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COSMICAL ELECTRODYNAMICS

FUNDAMENTAL PRINCIPLES

BY

HANNES ALFVÉN

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PREFACE TO THE SECOND EDITION

THE purpose of the first edition of *Cosmical Electrodynamics* was to draw attention to a field of research in an early state of its development. The first four chapters of the book were devoted to the fundamental principles of plasma physics and magneto-hydrodynamics whereas the last three chapters treated applications to (what is now called) the magnetosphere and to interplanetary space, to solar physics, and to cosmic radiation. (The application to the cosmogonic problem was later treated in a separate book.)

During the fifteen years which have elapsed since the first edition was written, the subject has been developed by two of the largest research efforts of our time: *thermonuclear research* has increased our knowledge of magneto-hydrodynamics and plasmas, and a major part of *space research* has been devoted to the exploration of the magneto-hydrodynamic conditions in space around the earth.

This has put the present authors in a difficult position with reference to a second edition. In order not to make the present book too thick we have confined ourselves to the fundamental principles which were covered by the first four chapters of the first edition. It is hoped that the applications will be covered by following books.

One of the most important but also most difficult tasks of both thermonuclear research and the exploration of the magneto-hydrodynamics of space has been to establish a good contact between theory on one side and experiments or observations on the other. Therefore in this book the emphasis has been put on the physical more than on the mathematical aspect. The exactness of the derivation of a formula has often been sacrificed if this has been thought necessary in order to make the reading easier for scientists who concentrate on experiments or observations.

Like the first edition the present book has been written as result of innumerable discussions with Professor Nicolai Herlofson. We also want to thank several members of the staff of this institute, and Dr. C. T. Chang, Risø, Denmark, for advice and criticism.

H. A.
C.-G. F.

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1

GENERAL SURVEY

1.1. Introduction

In cosmical physics electromagnetic processes have recently attracted a rapidly increasing interest, and it is now generally realized that they are of fundamental importance for many different phenomena. In *the interior of the earth* there exist electromagnetic processes by which the earth's general magnetic field is generated. In the *ionosphere* electric currents change the earth's magnetic field, especially during *magnetic storms*, and also produce luminous phenomena, *aurorae*, in certain regions around the geomagnetic poles. These phenomena are connected with other electromagnetic phenomena at greater height. The earth's magnetic field controls the conditions within a region out to about ten times the earth's radius. In this region, sometimes referred to as the *magnetosphere*, a complicated and rapidly varying system of currents exists as found by space-research measurements. In certain regions (the radiation belts or Van Allen belts) there is also a flux of high-energy charged particles, which are trapped in the magnetic field.

The conditions in the ionosphere and the magnetosphere of the earth are influenced by the electromagnetic state in *interplanetary space*, which in turn is affected by the sun. There are a number of electro magnetic phenomena at the sun, which are referred to as 'solar activity' (sunspots, prominences, solar flares, etc.). Also in the *deeper layers of the sun* electromagnetic phenomena must be of importance, as is indicated by the existence of a general solar magnetic field. In other stars electromagnetic phenomena are also of importance, most conspicuously in the *magnetic variable stars*, where fields of up to more than 30 kgauss have been observed. Also in *interstellar space*, and in *intergalactic space*, electromagnetic phenomena may be of decisive importance, although not very much is yet known with certainty.

If we want to understand all these phenomena and make physical theories about them we must first realize that our present knowledge of physics is mainly based on experience gained in the laboratory When we try to apply to cosmic phenomena the laws in which this experience is condensed, we make an enormous extrapolation, the legitimacy of which can be checked only by comparing the theoretical results with observations. Classical mechanics was once extrapolated into the realm of astronomy so successfully that only the most refined observations of the last decades have revealed phenomena for which it does not hold. The application of atomic theory, especially spectroscopy, to cosmic phenomena has proved equally successful. In fact, classical mechanics and spectroscopy have been two invaluable tools in exploring the universe around us.

When *classical electrodynamics* had been developed during the nineteenth century, it could be directly applied only to some special problems in cosmical physics. A more general application was not possible until classical electrodynamics had been combined with *hydrodynamics* to form *magneto-hydrodynamics*, which further must be combined with *plasma physics* in order to make possible a deeper understanding of electromagnetic phenomena in cosmical physics.

At an early stage of the study of electromagnetism in cosmical physics attention was drawn to some phenomena, e.g. the earth's permanent magnetic field and the generation of cosmic radiation, which seemed so mysterious that some authors believed that new natural laws must be invoked in order to explain them. Further study has given no support for this view, and we have now at least qualitatively plausible theories for these phenomena. There is at present little reason to doubt that all common physical laws hold up to lengths of the order of the 'radius of the universe' and times of the order of the 'age of the universe', limits given by the theory of general relativity.

The basic reason why electromagnetic phenomena are so important in cosmical physics is that there exist celestial magnetic fields which affect the motion of charged particles in space. Under certain conditions electromagnetic forces are much stronger than gravitation. In order to illustrate this, let us suppose that a particle moves at the earth's solar distance $R_♁$ (the position vector being $\mathbf{R}_♁$) with the earth's orbital velocity $\mathbf{v}$. If the particle is a neutral hydrogen atom, it is acted upon only by the solar gravitation (the effect of a magnetic field upon a possible atomic magnetic moment being negligible). If M is the solar and m the atomic mass, and γ is the constant of gravitation, this force is

$$\mathbf{f} = -\gamma M m \mathbf{R}_♁ / R_♁^3.$$

If the atom becomes singly ionized, the ion as well as the electron (charge $e = \pm 4{\cdot}8 \times 10^{-10}$ e.s.u.) is subject to the force

$$\mathbf{f}_m = e(\mathbf{v}/c) \times \mathbf{B}$$

from an interplanetary magnetic field which near the earth's orbit is **B**. The strength of the interplanetary magnetic field is of the order of 10^{-4} gauss, which gives

$$f_m/f \approx 10^7.$$

This illustrates the enormous importance of interplanetary and interstellar magnetic fields, compared to gravitation, as long as the matter is ionized.

On the other hand, as $\mathbf{f}_m$ is oppositely directed for electrons and ions, the forces on electrons and ions may in many cases cancel each other. For example, if we consider an ionized cloud, containing the same number of electrons and ions, the resulting magnetic force on the cloud becomes zero to a first approximation. (Second-order effects, e.g. due to the inhomogeneity of the magnetic field, may still be important.) The motion of the ionized cloud in the magnetic field produces a separation of the ions and electrons, but the resulting polarization causes an electric field which limits the separation. Under certain conditions the electric field may produce currents in adjacent conductors so that very complicated phenomena occur.

The examples above demonstrate, on one hand, the importance of electromagnetic forces in cosmic physics, and on the other the complexity of the electromagnetic phenomena. The rest of the present chapter is devoted to discussions of the magnetic and electric fields in cosmic physics. In Chapter 2 we shall treat the motion of a single particle in such fields. A coupling between electromagnetic fields and motion of electrically conducting matter is produced very generally under cosmical conditions (cf. also Dungey, 1958). This coupling gives rise to the complicated phenomena of magneto-hydrodynamics, which will be treated in Chapter 3. When the electrically conducting material consists of ionized gas (plasma), the thermal motion of the individual particles produces several important phenomena (diamagnetism, ambipolar diffusion, etc.). Such phenomena fall outside the scope of magneto-hydrodynamics taken in a restricted sense and are discussed in the chapters on plasma physics (Chapters 4 and 5).

Electromagnetic phenomena are likely to have been of decisive importance at the formation of the solar system. A treatment of this problem has been given in a separate monograph (Alfvén, 1954) and in recent articles (Alfvén, 1962; Alfvén and Wilcox, 1962).

1.2. Magnetic fields in cosmic physics

Since the geomagnetic field resembles the field from a magnetic dipole, we shall state here some of the properties of a *dipole field* for later reference. Let (r, θ, φ) be the coordinates of a spherical coordinate system centred in the dipole and having its axis parallel to the magnetic moment $\mathbf{a}$, Fig. 1.1. We also introduce the latitude $\lambda = \frac{1}{2}\pi - \theta$. The magnetic field is then given by

$$\mathbf{B} = -\operatorname{grad}\psi, \tag{1}$$

where

$$\psi = \frac{\mathbf{ar}}{r^3} = \frac{a \sin\lambda}{r^2}. \tag{2}$$

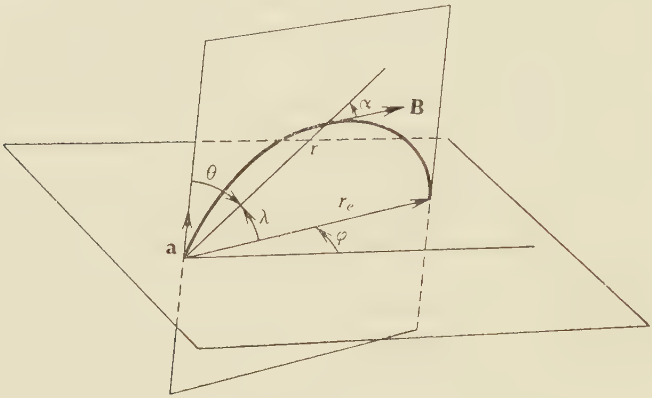


FIG. 1.1. Magnetic line of force from a dipole **a**.

The components are:

$$B_r = B_p \sin\lambda, \tag{3}$$

$$B_\lambda = -\tfrac{1}{2}B_p \cos\lambda, \tag{4}$$

$$B_\varphi = 0,$$

and the total field strength:

$$B = \sqrt{(B_r^2 + B_\lambda^2 + B_\varphi^2)} = \tfrac{1}{2}B_p \phi = a\phi/r^3, \tag{5}$$

where

$$B_p = 2a/r^3 \tag{6}$$

and

$$\phi = \sqrt{(1 + 3\sin^2\lambda)}. \tag{7}$$

B_r represents the 'vertical' and B_λ the 'horizontal' component of the field.† A magnetic line of force has the equation

$$r = r_e \cos^2\lambda, \tag{8}$$

$$\varphi = \text{const},$$

where r_e is the distance from the origin to the point of intersection with the equatorial plane ($\lambda = 0$). The angle α between the line of force and

† In the geomagnetic field the 'vertical component' is counted positive if directed downwards.

the radius vector is given by

$$\tan\alpha = \tfrac{1}{2}\cot\lambda, \tag{9}$$

or

$$\sin\alpha = \frac{\cos\lambda}{\phi}, \tag{10}$$

$$\cos\alpha = \frac{2\sin\lambda}{\phi}. \tag{11}$$

The 'inclination' of the field is $\frac{1}{2}\pi-\alpha$.

The total strength of the field along a given line of force can also be written

$$B = \frac{a}{r^3}\phi = \frac{a}{r_e^3}(\cos\lambda)^{-6}\phi = \frac{a}{r_e^3}\eta, \tag{12}$$

where

$$\eta = \frac{\sqrt{(1+3\sin^2\lambda)}}{\cos^6\lambda}. \tag{13}$$

In a Cartesian coordinate system (x, y, z) we have

$$B_x = 3xz\frac{a}{r^5}, \tag{14}$$

$$B_y = 3yz\frac{a}{r^5}, \tag{15}$$

$$B_z = (3z^2-r^2)\frac{a}{r^5}, \tag{16}$$

where

$$r^2 = x^2+y^2+z^2.$$

The *geomagnetic field* is in the first approximation a dipole field. If a dipole situated at the centre of the earth is fitted to the geomagnetic data, the dipole moment becomes $a = 8{\cdot}1\times10^{25}$ gauss cm^3, which corresponds to a polar field strength of 0·62 gauss. The axis of this '*centric*' *dipole* cuts the earth's surface at the antipodal points 78·3° N. 69·0° W. and 78·3° S. 111·0° E. (Chapman and Bartels, 1940, pp. 645 and 648).

The magnetic dipole that gives the best approximation to the geomagnetic field is *eccentric*. It is displaced 342 km from the earth's centre towards the point 6·5° N. 161·8° E., and its axis intersects the earth's surface at two points, 76·3° S. 121·2° E., and 80·1° N. 82·7° W. (Chapman and Bartels, 1940, pp. 651–2). More recent calculations indicate a motion of the eccentric dipole away from the centre of the earth (see Singer, 1958, p. 239).

For more accurate approximation to the geomagnetic field a spherical harmonic expansion can be used (see Chapman and Bartels, 1940, pp. 639 ff.). This method was originally employed by Gauss (1838). Gauss's analysis showed that apart from 'disturbances' which at the surface seldom exceed 1 per cent, the main geomagnetic field is due to sources within the earth.

If external sources are neglected, the field can be written in the form

$$\mathbf{B} = -\operatorname{grad}\psi, \tag{17}$$

where

$$\psi = R_e \sum_{n=1}^{n=\infty} \left(\frac{R_e}{r}\right)^{n+1} T_n, \tag{18}$$

and

$$T_n = \sum_{m=0}^{m=n} (g_n^m \cos m\varphi + h_n^m \sin m\varphi) P_n^m(\theta). \tag{19}$$

In these expressions r, θ, and φ are defined according to Fig. 1.1. R_e is the mean radius of the earth and $P_n^m(\theta)$ are, essentially, the associated Legendre functions (see Chapman and Bartels, 1940, p. 609). The coefficients g_n^m and h_n^m determine the strengths and orientations of the multipoles. Since the main magnetic field of the earth is subject to a slow (secular) variation, the values of the coefficients change slightly with time. Analyses have been made for the field of epoch 1945 (Vestine *et al.*, 1947; Hultqvist, 1958 *a*, *b*) and for that of epoch 1955 (Finch and Leaton, 1957). Mapping of the geomagnetic field is now being made by means of magnetometers carried in artificial satellites.

Currents in the ionosphere and in space around the earth modify the magnetic field, producing rapid changes ('magnetic storms') and also slow, semi-permanent changes. The changes in the surface field produced by such currents seldom exceed 1 per cent of the field strength, but in the outer parts of the magnetosphere the relative changes may be much larger.

Magnetic fields may exist on other *planets*. The radio emission from Jupiter (Radhakrishnan and Roberts, 1960; and others) indicates that this planet has a magnetic field. Recent space-probe measurements down to a distance of 41 000 km from the centre of Venus did not indicate any magnetic field from Venus (Smith *et al.*, 1963).

According to results from the space probe Lunik 2 the *moon* has no magnetic field in excess of 10^{-3} gauss.

Solar magnetic fields were first discovered in sunspots (Hale, 1908). A decade later a general field was observed (Hale *et al.*, 1918). Since then the observational technique has been very much refined, and extensive measurements of the photospheric Zeeman effect have been made (for a survey see H. W. Babcock and H. D. Babcock, 1955).

Three types of field are found in the photosphere: (1) the general field, which is observable at high heliographic latitudes; (2) bipolar magnetic regions, which, when the field strengths are large, are associated with spots, flares, coronal streamers, etc.; (3) unipolar magnetic regions occurring at low latitudes. The bipolar fields can be very strong (up to 4000 gauss). If the measurements are interpreted in a straightforward way, the general field and the fields of unipolar regions have intensities of the order of 1 gauss, and the general field, which before 1957 was antiparallel to the geomagnetic field, has changed its polarity, so that since 1958 the two fields are parallel (H. D. Babcock, 1959). However, the interpretation of Zeeman-effect measurements of weak magnetic fields in the strongly magneto-turbulent photosphere is questionable (see Alfvén and Lehnert, 1956). It is also possible that most of the flux through the solar surface is bunched to filaments too narrow to be observed with the present technique, and that the observations apply essentially to the interfilamentary medium (see § 5.5.3). It seems fair to state that our knowledge about the sun's general magnetic field is still not very certain.

The magnetic field inside the sun is not accessible to direct observation. The progression of the sunspot zone, if interpreted according to the magneto-hydrodynamic theory of sunspots (Alfvén, 1943), indicates the existence of a general magnetic field with a magnetic moment $a = 2{\cdot}1 \times 10^{33}$ gauss cm^3 in the region between the photosphere and the centre. An independent check of this result is very desirable.

The ray structure of the corona is generally interpreted as due to the solar magnetic field. This extends far out from the surface of the sun but seems not to approach a dipole field at great distances. The reason for this is that currents in interplanetary space are not negligible. Such currents produce an *interplanetary magnetic field*. Due to the motion of the medium this magnetic field also produces an *interplanetary electric field*. The interplanetary electromagnetic conditions control the phenomena near the earth and are responsible for the production of magnetic storms and aurorae. They also produce intensity variations in cosmic radiation, and may play a role for the generation of cosmic radiation.

The first *stellar magnetic field* was observed by H. W. Babcock in 1947 (in the star 78 Virginis). Since then a great number of such observations have been made, and magnetic fields now seem to be a general characteristic of a certain category of stars, the rapidly rotating A-stars with a convective core (H. W. Babcock, 1958). The stellar fields are in most cases variable and in some cases very strong. Thus in the star HD

215441 the field strength reaches about 34000 gauss (H. W. Babcock, 1960). The discovery of the magnetic variable stars has opened a new field of astrophysics, and the theoretical study of this field is still in an early stage.

The idea of a *galactic magnetic field* was first introduced to explain the isotropy of cosmic radiation (Alfvén, 1937; Pikelner, 1953) and has later been invoked in theories of cosmic-ray acceleration (Fermi, 1954, cf. § 2.7) and in theories of spiral arms (Chandrasekhar and Fermi, 1953; Elvius and Herlofson, 1960). Evidence of weak interstellar fields was provided by the discovery of the polarization of light from distant stars (Hiltner, 1949; Hall, 1949). The polarization may be caused by interstellar grains which have been aligned by a magnetic field, although there may also be other possible ways of explaining the polarization (Spitzer and Tukey, 1951; Davis and Greenstein, 1951; Gold, 1952; Henry, 1958). Part of the cosmical noise observed by radio telescopes may consist of synchrotron radiation of relativistic electrons moving in large-scale magnetic fields (Alfvén and Herlofson, 1950; Shklovsky, 1952, 1953*a*, *b*, 1960, cf. § 2.8). Thus the observed corona of radio emission enveloping the galaxy provides an indication of extensive weak magnetic fields not only in the galactic plane but also at great distances away from it (Baldwin, 1955; Burbidge, 1956). However, there are also other mechanisms by which the thin interstellar plasma may emit radio noise (see, for example, the monograph by Shklovsky, 1960).

All estimates of the strength of galactic fields are as yet very uncertain. The value 10^{-6}–10^{-5} gauss is often quoted as a probable order of magnitude. One reason for this value is that it gives a magnetic energy density $B^2/8\pi = 4\times10^{-14}$–$4\times10^{-12}$ erg/cm³ which is of the same order of magnitude as the kinetic energy density $\frac{1}{2}\rho v^2$ of the interstellar medium, assuming $\rho \approx 10^{-24}$ g/cm³ and $v = 3\times10^5$–3×10^6 cm/sec (Chandrasekhar and Fermi, 1953; see also Spitzer, 1954). Direct observational results neither confirm nor contradict this estimate (R. D. Davies *et al.*, 1960).

1.3. Induced electric fields

In the presence of a magnetic field an electric field is defined only in relation to a certain coordinate system. If in a system 'at rest' the electric and magnetic fields are $\mathbf{E}$ and $\mathbf{B}$, we can calculate by means of relativistic transformation formulae the fields $\mathbf{E}'$, $\mathbf{B}'$ in a system which moves in relation to the first with the velocity $\mathbf{v}$. The components parallel to $\mathbf{v}$ remain unchanged, but the components perpendicular to

v are transformed in the following way:

$$\mathbf{E}' = \frac{\mathbf{E}+(\mathbf{v}/c)\times\mathbf{B}}{\sqrt{(1-v^2/c^2)}}, \tag{1}$$

$$\mathbf{B}' = \frac{\mathbf{B}-(\mathbf{v}/c)\times\mathbf{E}}{\sqrt{(1-v^2/c^2)}}. \tag{2}$$

($\mathbf{D} = \epsilon\mathbf{E}$, $\mathbf{B} = \mu\mathbf{H}$; reduced in a vacuum to $\mathbf{D} = \mathbf{E}$, $\mathbf{B} = \mathbf{H}$.)

The astronomical velocities are much smaller than the velocity of light (c). Electrostatic fields are of little importance except in special regions (cf. §§ 4.2.3 and 5.1.3). Hence the electric fields are usually secondary to the magnetic fields, which, according to (1), means that the electric fields are much weaker than the magnetic fields. Consequently in cosmic physics we can usually to a good approximation write

$$\mathbf{E}' = \mathbf{E}+(\mathbf{v}/c)\times\mathbf{B}, \tag{3}$$

$$\mathbf{B}' = \mathbf{B} \tag{4}$$

(where also the components parallel to **v** are included in the vectors).

Thus magnetic fields are independent of the choice of coordinate system; there is no difference between the magnetic field in a moving system and the magnetic field in a system 'at rest'. On the other hand, electric fields depend on the coordinate system from which they are measured. *To speak of an electric field without defining exactly the coordinate system to which it refers is meaningless.*

1.3.1. *Induced electric field in uniformly moving matter*

Since cosmical clouds of ionized gas (plasma, cf. Chapters 4 and 5) are generally magnetized, their motion produces induced electric fields according to the formula (3). These induced electric fields are of great importance in many astrophysical problems. For example the motion of the magnetized interplanetary plasma produces electric fields that are essential for the production of aurora and magnetic storms (cf. § 2.6), certain cosmic-ray variations, etc. In this section we shall illustrate the basic features of electric fields associated with such motion by a simple example, namely a plane, solid, infinitely conductive slab which is penetrated by a magnetic field and moves parallel to itself.

Let x, y, z (Fig. 1.2) be the axes of a fixed Cartesian coordinate system S (with unit vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, $\hat{\mathbf{z}}$), in which the slab occupies the volume

$$|x| < a$$

and moves with the velocity

$$\mathbf{v} = v_0\,\hat{\mathbf{y}}, \tag{5}$$

where

$$v_0 \ll c.$$

Let another coordinate system S' move with the slab and have its coordinate axes x', y', z' and unit vectors $\hat{\mathbf{x}}'$, $\hat{\mathbf{y}}'$, $\hat{\mathbf{z}}'$ parallel to those of S.

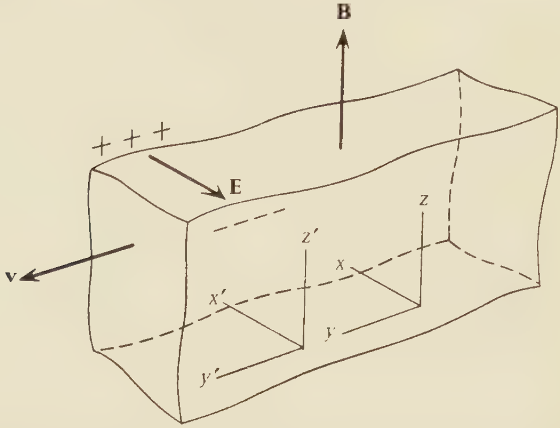


FIG. 1.2. Electromagnetic induction in a slab of electrically conducting matter moving in a magnetic field.

Denote by $\mathbf{E}$, $\mathbf{B}$, etc., quantities measured in S and by $\mathbf{E}'$, $\mathbf{B}'$, etc., the corresponding quantities measured in S'. A magnetic field

$$\mathbf{B} = \begin{cases} B_0 \hat{\mathbf{z}} & (|x| < a) \\ 0 & (|x| > a) \end{cases} \tag{6}$$

is assumed to exist. It is associated with the surface currents

$$\mathbf{i}_s = \begin{cases} -\dfrac{cB_0}{4\pi}\hat{\mathbf{y}} & (x = -a) \\[2ex] \dfrac{cB_0}{4\pi}\hat{\mathbf{y}} & (x = a). \end{cases} \tag{7}$$

The slab is assumed to have infinite conductivity, and therefore the electric field $\mathbf{E}'$ in the coordinate system S' following the slab must be zero:

$$\mathbf{E}' = \mathbf{E} + (\mathbf{v}/c) \times \mathbf{B} = 0. \tag{8}$$

Hence

$$\mathbf{E} = -(\mathbf{v}/c) \times \mathbf{B} = -\frac{v_0 B_0}{c}\hat{\mathbf{x}} \quad (|x| < a). \tag{9}$$

The electric lines of force end in surface charges of density

$$q_s = \begin{cases} -\dfrac{v_0 B_0}{4\pi c} & (x = -a), \\[2ex] \dfrac{v_0 B_0}{4\pi c} & (x = a), \end{cases} \tag{10}$$

and outside the slab there is no electric field:

$$\mathbf{E} = 0 \quad (|x| > a). \tag{11}$$

In the moving system S' the magnetic field is the same as in S, whereas (8), (3), and (11) give for the electric field in S':

$$\mathbf{E}' = 0 \quad (-\infty < x < \infty). \tag{12}$$

As the magnetic field is the same in both coordinate systems, the current densities i' and i must also be equal. On the other hand $\mathbf{E}'$ differs from $\mathbf{E}$, and from (12) it follows that the electric charge density vanishes identically in S'. The fact that the charge densities are different in the two coordinate systems is a relativistic effect. It follows directly from the relativistic transformation formulae that

$$\mathbf{i}'_s = (\mathbf{i}_s - \mathbf{v}q_s)/\sqrt{(1-v^2/c^2)}, \tag{13}$$

$$q'_s = (q_s - \mathbf{vi}/c^2)/\sqrt{(1-v^2/c^2)} \tag{14}$$

(see, for example, Becker and Sauter, 1957, p. 264), which in our case ($v \ll c$, $E \ll B$) reduce to

$$\mathbf{i}'_s = \mathbf{i}_s, \tag{15}$$

$$q'_s = q_s - (\mathbf{vi})/c^2. \tag{16}$$

The physical explanation of the difference in net charge density is that the presence of an electric current implies a relative motion between the positive and the negative particles. Owing to the relativistic contraction the density of the positive particles depends on the coordinate system from which it is seen, and the same is true for the negative particles. In our case the densities are equal when seen from the system $x'y'z'$, but they differ in the xyz system, and this difference is enough to cause a considerable surface charge. (Cf. § 5.8.1.)

