$ for all $x, y \in A$. (for our purpose we consider only rings that commute)*
4. *$\exists 1 \in A$ such that $x1 = 1x = x$ for all $x \in A$. The identity element is unique.*

If $1 = 0$ then for any $x \in A$ we have $x = x1 = x0 = 0$ so A has only one element 0.

Definition 3.1.2. *A ring homomorphism is a mapping f of a ring A into a ring B such that*

1. *$f(x+y) = f(x)+f(y)$ so that f is a homomorphism of abelian groups.*
2. *$f(xy) = f(x)f(y)$,*
3. *$f(1) = 1$*

Definition 3.1.3. *A subset S of a ring A is a subring of A if S is closed under addition and multiplication and contains th identity element of A.*

3.2 Ideals and Quotients

Definition 3.2.1. *An ideal $\mathfrak{a}$ of a ring A is a subset of A which is an additive subgroup and is such that $A\mathfrak{a} \subseteq \mathfrak{a}$.*

Definition 3.2.2. *The quotient group $A/\mathfrak{a}$ inherits a uniquely defined multiplication from A which makes it into a ring. Called the quotient ring $A/\mathfrak{a}$. The elements of $A/\mathfrak{a}$ are the cosets of $\mathfrak{a}$ in A, and the mapping $\phi : A \to A/\mathfrak{a}$ which maps each $x \in A$ to its coset $x + \mathfrak{a}$ is a surjective ring homomorphism.*

Proposition 3.2.3. *There is a one-to-one order-preserving correspondence between the ideals $\mathfrak{b}$ of A which contain $\mathfrak{a}$ and the ideals $\bar{\mathfrak{b}}$ of $A/\mathfrak{a}$, given by $\mathfrak{b} = \phi^{-1}(\bar{\mathfrak{b}})$.*

3.3 Zero-Divisors, Nilpotent Elements, Units

Definition 3.3.1. *A zero-divisor in a ring A is an element x for which there exists $y \neq 0$ in A such that $xy = 0$. A ring with no zero-divisors $\neq 0$ is called an *integral domain*.*

Definition 3.3.2. *An element $x \in A$ is nilpotent if $x^n = 0$ for some $n > 0$. A nilpotent element is a zero-divisor. A *unit* in A is an element x such that $xy = 1$ for some $y \in A$. The element y is then uniquely determined by x and is written x^{-1}. The units in A form a multiplicative abelian group.*

Definition 3.3.3. *The multiples ax of an element $x \in A$ form a principal ideal, denoted by (x) or Ax. x is a unit $\Leftrightarrow (x) = A = (1)$. The zero ideal (0) is usually denoted by 0. A *field* is a ring A in which $1 \neq 0$ and every non-zero element is a unit. Every field is an integral domain.*

Proposition 3.3.4. *Let A be a ring $\neq 0$. Then the following are equivalent:*

1. *A is a field*
2. *the only ideals in A are 0 and (1).*
3. *every homomorphism of A into a non-zero ring B is injective.*

3.4 Prime and Maximal Ideals

Definition 3.4.1. *An ideal $\mathfrak{p}$ in A is prime if $\mathfrak{p} \neq (1)$ and if $xy \in \mathfrak{p} \implies x \in \mathfrak{p}$ or $y \in \mathfrak{p}$. An ideal $\mathfrak{m}$ in A is maximal if $\mathfrak{m} \neq (1)$ and if there is no ideal $\mathfrak{a}$ such that $\mathfrak{m} \subset \mathfrak{a} \subset (1)$. Equivalently:*

$$\mathfrak{p} \text{ is prime } \Leftrightarrow A/\mathfrak{p} \text{ is an integral domain}$$

$$\mathfrak{m} \text{ is maximal } \Leftrightarrow A/\mathfrak{p} \text{ is a field}$$

Hence a maximal ideal is prime. But the converse is not true in general.

