Propagator and quasi-energy

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First, we will review the one-dimensional lattice structure and its energy bands. The Schrödinger equation is:

$$\hat{H}\phi_q^{(b)}(x) = E_q^{(b)}\phi_q^{(b)}(x) \quad \text{with} \quad \hat{H} = \frac{\hat{p}^2}{2M} + V_L(x) \tag{1}$$

where $V_L(x) = V_0 \cos^2(k_L x)$ is the lattice potential. According to Bloch's theorem, the eigenstates of this Hamiltonian could be written as:

$$\phi_q^{(b)}(x) = e^{iqx/\hbar} u_q^{(b)}(x) \tag{2}$$

And, we can use discrete Fourier sums to overwrite the potential and periodic function $u_q^{(b)}(x)$:

$$V_L(x) = \sum_m V_m e^{2ik_L m x} \quad \text{and} \quad u_q^{(b)}(x) = \sum_n c_n^{(b,q)} e^{2ik_L n x} \tag{3}$$

In this case, the lattice potential is given by

$$V_L(x) = \frac{1}{4} V_0 \left(e^{2ik_L x} + e^{-2ik_L x} + 2\right) \tag{4}$$

Inserting the above Fourier sums into Schrödinger equation, we can obtain:

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dk^2} + V_L(x)\right] \sum_n C_n^{(b,q)} e^{i(q/\hbar + 2k_L n)x} = E_q^{(b)} \sum_n C_n^{(b,q)} e^{i(q/\hbar + 2k_L n)x} \tag{5}$$

then

$$\begin{aligned} \left[\frac{\hbar^2}{2m}(q/\hbar + 2k_L n)^2 + \frac{1}{4} V_0 \left(e^{2ik_L x} + e^{-2ik_L x} + 2\right)\right] \sum_n C_n^{(b,q)} e^{i(q/\hbar + 2k_L n)x} \\ = E_q^{(b)} \sum_n C_n^{(b,q)} e^{i(q/\hbar + 2k_L n)x} \end{aligned} \tag{6}$$

We can express this equation in matrix form as:

$$\sum_{n'} H_{n,n'} c_{n'}^{(b,q)} = E_q^{(b)} c_n^{(b,q)} \quad \text{with} \quad H_{n,n'} = \begin{cases} \left(2n + \frac{q}{\hbar k_L}\right)^2 E_r + V_0/2 & \text{if } |n - n'| = 0 \\ V_0/4 & \text{if } |n - n'| = 1 \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

where,

$$E_r = \frac{\hbar^2 k_L^2}{2M} \tag{8}$$

The eigenenergies $E_q^{(b)}$ and eigenvectors $C_n^{(b,q)}$ can be determined by numerically diagonalizing the Hamiltonian $H_{n,n'}$.

Second, in a time dependent system, we assume the Hamiltonian has the form:

$$H(t+T) = H(t) \tag{9}$$

The period is T . Now, the system has time translation symmetry.Schrödinger equation can be

$$(H - i\partial_t)\,|\psi\rangle = 0 \tag{10}$$

From Floquet theory, the eigensolutions can be written

$$\psi(t) = e^{-i\epsilon t}u(t) \tag{11}$$

where,

$$u(t+T) = u(t) \tag{12}$$

It is a time periodic function, meets the eigenfunction

$$(H - i\partial_t)\,|u(t)\rangle = \epsilon|u(t)\rangle \tag{13}$$

Eigenstates $|\psi(t)\rangle$ is the eigensolution of time evolution operator

$$U(T+t,t)|\psi(t)\rangle = |\psi(t+T)\rangle = e^{-i\mathcal{E}T}|\psi(t)\rangle \tag{14}$$

where time evolution operator is

$$U(T+t,t) \equiv \hat{\mathrm{T}}\exp\left[-i\int_t^{T+t} H\left(t'\right)dt'\right] \tag{15}$$

Similar to the quasi-momentum in the space periodic potential, ϵ is called quasi-energy here.

Quasi-energy of time-periodic system can be obtained by diagonalization propagator $U(t,0)$. Schrödinger equation is

$$i\partial_t|\psi(t)\rangle = H|\psi(t)\rangle \tag{16}$$

and we can get

$$i\frac{|\psi(t+dt)\rangle - |\psi(t-dt)\rangle}{2\cdot dt} = H|\psi(t)\rangle \tag{17}$$

then

$$|\psi(t+dt)\rangle = |\psi(t-dt)\rangle + 2\cdot i\cdot dt\cdot H|\psi(t)\rangle \tag{18}$$

From initial state to time T, $|\psi\rangle$ derived from $|\psi(0)\rangle$ to $|\psi(t)\rangle$

$$|\psi(t)\rangle = e^{-i\epsilon T}|\psi(0)\rangle \tag{19}$$

also

$$|\psi(t)\rangle = U(0+T,0)|\psi(0)\rangle \tag{20}$$

where $U(t)$ is the time evolution operator. So, we can figure out the quasi-energy by the above theory.

The following is concrete calculation process. The Hamiltonian is

$$\hat{H}(t) = \frac{\hat{p}^2}{2M} + V_0 \cos^2(k_L x) - F_0 \sin(\omega t) \tag{21}$$

We can express this equation in matrix form as

$$H_{n,n'} = \begin{cases} \left(2n + \frac{q}{\hbar k_L}\right)^2 E_r + V_0/2 - F_0 \sin(\omega t) & \text{if } |n-n'| = 0 \\ V_0/4 & \text{if } |n-n'| = 1 \\ 0 & \text{otherwise.} \end{cases} \tag{22}$$

Then, use the Eq(18) to derived the evolution operator.

We can rewrite the Hamiltonian of Eq(21) by using identity $\sin(\omega t) = 1/(2i)\left(e^{i\omega t} - e^{-i\omega t}\right)$ as following:

$$\hat{H}(t) = \sum_{m=-1}^{1} \hat{H}_m e^{im\omega t} \tag{23}$$