Further, it should be noted that the magnetic field is almost the same in the moving system and in the system at rest. This means that it is not important how the sources of the magnetic field are located. We may speak of a 'frozen-in' magnetic field which is 'carried' by the plasma, but such a field cannot be distinguished from a field which derives from sources in the system at rest. (In non-stationary cases these sources must of course be time-variable.)

1.3.2. *Induced electric fields in rotating bodies*

One of the consequences of the formulae (3) and (4) is that the rotation of a conductor in a magnetic field produces an electric field in the system at rest. This phenomenon is well known from laboratory experiments and is usually called 'homopolar' or 'unipolar' induction.

It was first studied by Faraday, and attracted much interest during the last century because it was thought that by investigating this subject it should be possible to ascertain whether the magnetic lines of force from a rotating magnet rotate with the magnet or not. The result was that it has no meaning to speak about the 'motion' of a magnetic field. It is irrelevant whether the sources from which the field derives move or not.

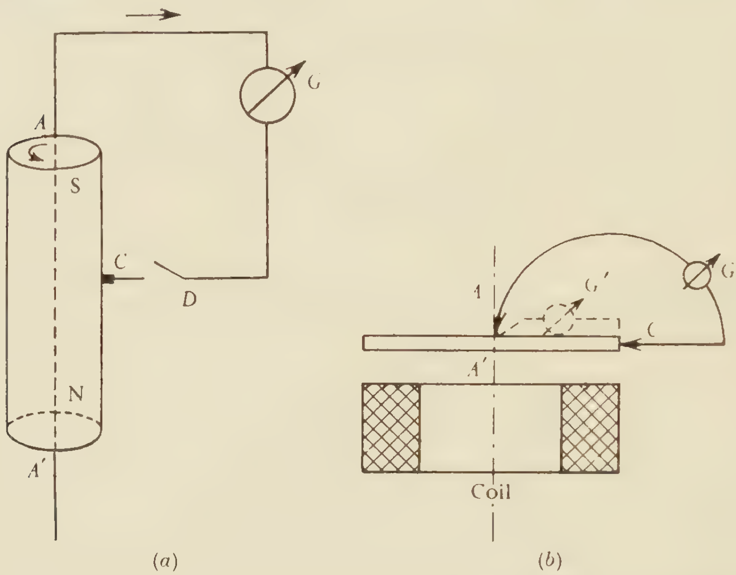


FIG. 1.3. Unipolar inductors. (*a*) When a bar-magnet NS is rotated around its axis AA', a current is obtained in a fixed circuit connecting the axis with a sliding contact C. (*b*) Unipolar inductor consisting of a rotating copper disk situated in a magnetic field from a coil.

A simple unipolar inductor is obtained by rotating a cylindrical bar-magnet NS around its axis AA' (see Fig. 1.3 (*a*)). A fixed wire $AGDC$ connects the axis with a sliding contact C at the middle of the bar. If the switch D is closed the galvanometer G indicates a current as soon as the magnet rotates. It is not necessary that the rotating body should be a permanent magnet. Any conductor will do, if only a magnetic field is established in some way, for example as in the arrangement of Fig. 1.3 (*b*).

In cosmical bodies the magnetic field generally derives from currents in the body itself. We shall briefly discuss the electric fields induced by rotation in such a body considering the simple example of a rigidly rotating, electrically conducting sphere (radius R) surrounded by vacuum. We shall limit the treatment to the case where the rotational velocity $v = \omega r$ is much smaller than c so that v^2/c^2 can be neglected compared to unity. For the general form of the electromagnetic equations in rotating coordinate systems see Möller, 1952, p. 302.

Let S be a *non-rotating coordinate system* with its origin at the centre of the sphere and $\mathbf{r}$ the position vector of an arbitrary point. When the sphere is set into motion with an angular frequency $\boldsymbol{\omega}$, an electron situated at a point $\mathbf{r}$ in the sphere moves with a velocity given by the vector

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r} \tag{17}$$

and experiences a Lorentz force

$$\mathbf{F} = -e\frac{\boldsymbol{\omega}\times\mathbf{r}}{c}\times\mathbf{B}.$$

This force tends to displace the electrons in the conductor until the resulting charge separation has established a balancing electric field, so that the resultant force on each electron vanishes. The balancing electric field, inside the sphere, is

$$\mathbf{E} = -\frac{\boldsymbol{\omega}\times\mathbf{r}}{c}\times\mathbf{B} \quad (r < R). \tag{18}$$

In a *rotating coordinate system* S' following the motion of the sphere the electric field vanishes:

$$\mathbf{E}' = \mathbf{E}+(\mathbf{v}/c)\times\mathbf{B} = \mathbf{E}+\frac{\boldsymbol{\omega}\times\mathbf{r}}{c}\times\mathbf{B} = 0 \quad (r < R).$$

As in this system the electrons are at rest, the magnetic force vanishes as well, and the resultant force on them is zero as it should be.

Returning to the rest system S we can use (18) to calculate the net space charge density q in the sphere. It is

$$q = \frac{1}{4\pi}\operatorname{div}\mathbf{D} = \frac{1}{4\pi}\operatorname{div}\mathbf{E}$$

$$= -\frac{1}{2\pi c}(\boldsymbol{\omega}\mathbf{B})+\frac{\boldsymbol{\omega}\times\mathbf{r}}{4\pi c}\operatorname{curl}\mathbf{B}.$$

Utilizing the Maxwell equation

$$\operatorname{curl}\mathbf{B} = \frac{4\pi}{c}\mathbf{i}$$

(we have assumed $\mathbf{H} = \mathbf{B}$) we obtain the result

$$q = -\frac{1}{2\pi c}(\boldsymbol{\omega}\mathbf{B})+\frac{(\boldsymbol{\omega}\times\mathbf{r})}{c^2}\mathbf{i} \quad (r < R). \tag{19}$$

(In addition to the space charge density q in the interior there may be a distribution of surface charge q_s. The total charge remains zero because the assumed vacuum outside prevents charges from escaping.) In the vacuum there are no charges present,

$$q = 0 \quad (r > R), \tag{20}$$

and the electric field there is determined by the distribution of the charges in the sphere (and on its surface). As **E** derives from charges only (and not time-variable magnetic fields) it is everywhere irrotational.

If an external circuit *at rest* is connected by sliding contacts to two points on the sphere with different potentials in the system S, the arrangement will act as a unipolar generator just like those shown in Fig. 1.3. For in the closed circuit consisting of the external immobile circuit, where $\mathbf{E} \neq 0$, and the moving sphere, where $\mathbf{E}' = 0$, there is a resultant e.m.f. equal to the potential difference (in the system S) between the sliding contacts.

Let us apply the above considerations to the special case where the external magnetic field is a *dipole field*,

$$\left.\begin{aligned} B_r &= B_0\left(\frac{R}{r}\right)^3 \sin\lambda \\ B_\lambda &= -\tfrac{1}{2}B_0\left(\frac{R}{r}\right)^3 \cos\lambda \end{aligned}\right\}, \tag{21}$$

where B_0 is the polar field strength at the radius R. The potential difference between the latitude λ and the equator is

$$V = \int\limits_0^\lambda (-E_\lambda)R\,d\lambda = \frac{\omega R^2 B_0}{2}(\cos^2\lambda - 1)\ \text{e.s.u.} \tag{22}$$

For the earth we have $R = 0{\cdot}64\times 10^9$ cm, $\omega R = 0{\cdot}5\times 10^5$ cm sec^{-1}, and $B_0 = -0{\cdot}6$ gauss. The total voltage between the pole ($\lambda = \frac{1}{2}\pi$) and the equator ($\lambda = 0$) is 10^5 V, and the value of the horizontal component of the electric field at latitude 45° is 150 μV/cm.

If the surrounding space is empty, the external electric potential is found by fitting the general solution of Laplace's equation $\Delta V = 0$ to the boundary values given by (22). It turns out to be a quadrupole field (Davis, 1947). However, the presence of even a small density of charged particles may change the situation very much, cf. Chapter 5.

1.4. Approximate equality of positive and negative space charge

Consider a sphere of radius R containing N_1 positive charges, e, and N_2 negative charges, $-e$, per unit volume. The electrostatic potential at its surface is

$$V = \frac{4\pi}{3}(N_1 - N_2)eR^2.$$

When we deal with a problem in, for example, the solar corona, we can be sure that there cannot be a potential of say 3×10^{10} V ($= 10^8$ e.s.u.). (A sphere with radius R equal to 10^9 cm is only a small part of the corona.)

Inserting the value of R and putting $e = 4{\cdot}8 \times 10^{-10}$ e.s.u., the condition $V < 10^8$ gives

$$N_1 - N_2 < 0{\cdot}05 \text{ particles cm}^{-3}.$$

As $N_2 \approx 10^8$ cm^{-3}, we find

$$\frac{N_1 - N_2}{N_2} < 0{\cdot}5 \times 10^{-9}.$$

Hence even if there are only $10^9 + 1$ electrons for 10^9 protons, an impossibly high voltage is produced. Similar results are obtained for almost all other cosmic problems.

From the study of electric discharges in gases it is also known that the number of positive and of negative particles must be approximately the same, as soon as the density of charged particles is large. As we shall see in Chapter 4 (§ 4.2.3) the quantitative condition for approximate equality between positive and negative charge density is that the Debye distance is small. This condition is characteristic for a *plasma*.

REFERENCES

ALFVÉN, H., 1937: 'Cosmic Radiation as an Intra-galactic Phenomenon', *Ark. f. mat., astr. o. fys.* **25 B**, no. 29.

—— 1943: 'On Sunspots and the Solar Cycle', ibid. **29 A**, no. 12.

—— and HERLOFSON, N., 1950: 'Cosmic Radiation and Radio Stars', *Phys. Rev.* **78**, 616.

—— 1954: *On the Origin of the Solar System*, Clarendon Press, Oxford.

—— and LEHNERT, B., 1956: 'The Sun's General Magnetic Field', *Nature*, **178**, 1339.

—— 1962: 'On the Mass Distribution in the Solar System', *Astrophys. J.* **136**, 1005.

—— and WILCOX, J., 1962: 'On the Origin of the Satellites and the Planets', *Astrophys J.* **136**, 1016.

BABCOCK, H. D., 1959: 'The Sun's Polar Magnetic Field', ibid. **130**, 364.

BABCOCK, H. W., 1947: 'Remarks on Stellar Magnetism', *Publ. Astr. Soc. Pacific*, **59**, 112, no. 348. 'The Reversing Magnetic Field of BD–18°3789', ibid. 260, no. 350.

—— and BABCOCK, H. D., 1955: 'The Sun's Magnetic Field', *Astrophys. J.* **121**, 349.

—— 1958: 'A Catalog of Magnetic Stars', ibid., Suppl. no. 30.

—— 1958: 'Magnetic Fields of the A-type Stars', ibid. **128**, 508.

Babcock, H. W., 1960: 'The 34-Kilogauss Magnetic Field of HD 215441', ibid. **132,** 521.

Baldwin, J. E., 1955: 'The Distribution of the Galactic Radio Emission', *M.N.R.A.S.* **115,** 690.

Becker, R., and Sauter, F., 1957: *Theorie der Elektrizität*, Bd. I, Teubner, Stuttgart, p. 264.

Burbidge, G. R., 1956: 'The Halo Distribution of Radio Emission and extended Magnetic Fields in our Galaxy and M 31', *Astrophys. J.* **123,** 491.

Chandrasekhar, S., and Fermi, E., 1953: 'Magnetic Fields in Spiral Arms', ibid. **118,** 113.

Chapman, S., and Bartels, J., 1940: *Geomagnetism*, Clarendon Press, Oxford.

Davies, R. D., Slater, C. H., Shuter, L. H., and Wild, P. A. T., 1960: 'A New Limit to the Galactic Magnetic Field set by Measurement of the Zeeman Splitting of the Hydrogen Line', *Nature*, **187,** 1088.

Davis, L., 1947: 'Stellar Electromagnetic Fields', *Phys. Rev.* **72,** 632.

—— and Greenstein, J. L., 1951: 'The Polarization of Starlight by Aligned Dust Grains', *Astrophys. J.* **114,** 206.

Dungey, J. W., 1958: *Cosmic Electrodynamics*, Cambridge University Press.

Elvius, A., and Herlofson, N., 1960: 'On the Dynamics of Galaxies with Large-scale Magnetic Fields', ibid. **131,** 304.

Fermi, E., 1954: 'Galactic Magnetic Fields and the Origin of Cosmic Radiation', ibid. **119,** 1.

Finch, H. F., and Leaton, B. R., 1957: 'The Earth's Main Magnetic Field, Epoch 1955', *M.N.R.A.S.*, Suppl. **7,** 314.

Gauss, C. F., 1838: 'Allgemeine Theorie des Erdmagnetismus', *Resultate magn. Verein*, 1838. Reprinted in *Werke*, **5,** 121–93.

Gold, T., 1952: 'The Alignment of Galactic Dust', *M.N.R.A.S.* **112,** 215.

Hale, G. E., 1908: 'On the Probable Existence of a Magnetic Field in Sunspots. Mt. Wilson Contr. No. 30', *Astrophys. J.* **28,** 315.

Hale, G. E., Seares, F. H., von Maanen, A., and Ellerman, F., 1918: 'The General Magnetic Field of the Sun. Mt. Wilson Contr. No. 148', ibid. **47,** 206.

Hall, J. S., 1949: 'Observations of the Polarized Light from Stars', *Science*, **109,** 166.

Henry, J., 1958: 'Polarization of Starlight by Ferro-magnetic Particles', *Astrophys. J.* **128,** 497.

Hiltner, W. A., 1949: 'On the Presence of Polarization in the Continuous Radiation of Stars. II', ibid. **109,** 471.

Hultqvist, B., 1958*a*: 'The Spherical Harmonic Development of the Geomagnetic Field, Epoch 1949, transformed into Rectangular Geomagnetic Coordinate Systems', *Ark. Geophys.* **3,** 53.

—— 1958*b*: 'The Geomagnetic Field Lines in Higher Approximation', ibid. **3,** 63.

Möller, C., 1952: *The Theory of Relativity*, Clarendon Press, Oxford.

Pikelner, S. P., 1953: 'Kinematic Properties of the Interstellar Gas in Connection with the Isotropy of the Cosmic Radiation', *Dokl. Akad. Nauk, USSR*, **88,** 229.

Radhakrishnan, V., and Roberts, J. A., 1960: 'Polarization and Angular Extent of the 960-Mc/s Radiation from Jupiter', *Phys. Rev. Letters*, **4,** 493.

Shklovsky, I. S., 1952: 'On the Nature of the Galactic Radio Noise', *Astron. J. USSR*, **29,** 418.

SHKLOVSKY, I. S., 1953*a*: 'The Problem of Cosmical Radio Noise', ibid. **30,** 15.

—— 1953*b*: 'Radio Astronomy and the Problem of the Origin of Cosmic Rays', ibid. **30,** 577.

—— 1960: *Cosmic Radio Waves*, Harvard University Press, Cambridge, Mass.

SINGER, S. F., 1958: 'The Primary Cosmic Radiation and its Time Variations', *Progress in Elementary Particle and Cosmic Ray Physics*, **4,** 205–335, North-Holland Publ. Co.

SMITH, J., DAVIS, L., COLEMAN, P. J., and SONETT, P., 1963: 'Mariner II Preliminary Reports on Measurements of Venus: Magnetic Field', *Science*, **139,** 909.

SPITZER, L., and TUKEY, J. W., 1951: 'A Theory of Interstellar Polarization', *Astrophys. J.* **114,** 187.

—— 1954: 'Behaviour of Matter in Space', ibid. **120,** 1.

VESTINE, E. H., LANGE, I., LAPORTE, L., and SCOTT, W. E., 1947: 'The Geomagnetic Field, its Description and Analysis', *Carn. Inst. Wash. Publ.* **580.**

2

ON THE MOTION OF CHARGED PARTICLES IN MAGNETIC FIELDS

2.1. Introduction

THE first to appreciate fully the paramount importance to cosmic physics of the problems of the motion of charged particles in magnetic fields were Birkeland and Störmer. Inspired by Birkeland's terrella experiment in the 1890's, Störmer, during the first half of the twentieth century, devoted a long series of mathematical papers especially to motion in a magnetic dipole field (for summaries see Birkeland, 1908; Störmer, 1955). This forms the foundation of all later work on these problems. However, even with modern electronic computers it is difficult to carry through the numerical calculations when the particle makes a very large number of loops in the magnetic field. In the geomagnetic field this occurs for all particles below the momentum range of cosmic rays. Particles below the cosmic-ray range make thousands or millions of loops in the geomagnetic field, and a numerical integration is practically impossible.

When the path makes many loops, the size of one loop is in general small compared with the extension of the magnetic field. Hence during a single turn the particle moves in an approximately homogeneous field. In order to calculate the motion it is advantageous to start with the motion in a homogeneous field and introduce the inhomogeneity as a perturbation. This *perturbation method* which is developed in §§ 2.2 and 2.3 is especially suited for low-energy particles. In the geomagnetic field it is applicable to almost all problems where the momentum is below the cosmic-ray range. For recent surveys of the method see, for example, Allis (1960), Northrop (1961), and Lehnert (1963 *b*).

In the following paragraphs the motion in a dipole field is treated by Störmer's method (§ 2.4), as well as by the perturbation method (§ 2.5), and the results are compared.

The orbits of cosmic rays may also be found by direct experiment in which very thin electron beams are produced and their paths studied in the dipole field from a magnetized sphere (§ 2.4.1). Thus the three methods, *numerical integration*, *model experiment*, and *perturbation method* are complementary to each other.

Low-energy particles in the geomagnetic field are affected very much by an *electric field*, even if this is rather weak. Suppose for example that a 1 keV electron in the absence of any electric field drifts in a circular orbit in the equatorial plane at a distance of three times the earth's radius. If an electric field as weak as 1 μV/cm is applied, its orbit is changed so drastically that it leaves the earth's magnetic field and drifts to infinity (cf. § 2.6.3). Hence in studies of the behaviour of low-energy particles in the earth's magnetic field, it is often essential to take account of electric fields (§ 2.6).

In fluctuating magnetic fields charged particles can be accelerated under certain conditions. Such acceleration is important for the production of cosmic radiation, the radiation belts, etc. The fundamentals of these processes are discussed in § 2.7.

During its motion in magnetic fields a charged particle also emits radiation. The characteristics of this radiation is briefly outlined in the final paragraph (§ 2.8).

2.2. Motion of a charged particle in a homogeneous static magnetic field

2.2.1. *Unperturbed motion*

Consider a particle of rest mass m and charge e in a homogeneous magnetic field $\mathbf{B}$. The velocity $\mathbf{v}$ of the particle has the component $\mathbf{v}_{\|}$ parallel to the magnetic field and the component $\mathbf{v}_{\perp}$ perpendicular to it. The momentum of the particle is

$$\mathbf{p} = \gamma m \mathbf{v}, \tag{1}$$

where
$$\gamma = \frac{1}{\sqrt{(1-v^2/c^2)}} = \sqrt{\left(1+\frac{p^2}{m^2c^2}\right)}, \tag{2}$$

and $v = |\mathbf{v}| = (v_{\|}^2+v_{\perp}^2)^{\frac{1}{2}}$. The equation of motion is

$$\frac{d\mathbf{p}}{dt} = \frac{e}{c}\,\mathbf{v}\times\mathbf{B}. \tag{3}$$

Since the force term on the right-hand side of (3) is perpendicular to $\mathbf{v}$, the absolute value of the momentum and hence also v remains constant. This makes it easy to solve (3), and it is found that the motion is a superposition of a motion with constant velocity parallel to the magnetic field ($v_{\|}$ = const) and a circular motion in the plane perpendicular to it. The instantaneous centre of the circular motion will be referred to as the *centre of gyration*. In the projection of the path upon the plane perpendicular to **B** the radius of curvature $\boldsymbol{\rho}$ (a vector from the position of the particle to the centre of gyration) is given by

$$\gamma m \omega^2 \boldsymbol{\rho} = \frac{e}{c} \mathbf{v} \times \mathbf{B}, \tag{4}$$

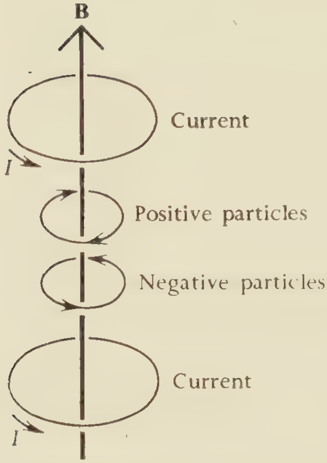


FIG. 2.1. In a magnetic field **B** produced by currents I, negative particles rotate in the same direction as I, positive particles in the opposite direction.

where ω is the angular frequency of the circular motion. Because

$$v_\perp = \omega \rho, \tag{5}$$

we obtain from (1) and (4)

$$\omega = \frac{|e| B}{\gamma m c}, \tag{6}$$

$$\boldsymbol{\rho} = \frac{c \gamma m}{e B^2} \mathbf{v} \times \mathbf{B} = \frac{c}{e B^2} \mathbf{p} \times \mathbf{B}, \tag{7}$$

and $$\rho = \frac{\gamma m v_\perp c}{|e| B} = \frac{p_\perp c}{|e| B}. \tag{8}$$

(Note that the symbol e includes also the sign of the charge.) The radius ρ is generally called the *Larmor radius* and the product $B\rho = cp_\perp/|e|$ is called *the magnetic rigidity*. (When $v_\parallel \neq 0$, so that the path is a helix, the Larmor radius ρ is different from the radius of curvature of the path.)

The circling motion of the charged particle produces a magnetic field which has a time average equal to the field from a circular current

$$\bar{I} = |e|/T_g, \tag{9}$$

where T_g is the period of gyration,

$$T_g = \frac{2\pi}{\omega} = \frac{2\pi \gamma m c}{|e| B}. \tag{10}$$

Hence the particle is equivalent to a magnet with a *magnetic moment*

$$\mu = \pi \rho^2 I / c = \frac{p_\perp^2}{2 \gamma m B} \tag{11}$$

directed antiparallel to **B** (cf. Fig. 2.1). The magnetic flux through the circular path amounts to

$$\phi = \pi \rho^2 B = \frac{\pi c^2}{e^2} \cdot \frac{p_\perp^2}{B} = \frac{2\pi \gamma m c^2}{e^2} \mu. \tag{12}$$

In the non-relativistic limit $v \ll c$, γ is unity, and the expression (11) reduces to

$$\mu = \frac{m v_\perp^2}{2B} = \frac{W_\perp}{B}. \tag{13}$$

Here $W_\perp$ is the kinetic energy due to the motion perpendicular to the lines of force.

2.2.2. *Motion perturbed by the action of non-magnetic forces or isolated magnetic inhomogeneities. The guiding centre*

If the motion of the particle is perturbed by the action of non-magnetic forces or if the magnetic field contains inhomogeneities, it is often convenient to use the concept of *guiding centre*. We define the guiding centre as the point

$$\mathbf{r}_c = \mathbf{r}+\boldsymbol{\rho} = \mathbf{r}+\frac{c}{eB^2}\mathbf{p}\times\mathbf{B}, \tag{14}$$

where $\mathbf{r}$ and $\mathbf{p}$ are the position and momentum of the particle and $\mathbf{B}$ is the magnetic field. It follows from (7) that when the only force acting is that from the homogeneous magnetic field, the guiding centre $\mathbf{r}_c$ coincides with the centre of gyration. This means that *if we take away the disturbing force for a short time, the particle moves in a circle around the guiding centre during this short time.*

As we shall find, it is often easy to calculate the motion of the guiding centre due to the action of a disturbing force. If the particle moves a distance that is very large compared to ρ, it is in many cases not necessary to know the detailed orbit of a particle. We can be satisfied if we can compute the motion of the guiding centre, because we know that the particle is spiralling near the guiding centre.

Impacts

Suppose that a charged particle moving in a homogeneous static magnetic field $\mathbf{B}$ experiences an impact. This means that during a small interval of time $\Delta t \ll T_g$ there acts upon it a large force $\mathbf{f}$. Let $\mathbf{f}_\parallel$ and $\mathbf{f}_\perp$ be the components parallel and perpendicular to the magnetic field. If

$$\mathbf{f}_\perp \gg \left|\frac{e}{c}\mathbf{v}\times\mathbf{B}\right|, \tag{15}$$

the momentum of the particle changes to

$$\mathbf{p}' = \mathbf{p}+\Delta\mathbf{p}, \tag{16}$$

where

$$\Delta\mathbf{p} = \int\limits_t^{t+\Delta t} \mathbf{f}\,dt. \tag{17}$$

Before and after the impact the guiding centre is at rest (or has a uniform motion parallel to the field lines). During the brief impact the position vector $\mathbf{r}$ does not change appreciably, but the momentum vector $\mathbf{p}$ does, and so the effect is to change the position of the guiding centre by the amount

$$\Delta\mathbf{r}_c = \frac{c}{eB^2}\Delta\mathbf{p}\times\mathbf{B} = -\frac{c}{eB^2}\mathbf{B}\times\int\limits_t^{t+\Delta t}\mathbf{f}\,dt. \tag{18}$$

(See Fig. 2.2.) The same result applies also to the centre of gyration (§ 2.2.3).