1. If $f : A \to B$ is a ring homomorphism and $\mathfrak{q}$ is a prime ideal in B, then $f^{-1}(\mathfrak{q})$ is a prime ideal in A, for $A/f^{-1}(\mathfrak{q})$ is isomorphic to a subring of $B/\mathfrak{q}$ and hence has no zero-divisor $\neq 0$.

Theorem 3.4.2. *Every commutative ring A (with identity) $\neq 0$ has at least one maximal ideal.*

Corollary 3.4.2.1. *Every non-unit of A is contained in a maximal ideal.*

Definition 3.4.3. *A ring A with exactly one maximal ideal $\mathfrak{m}$ is called a local ring. The field $k = A/\mathfrak{m}$ is called the residue field of A.*

Exercise 3.4.4. *A local ring contains no idempotent $\neq 0, 1$. (Ex 12)*

Proposition 3.4.5.

1. *Let A be a ring and $\mathfrak{m} \neq (1)$ an ideal of A such that every $x \in A - \mathfrak{m}$ is a unit in A. Then A is a local ring and $\mathfrak{m}$ its maximal ideal.*

2. *Let A be a ring and $\mathfrak{m}$ a maximal ideal of A, such that every element of $\mathfrak{1} + \mathfrak{m}$ (every $1 + x$ where $x \in \mathfrak{m}$) is a unit in A. Then A is a local ring.*

Example 3.4.6. *$A = k[x_1, \cdots, x_n], k$ is a field. Let $f \in A$ be an irreducible polynomial. By unique factorization, the ideal (f) is prime.*

Example 3.4.7. *$A = Z$. Every ideal in Z is of the form (m) for some $m \geq 0$. The ideal (m) is prime $\Leftrightarrow \mathfrak{m} = 0$ or a prime number. All the ideals (p), where $\mathfrak{p}$ is a prime number are maximal: $Z/(p)$ is a field of p elements.*

3.5 Nilradical and Jacobson Radical

Proposition 3.5.1. *The set $\mathfrak{N}$ of all nilpotent elements in a ring A is an ideal and $A/\mathfrak{N}$ has no nilpotent element $\neq 0$.*

Definition 3.5.2. *The ideal $\mathfrak{N}$ is called the nilradical of A. The following proposition gives an alternative definition of $\mathfrak{N}$:*

Proposition 3.5.3. *The nilradical of A is the intersection of all the prime ideals of A.*

Definition 3.5.4. *The Jacobson radical of $\mathfrak{R}$ of A is defined to be the intersection of all the maximal ideals of A. It can be characterized as follows:*

Proposition 3.5.5. *$x \in \mathfrak{R} \Leftrightarrow 1 - xy$ is a unit in A for all $y \in A$.*

Exercise 3.5.6. *Let x be a nilpotent element of a ring A. Show that $\mathfrak{1}+\mathfrak{x}$ is a unit of A. Deduce that the sum of a nilpotent element and a unit is a unit. (Ex 1)*

Exercise 3.5.7. *Let A be a ring in which every element x satisfies $x^n = x$ for some $n > 1$ (depending on x). Show that every prime ideal in A is maximal.*

3.6 Some properties of ideals

Definition 3.6.1. *If $\mathfrak{a}, \mathfrak{b}$ are ideals in a ring A, their *ideal quotient* is*

$$(\mathfrak{a} : \mathfrak{b}) = \{x \in A : x\mathfrak{b} \subseteq \mathfrak{a}\}$$

*which is an ideal. In particular, $(0 : \mathfrak{b})$ is called the *annihilator* of $\mathfrak{b}$ and is also denoted by $Ann(\mathfrak{b})$.*

Definition 3.6.2. *If $\mathfrak{a}$ is any ideal of A, the radical of $\mathfrak{a}$ is*

$$r(\mathfrak{a}) = \{x \in A : x^n \in \mathfrak{a} \text{ for some } n > 0\}$$

If $\phi : A \to A/\mathfrak{a}$ is the standard homomorphism, then $r(\mathfrak{a}) = \phi^{-1}(\mathfrak{N}_{A/\mathfrak{a}})$ and hence $r(\mathfrak{a})$ is an ideal by (1.7).