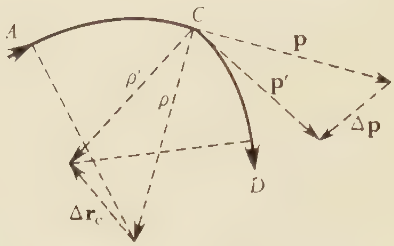


FIG. 2.2. A change in momentum $\mathbf{p}$ produces a displacement $\Delta\mathbf{r}_c$ of the guiding centre.

Continuous non-magnetic force

If instead $\mathbf{f}$ is a continuous force, the guiding centre experiences a continuous displacement, i.e. it drifts with a certain velocity, which we denote by $\mathbf{U}$. Differentiation of (14) gives

$$\mathbf{U} = \frac{d\mathbf{r}_c}{dt} = \mathbf{v}+\frac{c}{eB^2}\frac{d\mathbf{p}}{dt}\times\mathbf{B} \tag{19}$$

(the magnetic field being homogeneous and constant in time). Introducing the equation of motion,

$$\frac{d\mathbf{p}}{dt} = \mathbf{f}+\frac{e\mathbf{v}}{c}\times\mathbf{B} \tag{20}$$

into (19) and using the vector identity

$$(\mathbf{v}\times\mathbf{B})\times\mathbf{B} = \mathbf{B}(\mathbf{vB})-\mathbf{v}B^2 = \mathbf{v}_{\parallel}B^2-\mathbf{v}B^2 = -\mathbf{v}_{\perp}B^2,$$

we get

$$\mathbf{U} = \mathbf{v}_{\parallel}-\frac{c}{eB^2}\mathbf{B}\times\mathbf{f}. \tag{21}$$

The motion of the guiding centre can alternatively be depicted as produced by a large number of small impacts, each producing a displacement Δr_c according to (18). The velocity perpendicular to the magnetic field is $\Delta\mathbf{r}_c/\Delta t$ which gives (21).

If we separate $\mathbf{U}$ into the components $\mathbf{U}_\perp$ and $\mathbf{U}_\parallel$, which are perpendicular and parallel to $\mathbf{B}$, the result is

$$\mathbf{U}_\perp = -\frac{c}{eB^2}\,\mathbf{B}\times\mathbf{f}, \tag{22}$$

$$\frac{d}{dt}(\gamma m U_\parallel) = f_\parallel. \tag{23}$$

Fig. 2.3 illustrates the motion of the particle and the guiding centre in typical cases.

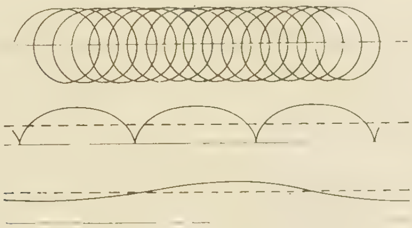


FIG. 2.3. Paths of the particle (———), the guiding centre (———), and the centre of gyration (- - - - -) when the force $\mathbf{f}$ is constant. Three cases are shown corresponding to different values of the ratio of the drift velocity, $u_\perp = cf/eB$, to the circling velocity $v'_\perp$. (a) $u_\perp/v'_\perp < 1$; (b) $u_\perp/v'_\perp = 1$; (c) $u_\perp/v'_\perp > 1$.

Magnetic inhomogeneities

Suppose that a charged particle moves in a static magnetic field $\mathbf{B}_0$, which is homogeneous except in a certain region (extending uniformly in the direction parallel to $\mathbf{B}$). In the exceptional region (see Fig. 2.4) the magnetic field is parallel to $\mathbf{B}_0$ but has the strength B_1. When the particle enters this region at A, the Larmor radius changes from ρ_0 to ρ_1 so that the guiding centre jumps from A' to P. When the particle leaves the field $\mathbf{B}_1$ at C, the guiding centre jumps to C'. Hence the effect of the inhomogeneity is that the guiding centre is displaced the distance $\Delta\mathbf{r}_c$ from A' to C'. The figure shows that we have

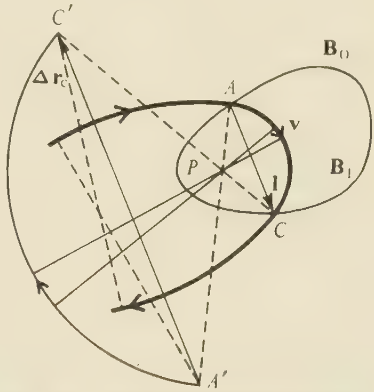


FIG. 2.4. Motion of the guiding centre when the particle passes a region of stronger magnetic field. (The lines of force are assumed to be straight and perpendicular to the plane of the figure.)

$$\Delta\mathbf{r}_c = -\mathbf{l}\,\frac{\rho_0-\rho_1}{\rho_1} = -\mathbf{l}B_1\left(\frac{1}{B_0}-\frac{1}{B_1}\right),$$

or

$$\Delta\mathbf{r}_c = -\mathbf{l}\,\frac{\Delta B}{B_0}, \tag{24}$$

where $\Delta B = B_1 - B_0$, and $\mathbf{l}$ is the distance between the point A where the particle enters the field $\mathbf{B}_1$ and the point C where it returns to the field $\mathbf{B}_0$.

In the more general case, where B may vary in the inhomogeneous region, we subdivide the latter into subregions so small that in each of them B is approximately constant and $\mathbf{l}$ approximately equal to the (projected) path element $\Delta\mathbf{s}$. Then $\Delta\mathbf{s}/\Delta t$ is approximately equal to $\mathbf{v}_\perp$ and we conclude from (24) that

$$\mathbf{U}_\perp = -\mathbf{v}_\perp \frac{\Delta B}{B_0}. \tag{25}$$

2.2.3. *The motion of the centre of gyration. Inertia force*

In the case of continuous forces the average motion can also be derived by the following alternative approach, which consists in finding a moving coordinate system relative to which the path of the particle is circular.

Let the non-magnetic force $\mathbf{f}$ acting on the particle be due in part to an electric field $\mathbf{E}$ and in part to other forces $\mathbf{f}^o$, so that we have, in a coordinate system S, which is 'at rest',

$$\mathbf{f} = \mathbf{f}^o + e\mathbf{E}. \tag{26}$$

By means of the formulae in § 1.3 we make a transformation to a coordinate system S' moving with a certain velocity which we denote by $\mathbf{u} = \mathbf{u}_\perp + \mathbf{u}_\parallel$. Confining ourselves to the non-relativistic case we assume

$$v \ll c, \qquad u \ll c. \tag{27, 28}$$

Then the square root appearing in 1.3 (1) and 1.3 (2), which is a relativistic correction, can be put equal to unity, and we have, as in 1.3 (3) and 1.3 (4),

$$\mathbf{E}' = \mathbf{E} + (\mathbf{u}/c) \times \mathbf{B}, \tag{29}$$

$$\mathbf{B}' = \mathbf{B} - (\mathbf{u}/c) \times \mathbf{E} \approx \mathbf{B}. \tag{30}$$

When the moving system is accelerated, we must introduce the inertia force

$$\mathbf{f}^{i} = -m\frac{d\mathbf{u}}{dt}. \tag{31}$$

This force is usually small and may often be neglected. In the moving system the particle is acted upon by the magnetic field

$$\mathbf{B}' \approx \mathbf{B}$$

and by the non-magnetic force

$$\mathbf{f}' = \mathbf{f}^{\circ} + \mathbf{f}^{\mathrm{i}} + e\mathbf{E}', \tag{32}$$

or because of (26) and (29)

$$\mathbf{f}' = \mathbf{f} + e(\mathbf{u}/c) \times \mathbf{B} + \mathbf{f}^{\mathrm{i}}.$$

The condition that the particle should move in a circle relative to the moving coordinate system S' is that the non-magnetic force $\mathbf{f}'$ must vanish. Putting

$$\mathbf{f}' = 0 \tag{33}$$

in (32) we have
$$e(\mathbf{u}/c) \times \mathbf{B} = -(\mathbf{f} + \mathbf{f}^{\mathrm{i}}). \tag{34}$$

If this equation is multiplied vectorially by $\mathbf{B}$, it becomes

$$\mathbf{u}_{\perp} = -\frac{c}{eB^2}\mathbf{B} \times (\mathbf{f} + \mathbf{f}^{\mathrm{i}}). \tag{35}$$

(because $\mathbf{B} \times (\mathbf{u} \times \mathbf{B}) \equiv \mathbf{u}B^2 - \mathbf{B}(\mathbf{uB}) \equiv B^2\mathbf{u}_{\perp}$). If (34) is instead multiplied scalarly by $\mathbf{B}$, it becomes

$$\mathbf{B}(\mathbf{f} + \mathbf{f}^{\mathrm{i}}) = 0. \tag{36}$$

In the coordinate system S' no force, except the force from the magnetic field, is acting, and so the particle moves in a circle. The centre of this circle is called the *centre of gyration*. Thus the velocity $\mathbf{u} = \mathbf{u}_{\perp} + \mathbf{u}$ of the coordinate system S' represents the *velocity of the centre of gyration* relative to the fixed coordinate system S. The velocity $\mathbf{u}$ obeys the differential equations (35) and (36), where

$$\mathbf{f} = \mathbf{f}_0 + e\mathbf{E} \tag{37}$$

and
$$\mathbf{f}^{\mathrm{i}} = -m\frac{d\mathbf{u}}{dt}. \tag{38}$$

In many important cases $\mathbf{f}^{\mathrm{i}}$ is small and can be neglected in (35).

Let us first consider the case where the inertia term can be neglected. Then the gyro-centre velocity $\mathbf{u}_{\perp}$ given by (35) is equal to the guiding-centre velocity $\mathbf{U}_{\perp}$ (equation (22)). Relative to the coordinate system S' the particle moves in a circle, and its velocity, $\mathbf{v}'_{\perp}$, has a constant modulus $v'_{\perp}$. The corresponding kinetic energy is $W'_{\perp}$. In the fixed system the velocity perpendicular to $\mathbf{B}$ is given by

$$\mathbf{v}_{\perp} = \mathbf{u}_{\perp} + \mathbf{v}'_{\perp}, \tag{39}$$

and the corresponding kinetic energy is

$$W_\perp = \tfrac{1}{2}m\mathbf{v}_\perp^2 = \tfrac{1}{2}m(\mathbf{u}_\perp+\mathbf{v}'_\perp)^2 = W'_\perp+\tfrac{1}{2}mu_\perp^2+m(\mathbf{u}_\perp\,\mathbf{v}'_\perp). \tag{40}$$

When (40) is averaged over one gyro period, the scalar product cancels, and we obtain for the mean value of $W_\perp$

$$W_\perp = W'_\perp+\tfrac{1}{2}mu_\perp^2. \tag{41}$$

If $\mathbf{f}$ derives from a potential, the drift, which is perpendicular to $\mathbf{f}$, follows an *equipotential line*.

Fig. 2.3 illustrates typical paths of a charged particle drifting perpendicular to $\mathbf{B}$ under the influence of a constant force. The corresponding paths of the guiding centre and of the centre of gyration are also shown. It is seen from the figure that when the drift velocity is comparable to the circling velocity, the positions (although not the velocities) of the centres are different. Whenever the drift velocity is small compared to the circling velocity, the guiding centre and the centre of gyration nearly coincide, so that no distinction need be made between the two.

When the inertia term is taken into account, the gyration-centre velocity $\mathbf{u}$ (although not the guiding-centre velocity $\mathbf{U}$) has also a component in the direction of $d\mathbf{f}/dt$. Hence the average energy in the fixed coordinate system changes at the rate

$$\frac{dW_\perp}{dt} = (\mathbf{f}\mathbf{u}_\perp) = -\frac{c}{eB^2}\,\mathbf{f}\{\mathbf{B}\times(\mathbf{f}+\mathbf{f}^{\mathrm{i}})\}. \tag{42}$$

Using the vector identity

$$\mathbf{A}_1\{\mathbf{B}\times(\mathbf{A}_1+\mathbf{A}_2)\} = -\mathbf{A}_2\{\mathbf{B}\times(\mathbf{A}_1+\mathbf{A}_2)\},$$

we find from (35) and (38)

$$\frac{dW_\perp}{dt} = \frac{c}{eB^2}\,\mathbf{f}^{\mathrm{i}}\{\mathbf{B}\times(\mathbf{f}+\mathbf{f}^{\mathrm{i}})\} = m\mathbf{u}_\perp\frac{d\mathbf{u}_\perp}{dt} = \frac{d}{dt}\left(\frac{m\mathbf{u}_\perp^2}{2}\right). \tag{43}$$

Hence the centre of gyration can be displaced to another equipotential line by a change in $\mathbf{f}$. The difference in energy between the two equipotential lines is equal to the change in kinetic energy due to the change in drift velocity. The circling velocity, i.e. the velocity in relation to the moving system, remains constant. This result applies only in a homogeneous static magnetic field. If the magnetic field strength varies in space or time, the circling velocity need no longer be constant, cf. § 2.7.

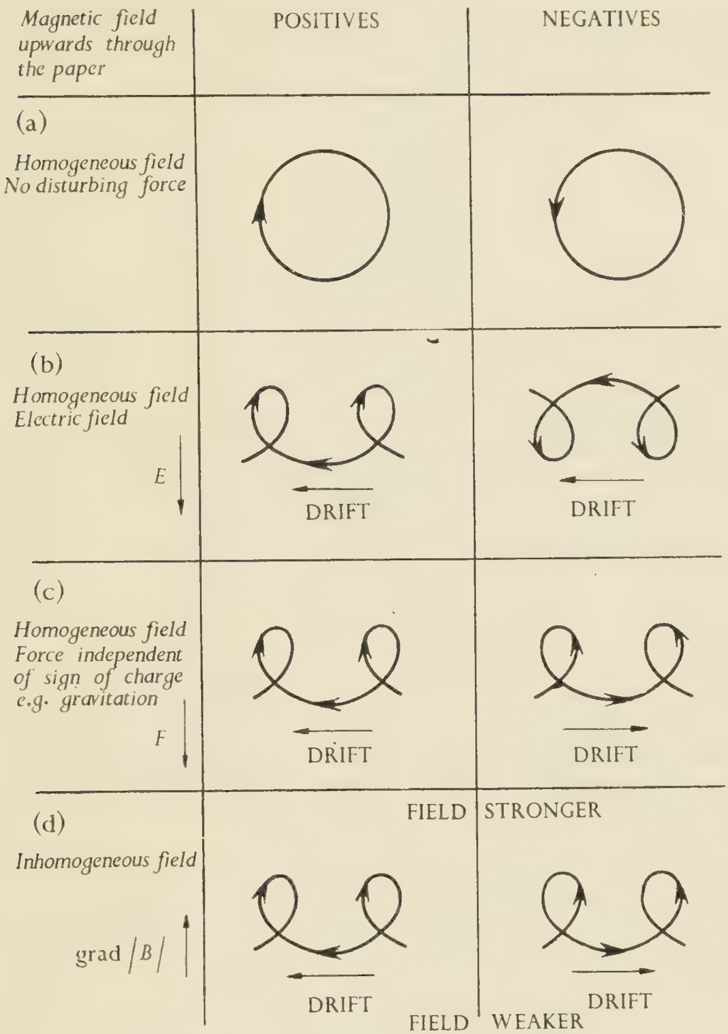


FIG. 2.5. Drifts of charged particles in a magnetic field. If the magnetic field is inhomogeneous, the radius of curvature is smaller where the field is strong than where it is weak. Hence the circle, in which the particle moves in the case of a homogeneous field, is changed into the curve shown above. When the inhomogeneity is small, the motion consists of a circular motion superimposed by a translational motion ('drift') perpendicular to the magnetic gradient. An electric field E, or another force F, also changes the curvature with the results shown above.

2.3. Slowly varying magnetic field

When the magnetic field has a slow variation with the space coordinates and with time, the motion of a charged particle can be conveniently treated by a *perturbation method*. The condition for this method to be applicable is that the change in the magnetic field during one gyration is small. We shall therefore assume that

$$\frac{T_g}{B}\frac{\partial B}{\partial t} \ll 1, \tag{1}$$

$$\frac{\rho|(\operatorname{grad} B)_\perp|}{B} \ll 1, \tag{2}$$

and

$$\frac{T_g v_\parallel |(\operatorname{grad} B)_\parallel|}{B} \ll 1, \tag{3}$$

where T_g is the gyro period given by 2.2.1 (10).

We shall consider separately the different kinds of perturbations that are of interest. Except in the simple cases treated in §§ 2.3.1 and 2.3.2 the analysis will be limited to the non-relativistic case $v \ll c$.

2.3.1. *The magnetic field varies with time*

In this case an electric field $\mathbf{E}$ is induced according to the Maxwell equation

$$\operatorname{curl} \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}.$$

The electric field, the structure of which depends on boundary conditions, acts in two ways. On the one hand it changes the energy of the particle in a way which will be examined in this section. On the other hand it causes a drift motion which has to be accounted for by the method developed in § 2.2.

Treating the general case, where the particle may have relativistic velocity we can write the change of energy during one turn

$$\Delta W = \Delta(\gamma mc^2) = -e\oint \mathbf{E}\,d\mathbf{s} = \frac{\pi\rho^2|e|}{c}\frac{\partial B}{\partial t}, \tag{4}$$

because

$$\oint \mathbf{E}\,d\mathbf{s} = \iint \operatorname{curl} \mathbf{E}\,d\mathbf{S} = -\frac{1}{c}\frac{\partial \phi}{\partial t}, \tag{5}$$

where

$$\phi = \pi\rho^2 B \tag{6}$$

is the flux through the circular path of the particle. (The negative sign in (4) derives from the fact that a positive particle gyrates in a direction opposite to that in which the integral is to be taken.) Since the number of turns per second is $1/T_g$, the average time derivative of the energy is

$$\frac{d}{dt}(\gamma mc^2) = \frac{\Delta(\gamma mc^2)}{T_g}. \tag{7}$$

Using (4) and 2.2.1 (8) and (10) we obtain from (7)

$$\frac{d(\gamma m)}{dt} = \frac{p_\perp^2}{2\gamma mc^2 B}\frac{\partial B}{\partial t}. \tag{8}$$

On the other hand, differentiation of 2.2.1 (2) gives

$$\frac{d(\gamma m)}{dt} = \frac{1}{2\gamma mc^2}\frac{dp^2}{dt} = \frac{1}{2\gamma mc^2}\frac{dp_\perp^2}{dt} \tag{9}$$

because the 'betatron acceleration' (7) acts only on the momentum component $p_\perp$. When $\partial B/\partial t = dB/dt$ (time variation only), (8) and (9) give

$$\frac{p_\perp^2}{B} = C. \tag{10}$$

Comparison with formula 2.2.1 (12) shows that the 'integration constant' C is proportional to the orbital magnetic flux ϕ. In the non-relativistic limit (10) can be written

$$\frac{mv_\perp^2}{2B} = \frac{W_\perp}{B} = C/2m = \text{const}, \tag{11}$$

where $W_\perp$ stands for the kinetic energy due to motion perpendicular to the lines of force. In this case the magnetic moment μ, see 2.2.1 (11), as well as the flux is constant.

The invariant $p_\perp^2/B$ is one of the three 'adiabatic invariants' of the motion of a charged particle. These invariants will be discussed below (§ 2.3.7).

2.3.2. *The gradient of the magnetic field has a component in the direction of the field*

If we place a Cartesian coordinate system with the z-axis parallel to the magnetic field at the origin, we have $\partial B/\partial z \neq 0$. We consider a small region of the order ρ around the origin, and introduce cylindrical coordinates (R, φ, z) with the z-axis parallel to $\mathbf{B}$. The condition

$$\operatorname{div} \mathbf{B} = 0 \tag{12}$$

gives (if $\partial B_\varphi/\partial\varphi = 0$)

$$\frac{1}{R}\frac{\partial}{\partial R}(RB_R) + \frac{\partial B_z}{\partial z} = 0, \tag{13}$$

where B_R, B_z, and B_φ are the components of $\mathbf{B}$. We can put

$$\frac{\partial B_z}{\partial z} = \frac{\partial B}{\partial z} = \text{const}$$

inside the circle with radius $R = \rho$. If we assume $\partial B_R/\partial\varphi = 0$, which essentially means neglecting the gradient perpendicular to the field (this is treated in § 2.3.3) (13) can be integrated to

$$B_R = -\tfrac{1}{2}R\frac{\partial B}{\partial z}. \tag{14}$$

As the particle moves with the velocity $v_\perp$ in the $R\varphi$-plane at the distance $R = \rho$ from the z-axis, it is subject to an average force

$$f_z^{\mathrm{m}} = \frac{|e|}{c}v_\perp B_R = -\frac{|e|v_\perp \rho}{2c}\frac{\partial B}{\partial z} \tag{15}$$

in the direction of the z-axis. On application of 2.2.1 (1) and 2.2.1 (8) the expression (15) becomes

$$f_z^{\mathrm{m}} = -\frac{p_\perp^2}{2\gamma m B}\frac{\partial B}{\partial z}. \tag{16}$$

During the motion of the particle in the z-direction its momentum component $p_\parallel$ changes at the rate

$$\frac{dp_\parallel}{dt} = f_z^{\mathrm{m}}. \tag{17}$$

If only magnetic forces are acting, the energy, which is

$$\{m^2c^4+(p_\parallel^2+p_\perp^2)c^2\}^{1/2},$$

remains constant, and hence

$$\frac{d}{dt}p_\parallel^2 = -\frac{d}{dt}p_\perp^2. \tag{18}$$

Equation (17) can be written

$$\frac{d}{dt}\frac{p_\parallel^2}{2} = p_\parallel f_z^{\mathrm{m}} = \gamma m\frac{dz}{dt}f_z^{\mathrm{m}}. \tag{19}$$

If (16) and (18) are introduced into (19) the result is

$$\frac{1}{p_\perp^2}\frac{dp_\perp^2}{dt} = \frac{1}{B}\frac{dB}{dt}, \tag{20}$$

where d/dt denotes the rate of change due to the motion of the gyration centre along magnetic field lines. Equation (20) implies

$$\frac{d}{dt}(p_\perp^2/B) = 0. \tag{21}$$

It follows that when the gyro centre of the particle moves along the field lines, $p_\perp$ and B vary in such a way that the quantity

$$p_\perp^2/B = C \tag{22}$$

remains constant.

Specializing to the non-relativistic case we can write the equation for the gyro-centre motion parallel to $\mathbf{B}$:

$$\left(m\frac{d\mathbf{u}}{dt}\right)_{\parallel} = f_z^{\mathrm{m}},$$

or, which is equivalent, $$(\mathbf{f}^{\mathrm{m}}+\mathbf{f}^{\mathrm{i}})\mathbf{B} = 0, \tag{23}$$

where $$\mathbf{f}^{\mathrm{m}} = -\mu\,\mathrm{grad}\,B \tag{24}$$

and $$\mathbf{f}^{\mathrm{i}} = -m\frac{d\mathbf{u}}{dt}. \tag{25}$$

2.3.3. *The gradient of the magnetic field has a component perpendicular to the field*

Let the coordinate system be chosen in such a way that the z-axis coincides with the direction of $\mathbf{B}$ and the y-axis with the direction of grad B. Then $\partial B/\partial x = 0$. We put

$$\frac{1}{B_0}\left(\frac{\partial B}{\partial y}\right)_0 = \frac{1}{l_{\mathrm{c}}}. \tag{26}$$

The subscript 0 denotes the value at the origin. The condition (2) now means

$$\frac{\rho}{l_{\mathrm{c}}} \ll 1. \tag{27}$$

In this section we assume in addition $\partial\mathbf{B}/\partial z = 0$, which implies straight lines of force. (We may remark that when the lines of force are straight, a non-vanishing $\partial B/\partial y$ requires non-vanishing curl $\mathbf{B}$.) A curvature in the lines of force is of importance if $v_{\parallel} \neq 0$, because it causes a centrifugal force. This is treated in § 2.3.5.