Exercise 3.6.3.

1. $r(\mathfrak{a}) \supseteq \mathfrak{a}$
2. $r(r(\mathfrak{a})) = r(\mathfrak{a})$
3. $r(\mathfrak{a}\mathfrak{b}) = r(\mathfrak{a} \cap \mathfrak{b}) = r(\mathfrak{a}) \cap r(\mathfrak{b})$
4. $r(\mathfrak{a}\mathfrak{b}) = (1) \Leftrightarrow \mathfrak{a} = (1)$
5. $r(\mathfrak{a} + \mathfrak{b}) = r(r(\mathfrak{a}) + r(\mathfrak{b}))$
6. *If $\mathfrak{p}$ is prime, $r(\mathfrak{p}^n) = \mathfrak{p}$ for all $n > 0$.*

Proposition 3.6.4. *The radical of an ideal $\mathfrak{a}$ is the intersection of the prime ideals which contain $\mathfrak{a}$.*

Proposition 3.6.5. *D = set of zero-divisors of $A = \bigcup_{x \neq 0} r(Ann(x))$.*

Proposition 3.6.6. *Let $\mathfrak{a}, \mathfrak{b}$ be ideals in a ring A such that $r(\mathfrak{a}), r(\mathfrak{b})$ are coprime. Then $\mathfrak{a}, \mathfrak{b}$ are coprime.*

Exercise 3.6.7. *Let $\mathfrak{a}$ be an ideal $\neq (1)$ in a ring A. Show that $\mathfrak{a} = r(\mathfrak{a}) \Leftrightarrow \mathfrak{a}$ is an intersection of prime ideals. (Ex 9)*

Exercise 3.6.8. *Let A be a ring and let X be the set of all prime ideals of A. For each subset E of A, let $V(E)$ denote the set of all prime ideals of A which contain E. Prove that*

1. *if $\mathfrak{a}$ is the ideal generated by E, then $V(E) = V(\mathfrak{a}) = V(r(\mathfrak{a}))$.*
2. $V(0) = X, V(1) = \varnothing$.

3. *if $(E_i)_{i\in I}$ is an family of subsets of A, then*
$$V(\bigcup_{i\in I} E_i) = \bigcap_{i\in I} V(E_i)$$

4. $V(\mathfrak{a}\cap\mathfrak{b}) = V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a})\cup V(\mathfrak{b})$ *for any ideals $\mathfrak{a},\mathfrak{b}$ of A.*

These results show that the sets $V(E)$ satisfy the axioms for closed sets in a topological space. The resulting topology is called the Zariski Topology. The topological space X is called the prime spectrum of A, and is written Spec(A). (Ex 15)

Exercise 3.6.9. *Draw pictures of Spec$(\mathbb{Z})$, Spec$(\mathbb{R})$, Spec$(\mathbb{C}[x])$, Spec$(\mathbb{R}[x])$, Spec$(\mathbb{Z}[x])$. (Ex 16)*

Exercise 3.6.10. *For each $f\in A$, let X_f denote the complement of $V(f)$ in $X =$ Spec(A). The sets X_f are open. Show that they form a basis of open sets for the Zariski topology, and that*

1. $X_f\cap X_g = X_{fg}$
2. $X_f = \varnothing \Leftrightarrow f$ *is nilpotent.*
3. $X_f = X \Leftrightarrow f$ *is a unit.*
4. $X_f = X_g \Leftrightarrow r((f)) = r((g))$.
5. *X is a quasi-compact (that is, every open covering of X has a finite subcovering).*
6. *More generally, each X_f is quasi-compact.*
7. *An open subset of X is quasi-compact if and only if it is a finite union of sets X_f.*

The sets X_f are called basic open sets of $X =$ Spec(A). (Ex 17)

Exercise 3.6.11. *For psychological reasons it's sometimes convenient to denote a prime ideal of A by a letter such as x or y when thinking of it as a point of $X =$ Spec(A). When thinking of x as a prime ideal of A, we denote it by $\mathfrak{p}_x$ (logically, of course, it is the same thing). Show that*

1. *the set $\{x\}$ is closed (we say that x is a "closed point") in Spec$(A) \Leftrightarrow \mathfrak{p}_x$ is maximal.*
2. $\{\bar{x}\} = V(\mathfrak{p}_x)$.
3. $y \in \{\bar{x}\} \Leftrightarrow \mathfrak{p}_x \subseteq \mathfrak{p}_y$.
4. *X is a T_0-space (this means that if x, y are distinct points of X, then either there is a neighborhood of y which does not contain A).*

(Ex 18)

Chapter 4

Modules

4.1 Modules and Module homomorphisms

Definition 4.1.1. *Let M be an abelian group and let A be a commutative ring that acts *linearly* on M. If we denote $\mu(a,x) = ax$, where $a \in A, x \in M$ and the following axioms are satisfied,*

1. $a(x+y) = ax + ay$
2. $(a+b)x = ax + bx$
3. $(ab)x = a(bx)$
4. $1x = x$, *($\forall a, b \in A, \forall x, y \in M$.)*

Then (M, μ) is called a module

4.2 Submodules and Quotient Modules

Definition 4.2.1. *A submodule M' of M is a subgroup that is closed under multiplication by elements of A. The quotient M/M' is also an A-module is the action of A on the quotient is $a(x+M') = ax + M'$.*

Definition 4.2.2. *If $f : M \to N$ A-module homomorphism then,*

1. $Ker(f) = \{x \in M : f(x) = 0\}$
2. $Im(f) = f(M)$
3. *The cokernel of f is denoted by, $Coker(f) = N/Im(f)$*

Proposition 4.2.3. *There is a one-to-one order preserving correspondence between the submodules of M containing the submodule M' and the submodules of the quotient M/M'. The correspondence for ideals is a special case of this proposition.*

Definition 4.2.4. *(Induced homomorphism) If $M' \subseteq Ker(f)$. Define $\bar{f} : M/M' \to N$ as $\bar{f}(\bar{x}) = f(x)$. Then it is a homomorphism and it is said to be induced by f. Specially we have, $Ker(\bar{f}) = Ker(f)/M'$.*

If we take $Ker(f) = M'$ we have that $M/Ker(f) \cong Im(f)$.

Proposition 4.2.5. 1. *If $L \supseteq M \supseteq N$ are A-modules, then*

$$\frac{L/N}{M/N} = L/M.$$

2. *If M_1, M_2 are submodules of M, then*

$$(M_1 + M_2)/M_1 \cong M_2/(M_1 \cap M_2).$$

4.2.1 Some important notes

1. The product aM where a is an ideal is the set of all finite sums $\sum a_i x_i$ with $a_i \in a, x_i \in M$.
2. If N, P are submodules of M then $(N : P) = \{a \in A : aP \subseteq N\}$.
3. The annihilator, $Ann(M) = (0, M) = \{a \in A : aM = 0\}$
4. An A-module is called *faithful* if $Ann(M) = 0$.
5. If $a \subseteq Ann(M)$ we can define M as an A/a-module by setting $\bar{x}m = xm$, $\bar{x} \in A/a, m \in M$. This is true as $x, y \in \bar{x}$ then $x - y \in a$ hence $(x - y)m = 0 \implies xm = ym$. So it's indepenent of the choice of the representative of $\bar{x}$.
6. If $Ann(M) = a$, then M is faithful as an $A/a-$module.

4.3 Direct Sum and Product

Definition 4.3.1. *(Direct Sum) Given two A-moduels M and N their direct product denoted by $M \oplus N$ is defined as the set of all pair (x, y) such that $x \in M$ and $y \in N$ and is an A-module by,*

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$

$$a.(x, y) = (ax, ay)$$

If $(M_i)_{i \in I}$ is a family of A-modules then the direct sum $\oplus_{i \in I} M_i$ is the set of all $(x_i)_{i \in I}$ such that all but finitely many x_i are zero.