To a first approximation the particle moves in a circle around the origin, so that for a positive particle the coordinates are

$$x = \rho\cos\omega t, \qquad y = -\rho\sin\omega t, \tag{28}$$

and the velocity components

$$v_x = -\omega\rho\sin\omega t, \qquad v_y = -\omega\rho\cos\omega t. \tag{29}$$

Hence the magnetic field in which it moves changes periodically so that at the time t the field strength is

$$B = B_0+\Delta B = B_0+\frac{\partial B}{\partial y}y = B_0\left(1-\frac{\rho}{l_{\mathrm{c}}}\sin\omega t\right),$$

which gives $$\frac{\Delta B}{B_0} = -\frac{\rho}{l_{\mathrm{c}}}\sin\omega t. \tag{30}$$

When the particle moves approximately in a circle with the velocity $\mathbf{v}_\perp$, its guiding centre moves with the velocity given by 2.2.2 (25), namely

$$\mathbf{U}_\perp = -\mathbf{v}_\perp \frac{\Delta B}{B_0}. \tag{31}$$

According to (30) this can be written

$$\mathbf{U}_\perp = \mathbf{v}_\perp \frac{\rho}{l_c} \sin \omega t, \tag{32}$$

or in component form

$$U_x = -\omega\rho \frac{\rho}{l_c} \sin^2\omega t, \tag{33}$$

$$U_y = \omega\rho \frac{\rho}{l_c} \sin \omega t \cos \omega t. \tag{34}$$

Averaging these expressions over a gyro period we find

$$\overline{U}_x = -\frac{\omega\rho^2}{2l_c} = -\frac{v_\perp \rho}{2l_c} = -\frac{v_\perp \rho}{2B_0}\left(\frac{\partial B}{\partial y}\right)_0, \tag{35}$$

$$\overline{U}_y = 0. \tag{36}$$

In case the drift velocity is much slower than the circling velocity, the path consists of nearly circular loops, cf. Fig. 2.3 (a), and we *need not distinguish between the guiding centre and the centre of gyration. In the following we shall use the symbol* **u** *for the velocity of either of the centres.*

Using (27), 2.2.1 (1), (8), and (11), we can rewrite (35) and (36) in terms of μ and grad B. It becomes

$$\mathbf{u}_\perp = -\frac{c}{eB^2} \mathbf{B} \times \mathbf{f}^{\mathrm{m}}, \tag{37}$$

where

$$\mathbf{f}^{\mathrm{m}} = -\mu \operatorname{grad} B. \tag{38}$$

2.3.4. *The resultant average velocity*

The resultant velocity under the influence of both magnetic perturbations (treated in §§ 2.3.1 to 2.3.3) and non-magnetic forces (§ 2.2) is found by superposition. We obtain for the drift perpendicular to the field

$$\boxed{\mathbf{u}_\perp = -\frac{c}{eB^2} \mathbf{B} \times (\mathbf{f} + \mathbf{f}^{\mathrm{m}} + \mathbf{f}^{\mathrm{i}})} \tag{39}$$

where the non-magnetic force

$$\mathbf{f} = \mathbf{f}^0 + e\mathbf{E} \tag{40}$$

may be due partly to an electric field $\mathbf{E}$ and partly to other forces $\mathbf{f}^0$, and where

$$\mathbf{f}^{\mathrm{m}} = -\mu \operatorname{grad} B, \tag{41}$$

$$\mathbf{f}^{\mathrm{i}} = -m\frac{d\mathbf{u}}{dt}. \tag{42}$$

For the motion along the lines of force we have

$$\boxed{\mathbf{B}(\mathbf{f}+\mathbf{f}^{\mathrm{m}}+\mathbf{f}^{\mathrm{i}}) = 0} \tag{43}$$

Fig. 2.5 illustrates schematically various types of drift motion.

The inertia term is in many cases small enough to be neglected. However, in a plasma (cf. Chapters 4 and 5) it may be quite important, even if it is relatively small, because it tends to produce a separation of positive and negative charges. This separation creates an electric field, which in its turn can influence the drift decisively.

The force $\mathbf{f}$ may be due to an electric field or to gravitation. It may also be composed of a series of impacts (compare § 2.2.2). For example, when the particle is part of a gas which is subject to a pressure gradient, we may have

$$\mathbf{f} = -(1/n)\operatorname{grad} p, \tag{44}$$

where n is the number of particles per unit volume and p is the pressure.

2.3.5. *Effect of motion along curved lines of force*

In § 2.3.3 we assumed the lines of force to be straight. Therefore the drift perpendicular to the magnetic field was independent of the simultaneous motion along the lines of force. When this restriction is removed, the guiding centre will in general follow a curved path and as a consequence centrifugal forces occur. A centrifugal force influences the drift perpendicular to the magnetic field because it enters into the term $\mathbf{f}^{\mathrm{i}}$ of the general drift equation (39). In case the velocity $v_{\|}$ ($= u_{\|}$) along the lines of force is much larger than the drift velocity $u_{\perp}$ the most important contribution of the inertia term in (39) is the centrifugal force associated with $v_{\|}$. We shall consider this important case in some detail and derive an explicit expression for the inertia term.

Consider a moving local cartesian coordinate system (unit vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$) which follows the motion of the guiding centre and is oriented in such a way that the z- and y-axes are parallel to the tangent and the principal normal of the magnetic field line through the origin. Then $\mathbf{B} = B\hat{\mathbf{z}}$ at the origin.

In the coordinate system just defined, we introduce the centrifugal force due to $\mathbf{v}_{\parallel}$, which is

$$\mathbf{f}^{c} = -\frac{m v_{\parallel}^{2}}{R}\hat{\mathbf{y}}, \tag{45}$$

where R is the radius of curvature of the line of force. The value of R is

$$R = \frac{B}{|\partial B_y/\partial z|} \tag{46}$$

and the centrifugal force is

$$\mathbf{f}^{c} = -\frac{m v_{\parallel}^{2}}{B}\left(\frac{\partial B_y}{\partial z}\right)\hat{\mathbf{y}}. \tag{47}$$

The contribution it gives to the drift velocity, is, according to (39), directed along the x-axis and has the value

$$u_x^{c} = -\frac{c}{eB}\frac{m v_{\parallel}^{2}}{B}\frac{\partial B_y}{\partial z} = \frac{c}{eB}\frac{m v_{\parallel}^{2}}{B}(\operatorname{curl}\mathbf{B}-\operatorname{grad}B\times\hat{\mathbf{z}})_x. \tag{48}$$

Assuming currents to be negligible ($\operatorname{curl}\mathbf{B} = 0$) we get from (48) and 2.3.4 (39) the total drift velocity perpendicular to the magnetic field

$$\mathbf{u}_{\perp} = -\frac{c}{eB^{2}}\mathbf{B}\times(\mathbf{f}+\mathbf{f}_1^{m}), \tag{49}$$

$$\mathbf{f}_1^{m} = -\frac{m}{B}\left(\frac{v_{\perp}^{2}}{2}+v_{\parallel}^{2}\right)\operatorname{grad}B. \tag{50}$$

(Cf. the discussion by Cowling 1942.)

2.3.6. *The magnetic mirror*

The longitudinal motion of the guiding centre is very similar to the motion of a particle in a potential field. Consider a charged particle moving in a *static* magnetic field $\mathbf{B}$ in the presence of an electric field with the potential function V. If, as usual, we denote the momentum components parallel and perpendicular to $\mathbf{B}$ by $p_{\parallel}$ and $p_{\perp}$, the conservation of energy (non-relativistic case) implies

$$\frac{p_{\parallel}^{2}+p_{\perp}^{2}}{2m}+eV = \text{const} = \frac{p_{\parallel 0}^{2}+p_{\perp 0}^{2}}{2m}+eV_0, \tag{51}$$

where the subscript 0 refers to a given initial position of the particle. When the perturbation method is applicable, we have

$$\mu = \frac{p_{\perp}^{2}}{2mB} = \frac{p_{\perp 0}^{2}}{2mB_0} = \text{const}. \tag{52}$$

From (51) and (52) we get

$$\frac{p_{\parallel}^{2}-p_{\parallel 0}^{2}}{2m} = -\mu(B-B_0)-e(V-V_0). \tag{53}$$

Differentiation of (53) along the magnetic lines of force gives

$$\frac{dp_{\|}}{dt} = -\mu\frac{\partial B}{\partial s} - e\frac{\partial V}{\partial s}. \tag{54}$$

Equations (53) and (54) are exactly similar to the equation of motion for a particle in a potential field of the form

$$\phi = \mu B + eV. \tag{55}$$

(Since μ is different for different particles, so is the equivalent potential ϕ.)

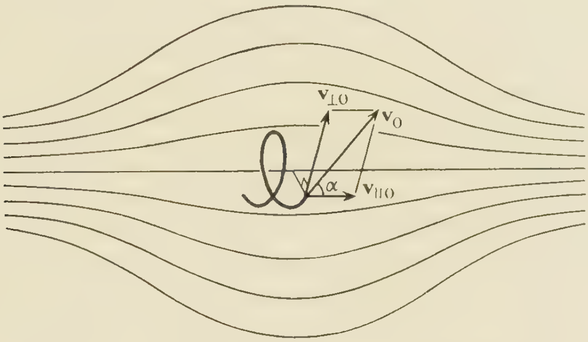


FIG. 2.6. Particle moving between magnetic mirrors.

The kinetic energy of the particle may change during the motion. As (51) shows this is possible if and only if the guiding centre is displaced from one equipotential surface of the electric field to another. In other words, the kinetic energy is a function of the space coordinates only.

Consider the magnetic field configuration of Fig. 2.6 and a charged particle, which at the point of minimum magnetic field strength (say B_0) has the velocity components

$$v_{\perp 0} = v_0 \sin\alpha, \tag{56}$$

$$v_{\| 0} = v_0 \cos\alpha. \tag{57}$$

Then, according to (52), (53), (56), and (57) and the relation

$$\mathbf{p} = m\mathbf{v},$$

the longitudinal velocity $v_{\|}$ is given by

$$v_{\|} = v_0 \sqrt{\left(1 - \frac{B}{B_0}\sin^2\alpha - \frac{2e(V - V_0)}{mv_0^2}\right)}. \tag{58}$$

In the absence of an electric field, the formula is simplified to

$$v_{\parallel} = v_0\sqrt{(1-B/B_1)} \tag{59}$$

with

$$B_1 = B_0/\sin^2\alpha. \tag{60}$$

When the guiding centre has reached the region where $B = B_1$, $v_{\parallel}$ passes through zero, which means that the particle turns back into regions of weaker field. Because of this reflection phenomenon a region with converging magnetic field lines is often referred to as a *magnetic mirror*. In the region between two such mirrors (see Fig. 2.6) a particle for which α exceeds a certain critical value will be reflected at both ends of the region and will thus be trapped. The alternative possibility, escape through a mirror, will occur if the velocity vector makes a small enough angle with the field, i.e. if it lies in the *escape cone* (or 'loss cone') of velocity space.

Magnetic trapping of this kind occurs in the dipole-like geomagnetic field as discussed in § 2.5. Such trapping is responsible for the existence of the earth's radiation belts (Van Allen belts).

If *the magnetic field varies with time*, the electric field has a non-vanishing curl and cannot have a potential. Hence the above results are no longer valid. However, the average magnetic force in the longitudinal direction is still given by $-\mu\partial B/\partial s$ (see 2.3.2 (16) and (17)), and instead of (54) we have

$$\frac{dp_{\parallel}}{dt} = -\mu\frac{\partial B}{\partial s}+eE_{\parallel}. \tag{61}$$

The magnetic mirror effect is the same as before, but the charged particle may now be accelerated by rotational electric fields (betatron acceleration) and hence its kinetic energy has no simple relation to its spatial position.

If a particle experiences repeated reflections by magnetic mirrors approaching each other, it gains energy. This process, which will be discussed in §§ 2.3.7 and 2.7.1, is the basis of Fermi's theory of the origin of cosmic rays (cf. § 2.7.1).

2.3.7. *The adiabatic invariants*

I. *The orbital flux* (*and magnetic moment*)

In §§ 2.3.1 and 2.3.2 it was shown in two special cases that when B changes slowly (in the sense of 2.3 (1) to (3)) the quantity $C = p_{\perp}^2/B$ is constant to the first order. This quantity is proportional to the 'orbital magnetic flux' ϕ encircled by the charged particle in one gyration (see 2.2.1 (12))

$$\phi = \frac{\pi c^2}{e^2}\frac{p_{\perp}^2}{B}.$$

The magnetic moment associated with the circular motion of the particle is

$$\mu = \frac{1}{2\gamma m}\frac{p_\perp^2}{B},$$

see 2.2.1 (11). Since for a relativistic particle the factor γ depends on the momentum, the magnetic moment is invariant only in the non-relativistic limit $v \ll c$, and there it can also be written

$$\mu = \frac{W'_\perp}{B}, \tag{62}$$

where $W'_\perp$ is the kinetic energy corresponding to the circling velocity $v'_\perp$ (as measured in a coordinate system following the drift motion).

In § 2.3.4 was given the general non-relativistic formula, 2.3.4 (39), for drift motion across the lines of force. As an application we shall now show that the magnetic moment remains constant when the value of B, at the position of the particle, changes due to such drift motion.

As the particle drifts across the lines of force, the field strength at the instantaneous position of the gyration centre changes at the rate

$$\frac{dB}{dt} = \mathbf{u}_\perp \operatorname{grad} B = -\frac{c}{eB^2}\{\mathbf{B}\times(\mathbf{f}+\mathbf{f}^{\mathrm{m}}+\mathbf{f}^{\mathrm{i}})\}\operatorname{grad}\mathrm{B}. \tag{63}$$

Here $\mathbf{f}^{\mathrm{m}} = -\mu \operatorname{grad} B$. For an arbitrary vector $\mathbf{A}$ we have

$$(\mathbf{B}\times\mathbf{A}).\mathbf{A} = 0.$$

Hence
$$\frac{dB}{dt} = -\frac{c}{eB^2}\{\mathbf{B}\times(\mathbf{f}+\mathbf{f}^{\mathrm{i}})\}\operatorname{grad} B. \tag{64}$$

At the same time the average energy changes. As the force from the magnetic field acts perpendicular to $\mathbf{v}$, it produces no energy change. Thus the average rate of change of energy, referred to the fixed system, is

$$\frac{dW_\perp}{dt} = (\mathbf{u}_\perp\,\mathbf{f}). \tag{65}$$

The average energy referred to the moving system is, according to 2.2.3 (41), given by
$$W'_\perp = W_\perp - \tfrac{1}{2}mu_\perp^2.$$

Differentiating this we obtain, using (65) and 2.2.3 (38),

$$\frac{dW'_\perp}{dt} = \mathbf{u}_\perp\,\mathbf{f} - m\mathbf{u}_\perp\frac{d\mathbf{u}_\perp}{dt} = \mathbf{u}_\perp(\mathbf{f}+\mathbf{f}^{\mathrm{i}}).$$

If, further, the expression 2.3.4 (39) for $\mathbf{u}_\perp$ is introduced, we get

$$\frac{dW'_\perp}{dt} = -\frac{c}{eB^2}\{\mathbf{B}\times(\mathbf{f}+\mathbf{f}^{\mathrm{m}}+\mathbf{f}^{\mathrm{i}})\}(\mathbf{f}+\mathbf{f}^{\mathrm{i}}) \tag{66}$$

or, since $\{\mathbf{B}\times(\mathbf{A}_1+\mathbf{A}_2)\}\mathbf{A}_1 = -(\mathbf{B}\times\mathbf{A}_1)\mathbf{A}_2$,

$$\frac{dW'_\perp}{dt} = \frac{c}{eB^2}\{\mathbf{B}\times(\mathbf{f}+\mathbf{f}^{\mathrm{i}})\}\mathbf{f}^{\mathrm{m}}. \tag{67}$$

Since $\mathbf{f}^{\mathrm{m}} = -\mu \operatorname{grad} B = -(W'_\perp/B)\operatorname{grad} B$, we have, from (64) and (67),

$$\frac{dW'_\perp}{dB} = \frac{W'_\perp}{B}, \tag{68}$$

showing that $W'_\perp/B$ remains constant.

The quantity $W'_\perp/B$ (in the relativistic case $p_\perp^2/B$) is an *adiabatic invariant* (see, for example, Landau and Lifshitz, 1960). This means that its variation due to a given change in B can be made *arbitrarily* small by letting the change in B take place *sufficiently slowly* in space and time. The 'slowness' of the change can be expressed by a suitable parameter, say δ, which in the case of the magnetic moment is essentially the maximum relative change of $\mathbf{B}$ during one gyration.

Analysis of higher order perturbations has shown that there exists a more general invariant (reducing in first order to μ) that is adiabatic to all orders in δ. This result refers to an asymptotic expansion and does not mean that the magnetic moment is exactly constant. It only implies that the deviation from invariance goes to zero very rapidly as δ goes to zero. For a survey of the theory of higher-order invariance see, for example, Kruskal, 1962.

II. *The longitudinal invariant. The Fermi process*

Consider for a moment a purely mechanical problem. Suppose that a particle moves with a velocity $v_\parallel$ (momentum $p_\parallel = mv_\parallel$) along the z-axis of a given coordinate system, and that perpendicular to the z-axis there are two perfectly reflecting planes situated at $z = \pm\frac{1}{2}D$ and moving with the velocity $\mp U$ ($\ll v_\parallel$). When hitting any of these planes the particle makes an elastic collision. At each collision the particle changes its momentum by the amount

$$\Delta p_\parallel = 2mU. \tag{69}$$

The result (69) is valid also in the relativistic case, if for m we use the relativistic mass of the particle, $\gamma m = W/c^2 = \{m^2+(p_\parallel^2+p_\perp^2)c^{-2}\}^{1/2}$.

As collisions occur at intervals $\Delta t = D/v_\parallel$, the particle changes its momentum at the average rate

$$dp_\parallel/dt = \Delta p_\parallel/\Delta t = 2mUv_\parallel/D. \tag{70}$$

As $2U = -dD/dt$, and $mv_{\parallel} = p_{\parallel}$ we obtain

$$\frac{dp_{\parallel}}{p_{\parallel}}+\frac{dD}{D} = 0,$$

showing that

$$p_{\parallel} D = \text{const.} \tag{71}$$

This means that when the reflecting planes approach each other so that D diminishes, the velocity increases.

The motion of the particle between the slowly approaching planes is a simple special case of motion in a slowly variable potential trough. It is well known from classical mechanics that for such motion the quantity

$$J = \oint p_{\parallel}\, ds = \oint mv_{\parallel}\, ds, \tag{72}$$

of which (71) is a special case, is an adiabatic invariant (see, for example, Landau and Lifshitz, 1960; cf. also Vandervoort, 1961).

We now turn our attention to charged particles in magnetic fields. If a charged particle is trapped between two magnetic mirrors, the motion of its guiding centre along the lines of force is similar to the motion of a particle in a potential trough (this was shown in § 2.3.6), and there exists an invariant corresponding to (72), namely

$$J = \oint mu_{\parallel}\, ds, \tag{73}$$

where the velocity $u_{\parallel}$ and the path element ds now refer to the longitudinal motion of the guiding centre. The quantity given by (73) is called the *longitudinal invariant* (see, for example, Northrop and Teller, 1960). When (58) is valid (73) can be written

$$J = p_0 \oint \sqrt{\left(1-\frac{B}{B_0}\sin^2\alpha-\frac{2me(V-V_0)}{p_0^2}\right)}\, ds, \tag{74}$$

and if only magnetic forces are acting we have, by (59) and (60),

$$J = p_0 \oint \sqrt{(1-B/B_1)}\, ds. \tag{75}$$

The longitudinal invariance requires a slow variation in the magnetic field. This implies that the field should not change appreciably during one period, T_{L}, of the longitudinal oscillation, i.e.

$$\frac{T_{\mathrm{L}}}{B}\left|\frac{\partial B}{\partial t}\right| \ll 1. \tag{76}$$

Because $T_{\mathrm{L}} \gg T_g$, this is a much more stringent condition of slow field variation than 2.3 (1) to (3).

Let us apply the longitudinal invariant to the following simple case. A magnetic field is homogeneous, $B = B_0$ between $-z_0$ and $+z_0$. For $z > +z_0$ the field at the axis of symmetry is

$$B = B_0+b(z-z_0)$$

and for $z < -z_0$ the conditions are symmetric. We assume that

$$bz_0 \gg B_0. \tag{77}$$

A particle spirals around the axis of symmetry and is reflected by the two mirrors, which because of (77) are situated very close to $+z_0$ and $-z_0$. Neglecting the small contributions from the regions $z > z_0$ and $z < -z_0$ to the integral in (75), we find

$$J = p_0\sqrt{(1-B_0/B_1)}.2z_0 = 2z_0 p_\parallel, \tag{78}$$

where $\quad p_\parallel = p_0 \cos\alpha = p_0\sqrt{(1-\sin^2\alpha)} = p_0\sqrt{(1-B_0/B_1)}.$

Suppose now that the mirrors approach each other very slowly, so that after some time they are situated at $\pm z_0'$. As J is an invariant we have

$$p_\parallel' = p_\parallel \frac{z_0}{z_0'}. \tag{79}$$

The invariant $p_\perp^2/B$ also remains constant so that

$$p_\perp' = p_\perp, \tag{80}$$

which means that the path always encloses the same magnetic flux. The angle between the trajectory and the z-axis changes in the following way:

$$\tan\alpha' = \frac{p_\perp'}{p_\parallel'} = \frac{z_0'}{z_0}\frac{p_\perp}{p_\parallel} = \frac{z_0'}{z_0}\tan\alpha. \tag{81}$$

When the mirror separation has been reduced from $2z_0$ to $2z_0'$, the energy of the particle has increased from

$$W = \frac{1}{2m}(p_\perp^2+p_\parallel^2) \tag{82}$$

to

$$W' = \frac{1}{2m}(p_\perp'^2+p_\parallel'^2) = \frac{1}{2m}\left\{p_\perp^2+\left(\frac{z_0}{z_0'}\right)^2 p_\parallel^2\right\}. \tag{83}$$

By means of the longitudinal invariant we have obtained the result (83) without looking into how the acceleration takes place. If we study the details of the acceleration we shall find that a 'collision' with a magnetic mirror moving with the velocity U, brings about an energy change

$$\Delta W = 2Up_\parallel, \tag{84}$$

just as in the case with the perfectly reflecting planes. In the present connexion the mirror velocity U is defined as the velocity of a local moving coordinate system in which $\partial\mathbf{B}/\partial t = 0$. (This is a meaningful definition as long as the structure of each mirror does not change appreciably. It is an example where the 'velocity of a magnetic field' is a useful concept.)

The acceleration of the particle colliding with a moving magnetic mirror is in fact intimately related with betatron acceleration. We shall deduce the expression (84) in a way that throws light on this relationship.

In terms of electric and magnetic fields in a coordinate system 'at rest' the 'moving mirror' is a region where the magnetic field has a radial component B_R, a z-component B_z, and an azimuthal electric field E_φ associated with the time variation of B_z and having the value

$$E_\varphi = (U/c)B_R. \tag{85}$$

A charged particle which enters the mirror region with a momentum $p_\parallel$ experiences a force

$$f_z = -\frac{e\,v_\varphi}{c} B_R \tag{86}$$

which retards the axial velocity. After a time t given by

$$-2p_\parallel = \int_0^t f_z\,dt = -\frac{e}{c}\int_0^t v_\varphi\,B_R\,dt \tag{87}$$

the particle has changed its axial momentum from $p_\parallel$ to $-p_\parallel$ and is thrown out of the mirror again. During the stay in the mirror the particle is also subject to an electric force eE_φ, which, in contrast to the magnetic forces, changes its energy (by betatron acceleration). The energy change amounts to

$$\Delta W = \int_0^t eE_\varphi\,v_\varphi\,dt = \frac{eU}{c}\int_0^t v_\varphi\,B_R\,dt = 2Up_\parallel. \tag{88}$$

Although this energy is primarily fed into the momentum component $p_\perp$, it finally appears as an increase in $p_\parallel$, the transfer taking place by magnetic forces, which do not change $p_\perp^2+p_\parallel^2$. Thus when the particle is out of the mirror we have

$$2Up_\parallel = \frac{\Delta p_\parallel^2}{2m} = \frac{p_\parallel\,\Delta p_\parallel}{m}, \tag{89}$$

so that $\Delta p_\parallel = 2mU$ as before.

The process by which a particle increases its energy when reflected against a moving mirror is the basic phenomenon of the *Fermi process* for acceleration of charged particles, cf. § 2.7.1.

The longitudinal invariant can be used for defining a *coordinate system* which is suitable for mapping distributions of geomagnetically trapped particles (McIlwain 1961).

III. *The flux invariant*

When the guiding centre is situated off the axis of a rotationally symmetric mirror field, the radial gradient of the magnetic field causes a drift around this axis in addition to the oscillatory motion along the lines of force. (Compare the discussion in § 2.5.2 on particle motion in the earth's dipole field.) In the absence of electric fields and other disturbances the guiding centre traces out a rotational surface, the *longitudinal invariant surface*. The magnetic flux enclosed by this surface is also an adiabatic invariant (Northrop and Teller, 1960).

The requirement of slow field variation is in this case of the same form as (76) but with T_L replaced by the still larger time T_S needed for the guiding centre to drift once around the longitudinal invariant surface. For an electron of 10^5 eV energy oscillating in the geomagnetic field along lines of force between the north and south auroral zones (latitude about 68°), we find from equations 1.2 (8) and 2.5.2 (15) and Figs. 2.8 and 2.9 that T_S is about 4000 times larger than T_L, which is of the order of seconds (its value depends on the angle α, cf. 2.3.6 (57)). In comparison to this the gyro period is very short, less than a millisecond.

2.3.8. *Experiments and observations*

The experimental study of the single-particle drift motions treated above requires that the density of particles is so small that collisions are unimportant. The trapping of charged particles between magnetic mirrors has been investigated experimentally. An extremely long confinement time has been observed by Gibson, Jordan, and Lauer (1960) who found that positrons (from radioactive decay) remained trapped for more than 10 sec. This means that the particles performed more than 10^{10} gyrations, and of the order of 10^8 longitudinal oscillations before they finally escaped due to cumulative small-angle scattering against residual gas molecules. Similar experiments have also been performed by Rodionov (1959).

In the geomagnetic field particles are trapped very efficiently. In the earth's radiation belts (Van Allen belts) electrons of energies up to at least some hundred kilovolts and ions at as much as one hundred megavolts are trapped during weeks or months. Experiments with injection of charged particles from an atom bomb exploding above the atmosphere show that due to the inhomogeneity of the dipole field the charged particles released by the explosion drift around the earth and form a shell (Project 'Argus', Christofilos, 1959). The trapping of charged particles in the terrestrial field is discussed in § 2.6.3.

2.4. Motion in a magnetic dipole field. Störmer's method

The general case of the motion of a charged particle in a dipole field has been treated by Störmer. Only a brief account will be given here. For a detailed study the reader is referred to Störmer's monograph, *The Polar Aurora* (Störmer, 1955).