Definition 4.3.2. *(Direct Product) The direct product of a family of A-modules denoted by $\prod_{i \in I} M_i$, is exactly the same as the direct sum if I is finite. Otherwise the only difference is that $(x_i)_{i \in I}$ does not have the condition that x_i must be zero for all but finitely many x_i.*

4.4 Finitely Generated Module

Definition 4.4.1. *A module M is said to be finitely generated if it can be expressed as $M = \sum_{i \in I} Ax_i$. That is each $m \in M$ can be expressed as a finite linear combination of elements in A with x_i. If the number of x_i is finite then the module is said to be a finitely generated A-module.*

4.4.1 Free Module

Definition 4.4.2. *A finitely generated free A-module is a module that's isomorphic to $A \oplus \cdots \oplus A$ denoted as $A^{(n)}$ (n summands). If we remove the restriction of being finite a free A-module is isomorphic to a module of the form $\oplus_{i \in I} M_i$, where M_i are A-modules isomorphic to A. * A^0 is the zero module denoted by 0.*

Proposition 4.4.3. *M is a finitely generated A-module if and only if M is isomorphic to a quotient of A^n for some $n > 1$.*

Definition 4.4.4. *Let M be a finitely generated A-module and $a \subseteq A$ be an ideal of M. If ϕ is an endomorphism of M such that $\phi(M) \subseteq aM$ then ϕ satisfies a polynomial equation in ϕ,*

$$\phi^n + a_1\phi^{n-1} + \cdots + a^n = 0,$$

for some $a_i \in a$.

Corollary 4.4.4.1. *If M is a finitely generated module such that for an ideal $a \subseteq A$, $aM = M$ then there exists an $x \equiv 1 (mod\ a)$ such that $xM = 0$.*

Proposition 4.4.5. *(Nakayama's lemma) If M is a finitely generated module and $a \subseteq A$ is an ideal of A contained in the Jacobson radical of A then $aM = M$ implies that $M = 0$.*

Proposition 4.4.6. *Suppose M is finitely generated A-module and a is an ideal contained in the Jacobson radical of A then $aM = M + N$ implies that $M = N$ for any submodule N of M.*

Proposition 4.4.7. *Let x_i be element of M whose image in M/mM are the basis of this vector space. Then the x_i generate M.*

4.5 Exact Sequence

Definition 4.5.1. *A sequence of A-modules and A-homomorphisms,*

$$\cdots \to M_{i-1} \to^{f_i} M_i \to^{f_{i+1}} M_{i+1} \to \cdots$$

is said to be exact at M_i if, $Ker(f_{i+1}) = Imf_i$. The sequence is exact if it is exact at each M_i.

1. $0 \to M' \to^f M$ is exact $\Leftrightarrow$ f is inective.

2. $M \to^g M'' \to 0$ is exact $\Leftrightarrow$ g is surjective.

3. $0 \to M' \to^f M \to^g M'' \to 0$ is exact $\Leftrightarrow$ f is injective and g is surjective. But also $Coker(f) = M/f(M')$ is isomorphic to M''.

4. This type of exact sequence is also called a short exact sequence. Any exact sequence can be split up into a collection of short exact sequences. If $N_i = Im(f_i) = Ker(f_{i+1})$ we have the short exact sequence $0 \to N_i \to M_i \to N_{i+1} \to 0$ for each i.

4.6 Tensor Product of Modules

Definition 4.6.1. *(Tensor product) The tensor product of two A-modules M and N is the A-module $M \otimes N$ of all linear combination of the pair $x \otimes y$ with coefficients in A along with the following properties,*

1. $(x + x') \otimes y = x \otimes y + x' \otimes y$

2. $x \otimes (y + y') = x \otimes y + x \otimes y'$

3. $a.x \otimes y = x \otimes a.y = a(x \otimes y)$

Properties:

1. $0 \otimes x = 0$

2. If x_i generates M and y_i generates N then $x_i \otimes y_i$ generates $M \otimes N$.

3. Let $x \in M$ and $y \in N$ if $M' \subseteq M$ and $N' \subseteq N$ are submodules then $x \otimes y \in M \otimes N$ is not the same as $x \otimes y \in M' \otimes N'$

4. To be specific the tensor product of A-modules is denoted by $M \otimes_A N$ but if the context is clear we can write $M \otimes N$.

Note: The correct definition of a tensor product is by a universal property that there's a one to one correspondence between the bilinear maps $M \times N \to P$ and the A-linear map $M \otimes N \to P$. But this is not important nor useful at this stage. Whatever it is is ultimately exactly equivalent to the definition above.

The following isomorphisms are considered canonical.

Proposition 4.6.2. *Let M,N,P be A-modules. Then there exists unique isomorphism*

1. $M \otimes N \to N \otimes N$, *(given by $x \otimes y \mapsto y \otimes x$)*

2. $(M \otimes N) \otimes P \to M \otimes (N \otimes P) \to M \otimes N \otimes P$, *(given by* $(x \otimes y) \otimes z \mapsto x \otimes (y \otimes z) \mapsto x \otimes y \otimes z$*)*

3. $(M \oplus N) \otimes P \to (M \otimes P) \oplus (N \otimes P)$, *given by* $(x, y) \otimes z \mapsto (x \otimes z, y \otimes z)$

4. $A \otimes M \to M$, *given by* $a \otimes x \mapsto ax$.

4.7 Restriction and extension of scalars

Definition 4.7.1. *Let $f : A \to B$ be a ring homomorphism and N be a B-module then the A-module defined by the action $f(a)x$ where $a \in A$ and $x \in N$ is called the A-module obtained by **restriction of scalars**. The homomorphism induces an A-module structure on B. Then the module $M_B = B \otimes M$ where B is considered as an A-module is said to be obtained from M by the extension of scalars. * M_B also has a B-module structure as well by $b(b' \otimes x) = bb' \otimes x$.*

Proposition 4.7.2. *Let N be a finitely generated B-modules and B be a finitely generated A-module then N is a finitely generated as an A-module.*

Proposition 4.7.3. *If M is finitely generated as an A-module then M_B is finitely generated as a B-module!*

4.8 Exactness Properties of the Tensor Product

The set S of A-bilinear mappings $M \times N \to P$ is in natural 1-1 correspondence with $Hom(M, Hom(N, P))$ but by the definition of the Tensor product S is in 1-1 correspondence with the A-linear maps $M \otimes N \to P$. There fore we have,

$$Hom(M \otimes N, P) \cong Hom(M, Hom(N, P))$$

For a A-module N let $T(M) = M \otimes N$ be a functor that takes modules to modules and let $U(P) = Hom(N, P)$. Then by the correspondence formula above we have,

$$Hom(T(M), P) = Hom(M, U(P))$$

This condition means exactly in the language of category theory that T is the left-adjoint of U and that U is the right-adjoint of T.

Note: The following proposition says that any functor which is left adjoint is right exact and any functor which is right adjoint is left exact.

Proposition 4.8.1. *If, $M' \to^f M \to^g M'' \to 0$ is an exact sequence of A-modules and homomorphisms then for any A-module N,*

$$M' \otimes N \to^{f\otimes 1} M \otimes N \to^{g\otimes 1} M'' \otimes N \to 0$$

is an exact sequence as well where 1 denotes the identity map taking N to N.