If the magnetic field from a dipole with moment $\mathbf{a}$ is $\mathbf{B}$, and e, m, and $\mathbf{v}$ are the charge, mass, and velocity of the particle, the equation of motion is

$$m\frac{d\mathbf{v}}{dt} = \frac{e}{c}\mathbf{v}\times\mathbf{B}. \tag{1}$$

Introducing cylindrical coordinates (R, φ, z), with the z-axis parallel to the dipole, we have for the components of (1)

$$\frac{d^2R}{dt^2} - R\left(\frac{d\varphi}{dt}\right)^2 = \frac{e}{mc}B_z\,R\frac{d\varphi}{dt}, \tag{2}$$

$$\frac{d^2z}{dt^2} = -\frac{e}{mc}B_R\,R\frac{d\varphi}{dt}, \tag{3}$$

$$\frac{d}{dt}\left(R^2\frac{d\varphi}{dt}\right) = \frac{eR}{mc}\left(B_R\frac{dz}{dt} - B_z\frac{dR}{dt}\right). \tag{4}$$

We put $\mathbf{ds} = \mathbf{v}\,dt$ and introduce the Störmer length c_{St},

$$c_{\mathrm{St}} = \left(\frac{|e|a}{mcv}\right)^{\frac{1}{2}} = \left(\frac{|e|a}{cp}\right)^{\frac{1}{2}} = \left(\frac{a}{B\rho}\right)^{\frac{1}{2}}, \tag{5}$$

where $p = mv$ is the momentum. We shall discuss the case of a positive particle. For negative particles the paths are the mirror images, with respect to a plane through the z-axis, of the paths of positive particles.

For the purely mathematical discussion it is convenient to express all lengths in c_{St} as unit, but for the physical application it is preferable to have this quantity included in the formulae.

As long as only magnetic forces are acting, the following formulae remain valid for relativistic particles, if for m we use the relativistic mass.

Equations (4) and 1.2 (14) to (16) give for a positive charge

$$\frac{d}{ds}\left(R^2\frac{d\varphi}{ds}\right) = c_{\mathrm{St}}^2\left(\frac{3R^2z}{r^5}\frac{dz}{ds} + \frac{r^2-3z^2}{r^5}R\frac{dR}{ds}\right)$$

$$= -c_{\mathrm{St}}^2\left\{\frac{\partial}{\partial z}\left(\frac{R^2}{r^3}\right)\frac{dz}{ds} + \frac{\partial}{\partial R}\left(\frac{R^2}{r^3}\right)\frac{dR}{ds}\right\}$$

with $r = (z^2+R^2)^{\frac{1}{2}}$. After integration we obtain

$$\frac{R^2}{c_{\mathrm{St}}}\frac{d\varphi}{ds} = -c_{\mathrm{St}}\frac{R^2}{r^3}-2\gamma, \tag{6}$$

where 2γ is an integration constant proportional to the angular momentum at infinity. Observing that

$$R^2\left(\frac{d\varphi}{ds}\right)^2+\left(\frac{dR}{ds}\right)^2+\left(\frac{dz}{ds}\right)^2 = 1,$$

we obtain from (2), (3), and (6)

$$\frac{1}{c_{\mathrm{St}}^4}\frac{d^2R}{ds^2} = \left(\frac{2\gamma}{c_{\mathrm{St}}R}+\frac{R}{r^3}\right)\left(\frac{2\gamma}{c_{\mathrm{St}}R^2}+\frac{3R^2}{r^5}-\frac{1}{r^3}\right), \tag{7}$$

$$\frac{1}{c_{\mathrm{St}}^4}\frac{d^2z}{ds^2} = \left(\frac{2\gamma}{c_{\mathrm{St}}R}+\frac{R}{r^3}\right)\frac{3Rz}{r^5}, \tag{8}$$

$$\left(\frac{dR}{ds}\right)^2+\left(\frac{dz}{ds}\right)^2 = 1-\left(\frac{2\gamma}{c_{\mathrm{St}}R}+\frac{R}{r^3}\right)^2 c_{\mathrm{St}}^4. \tag{9}$$

Putting the velocity component $Rd\varphi/dt$ $(= Rv d\varphi/ds)$ equal to $v\sin\vartheta$, we have

$$\sin\vartheta = R\frac{d\varphi}{ds} = -\left(\frac{c_{\mathrm{St}}^2 R}{r^3}+\frac{2\gamma c_{\mathrm{St}}}{R}\right). \tag{10}$$

As $|\sin\vartheta| \leqslant 1$, we obtain

$$\left|2\gamma\frac{c_{\mathrm{St}}}{R}+\frac{R}{r}\frac{c_{\mathrm{St}}^2}{r^2}\right| \leqslant 1. \tag{11}$$

Störmer has shown that the value of γ determines the character of the orbits. If $\gamma > 0$ the orbits never reach the dipole. For $-1 \leqslant \gamma \leqslant 0$, particles can move up to the dipole from infinity. One of the orbits for $\gamma = -1$ is a circle in the equatorial plane with $r = c_{\mathrm{St}}$. If $\gamma < -1$, there are two different allowed regions, one outer region ($r > c_{\mathrm{St}}$) far away from the dipole and one inner region ($r < c_{\mathrm{St}}$) close to the dipole. The latter group contains those periodic (or quasi-periodic) orbits which may be treated by the perturbation method.

The conditions in the equatorial plane ($r = R$) are simple. Particles reach the equator at an angle ϑ given by (10)

$$\sin\vartheta = -\frac{c_{\mathrm{St}}}{R}\left(2\gamma+\frac{c_{\mathrm{St}}}{R}\right). \tag{12}$$

If the momentum is given, c_{St} is defined. For $c_{\mathrm{St}} > R$ and for a given value of ϑ we have an orbit coming from infinity if $\gamma > -1$, and a periodic orbit (never leaving the neighbourhood of the earth) if $\gamma < -1$.

The boundary between orbits from infinity and periodical orbits is given by $\gamma = -1$. For $\sin\vartheta = +1$ we have

$$R = c_{\mathrm{St}}. \tag{13}$$

According to (5) this corresponds to a momentum

$$p_2 = \frac{ea}{cR^2} = 6\times10^{10}\left(\frac{R_{\mathrm{e}}}{R}\right)^2 \ \mathrm{eV}/c, \tag{14}$$

where R_{e} is the radius of the earth. Particles above this momentum can reach a point at the equator from all directions. For $\sin\vartheta = -1$ we obtain

$$R(1+\sqrt{2}) = c_{\mathrm{St}}.$$

The corresponding momentum is

$$p_1 = \frac{ea}{cR^2}(3-2\sqrt{2}) = 1{\cdot}0\times10^{10}\left(\frac{R_{\mathrm{e}}}{R}\right)^2 \ \mathrm{eV}/c. \tag{15}$$

Particles below this momentum cannot reach the equator at all. For p-values in the range $p_1 < p < p_2$, particles are allowed within a cone defined by ϑ according to (12).

As will be shown in the next section (§ 2.5) the problem of the motion of charged particles in cosmic magnetic fields can often be very much simplified by using the perturbation method.

Problems concerning *cosmic rays* are an exception to this rule, because at the high energies involved (10^9 to 10^{18} eV) the radius of curvature is not small compared to the radius of the earth. For the calculation of orbits one must then use Störmer's method and in general resort to numerical integration. The pioneering work by Störmer is summarized in his book (Störmer, 1955). Other numerical calculations have been made by Lemaître and Vallarta (1936 *a*, *b*), Firror (1954), Jory (1955), Lüst, Schlüter, and Katterbach (1956), Lüst (1957), McCracken, Rao, and Shea (1962), and many others. Recently reviews have been given by Singer (1958) and Vallarta (1961).

The knowledge of cosmic-ray orbits is of importance in several respects. Calculations of the lowest energy that a particle must have to reach an observer at a certain latitude are essential for the study of the latitude effect, from which it is possible to determine the momentum spectrum of cosmic rays. Further, it is necessary to know the deviation of the particle paths by the geomagnetic field in order to interpret the measurements of cosmic ray telescopes. If at a certain latitude a telescope is directed towards a certain point of the sky, the problem is to calculate the original direction of the particles registered by the telescope (Brunberg, 1958).

Although in many cases the geomagnetic field can be approximated to a dipole field, the influence of quadrupole and higher-multipole terms (§ 1.2) is often noticeable. As a consequence the intensity of cosmic radiation is not strictly a function of latitude only but has also some longitude dependence.

2.4.1. *Model experiments*

The orbits of cosmic rays in a dipole field can also be found by a scale model experiment, in which a narrow electron beam moves in the magnetic field from a 'terrella' and its path is measured (Brüche, 1931; Bennett and Hulburt, 1953, 1954 *a*, *b*). The method has been developed so that measurements can be made with a high degree of accuracy (Malmfors, 1945; Brunberg and Dattner, 1953; Brunberg, 1956, 1958) and in this way a comprehensive survey of cosmic ray orbits in a dipole field has been obtained.

Recently model experiments have been made using a terrella with a magnetic field that simulates also some of the deviations of the earth's magnetic field from that of a dipole (Quenby, Wenk, and Bland, 1962).

2.5. Motion in a magnetic dipole field. The perturbation method

2.5.1. *On the applicability of the perturbation method*

The condition for the perturbation method to be valid is given by 2.3 (1) to (3). In a time-constant dipole field this is essentially equivalent to the requirement that the Larmor radius ρ must be small compared with the distance r to the dipole. Let this condition be

$$\rho/r < 10^{-2} \tag{1}$$

and consider to what extent this restricts the use of the perturbation method in cosmic physics.

If the magnetic field B is due to a dipole at distance r, we have $B \geqslant ar^{-3}$. Further, we have according to 2.2.1 (8),

$$\rho = cp_{\perp}/|e|B \leqslant cr^3p/|e|a.$$

Consequently the criterion (1) is satisfied if

$$r < 0{\cdot}1(|e|a/cp)^{\frac{1}{2}} = 0{\cdot}1c_{\mathrm{St}}. \tag{2}$$

Fig. 2.7 is a graph of c_{St} as a function of energy for protons and electrons in the geomagnetic field. It shows that for electrons with energy as high as about 10^5 eV (which is probably the upper limit of energy of the auroral particles) the Störmer length is larger than 10^{11} cm, which means that the perturbation method is applicable out to a distance of 10^{10} cm from the dipole. However, the geomagnetic field resembles a dipole field only out to $0{\cdot}5–0{\cdot}8 \times 10^{10}$ cm. The electrons in the earth's radiation belts are situated at a distance of a few earth radii and have energies up to the order of 1 MeV. The protons in the inner radiation belt are situated at about 1·5 earth radii from the centre of the earth. The condition (2) may be violated for the highest-energy protons but

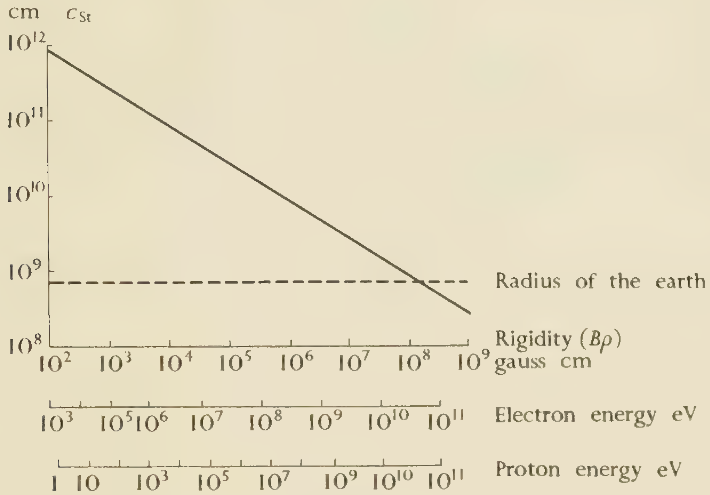


FIG. 2.7. Diagram of c_{St} for electrons and protons in the geomagnetic dipole field.

they are very few in number. For the lower-energy particles, which constitute the majority, the condition is reasonably well satisfied.

In addition to the energetic particles trapped in the geomagnetic field there are also low-energy particles present. These particles form a background plasma of much higher number density than the radiation particles. This plasma reaches the earth as magnetized beams from the sun (see §§ 2.6.3 and 5.8), and at its encounter with the geomagnetic field it produces the currents that are responsible for the magnetic storm effects. For these low-energy particles the perturbation method is always valid and forms a convenient tool for analysing the phenomena of magnetic storms and aurorae.

Consequently, in almost all problems of cosmic physics—except cosmic rays—the motion of charged particles can be treated by the perturbation method.

2.5.2. *The drift motion in a dipole field*

We first calculate for a dipole field the components of grad B parallel and perpendicular to the field lines (using 1.2 (5), (7), (10), and (11)):

$$(\operatorname{grad} B)_{\parallel} = \frac{\partial B}{\partial r}\cos\alpha - \frac{1}{r}\frac{\partial B}{\partial\lambda}\sin\alpha = -\frac{3B}{r}\frac{\sin\lambda(3+5\sin^2\lambda)}{(1+3\sin^2\lambda)^{\frac{3}{2}}}, \tag{3}$$

$$(\operatorname{grad} B)_{\perp} = \frac{\partial B}{\partial r}\sin\alpha + \frac{1}{r}\frac{\partial B}{\partial\lambda}\cos\alpha = -\frac{3B}{r}\frac{\cos\lambda(1+\sin^2\lambda)}{(1+3\sin^2\lambda)^{\frac{3}{2}}}, \tag{4}$$

cf. Fig. 1.1.

Next we introduce a local cartesian coordinate system with the z-axis parallel to B and the y-axis parallel to $(\operatorname{grad} B)_{\perp}$. Application of the formulae 2.3.5 (49) and (50) gives the total drift perpendicular to the field (including the effect of the centrifugal force):

$$r\cos\lambda\frac{d\varphi}{dt} = -u_x = \frac{c}{eB}\frac{m}{B}\left(\frac{v_{\perp}^2}{2}+v_{\parallel}^2\right)\frac{\partial B}{\partial y}, \tag{5}$$

$$u_y = 0. \tag{6}$$

For the motion along the lines of force we have by 2.3.4 (41) to (43)

$$\frac{du_z}{dt} = \frac{du_{\parallel}}{dt} = -\frac{v_{\perp}^2}{2B}\frac{\partial B}{\partial z}. \tag{7}$$

The integration of (7) leads to the result 2.3.6 (59), which we shall write here in the slightly different form

$$v_{\parallel} = \sqrt{\{2\mu(B_1-B)/m\}}, \tag{8}$$

where B_1 is the field strength at the mirror-points. Using 1.2 (12) and 1.2 (13) we can write

$$B = \frac{1}{\mu}\frac{mv_{\perp}^2}{2} = \frac{a\eta}{r_e^3}, \tag{9}$$

$$B_1 = \frac{1}{\mu}\frac{m(v_{\perp}^2+v_{\parallel}^2)}{2} = \frac{a\eta_0}{r_e^3} \tag{10}$$

On the other hand, we obtain from 1.2 (8)

$$v_{\parallel} = \frac{ds}{dt} = \frac{(dr^2+r^2\,d\lambda^2)^{\frac{1}{2}}}{dt} = r_e\cos\lambda(1+3\sin^2\lambda)^{\frac{1}{2}}\frac{d\lambda}{dt}. \tag{11}$$

Combination of (8) to (11) yields

$$\frac{d\lambda}{dt} = \left(\frac{2\mu a}{mr_e^5}\frac{\eta_0-\eta}{\cos^2\lambda(1+3\sin^2\lambda)}\right)^{\frac{1}{2}} \tag{12}$$

Further, introducing (4), (9), (10), and 1.2 (8), (13), we transform (5) into

$$\frac{d\varphi}{dt} = \frac{3c\mu}{er_e^2}\frac{1+\sin^2\lambda}{\cos^4\lambda(1+3\sin^2\lambda)^{\frac{1}{2}}}\frac{2\eta_0-\eta}{\eta}. \tag{13}$$

We can now compute the path of the centre of gyration or 'equivalent magnet', cf. p. 203. Introducing Störmer's unit of length

$$c_{\mathrm{St}} = \left(\frac{a|e|}{cmv}\right)^{\frac{1}{2}} = \left(\frac{ae^2r_e^3}{2c^2m\mu\eta_0}\right)^{\frac{1}{4}}. \tag{14}$$

and 1.2 (13) we find
$$\varphi-\varphi_0 = (r_e/c_{\mathrm{St}})^2 I_1, \tag{15}$$

with
$$I_1 = 3\int\limits_0^\lambda \frac{\cos^3\lambda(1+\sin^2\lambda)}{(1+3\sin^2\lambda)^{\frac{1}{2}}}\frac{1-\frac{1}{2}\eta(\lambda)/\eta_0}{\sqrt{\{1-\eta(\lambda)/\eta_0\}}}\,d\lambda, \tag{15'}$$

where r_e is the distance from the dipole to the points where the particle (or more exactly the 'equivalent magnet') crosses the equatorial plane ($\lambda = 0$); $\eta(\lambda)$ is defined by 1.2 (13) and η_0 is a constant (see (10)).

The parameter r_e is related to Störmer's constant γ. For the equatorial plane the condition 2.4 (11) can be written

$$-1 \leqslant \frac{c_{\mathrm{St}}}{R}\left(2\gamma+\frac{c_{\mathrm{St}}}{R}\right) \leqslant +1. \tag{16}$$

As R oscillates between the limits $r_e+\rho$ and $r_e-\rho$ we have ($r_e \lll c_{\mathrm{St}}$):

$$r_e/c_{\mathrm{St}} = -\tfrac{1}{2}\gamma^{-1} \tag{17}$$

and
$$\rho/r_e = (r_e/c_{\mathrm{St}})^2 = \tfrac{1}{4}\gamma^{-2}. \tag{17'}$$

The integral I_1 in (15′) is plotted in Fig. 2.8.

Equation (15) gives the path of the 'equivalent magnet'. The path of the particle itself is a spiral around the curve defined by (15). The spiral has the radius given by 2.2.1 (8). In most of the cases to which our perturbation method of calculation is applicable, (15) gives as much information about the motion of the particle as is wanted.

The motion defined by (12), (13), and (15) takes place on the surface defined by 1.2 (8) and is an oscillation through the equatorial plane $\lambda = 0$, combined with a rotation around the axis of the dipole. The amplitude of the oscillation is defined by the condition $\eta_0-\eta \geqslant 0$. The λ-value of the turning-point is given by

$$\eta_0 = \frac{(1+3\sin^2\lambda_0)^{\frac{1}{2}}}{\cos^6\lambda_0}. \tag{18}$$

Fig. 2.9 shows a comparison between one of the trajectories integrated by Störmer (1913) and the corresponding path found by the perturbation method (Alfvén, 1940).

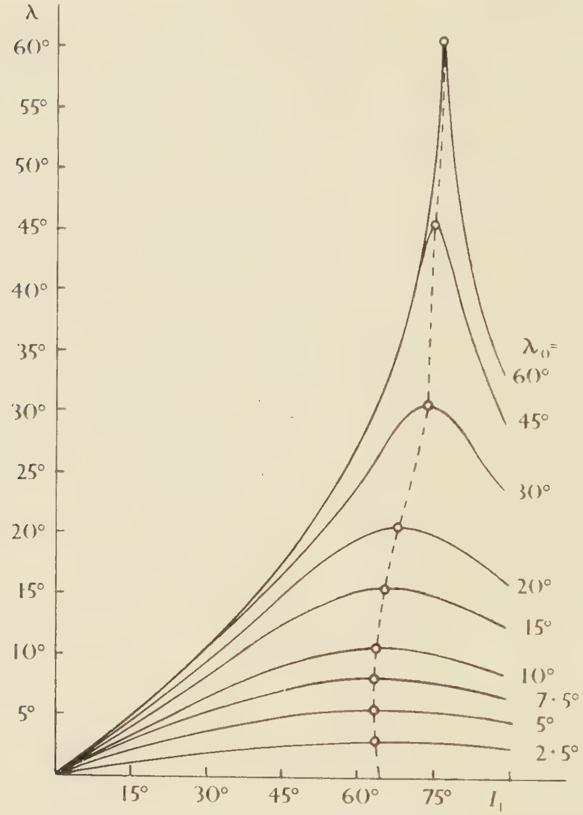


FIG. 2.8. Connexion between displacement in longitude (proportional to I_1) and latitude λ for a particle oscillating through the equatorial plane with amplitude λ_0.

2.5.3. *Motion close to the equatorial plane of a dipole field*

Of particular interest is the special case when the amplitude of the oscillation is small ($\lambda_0 \ll 1$). Then we have approximately

$$\eta = \frac{(1+3\sin^2\lambda)^{\frac{1}{2}}}{\cos^6\lambda} = 1+4{\cdot}5\lambda^2, \tag{19}$$

and in the same way $\eta_0 = 1+4{\cdot}5\lambda_0^2$. Putting these values into (12) and integrating, we obtain a harmonic oscillation

$$\lambda = \lambda_0 \sin\left(2\pi\frac{t-t_0}{T}\right), \tag{20}$$

where
$$T = \frac{2\pi}{3}\left(\frac{mr_e^5}{\mu a}\right)^{\frac{1}{2}} = \frac{2\pi}{3}\sqrt{2}\frac{r_e}{v}. \tag{21}$$

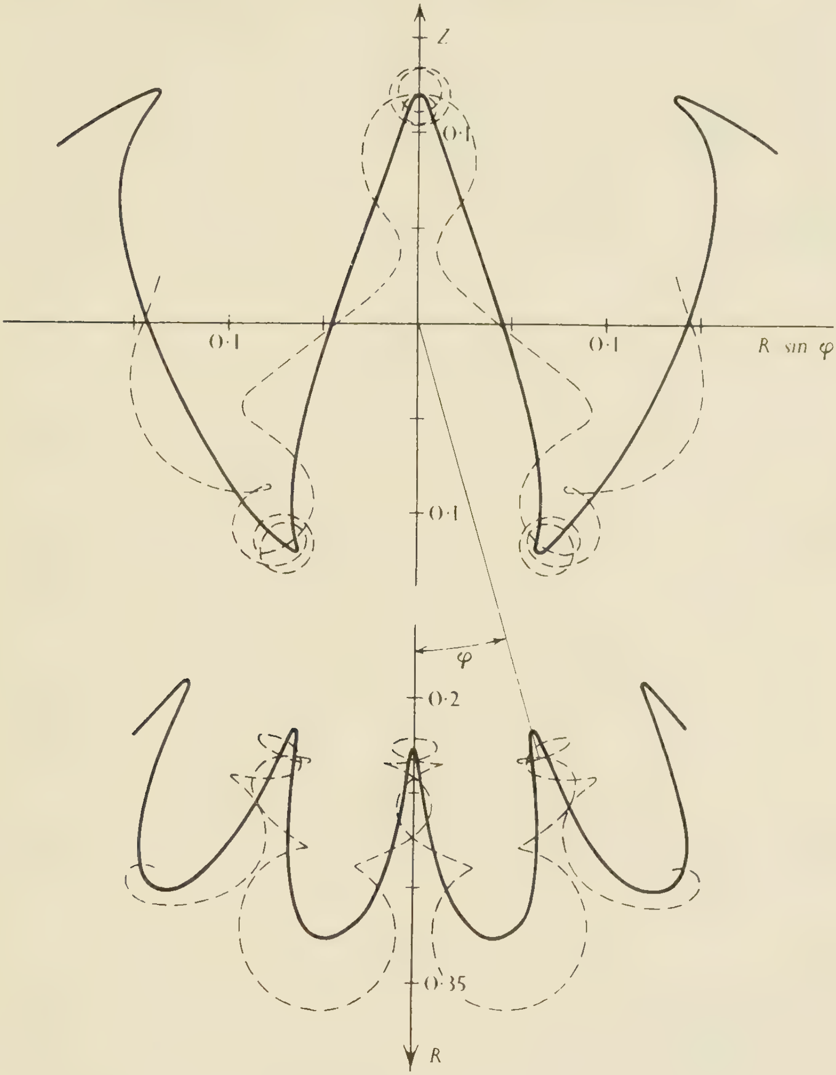


FIG. 2.9. Motion in dipole field calculated by Störmer and by the perturbation method. *Upper figure*: Projection upon a plane through the axis of the dipole. *Lower figure*: Projection upon the equatorial plane. ——— path of equivalent magnet. - - - - - path of the particle according to Störmer.

To the same approximation, φ increases at a constant rate:

$$\varphi = \frac{3c\mu}{er_e^2}(t-t_1) = \frac{3}{2}\frac{r_e}{c_{St}^2}v(t-t_1). \tag{22}$$

During the period T the increase in φ amounts to

$$\Phi = \pi\sqrt{2}\,r_e^2/c_{St}^2, \tag{23}$$

where c_{St} is given by 2.4 (5). For small amplitudes the value of $\Phi(c_{St}/r_e)^2$ is $\pi\sqrt{2}$ ($= 4{\cdot}44$), corresponding to 255°. The broken line in Fig. 2.8 represents the difference in longitude between the turning-point and the intersection of the orbit with the equatorial plane. Its value for $\lambda \to 0$ is $\frac{1}{4}\pi\sqrt{2}$ ($= 1{\cdot}11$), corresponding to 63·6°.