In general tensoring with an A-module does not turn an exact sequence into another exact sequence. But

Definition 4.8.2. *(Flat Modules) Let N be an A-moduels and let $T_N : M \mapsto M \otimes N$ be a functor. If T_N is exact, that is tensoring with N turns all exact sequences into exact sequences, then N is called a flat module.*

Proposition 4.8.3. *The following statements are equivalent, for an A-module N:*

1. *N is flat*
2. *If $0 \to M' \to M \to M'' \to 0$ is any exact sequences of A-moduels, the tensored sequence $0 \to M' \otimes N \to M \otimes N \to M'' \otimes N \to 0$ is exact.*
3. *If $f : M' \to M$ is injective, then $f \otimes 1 : M' \otimes N \to M \otimes N$ is injective.*
4. *If $f : M' \to M$ is injective and M, M' are finitely generated, then $f \otimes 1 : M' \otimes N \to M \otimes N$ is injective.*

Chapter 5

Rings and Modules of Fractions

5.1 Ring of Fractions

The standard rings of fraction of a ring A can only be defined when A is an integral domain. But we can generalise the notion to arbitary rings. Let A be a commutative ring and S be a *multiplicative subset* of A. That is $1 \in S$ and for every $x, y \in S$, $xy \in S$. Define the equivalence relation $\sim$ on $A \times S$ to be:

$$(a, s) \sim (b, t) \Leftrightarrow (at - bs)u = 0 \text{ for some } u \in S.$$

Definition 5.1.1. *Define the fraction a/s to be the equivalence class of (a, s) and let $S^{-1}A$ denote the set of these equivalence classes. Then $S^{-1}A$ is a ring with addition and multiplication defined by,*

$$(a/s) + (b/t) = (at + bs)st,$$
$$(a/s)(b/t) = ab/st.$$

$S^{-1}A$ is called the ring of fraction of A with respect to S. By f we mean the ring homomorphism $f : A \to S^{-1}A$ defined by $f(x) = x/1$.

Exercise 5.1.2. *Verify that these definitions are indepenent of the choices of representatives (a, s) and (b, t), and that $S^{-1}A$ satisfies the axioms of a commutative ring with identity.*

The ring $S^{-1}A$ and the homomorphism $f : A \to S^{-1}A$ have the following property:

1. $s \in S \implies f(s)$ is a unit in $S^{-1}A$.
2. $f(a) = 0 \implies as = 0$ for some $s \in S$.
3. Every element of $S^{-1}A$ is of the form $f(a)f(s)^{-1}$ for some $a \in A$ and some $s \in S$.

5.2 Localization

$S = A - \mathfrak{p}$, is a multiplicative subset of $A \Leftrightarrow \mathfrak{p}$ is prime. In this case write $A_{\mathfrak{p}}$ instead of $S^{-1}A$. **The process of passing from A to $A_{\mathfrak{p}}$ is called localisation at $\mathfrak{p}$.**

1. **$A_{\mathfrak{p}}$** is a local ring. To see this we construct an ideal $\mathfrak{m}$ in $A_{\mathfrak{p}}$ as the set of elements $a/s \in A_{\mathfrak{p}}$ such that $a \in \mathfrak{p}$. $\mathfrak{m}$ has the property that if $b/s \notin \mathfrak{m}$ then $b \notin \mathfrak{p}$ and hence $b \in S$ which makes b/s a unit. So if $A_{\mathfrak{p}}$ contains an ideal not in $\mathfrak{m}$ then it must contain a unit and so it's the whole ring. Therefore $\mathfrak{m}$ is the only maximal ideal of $A_{\mathfrak{p}}$.

2. $S^{-1}A$ is the zero ring if and only if $0 \in S$.

3. Let $f \in A$ and let $S = \{f^n\}_{n \geq 0}$. We denote by A_f, the ring of fractions $S^{-1}A$.

4. **Example:** If $\mathfrak{a}$ is an ideal in A then $S = 1 + \mathfrak{a}$ is multiplicatively closed.

5. Local ring of k^n along the variety V: Let $A = k[t_1, \cdots, k_n]$ where k is a field. $A_{\mathfrak{p}}$ is the set of all rational functions f/g where $g \notin \mathfrak{p}$. If $V = V(\mathfrak{p})$ is a vareity then $A_{\mathfrak{p}}$ can be identified with the ring of all rational function on k^n that are defined at almost all points in V.

5.3 Ring of fraction of Modules

$S^{-1}M$ is an $S^{-1}A$-module when M is an A-module. When $S = A - \mathfrak{p}$ we denote the ring of fractions by $M_{\mathfrak{p}}$.

If $u : M \to N$ is an A-module homomorphism. Then it gives rise to an $S^{-1}A$-module homomorphism $S^{-1}u : S^{-1}M \to S^{-1}N$, namely $S^{-1}u$ maps m/s to $u(m)/s$. We have $S^{-1}(v \circ u) = (S^{-1}v) \circ (S^{-1}u)$

Proposition 5.3.1. *S^{-1} is exact. If $M' \to_f M \to_g M''$ is exact at M then $S^{-1}M \to_{S^{-1}f} S^{-1}M \to_{S^{-1}g} S^{-1}M''$ is exact at $S^{-1}M$.*

If M' is a submodule of M then $S^{-1}M' \to S^{-1}M$ is injective and $S^{-1}M'$ is a submodule of $S^{-1}M$.

Corollary 5.3.1.1. *Formation of fractions commutes with formation of finite sums, finite intersections and quotients. Precisely, if N,P are submodules of an A-module M, then,*

1. $S^{-1}(N + P) = S^{-1}(N) + S^{-1}(P)$
2. $S^{-1}(N \cap P) = S^{-1}N \cap S^{-1}P$
3. $S^{-1}(M/N)$ *and* $S^{-1}M/S^{-1}N$ *are isomorphic.*

Proposition 5.3.2. *Let M be an A-module. Then $S^{-1}M$ is isomorphic to $S^{-1}A \otimes_A M$. That is there exists a unique isomorphism $f : S^{-1}A \otimes_A M \to S^{-1}M$ given by,*

$$f((a/s) \otimes m) = am/s \text{ for all } a \in A, m \in M \text{ and } s \in S$$

Corollary 5.3.2.1. *$S^{-1}A$ is a flat A-module.*

Proposition 5.3.3. *If M,N are A-modules then there is a unique isomorphism, $f : S^{-1}M \otimes_{S^{-1}A} S^{-1}N \to S^{-1}(M \otimes_A N)$ given by,*

$$f((m/s) \otimes (n/t)) = (m \otimes n)/st$$

If $\mathfrak{p}$ is any prime ideal then

$$M_{\mathfrak{p}} \otimes_{A_{\mathfrak{p}}} N_{\mathfrak{p}} \cong (M \otimes_A N)_{\mathfrak{p}}$$

5.4 Some examples of local properties

Proposition 5.4.1. *Let M be an A-moduel. Then the following are equivalent:*

1. *$M = 0$*
2. *$M_{\mathfrak{p}} = 0$ for all prime ideals $\mathfrak{p}$ of A*
3. *$M_{\mathfrak{m}} = 0$ for all maximal ideals $\mathfrak{m}$ of A*

Proposition 5.4.2. *Let $\phi : M \to N$ be an A-module homomorphism. Then the following are equivalent:*

1. *ϕ is injective*
2. *$\phi_{\mathfrak{p}} : M_{\mathfrak{p}} \to N_{\mathfrak{p}}$ is injective for each prime ideal $\mathfrak{p}$.*
3. *$\phi_{\mathfrak{m}} : M_{\mathfrak{m}} \to N_{\mathfrak{m}}$ is injective for each maximal ideal $\mathfrak{m}$. Similarily with "injective" replaced by "surjective" throughout.*

Note: Flatness is a local property.

Proposition 5.4.3. *For any A-module M, the following statements are equivalent:*

1. *M is a flat A-module*
2. *$M_{\mathfrak{p}}$ is a flat $A_{\mathfrak{p}}$-module for each prime ideal $\mathfrak{p}$.*
3. *$M_{\mathfrak{m}}$ is a flat $A_{\mathfrak{m}}$-module for each maximal ideal $\mathfrak{m}$*

Chapter 6

Integral Dependence

Bibliography