2.5.4. We now have to calculate the parameters η_0, r_e, and c_{St}. Suppose that the particle (mass $= m$) starts at the point (r', λ', φ') with the velocity $(v'_r, v'_\lambda, v'_\varphi)$. Then we have:

$$\phi' = (1+3\sin^2\lambda')^{\frac{1}{2}}, \tag{24}$$

$$v'_\parallel = \frac{2v'_r \sin\lambda' - v'_\lambda \cos\lambda'}{\phi'}, \tag{25}$$

$$v'_\perp = \left(\frac{(v'_r \cos\lambda' + 2v'_\lambda \sin\lambda')^2}{2a\phi'^3} + v'^2_\varphi\right)^{\frac{1}{2}}, \tag{26}$$

$$B' = \frac{a}{r'^3}\phi', \qquad \eta' = (\cos\lambda')^{-6}\phi', \tag{27}$$

$$\mu = \frac{mr'^3(v'_r \cos\lambda' + 2v'_\lambda \sin\lambda')^2 + \phi'^2 v'^2_\varphi}{2a\phi'}, \tag{28}$$

$$r_e = r'(\cos\lambda')^{-2}, \tag{29}$$

$$\eta_0 = \eta'\{1+(v'_\parallel/v'_\perp)^2\}, \tag{30}$$

$$c_{St} = \left(\frac{ae}{mc}\frac{1}{v'}\right)^{\frac{1}{2}}. \tag{31}$$

The particle spirals in a circle of radius ρ which moves according to what is said above. We have

$$\rho = \frac{r_e^3}{c_{St}^2\sqrt{\eta_0}}\frac{1}{\sqrt{\eta'}}. \tag{32}$$

2.5.5. In order to show the connexion between Störmer's equations and the perturbation method, we shall derive (22) through successive approximations, valid if $\gamma \ll -1$. We put

$$R_0 = r_e/c_{St} = -\tfrac{1}{2}\gamma^{-1}. \tag{33}$$

Consequently $R_0 \ll 1$.

As in the equatorial plane $z = 0$, we obtain from 2.4 (9)

$$\left(\frac{dR}{ds}\right)^2 = 1 - c_{St}^4\frac{1}{R^4}\left(1-\frac{R}{r_e}\right)^2. \tag{34}$$

We develop R into a series

$$R = R_0 c_{St}\{1+R_0^2 F(s)+R_0^4 G(s)\} \tag{35}$$

neglecting higher terms. Then we have (denoting d/ds by primes)

$$dR/ds = R_0^3 c_{\mathrm{St}}(F'+R_0^2 G') \tag{36}$$

and

$$1-R/R_0 c_{\mathrm{St}} = -R_0^2(F+R_0^2 G). \tag{37}$$

The first approximation solution of (34) is obtained from

$$R_0^6 c_{\mathrm{St}}^2 (F')^2 = 1-F^2 \tag{38}$$

which gives

$$F = \sin S, \tag{39}$$

where

$$S = \frac{s}{R_0^3 c_{\mathrm{St}}} \tag{40}$$

In order to obtain the second approximation we put (39) into (35), (36), and (37) and obtain from (34)

$$R_0^6 c_{\mathrm{St}}^2(F'^2+2R_0^2 F'G') = 1-(F^2+2R_0^2 FG)(1-4R_0^2 F). \tag{41}$$

Using (39) we get

$$G = 4-2\sin^2 S. \tag{42}$$

Consequently, (35) gives

$$R = R_0 c_{\mathrm{St}}\{1+R_0^2\sin S+R_0^4(4-2\sin^2 S)\}, \tag{43}$$

where terms of the order of R_0^6 in the expression in brackets are neglected.

If (33) is introduced, 2.4 (6) gives

$$\frac{d\varphi}{ds} = -\frac{c_{\mathrm{St}}^2}{r^3}\left(1-\frac{r}{R_0 c_{\mathrm{St}}}\right) = \frac{1}{R_0 c_{\mathrm{St}}}\frac{F+R_0^2 G}{(1+R_0^2 F+R_0^4 G)^3}$$

$$= \frac{1}{R_0 c_{\mathrm{St}}}(F+R_0^2 G)(1-3R_0^2 F) = \frac{1}{R_0 c_{\mathrm{St}}}\{\sin S+R_0^2(4-5\sin^2 S)\}. \tag{44}$$

Here terms of the order of R_0^4 have been neglected. Consequently we obtain

$$\frac{d\varphi}{ds} = \frac{1}{R_0 c_{\mathrm{St}}}\{\sin S+R_0^2(4-5\sin^2 S)\}. \tag{45}$$

This equation defines the motion of the particle. The motion of the equivalent dipole is the average of $d\varphi/ds$. As the average of $\sin S$ is zero and of $\sin^2 S$ is $\frac{1}{2}$, we obtain

$$\frac{d\varphi}{ds} = \frac{1}{c_{\mathrm{St}}} R_0(4-\tfrac{5}{2}) = \tfrac{3}{2}R_0\frac{1}{c_{\mathrm{St}}}. \tag{46}$$

The error is of the order of R_0^3.

This equation is identical with (22),

$$\frac{d\varphi}{dt} = \frac{3c\mu}{er_e^2}, \tag{47}$$

because

$$\frac{d}{dt} = v\frac{d}{ds}, \tag{48}$$

$$c_{St} = (ae/cmv)^{\frac{1}{2}}, \tag{49}$$

$$\mu = \tfrac{1}{2}mv^2r_e^2/a, \tag{50}$$

$$R_0 = r_e/c_{St}. \tag{51}$$

In a similar way the expression for T can be derived.

2.6. Influence of an electric field on the motion of charged particles in a magnetic field

Even a very weak electric field drastically changes the motion of low-energy particles in the geomagnetic field. From 2.3.4 (39) it is evident that the influence of an electric field can be neglected only if $f \ll f^m$, i.e.

$$|eE| \ll |\mu \operatorname{grad} B| = |e|V|\operatorname{grad} B|/B, \tag{1}$$

where $|e|V$ is the energy of the particle, associated with the motion perpendicular to $\mathbf{B}$.

If we introduce the length

$$l_c = B/\operatorname{grad} B, \tag{2}$$

which characterizes the distance in which the magnetic field varies, we find that

$$E \ll V/l_c \tag{3}$$

is the condition which permits us to neglect the influence of an electric field.

In the equatorial plane of a dipole field we have $l_c = R/3$, where R is the distance to the dipole. In the geomagnetic field we find

$$E \ll V\frac{3}{R} \tag{4}$$

or

$$E \ll 5\times 10^{-9}V\frac{R_e}{R} \text{ volt/cm}, \tag{4a}$$

where R_e is the radius of the earth. This means for example that all particles below $V = 2000$ eV are influenced very much by an electric field, unless E is much smaller than 10 μV/cm.

Although there is yet no direct measurement of electric fields in the earth's environment it is likely that electric fields are produced especially in connexion with magnetic storms.

It should also be noticed that the adiabatic drift of charged particles from interplanetary space into the geomagnetic field is possible only if an interplanetary magnetic field exists. This follows from the invariance of the quantity $C = p_\perp^2/B$ (for low-energy particles proportional to μ). If a particle has succeeded in penetrating into the dipole field, where $B \neq 0$, it has a finite value of C and hence it cannot come from a region with $B = 0$. The value of C determines how far the particle can penetrate (§ 2.6.1).

Earlier it has been assumed that the interplanetary magnetic field was zero, and this was the basis of the Chapman–Ferraro theory of magnetic storms. However, from space-probe measurements we now know that there exists an interplanetary magnetic field, the strength of which is usually of the order of 10^{-4} gauss ($= 10$ gamma).

2.6.1. *Motion in the equatorial plane*

We shall now treat a simplified model illustrating the drift of charged particles into a magnetic dipole field. We confine the analysis to the equatorial plane and assume that there is an interplanetary magnetic field B_0 which is homogeneous and has the same direction as the dipole field (z-direction), so that

$$B = B_0 + a/R^3, \tag{5}$$

where

$$R = (x^2+y^2)^{\frac{1}{2}}. \tag{6}$$

(Essential results of our analysis are valid even if the interplanetary field is not parallel with the dipole field, because the importance of B_0 is mainly to define the invariant C, or μ.)

We study the motion of a particle which at infinite distance has the momentum $p = mv_{\perp 0}$ so that the invariant magnetic moment is

$$\mu = \frac{mv_\perp^2}{2B} = \frac{mv_{\perp 0}^2}{2B_0}. \tag{7}$$

It is assumed that the component of $\mathbf{v}$ parallel to $\mathbf{B}$ is zero and that the electric field $\mathbf{E}$ is homogeneous and points in the x-direction. Then at $y = +\infty$ the particle drifts with the velocity

$$u_y = \frac{dy}{dt} = -cE/B_0. \tag{8}$$

A particle which at $y = +\infty$ has the x-coordinate x_0 and the energy $|e|V_0 = mv_{\perp 0}^2/2$ will later be found at the point (x, y) with the energy

$$|e|V = |e|V_0 + eE(x - x_0). \tag{9}$$

In the non-relativistic case we have

$$\frac{|e|V_0}{B_0} = \mu = \frac{|e|V_0+eE(x-x_0)}{B_0+aR^{-3}}$$

which gives
$$x_0-x = L^4R^{-3} \tag{10}$$

with
$$L = \left(\frac{\mu a}{|e|E}\right)^{\frac{1}{4}}. \tag{11}$$

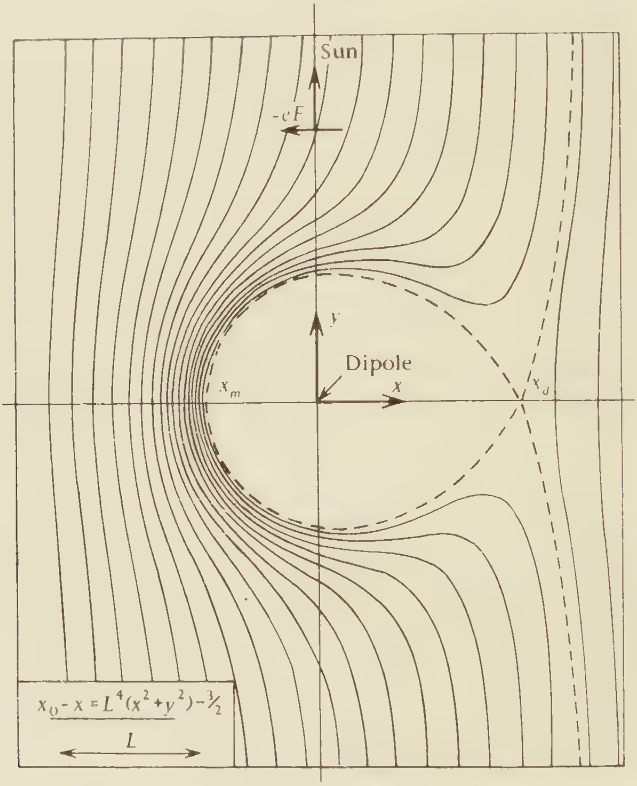


FIG. 2.10. Drift motion of electrons in the equatorial plane of a magnetic dipole field with a superimposed homogeneous magnetic field in the presence of a homogeneous electric field parallel to the equatorial plane. The curves show typical paths of gyration centres.

The curves corresponding to different values of x_0 are shown in Fig. 2.10. It is evident that electrons coming from the sun pass on the morning side if their x_0 value is larger than a certain critical value x_D, but on the evening side if $x_0 < x_D$. Consequently a stream of electrons becomes divided at a certain point x_d. The morning branch passes the earth at a rather large distance, but the evening branch encircles it and passes at a minimum distance x_m from the dipole. A simple calculation from (10) gives

$$x_D = (4\sqrt[4]{3})L/3 = 1{\cdot}76L, \tag{12}$$

$$x_d = \sqrt[4]{3}\,L = 1{\cdot}32L, \tag{13}$$

$$x_m = -0{\cdot}74L. \tag{14}$$

Between the two branches there is a 'forbidden region'. The size of this region is determined by the value of the length unit L which depends on μ, E, and a in accordance with (11). The limit of the forbidden region is indicated by a broken line in Fig. 2.10. Its equation is

$$\tfrac{4}{3}\sqrt[4]{3}\,L - x = L^4R^{-3}. \tag{15}$$

Inside the forbidden region particles move in closed orbits, which become more similar to circles near the origin. Particles moving in these orbits are captured, and cannot go to infinity. The Van Allen particles belong to this category.

Positive particles move in similar orbits which are the mirror images, with respect to the yz-plane, of the orbits in Fig. 2.10.

2.6.2. *Oscillations along the lines of force*

When the particles oscillate along the lines of force, the amplitude of the oscillations is determined by the longitudinal invariant

$$J = m\int v_{\parallel}\,ds. \tag{16}$$

We shall treat the case when the electric field parallel to $\mathbf{B}$ is zero: $E_{\parallel} = 0$.

With the help of 2.5.2 (11) and (12), and

$$ds = r_e\,d\lambda\cos\lambda(1+3\sin^2\lambda)^{\frac{1}{2}},$$

we find
$$J = m\int v_{\parallel}^2\,dt = mr_e^2\int \cos^2\lambda(1+3\sin^2\lambda)\frac{d\lambda}{dt}\,d\lambda$$

$$= \left(\frac{2m\mu a}{r_e}\right)^{\frac{1}{2}}\int_0^{\lambda_0}(1+3\sin^2\lambda)^{\frac{1}{2}}\cos\lambda\sqrt{(\eta_0-\eta)}\,d\lambda \tag{17}$$

or
$$J = \left(\frac{2m\mu a}{r_e^3}\right)^{\frac{1}{2}}\int_0^{s_0}\sqrt{(\eta_0-\eta)}\,ds, \tag{18}$$

where
$$\eta = (1+3\sin^2\lambda)^{\frac{1}{2}}\cos^{-6}\lambda. \tag{19}$$

For small amplitudes ($\lambda_0 \ll \frac{1}{2}\pi$) the integral in (17) is proportional to λ_0^2, so that $\lambda_0^2 r_e^{-\frac{1}{2}}$ must be invariant. Hence, when drifting closer to the dipole under the influence of the electric field the particles change their amplitude $s_0 = r_e \lambda_0$ of oscillation so that

$$s_0 = \text{const}\, r_e^{\frac{5}{4}}, \tag{20}$$

$$\lambda_0 = \text{const}\, r_e^{\frac{1}{4}}. \tag{21}$$

This means that the maximum value of $v_{\parallel}$

$$v_{\parallel}^{\max} \propto r_e^{-\frac{5}{4}}. \tag{22}$$

The maximum parallel energy

$$W_{\parallel}^{\max} \propto r_e^{-\frac{5}{2}}, \tag{23}$$

whereas $W_{\perp}$ is proportional to B and hence

$$W_{\perp} \propto r_e^{-3}. \tag{24}$$

We have seen that the value of $v_{\perp}$ in the dipole field is determined by the value which the magnetic moment has in the interplanetary field B_0, before the particle has reached the dipole field. Contrary to this the value of $v_{\parallel}$ during the motion in the dipole field is not determined by interplanetary conditions, because longitudinal oscillations are possible only when the particle has entered the magnetic-mirror region near the earth. The value of the longitudinal invariant is set at the transition from the interplanetary field to the dipole field, and is determined by the value of $v_{\parallel}$ when the particle begins its first oscillation in the dipole field.

2.6.3. *Allowed and forbidden regions for charged particles in a magnetic dipole field*

Although there are no direct measurements of electric fields in the magnetosphere, we should expect that at least under disturbed conditions electric fields are usually present. If this is the case, charged particles can pass between interplanetary space and the magnetosphere in two different ways:

(1) along *Störmer orbits*, if the particle energy is *above* a certain value;

(2) along *drift orbits*, if the particle energy is *below* a certain value.

We shall exemplify this by treating the motion in the equatorial plane of the field introduced in § 2.6.1,

$$B = B_0 + a/R^3,$$

$$E = E_x = \text{const}.$$

The problem can be considered from two different aspects: A, penetration of particles from infinity; B, capture of particles near the dipole.

A. *How far can interplanetary particles penetrate?* (Fig. 2.11)

1. *High-energy particles* (Störmer orbits). If B_0 is neglected, a particle from infinity with energy $|e|V_0$ can reach a distance R from a dipole a if its momentum is above the limit (see 2.4 (15))

$$p_1 = 1{\cdot}0\times 10^{10}(R_e/R)^2 \ \ \mathrm{eV}/c, \tag{25}$$

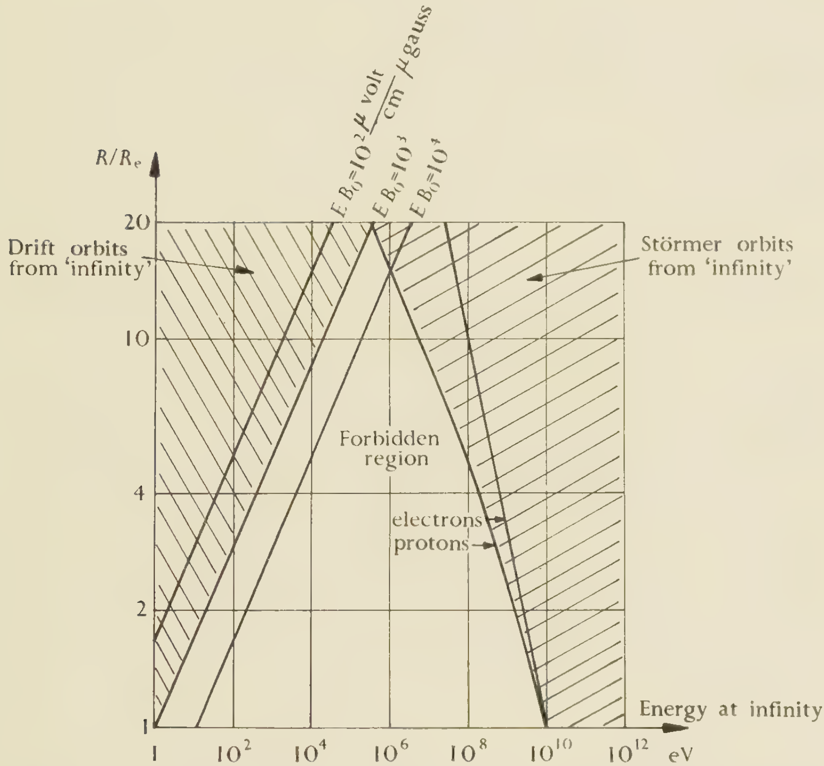


FIG. 2.11. Forbidden regions for charged particles moving in the equatorial plane of the geomagnetic dipole field and a superposed homogeneous interplanetary magnetic field (of strength B_0), when an electric field **E**, parallel to the equatorial plane, is present. The lines show the radius of the forbidden region as a function of particle energy. For the low-energy particles, which approach the dipole along drift orbits, the radius depends also on the product EB_0. Lines are drawn for three different values of this parameter.

where R_e is the radius of the earth. (If $B_0 \neq 0$ the allowed orbits do not go to infinity.)

2. *Drift orbits.* A low-energy particle with energy eV_0 can enter from infinity to a distance from the dipole which is (see 2.6.1 (11) and (14))

$$R \approx L = (a\mu/\,|e|E)^{\frac{1}{4}}, \tag{25a}$$

where the magnetic moment $\mu = |e|V_0/B_0$ is invariant. Introducing the radius of the earth $R_e = 6{\cdot}4\,.\,10^8$ cm and the dipole moment $a = 8{\cdot}1\,.\,10^{25}$ gauss cm^3, we obtain

$$\frac{R}{R_e} = 4{\cdot}7\,.\,10^{-3}\left(\frac{V_0}{EB_0}\right)^{\frac{1}{4}}, \tag{26}$$

where V_0 is the energy at infinity in eV, E is the electric field in V/cm; B_0 is the magnetic field in gauss. Example: Suppose that the interplanetary magnetic field is $5\gamma = 50\,\mu$gauss and the electric field is $10\,\mu$V/cm. If a particle has the energy 500 eV we find $V_0/EB_0 = 10^{12}$ in interplanetary space so that it can penetrate to 4·7 times the earth's radius.

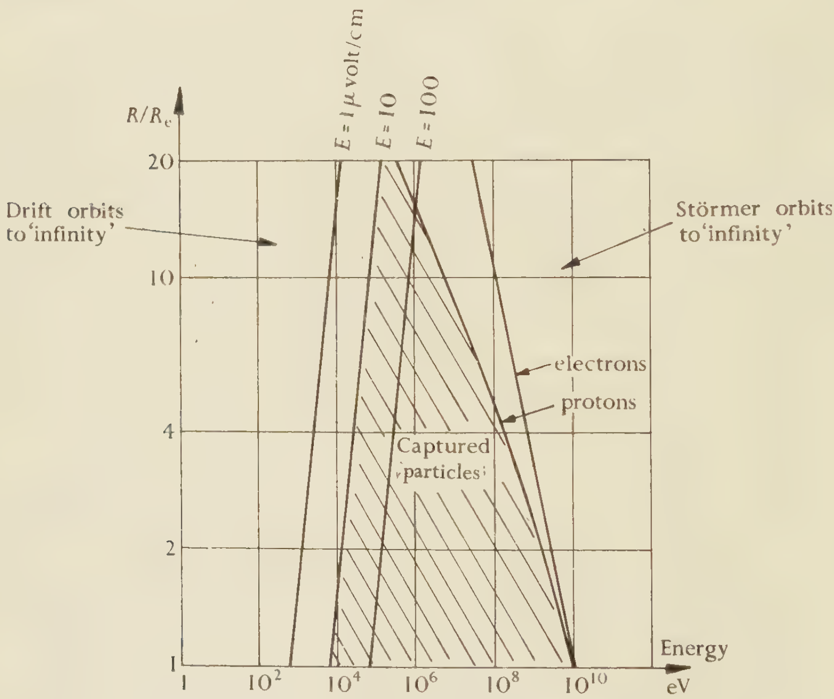


Fig. 2.12. Regions of periodic orbits in the equatorial plane. Cf. the caption to Fig. 2.11.

B. *Where can particles remain captured?* (Fig. 2.12)

As $\mu = |e|V/B$ is invariant a particle will increase its energy when drifting into the dipole field. The energy $|e|V_0$ at infinity should not be confused with the instantaneous energy $|e|V$ of the particle when moving in the field B, because work is done by the electric field during the drift motion. If we introduce V instead of V_0 we obtain from (25a)

$$R = \left(\frac{a|e|V}{|e|EB}\right)^{\frac{1}{4}} = \left(\frac{R^3 V}{E}\right)^{\frac{1}{4}}, \tag{27}$$

or simply

$$\frac{R}{R_e} = \frac{V}{R_e E} = 1 \cdot 6 . 10^{-9} \frac{V}{E}. \tag{28}$$

R is the maximum radius at which a particle with an energy of V eV can remain captured, when the electric field is E V/cm. Example: A 10^4 eV particle captured at sixteen times the earth's radius will drift to infinity as soon as the electric field exceeds 1 μV/cm.

2.7. Acceleration of charged particles in varying magnetic fields

In the polar aurora particle energies of 10^4–10^5 eV are observed. In the Van Allen belts energies of this magnitude, or even a few orders of magnitude larger, are found. In the cosmic radiation there are particles with energies up to 10^{18} eV. From this we conclude that there must exist acceleration mechanisms in space. Such mechanisms may conceivably be operating in the magnetosphere of the earth, in interplanetary, interstellar, and possibly also in intergalactic space. There are different opinions about the details of the mechanisms and about their location in space. However, it is generally agreed that the fundamental nature of the acceleration process is interaction of charged particles with varying magnetic fields, as first suggested by Swann, 1933. A discussion of the origin of cosmic rays and other corpuscular radiation is beyond the scope of this book. (Surveys of this subject have recently been given by Ginzburg and Syrovatskii, 1960, and by Morrison, 1961.)

2.7.1. *Survey of adiabatic momentum changes*

Charged particles moving in magnetic fields which change in time or space may, under certain conditions, be accelerated. We shall here limit ourselves to slowly varying magnetic fields so that the results derived in § 2.3 are valid.

The variation in the *momentum component* $p_\perp$ is simply determined by the variation of the magnetic field strength B where the particle is situated. This follows from the invariance of the quantity

$$C = p_\perp^2 / B. \tag{1}$$

In the general case B varies both in space and time. However, it is worth pointing out that acceleration of particles can occur even if the magnetic field is constant in time, provided it varies in space. A change in B takes place when the charged particle drifts into a region with different field strength. This drift can be produced by an electric field as in the example treated in § 2.6.1. As the particle drifts into a magnetic field of different strength, its guiding centre is displaced to a different equipotential surface of the electric field, and this accounts for the change in energy.

For the *momentum component* $p_{\|}$ the situation is more complicated. Let us distinguish between two cases.

A. Particles that are *not trapped* between two magnetic mirrors may now and then encounter a moving mirror. In so doing a particle with a momentum $p_{\|}$ parallel to $\mathbf{B}$ changes its energy by the amount

$$\Delta W = 2Up_{\|} \tag{2}$$

as shown in § 2.3.7. This energy change is positive or negative depending on the sign of U. As U is the velocity of the magnetized cloud with which the magnetic mirror is associated, its value is always much less than the velocity of light. For a highly relativistic particle the energy is

$$W \approx c(p_{\|}^2+p_{\perp}^2)^{\frac{1}{2}}, \tag{3}$$

and it follows that the relative energy change $\Delta W/W$ is very small:

$$|\Delta W/W| = 2Up_{\|}c^{-1}(p_{\|}^2+p_{\perp}^2)^{-\frac{1}{2}} < 2U/c. \tag{4}$$

The process by which a particle increases its energy when reflected against a moving mirror is the basic phenomenon of the *Fermi process* for acceleration of charged particles (Fermi, 1949, 1954; see also Davis, 1956). This has sometimes been thought to be the most important mechanism for acceleration of cosmic rays and other high-energy particles. However, the process is rather specialized because it implies that only the special type of electric field given in 2.3.7 (85) is active. In reality a charged particle can pick up energy from many other types of electrical field, which are necessarily produced by various motions and waves (including shock waves) in the interplanetary and interstellar plasma.

B. Particles that are *trapped* between two magnetic mirrors experience repeated encounters with them. The resultant effect of these can be found by means of the longitudinal invariant

$$J = \oint p_{\|}\,ds, \tag{5}$$

which was discussed in § 2.3.7. From (5) we can directly draw the general conclusion that if the mirrors come closer to each other, $p_{\|}$ increases and vice versa. For particles trapped between approaching mirrors the increase of $p_{\parallel}$ tends to decrease the angle α between $\mathbf{p}$ and $\mathbf{B}$ (Fig. 2.6), and if α becomes small enough, the particle escapes (cf. § 2.3.6). A change in mirror separation can occur in a time-varying magnetic field, where the magnetic mirrors approach or recede from each other, but also in a static but spatially varying magnetic field if the particles drift into a region with different mirror separation. An example of the latter case is given by the particle drift in the magnetosphere, which was discussed quantitatively in § 2.6.2.

It is worth noting that for particles moving in a *static* magnetic field 2.3.6. (53) is valid, so that

$$p_{\parallel}^2 = -p_{\perp}^2 - 2meV + \text{const.} \tag{6}$$

This shows that an increase in $p_{\parallel}^2$ is accompanied by an equal decrease in $p_{\perp}^2$ unless the particle is displaced to another equipotential surface in the electric potential field V. Only in the latter case does the total momentum change.

2.7.2. *Single-step acceleration*

An important example of a single-step acceleration is given by the interplanetary plasma penetrating into the *earth's magnetosphere* (§ 2.6). In so doing the plasma is magnetically compressed.

If a particle comes from an interplanetary field of 10^{-4} gauss into a region in the magnetosphere, where the field strength is, say, 10^{-2} gauss, the invariance of $p_{\perp}^2/B$ implies that $p_{\perp}^2$ increases by a factor of 100. Simultaneously the component $p_{\parallel}$ increases according to § 2.6.2. If the typical particle energy in the interplanetary plasma is a few hundred eV (corresponding to a temperature of a few million degrees) particle energies exceeding 10^4 eV are easily produced in the magnetosphere. In this connexion it should also be pointed out that electric fields parallel to $\mathbf{B}$ may exist both in the magnetosphere and in interplanetary space (cf. § 5.1.3). Acceleration of particles by such fields may be very important.

Acceleration of the kinds described above occur not only in the magnetosphere but also in *interplanetary and interstellar space*, where particles can get into stronger magnetic fields either by a drift or as a result of time variation in B. However, the reverse process, which results in energy decrease, occurs also.

2.7.3. *Multiple-step processes*

A single-step compression preserving the adiabatic invariants cannot increase p by more than a few orders of magnitude, because the invariance of $p_\perp^2/B$ would require impossibly large changes in B. Thus a process which accelerates particles up to cosmic-ray energies cannot be adiabatic. Neither is it likely that a one-step non-adiabatic process can achieve such a large acceleration. It is more likely that the acceleration is produced by a combination of adiabatic processes and scattering processes by which the adiabatic invariant changes. As a typical example of a process of this kind we shall discuss the so-called *magnetic pumping.* This consists of betatron acceleration in combination with non-adiabatic scattering against small-scale magnetic irregularities, and is a very general mechanism for acceleration of charged particles (Alfvén, 1949, 1950, 1954, 1959; Schlüter, 1957; Alfvén and Åström, 1958). A related mechanism has received attention in connexion with heating of plasmas in the laboratory (Berger *et al.*, 1958).

2.7.4. *Magnetic pumping*

Consider a particle whose momentum vector $\mathbf{p}$ has the components $p_\perp$ and $p_\parallel$ perpendicular and parallel to a homogeneous magnetic field. If the field strength changes slowly, the momentum components obey the relations

$$\frac{p_\perp^2}{B} = \text{const}, \tag{7}$$

$$p_\parallel = \text{const}, \tag{8}$$

as shown in § 2.3 above.

The component $p_\parallel$ is invariant and the value of p at any particular instant t depends only on its value at some earlier instant t_0 and the values of B at t_0 and t (but not on the values taken by B at times between t_0 and t). Thus in an increasing magnetic field the component of momentum perpendicular to the field (and consequently the total momentum) increases. However, if the magnetic field only fluctuates, there is no average energy increase as long as (7) and (8) are valid.

Scattering by irregularities

If, however, in the otherwise homogeneous magnetic field there are local irregularities where the magnetic field changes considerably over distances less than a Larmor radius, the behaviour of the particles is different. During the passage of a particle through such an irregularity, (7) and (8) are not valid. The momentum components change in an irregular way, and as a consequence of the change in $\mathbf{p}_\perp$ the guiding centre is displaced to another line of force. The motion along the lines of force will keep the particle from colliding repeatedly with the same irregularity and instead make it encounter other irregularities. The collisions therefore cause uncorrelated changes in the momentum vector of each particle and tend to randomize the distribution of momentum direction of a group of particles. The displacements of the guiding centre are equally uncorrelated and make the guiding centre of each individual particle perform a kind of two-dimensional random walk across the lines of force. Thus for a group of particles the effects of small-scale irregularities in the magnetic field will be:

(1) A tendency towards equipartition of momentum among the three degrees of freedom. This implies that, on the average,

$$p_\perp^2 = 2p_\parallel^2. \tag{9}$$

(2) A spatial diffusion of guiding centres across the lines of force (each guiding centre performing a random walk).

Systematic increase of momentum

If a redistribution of momentum between the different components takes place, the total momentum p can under certain conditions increase systematically, even if the fluctuating magnetic field B does not. This is the effect referred to as magnetic pumping. The redistribution of momentum can take place by collisions with small-scale magnetic inhomogeneities as discussed above. We shall first analyse a special type of field variation in order to bring out in a simple way the essential features of the magnetic pumping.

Consider a magnetic field that varies in the manner shown in Fig. 2.13, and assume that the isotropic momentum distribution is restored at a rate characterized by the relaxation time τ_2, which satisfies the condition:

$$T_g \ll \tau_1 \ll \tau_2 \ll \tau_0, \tag{10}$$

where T_g is the gyro period and τ_1 and τ_0 are defined in Fig. 2.13.

Consider a set of particles which have at some instant the square momentum p_0^2 equally distributed between the three degrees of freedom. In Fig. 2.13 are indicated five instants of time during the cycle of field fluctuation. The processes taking place in the corresponding intervals are the following:

In the short time interval (t_1, t_2) the perpendicular component of momentum increases in the proportion $k^{\frac{1}{2}}$, but no redistribution has time to take place. We are left at t_2 with an anisotropic momentum distribution with a total energy larger than the initial. In the next interval (t_2, t_3) isotropy is restored. This means that some of the energy gained during the betatron acceleration is fed into the parallel component of momentum, where it will stay intact during the subsequent field decrease. The field decrease in the interval (t_3, t_4) reduces again the perpendicular momentum in the proportion $k^{-\frac{1}{2}}$. But since the perpendicular momentum component is less than before the redistribution, the loss of energy during the decrease of field strength is less than

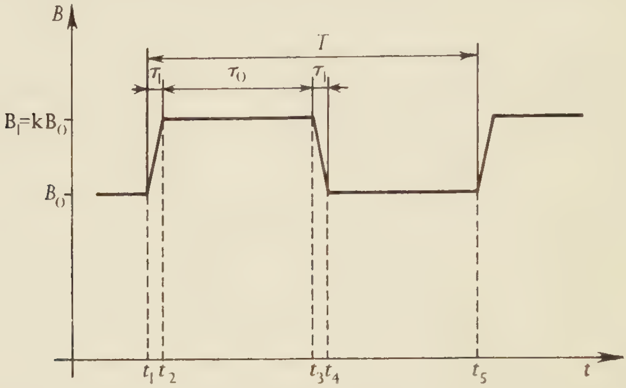


FIG. 2.13. Time variation of the magnetic field in the simple model of magnetic pumping.

the gain during the increase. Without further change in energy the distribution then becomes again isotropic during the interval (t_4, t_5). Therefore, at t_5, after a full cycle of field fluctuation, the total momentum (p_1) is larger than that at the beginning (p_0). The corresponding quantitative consideration is straightforward and is given in Table 2.1.

TABLE 2.1

Average values of the momentum components and the total momentum at various instants during one cycle of the magnetic field fluctuation shown in Fig. 2.13

t	$p_{\parallel}^2$	$p_{\perp}^2$	p^2
t_1	$\frac{1}{3}p_0^2$	$\frac{2}{3}p_0^2$	p_0^2
t_2	$\frac{1}{3}p_0^2$	$\frac{2k}{3}p_0^2$	$\left(\frac{1}{3}+\frac{2k}{3}\right)p_0^2$
t_3	$\frac{1}{3}\left(\frac{1}{3}+\frac{2k}{3}\right)p_0^2$	$\frac{2}{3}\left(\frac{1}{3}+\frac{2k}{3}\right)p_0^2$	$\left(\frac{1}{3}+\frac{2k}{3}\right)p_0^2$
t_4	$\frac{1}{3}\left(\frac{1}{3}+\frac{2k}{3}\right)p_0^2$	$\frac{2}{3k}\left(\frac{1}{3}+\frac{2k}{3}\right)p_0^2$	$\left(\frac{5}{9}+\frac{2k}{9}+\frac{2}{9k}\right)p_0^2$
t_5	$\frac{1}{3}\left(\frac{5}{9}+\frac{2k}{9}+\frac{2}{9k}\right)p_0^2$	$\frac{2}{3}\left(\frac{5}{9}+\frac{2k}{9}+\frac{2}{9k}\right)p_0^2$	$\left(\frac{5}{9}+\frac{2k}{9}+\frac{2}{9k}\right)p_0^2$

From the table we find that the resultant momentum after one cycle is (on average)

$$p_1 = \tfrac{1}{3}\left(5+2k+\frac{2}{k}\right)^{\frac{1}{2}} p_0 = \beta p_0. \tag{11}$$

The factor β is unity when k is unity and in all other cases β is larger than unity. (It is immaterial whether $k > 1$ or $k < 1$, that is, whether the field change begins by increasing or by decreasing.)

After a time t, long compared with T, approximately t/T cycles have passed, and the momentum has increased to

$$p(t) = p_0\beta^{t/T} = p_0 e^{t/\tau}, \tag{12}$$

where
$$\tau = \frac{T}{\ln\beta}. \tag{13}$$

Thus the momentum of the particle fluctuates, but in the mean it increases at a rate given by

$$\frac{dp}{p} = \frac{dt}{\tau}. \tag{14}$$

The momentum components given in Table 2.1 represent averages only. For individual particles the acceleration can be smaller or larger than that given in (5). Therefore the magnetic pumping also produces a spread in the momentum distribution, and *not all the particles* reach very high momenta.

The special type of field variation assumed above is not essential for the mechanism to work. Other types of field fluctuations, especially harmonic fluctuations, have been studied (Schlüter, 1957, in connexion with a laboratory plasma, where collisions between particles cause the momentum redistribution; and by Murty and Varma, 1958). Thus, for situations where the distribution is not too far from isotropic, Schlüter has shown that

$$\frac{d}{dt}\frac{W_\perp^2 W_\parallel}{B^2} = \frac{\nu}{3}\frac{W_\perp(2W_\parallel - W_\perp)^2}{B^2}, \tag{15}$$

where $W_\perp = p_\perp^2/2m$, $W_\parallel = p_\parallel^2/2m$, and ν is the collision frequency. The right-hand side is always positive except when $W_\perp = 2W_\parallel$ (which holds in equilibrium). Thus if we start from an equilibrium state and let the magnetic field vary in an arbitrary manner and finally return to its original value, the equilibrium which is established after this process always corresponds to a higher particle energy than the initial state.

In the special case of harmonic field fluctuations it has been shown that the momentum increases on the average exponentially so that

$$\frac{dp}{p} = \frac{dt}{\tau}, \tag{16}$$

which is similar to (14). For the case of small-amplitude oscillations with an angular frequency ω Schlüter (1957) gives the approximate relation

$$\frac{1}{\tau} \approx \frac{1}{9}\left(\frac{\Delta B}{B}\right)^2 \frac{\nu}{1+\nu^2/\omega^2}, \tag{17}$$

where ν is the collision frequency. This shows that the maximum relative increase of momentum per period of fluctuation occurs when

$$\nu = \omega. \tag{18}$$

If the collision frequency is too small or too large in relation to the frequency of fluctuation, the efficiency of the acceleration is decreased. This fact will have a selective effect resulting in preferential acceleration of such particles as have a favourable collision frequency.

In cosmical applications the fluctuations are of course neither of the type shown in Fig. 2.13, nor harmonic, but it should still be reasonable to expect that in the mean the momentum of the particles increases systematically according to a relation of the general form (16).

It is of interest to note that the acceleration mechanism discussed here is able to give a power spectrum of the same kind as the cosmic radiation has (Alfvén, 1959).

2.8. Radiation loss by charged particles moving in magnetic fields

Because of the acceleration $d\mathbf{v}/dt$ a charged particle moving with a varying velocity $\mathbf{v}$ emits electromagnetic radiation and thereby loses energy at the rate

$$-\frac{dW}{dt} = \frac{2e^2}{3c^3}\,\frac{(d\mathbf{v}/dt)^2-(\mathbf{v}\times d\mathbf{v}/dt)^2/c^2}{(1-v^2/c^2)^3}. \tag{1}$$

(See, for example, Panofsky and Phillips, 1956, p. 308.)

In the special case of circular motion we have

$$-\frac{dW}{dt} = \frac{2e^2}{3c^3}\,\frac{(d\mathbf{v}/dt)^2}{(1-v^2/c^2)^2}. \tag{2}$$

For a particle moving perpendicular to a magnetic field we can introduce 2.2.1 (1) to (3) into (2) to obtain the power of the *synchrotron radiation*:

$$-\frac{dW}{dt} = \frac{2}{3}\frac{e^4B^2}{m^2c^3}\left(\frac{W^2}{W_0^2}-1\right). \tag{3}$$

Here W denotes the relativistic energy $\gamma mc^2 = (1-v^2/c^2)^{-\frac{1}{2}}mc^2$ and W_0 the rest energy mc^2. The *decay time* for the kinetic energy $W-W_0$ is

$$T = \frac{W-W_0}{-dW/dt} = \frac{3c^5}{2e}\left(\frac{m}{e}\right)^3\frac{1}{1+W/W_0}\frac{1}{B^2}. \tag{4}$$

Introducing numerical values for an electron we get

$$T = \frac{5{\cdot}10.10^8}{B^2}\frac{1}{1+W/W_0}\,\text{sec}, \tag{5}$$

where $W_0 = 0{\cdot}511$ MeV and B is measured in gauss. In many cases this time is so long that the radiation loss by a single particle is negligible.

Synchrotron radiation

There are important differences between the radiation emitted by a slow (non-relativistic) particle ($W \ll W_0$) and a fast (relativistic) particle ($W \gg W_0$).

One difference concerns the *angular distribution* of the radiation. In the case $W \ll W_0$ ($v \ll c$) the intensity in a direction making an angle θ with the instantaneous acceleration vector is proportional to $\sin^2\theta$. In the case $W \gg W_0$ ($v \approx c$) the radiation is emitted preferentially in the forward direction, in a narrow cone with an opening angle of the order of magnitude W_0/W. The radiation is therefore essentially limited to the particle's orbital plane and reaches an observer in pulses.

It is also important to note that the *spectral distribution* of the radiation (2) is very different in the low-energy non-relativistic case $W \ll W_0$ (cyclotron radiation) and the high-energy case $W \gg W_0$ (synchrotron radiation). In the non-relativistic case the radiation has the frequency

$$\nu = \frac{\omega}{2\pi} = \frac{|e|B}{2\pi mc},$$

which corresponds to the frequency of the circular motion. In the high-energy case the spectrum is instead continuous and when $W \gg W_0$ it has a maximum at the frequency

$$\nu_m = \frac{1}{2\pi}\frac{|e|B}{mc}\left(\frac{W}{W_0}\right)^2 = \frac{1}{2\pi}\frac{|e|B}{\gamma mc}\left(\frac{W}{W_0}\right)^3. \tag{6}$$

Thus ultra-relativistic particles radiate mainly at frequencies exceeding the gyro frequency of a slow particle by a factor $(W/W_0)^2$, and its own gyro frequency by a factor $(W/W_0)^3$. In this way electrons moving in weak cosmic magnetic fields can radiate at radio or optical frequencies. For example, an electron of 3×10^9 eV energy moving in a magnetic field of 3×10^{-6} gauss has a gyro frequency of only about $1{\cdot}5\times 10^{-3}$ sec^{-1} but radiates mainly at a frequency of about 300 Mc/sec. It was suggested in 1950 (Alfvén and Herlofson, 1950; Kiepenheuer, 1950) that the non-thermal noise (in the 100 Mc/sec range) observed by radio telescopes could be explained as synchrotron radiation of relativistic electrons, and the theory is now well established. The subject of cosmic noise will not be treated in detail here, and the reader is referred to Shklovsky's monograph (Shklovsky, 1960).

Charged particles in a *plasma* can also emit radiation by *collective motion* of the particles (plasma oscillations).

REFERENCES

Alfvén, H., 1949: 'On the Solar Origin of Cosmic Radiation', *Phys. Rev.* **75**, 1732.

—— 1950: 'On the Solar Origin of Cosmic Radiation, II', ibid. **77**, 375.

—— 1954: 'On the Origin of Cosmic Radiation', *Tellus*, **6**, 232.

—— 1955: 'On the Electric Field Theory of Magnetic Storms and Aurorae', ibid. **7**, 50.

—— 1958: 'On the Theory of Magnetic Storms and Aurorae', ibid. **10**, 104.

—— 1959: 'Momentum Spectrum of Cosmic Radiation', ibid. **11**, 106.

—— and Herlofson, N., 1950: 'Cosmic Radiation and Radio Stars', *Phys. Rev.* **78**, 616.

—— and Åström, E., 1958: 'Energy Spectrum of Cosmic Radiation', *Nature*, **181**, 330.

Allis, W. P., 1956: 'Motions of Ions and Electrons', *Encyclopedia of Physics*, **21**, 383, Springer-Verlag, Berlin.

Bennett, W. H., and Hulburt, E. O., 1953: 'Magnetic Self-focusing of Auroral Protons', *Phys. Rev.* **91**, 1562.

—— —— 1954*a*: 'Theory of the Aurora based on Magnetic Self-focusing of Solar Ion Streams', ibid. **95**, 315.

—— —— 1954*b*: 'Magnetic Self-focused Solar Ion Streams as the Cause of Aurorae', *J. Atmos. Terrest. Phys.* **5**, 211.

Berger, J. M., Newcomb, W. A., Dawson, J. M., Frieman, E. A., Kulsrud, R. M., and Lenard, A., 1958: 'Heating of a Confined Plasma by Oscillating Electromagnetic Fields', *Phys. Fluids*, **1**, 301.

Birkeland, K., 1908: *The Norwegian Aurora Polaris Expedition 1902–1903*, Aschenhoug & Co., Christiania.

Brüche, E., 1931: 'Some New Theoretical and Experimental Results on the Aurora Polaris', *Terr. Magn. Atmos. Elect.* **36**, 41.

Brunberg, E.-Å., and Dattner, A., 1953: 'Experimental Determination of Electron Orbits in the Field of a Magnetic Dipole', *Tellus*, **5**, 135.

—— 1956: 'Cosmic Rays in the Terrestrial Magnetic Dipole Field', ibid. **8**, 215.

—— 1958: 'The Optics of Cosmic-ray Telescopes', *Ark. f. fys.* **14**, 195.

Christofilos, N. C., 1959: 'The Argus Experiment', *J. Geophys. Res.* **64**, 869; see also 'Symposium on the Scientific Effects of Artificially Introduced Radiations at High Altitudes'. Editor: Porter, R. W., *J. Geophys. Res.* **64**, 865, 1959.

Cowling, T. G., 1942: 'On Alfvén's Theory of Magnetic Storms and of the Aurora', *Terr. Magn. Atmos. Elect.* **47**, 209.

Davis, L., 1956: 'Modified Fermi Mechanism for the Acceleration of Cosmic Rays', *Phys. Rev.* **101**, 351.

Fermi, E., 1949: 'On the Origin of Cosmic Radiation', ibid. **75**, 1169.

—— 1954: 'Galactic Magnetic Fields and the Origin of Cosmic Radiation', *Astrophys. J.* **119**, 1.

Firror, J., 1954: 'Cosmic Radiation Intensity-time Variations and their Origin. IV. Increases associated with Solar Flares', *Phys. Rev.* **94**, 1017.

Gibson, G., Jordan, W. C., and Lauer, E. J., 1960: 'Containment of Positrons in a Mirror Machine', *Phys. Rev. Letters*, **5**, 141.

Ginzburg, V. L., and Syrovatskii, S. I., 1960: 'Present Status of the Question of the Origin of Cosmic Rays', *Soviet Physics USPEKHI*, **3**, 504.

Jory, F. S., 1955: 'Selected Cosmic-ray Orbits in the Earth's Magnetic Field', *Phys. Rev.* **103**, 1068.

Kiepenheuer, K. O., 1950: 'Cosmic Rays as the Source of General Galactic Radio Emission', ibid. **79**, 738.

Kruskal, M., 1962: 'Asymptotic Theory of Hamiltonian and Other Systems with all Solutions Nearly Periodic', *Nuclear Fusion*, 1962 Supplement, part 2, p. 775.

Landau, L. D., and Lifshitz, E. M., 1959: *The Classical Theory of Fields*, Pergamon Press.

—— —— 1960: *Mechanics*, Pergamon Press.

Lehnert, B., 1963: *Dynamics of Charged Particles*, North-Holland Publ. Co. (In press.)

Lemaître, G., and Vallarta, M. S., 1936*a*: 'On the Geomagnetic Analysis of Cosmic Radiation', *Phys. Rev.* **49**, 719.

—— —— 1936*b*: 'On the Allowed Cone of Cosmic Radiation', *Phys. Rev.* **50**, 493.

Lüst, R., 1957: 'Impact Zones for Solar Cosmic Ray Particles', *Phys. Rev.* **105**, 1827.

—— Schlüter, A., and Katterbach, K., 1955: 'The Paths of Cosmic Radiation Particles in the Earth's Magnetic Field', *Nachr. Akad. Wiss.*, Göttingen, Math.-Phys. Kl. **11a**, 128.

—— —— —— 1956: 'Die Bahnen von Teilchen der kosmischen Strahlung im Erdmagnetfeld', ibid. **11a**, Math.-Phys.-Chem. Abteilung, Nr. 8.

Malmfors, K. G., 1945: 'Determination of Orbits in the Field of a Magnetic Dipole with Applications to the Theory of the Diurnal Variation of Cosmic Radiation', *Ark. f. mat. astr. o. fys.* **32 A**, no. 8.

McCracken, K. G., Rao, U. R., and Shea, M. A., 1962: The Trajectories of Cosmic Rays in a High Degree Simulation of the Geomagnetic Field, M.I.T. Technical Report 77.

McIlwain, C. E., 1961: 'Coordinates for Mapping the Distribution of Magnetically Trapped Particles', *J. Geophys. Res.* **66**, 3681.

Morrison, P., 1961: 'The Origin of Cosmic Rays', *Encyclopedia of Physics*, **46/1**, Springer-Verlag, Berlin, p. 1.

Murty, G. S., and Varma, R. K., 1958: 'Acceleration of Cosmic Radiation', *Phys. Rev.* **112**, 1789.

Northrop, T. G., 1961: 'The Guiding Centre Approximation to Charged Particle Motion', *Ann. Phys.* **15**, 79.

—— and Teller, E., 1960: 'Stability of the Adiabatic Motion of Charged Particles in the Earth's Field', *Phys. Rev.* **117**, 215.

Panofsky, W. K. H., and Phillips, M., 1956: *Classical Electricity and Magnetism*, Addison-Wesley, Reading, Mass.

Quenby, J. J., Wenk, G. J., and Bland, C. J., 1962: 'Cosmic Ray Threshold Rigidities', *H.J. Phys. Soc. Japan*, **17**, Suppl. A–2, 449.

Rodionov, S. N., 1959: 'An Experimental Test of the Behaviour of Charged Particles in an Adiabatic Trap', *J. Nuclear Energy, Part C: Plasma Physics*, **1**, 247.

Schlüter, A., 1957: 'Der Gyro-Relaxations-Effekt', *Z. Naturforsch.* **12a**, 822.

Shklovsky, I. S., 1960: *Cosmic Radio Waves*, Harvard University Press, Cambridge, Mass.

Singer, F., 1958: 'The Primary Cosmic Radiation and its Time Variations', *Progress in Elementary Particle and Cosmic Ray Physics*, **4**, North-Holland Publ. Co.

Störmer, C., 1913: 'Sur un problème rélatif aux mouvements des corpuscules électriques dans l'espace cosmique', *Videnskapsselskabets Skr. Mat.-nat. Kl.*, no. 14, Kristiania, Oslo.

—— 1955: *The Polar Aurora*, Clarendon Press, Oxford.

Swann, W. F. G., 1933: 'A Mechanism of Acquirement of Cosmic Ray Energies by Electrons', *Phys. Rev.* **43**, 217.

Vallarta, M. S., 1961: 'Theory of the Geomagnetic Effects of Cosmic Radiatión', *Encyclopedia of Physics*, **46/1**, Springer-Verlag, Berlin.

Vandervoort, P. O., 1961: 'The Nonconstancy of the Adiabatic Invariants', *Ann. Phys.* **12**, 436.

3

MAGNETO-HYDRODYNAMICS

3.1. Introduction

If a magnetized medium moves, an electric field is produced. If the medium is electrically conducting and different parts of it move at different velocity, the electric field will produce currents (unless the motion satisfies certain special conditions). These currents interact with the magnetic field and produce forces which, under the conditions discussed in the following, are strong enough to change the state of motion of the medium appreciably. In this way hydrodynamic motion and electromagnetic phenomena are coupled. We cannot use ordinary hydrodynamics, nor ordinary electrodynamic theory, but must apply a combination which has been called *magneto-hydrodynamics* (with *hydromagnetics*, *magnetofluid dynamics*, and *magneto-dynamics* as the most common synonyms).

Ceteris paribus the coupling between electromagnetic and hydrodynamic phenomena increases with the linear scale of the phenomenon (§ 3.9 below). Hence there may be a strong coupling for large-scale phenomena but at the same time no appreciable coupling for small-scale phenomena. For example, in the ionosphere radio waves with a wavelength of say one metre do not produce any hydrodynamic motion, nor is the shock wave from an ordinary meteor of hydromagnetic character, but oscillations of a large part of the F-region with periods of the order of seconds or more must be treated by magneto-hydrodynamics, not by ordinary hydrodynamics. In the sun all phenomena which are large enough to be observed visually from the earth are hydromagnetic, and the same holds for interstellar clouds. Also in the interior of the stars, and in the fluid core of the earth, the coupling is strong enough. In fact, the only gaseous regions in the whole universe where ordinary hydrodynamics is valid without restriction, are the lower parts of the planetary atmospheres (below the ionospheres), where the electric conductivity is very small. To this should be added the oceans, lakes, etc., where also ordinary hydrodynamics is valid without appreciable complications from electromagnetic effects.

HYDROMAGNETIC WAVES

3.2. Qualitative discussion

3.2.1. *The case of incompressible fluids*

Consider an incompressible, electrically conducting, fluid penetrated by a homogeneous magnetic field **B**. Introduce a right-handed Cartesian coordinate system with the z-axis parallel to the lines of force. Assume that at a certain time the whole fluid is at rest with the exception of a pillar which has the intersection $ABCD$ with the zx-plane and extends indefinitely in the y-direction (Fig. 3.1). Let this pillar move with the velocity **v** in the direction of the y-axis.

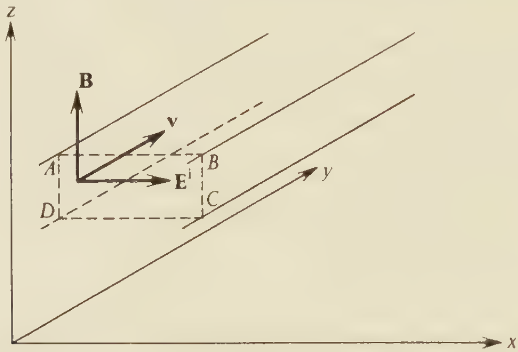


FIG. 3.1. When a pillar $ABCD$ moves in the y-direction, the magnetic field **B** causes an electric field $\mathbf{E}^{i}$.

The motion across the magnetic field induces an electric field

$$\mathbf{E}^{i} = (\mathbf{v}/c)\times\mathbf{B}$$

in the x-direction. Since the medium is electrically conducting, a current system is set up. The exact structure of this depends on the geometrical factors that influence the inductance and resistance. Fig. 3.2 illustrates the current system qualitatively.

In the presence of the magnetic field **B** the current density **i** produces a volume force

$$\mathbf{F} = (\mathbf{i}/c)\times\mathbf{B}.$$

As Figs. 3.1 and 3.2 show, this force is directed opposite to **v** in the moving pillar, i.e. it tends to brake the motion. In the adjacent parts of the stationary fluid it is directed parallel to **v**. The layer of the fluid that was moving initially is therefore retarded, and the layers above and below accelerated. *Thus the induced current system tends to transfer the initial motion to adjacent regions of the fluid.* Under the assumption of infinite conductivity the result will be that after some time the pillar $ABCD$ is at rest, while $EFBA$ and $DCGH$ (see Fig. 3.3) are moving in the direction of the y-axis. Later on the parts $IJKL$ and $MNOP$ are moving, while the remaining parts of the fluid are at rest. The fact

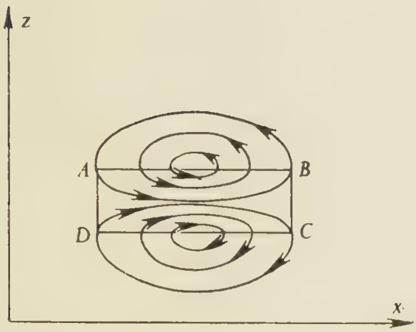


FIG. 3.2. Qualitative illustration of the current system.

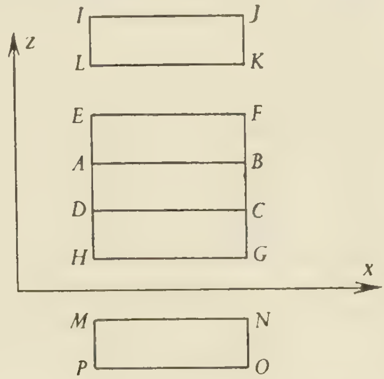


FIG. 3.3. Hydromagnetic waves transmit the state of motion of $ABCD$ to $EFBA$ and $DCGH$, and later to $IJKL$ and $MNOP$.

that the motion is propagated precisely in this way (namely, parallel to the lines of force and without distortion of the pillar) follows from the mathematical analysis below (§ 3.4).

3.2.2. *The case of compressible fluids*

Consider next a compressible electrically conducting medium situated in a homogeneous magnetic field parallel to the z-axis of a Cartesian coordinate system. Assume now that in some way a certain region, extending uniformly in the z-direction, is compressed perpendicular to $\mathbf{B}$. When the conductivity is high, it follows from the flux-conservation law, § 3.9, that the magnetic field strength is greater in the compressed region. The fact that the magnetic field varies in space means that there exists a current density given by the Maxwell equation

$$\operatorname{curl} \mathbf{H} = (4\pi/c)\mathbf{i},$$

where

$$\mathbf{H} = \mathbf{B}/\mu,$$

and μ is the magnetic permeability of the fluid. The current density produces a volume force, which is

$$\mathbf{F} = (\mathbf{i}/c) \times \mathbf{B} = (1/4\pi) \operatorname{curl} \mathbf{H} \times \mathbf{B}.$$

Since **B** and **H** have only z-components, the expression for **F** can be rewritten

$$\mathbf{F} = -\text{grad}\,\frac{B^2}{8\pi\mu}.$$

This shows that the magnetic field, when subject to lateral compression, responds by a restoring force equivalent to that from a pressure distribution given by the quantity $B^2/8\pi\mu$, the so-called *magnetic pressure.* As a consequence there exist compression waves—*magneto-sonic waves* or *magneto-acoustic waves*—that can propagate across the magnetic field by means of the combined action of the hydrostatic and magnetic pressures. Such waves were first analysed by Herlofson (1950), de Hoffman and Teller (1950), and Van de Hulst (1951). They will be discussed in § 3.8.

3.3. Fundamental equations

In order to formulate the problem mathematically, we must start with Maxwell's equations:

$$\text{curl}\,\mathbf{H} = \frac{1}{c}\left(4\pi\mathbf{i}+\frac{\partial\mathbf{D}}{\partial t}\right), \tag{1}$$

$$\text{curl}\,\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}, \tag{2}$$

$$\text{div}\,\mathbf{B} = 0, \tag{3}$$

and

$$\mathbf{B} = \mu\mathbf{H}, \tag{4}$$

$$\mathbf{i} = \sigma\{\mathbf{E}+(\mathbf{v}/c)\times\mathbf{B}\}, \tag{5}$$

where σ is the electrical conductivity. We must add to these equations the hydrodynamic equation of motion

$$\rho\frac{d\mathbf{v}}{dt} = \rho\mathbf{G}+(\mathbf{i}/c)\times\mathbf{B}-\text{grad}\,p, \tag{6}$$

and the continuity equation

$$-\frac{\partial\rho}{\partial t} = \text{div}(\rho\mathbf{v}). \tag{7}$$

Here ρ means the mass density, p the pressure, and **G** the non-electromagnetic forces acting per unit mass of the fluid. In (6) we have left out the viscous force terms, cf. § 3.9. For the analysis of the compressible case we need also a relation between pressure changes and the corresponding density changes. For an isotropic plasma undergoing a *reversible adiabatic* process we have

$$p = \text{const}\,\rho^\gamma, \tag{8}$$

where

$$\gamma = c_p/c_v \tag{9}$$

is the ratio of specific heats. This is shown in textbooks on thermodynamics (see, for example, Zemansky, 1957). In a magnetized plasma with small collision frequency the velocity distribution can be anisotropic so that the simple relation (8) is not applicable (see Chew, Goldberger, and Low, 1956). The present treatment will be limited to cases where (8) applies.

3.3.1. *The magnetic force term*

The magnetic force term in (6) can be rewritten by means of the familiar vector identity

$$\operatorname{grad}(\mathbf{ab}) = (\mathbf{a}\operatorname{grad})\mathbf{b}+(\mathbf{b}\operatorname{grad})\mathbf{a}+\mathbf{a}\times\operatorname{curl}\mathbf{b}+\mathbf{b}\times\operatorname{curl}\mathbf{a}.$$

Using also the relation (1) and assuming μ to be constant and $\partial\mathbf{D}/\partial t$ negligible we get the expression

$$\mathbf{F}_{\mathrm{m}} = (\mathbf{i}/c)\times\mathbf{B} = -\operatorname{grad}(B^2/8\pi\mu)+\frac{1}{4\pi\mu}(\mathbf{B}\operatorname{grad})\mathbf{B} \tag{10}$$

for the magnetic *volume force*. It is of some interest to give this force an alternative interpretation. If we consider an arbitrary volume V, the resultant force is obtained by integration over the volume. The volume integral can then be transformed to a surface integral and the resulting force thus reinterpreted in terms of a set of *equivalent surface forces*. (This is so because the magnetic force has the nature of a tensor divergence, see, for example, Stratton (1941, p. 97).) For the transformation we need the vector identities

$$\int_V \operatorname{grad}\varphi\, dV = \int_S \varphi\, \mathbf{ds} \tag{11}$$

and

$$\int_V (\mathbf{a}\operatorname{grad})\mathbf{b}\, dV = \int_S \mathbf{b}(\mathbf{a}\,\mathbf{dS})-\int_V \mathbf{b}\operatorname{div}\mathbf{a}\, dV, \tag{12}$$

where S is the surface of the volume V and $\mathbf{dS}$ is its surface element, φ is an arbitrary scalar function, and $\mathbf{a}$ and $\mathbf{b}$ arbitrary field vectors. Application of these identities to the expression (10) gives

$$\int_V \mathbf{F}_{\mathrm{m}}\, dV = \int_S \left(-\frac{B^2}{8\pi\mu}\right)\mathbf{dS}+\int_S \frac{\mathbf{B}(\mathbf{B}\,\mathbf{dS})}{4\pi\mu}. \tag{13}$$

Introducing the unit vectors $\hat{\mathbf{n}}$ perpendicular to the surface (and directed outward from the volume V) and $\hat{\mathbf{B}}$ parallel to the magnetic field (see Fig. 3.4), we have $\mathbf{dS} = \hat{\mathbf{n}}\, dS$, $\mathbf{B} = \hat{\mathbf{B}}B$ and

$$\mathbf{B}\,\mathbf{dS} = B\,dS(\hat{\mathbf{B}}\hat{\mathbf{n}}) = B\,dS\cos\theta,$$

where θ is the angle between the magnetic field lines and the normal to the surface element. Then we can write (13) in the form

$$\int \mathbf{F}_{\mathrm{m}}\,dV = \int\limits_S \frac{B^2}{8\pi\mu}(-\hat{\mathbf{n}}\,dS)+\int\limits_S \frac{B^2}{4\pi\mu}\hat{\mathbf{B}}(\cos\theta\,dS). \tag{14}$$

The first term on the right-hand side represents a *pressure of magnitude* $B^2/8\pi\mu$ and the last term represents a *tension of magnitude* $B^2/4\pi\mu$ *directed along the lines of force*, cf. Fig. 3.4.

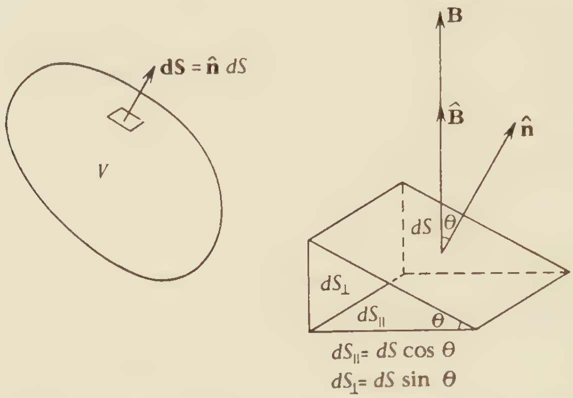


FIG. 3.4. Illustration of the interpretation of the equivalent surface forces.

3.4. Plane waves in incompressible fluid. Homogeneous magnetic field

In order to demonstrate the fundamental properties of hydromagnetic waves we shall treat the simple case of plane waves in an incompressible fluid with constant density ρ. Then we have

$$\operatorname{div}\mathbf{v} = 0. \tag{1}$$

Further we assume that the primary magnetic field $\mathbf{B}_0$ is homogeneous and parallel to the z-axis of an orthogonal coordinate system, and that the force term $\rho\mathbf{G}$ vanishes. The displacement current $(\partial\mathbf{D}/\partial t)/4\pi$ is assumed to be negligible in comparison with the conduction current $\mathbf{i}$.

The magnetic field is

$$\mathbf{B} = \mathbf{B}_0+\mathbf{b}, \tag{2}$$

where the field $\mathbf{b}$ is caused by the current $\mathbf{i}$. In order to study a plane wave propagated in the direction of $\mathbf{B}_0$ we assume that all vectors are independent of x and y but depend upon z and the time t.

This implies, according to 3.3 (1) and 3.3 (3), that we have $i_z = 0$ and $B_z = \text{const} = B_0$. Further, according to (1) we may put $v_z = 0$.

If we turn the coordinate system in such a way that $i_y = 0$, we obtain from 3.3 (1) and (4)

$$i_x = -\frac{c}{4\pi\mu}\frac{\partial b_y}{\partial z}, \tag{3}$$

$$i_y = i_z = 0, \tag{4}$$

$$b_x = \text{const} = 0,$$

$$B_z = B_0.$$

We introduce these values into 3.3 (6). As, according to our assumptions, $\operatorname{grad} p$ can have no components perpendicular to the z-axis, we obtain

$$\partial v_x/\partial t = 0; \qquad v_x = \text{const} = 0,$$

$$\frac{\partial v_y}{\partial t} = \frac{B_0}{4\pi\rho\mu}\frac{\partial b_y}{\partial z}, \tag{5}$$

$$v_z = 0,$$

and further

$$\frac{\partial p}{\partial z} = -\frac{1}{8\pi\mu}\frac{\partial(b_y^2)}{\partial z}. \tag{6}$$

Equation 3.3 (5) gives

$$\mathbf{E} = \frac{\mathbf{i}}{\sigma} - (\mathbf{v}/c)\times\mathbf{B},$$

or, with (4) and (5),

$$E_x = \frac{i_x}{\sigma} - \frac{v_y B_0}{c}, \tag{7}$$

$$E_y = E_z = 0.$$

Equation 3.3 (2) gives

$$\frac{\partial b_y}{\partial t} = -c\frac{\partial E_x}{\partial z}. \tag{8}$$

From (7) and (8) we obtain

$$\frac{\partial^2 b_y}{\partial t^2} = B_0\frac{\partial^2 v_y}{\partial t\partial z} - \frac{c}{\sigma}\frac{\partial^2 i_x}{\partial t\partial z}.$$

Introducing (3) and (5) we obtain

$$\frac{\partial^2 b_y}{\partial t^2} = \frac{B_0^2}{4\pi\mu\rho}\frac{\partial^2 b_y}{\partial z^2} + \frac{c^2}{4\pi\mu\sigma}\frac{\partial^3 b_y}{\partial z^2\partial t}. \tag{9}$$

If we abandon the requirement that all vectors are independent of x, it can be shown that (9) takes the slightly more general form

$$\frac{\partial^2 b_y}{\partial t^2} = \frac{B_0^2}{4\pi\rho\mu}\frac{\partial^2 b_y}{\partial z^2} + \frac{c^2}{4\pi\mu\sigma}\frac{\partial}{\partial t}\left(\frac{\partial^2 b_y}{\partial z^2} + \frac{\partial^2 b_y}{\partial x^2}\right). \tag{9 a}$$

3.4.1. *Infinite conductivity*

In the case $\sigma = \infty$ we get the simple equation

$$\frac{\partial^2 b_y}{\partial t^2} = \frac{B_0^2}{4\pi\mu\rho}\frac{\partial^2 b_y}{\partial z^2} \tag{9 b}$$

even if b_y varies with x in an arbitrary manner. This is a wave equation and represents waves propagating with velocity

$$V = \pm B_0/(4\pi\mu\rho)^{\frac{1}{2}}. \tag{10}$$

The *hydromagnetic velocity* V is independent of the frequency as well as the amplitude. Its value for some cases of interest is included in Table 3.2 on page 103.

The wave equation (9 b) has solutions of the form

$$b_y = A\sin\omega(t-z/V), \tag{11}$$

where the amplitude A may be an arbitrary function of x, and ω is the angular frequency. From (5), (3), (8), and (6) it then follows that

$$v_y = -(4\pi\mu\rho)^{-\frac{1}{2}}A\sin\omega(t-z/V), \tag{12}$$

$$i_x = \frac{c\omega}{B_0}\left(\frac{\rho}{4\pi\mu}\right)^{\frac{1}{2}}A\cos\omega(t-z/V), \tag{13}$$

$$E_x = \frac{B_0}{c}(4\pi\mu\rho)^{-\frac{1}{2}}A\sin\omega(t-z/V), \tag{14}$$

$$p = p_0-\frac{1}{8\pi\mu}A^2\sin^2\omega(t-z/V). \tag{15}$$

See Figs. 3.5 and 3.6. As the magnetic energy density of the wave is $b_y^2/8\pi\mu$ and the kinetic energy density is $\rho v_y^2/2$, we see from (11) and (12) that there is equipartition between kinetic and magnetic energy.

The magnetic lines of force, which in the absence of hydromagnetic waves were straight lines

$$x = x_0, \qquad y = y_0, \tag{16}$$

change their form as a consequence of the field $\mathbf{b}$ being superimposed on $\mathbf{B}_0$. As the angular coefficient of the lines of force is everywhere given by

$$\frac{dy}{dz} = \frac{b_y}{B_z} = \frac{b_y}{B_0}, \tag{17}$$

we find, by introducing (11) into (17) and integrating, that the lines of force are sine curves given by

$$x = x_0, \tag{18}$$

$$y = y_0+\frac{A}{\omega(4\pi\mu\rho)^{\frac{1}{2}}}\cos\omega(t-z/V). \tag{19}$$

If we differentiate y with respect to t, we find that the derivative is equal to the material velocity (12). This fact means that *in a certain sense* the magnetic lines of force 'move' with the same velocity as the fluid. This is often expressed by saying that the lines of force are 'frozen' into the fluid. However, this is a mode of expression that must be used with care, because in certain situations it can sometimes be misleading, cf. § 5.4.

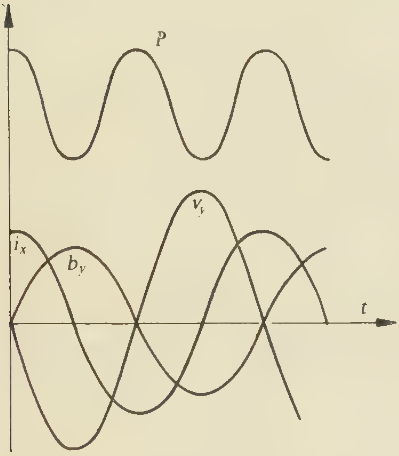


FIG. 3.5. Induced magnetic field b_y, velocity v_y, current i_x, and pressure p as functions of t in a sine wave.

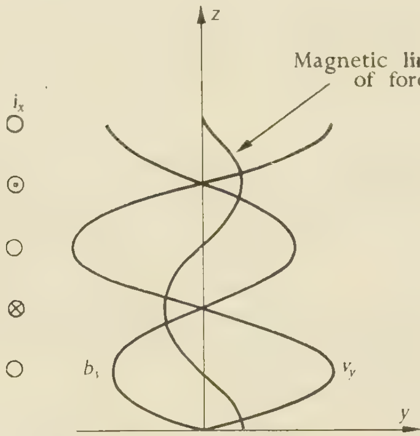


FIG. 3.6. Induced magnetic field b_y, velocity v_y, current i_x as functions of z at a certain moment. A magnetic line of force of the resultant field $\mathbf{B}_0+\mathbf{b}$ is also shown.

It should be noted that the wave equation (9 b) is satisfied by

$$b_y = f(t-z/V)g(x), \tag{19 a}$$

where f and g are arbitrary functions. Equation (19 a) shows that an arbitrary disturbance propagates parallel to $\mathbf{B}$ and with a velocity V given by (10). This is expected because all elementary waves of the kind (11) propagate at the same speed, and a wave of arbitrary form can be obtained by superposition. However, in compressible media other kinds of hydromagnetic waves exist by which disturbances can propagate across the magnetic field (§ 3.8).

3.4.2. *Finite conductivity*

Suppose that σ is finite and $\mathbf{b}$ and $\mathbf{v}$ are parallel to the y-axis as earlier. Suppose further that b and v are functions of z and t only. We write them as complex exponentials

$$b = b_0 e^{j(\omega t-kz)}, \tag{20}$$

$$v = v_0 e^{j(\omega t-kz)}, \tag{21}$$

where $j = \sqrt{(-1)}$, and ω, k, b_0, and v_0 are constants, all of which except ω may be complex numbers.

As $\partial^2/\partial t^2 = -\omega^2$, $\partial^2/\partial z^2 = -k^2$, and $\partial^3/\partial z^2\partial t = -j\omega k^2$, we obtain from (9) and (10)

$$\omega^2-\left(V^2+j\frac{c^2\omega}{4\pi\mu\sigma}\right)k^2 = 0 \tag{22}$$

or

$$k = \pm\frac{\omega}{V}\left(1+\frac{j\omega c^2}{4\pi\mu\sigma V^2}\right)^{-\frac{1}{2}} \tag{23}$$

or, for small damping, approximately

$$k = \pm\left(\frac{\omega}{V}-\frac{j\omega^2c^2}{8\pi\mu\sigma V^3}\right). \tag{24}$$

The distance z_0 in which the amplitude of the wave is reduced by $1/e$ is the inverse value of the imaginary component of k. Thus we have

$$z_0 = \frac{8\pi\mu\sigma V^3}{\omega^2c^2} = \frac{2\mu\sigma V}{\pi c^2}\lambda^2 = \frac{\mu^{\frac{1}{2}}\sigma B_0}{\pi^{\frac{3}{2}}\rho^{\frac{1}{2}}c^2}\lambda^2 \doteq \frac{\sigma B_0^3}{\pi^{\frac{1}{2}}c^2\rho^{\frac{3}{2}}\mu^{\frac{1}{2}}\omega^2}, \tag{25}$$

where $\lambda = 2\pi V/\omega$ is the wavelength. The expression (25) is approximately correct if $\lambda \ll z_0$.

Thus we can write

$$b = b_0\,e^{-z/z_0}e^{j\omega(t-z/V)} \tag{26}$$

and

$$v = v_0\,e^{-z/z_0}e^{j\omega(t-z/V)} \tag{27}$$

with

$$v_0 = -\frac{Vb_0}{B_0}\left(1-j\frac{\omega c^2}{8\pi\mu\sigma V^2}\right). \tag{28}$$

The latter expression is derived from (5), (20), and (21).

The imaginary part indicates the phase shift of the velocity v in relation to the magnetic disturbance field b. We can also write

$$v = v_0'\,e^{-z/z_0}e^{j\omega(t-z/V)-j\varphi}, \tag{29}$$

where, for small λ/z_0,

$$v_0' = -Vb_0/B_0 \tag{30}$$

and

$$\varphi = \frac{\omega c^2}{8\pi\mu\sigma V^2} = \frac{\omega c^2\rho}{2\sigma B_0^2}. \tag{31}$$

Note that in the extreme case of low conductivity, (22) reduces to the dispersion relation characterizing the skin effect in solid conductors:

$$k = (-j4\pi\mu\omega\sigma/c^2)^{\frac{1}{2}} = \pm(1-j)/\delta,$$

where

$$\delta = (2\pi\mu\omega\sigma/c^2)^{-\frac{1}{2}}$$

is the ordinary skin depth.

3.4.3. *Hydromagnetic waves as oscillations of the lines of force*

A simple picture of the hydromagnetic wave motion can be obtained by comparing the magnetic lines of force to elastic strings. As in a certain sense the lines of force move with the material, they could be regarded as material strings with masses equal to the fluid mass per line of force. The forces acting are, according to § 3.3.1, an isotropic pressure $B^2/8\pi\mu$, which has no influence on the transverse oscillations of the incompressible medium, and a tension $B^2/4\pi\mu$ along the lines of force. For an ordinary string, transverse oscillations in the direction of the y-axis are governed by the equation

$$m\frac{\partial^2 y}{\partial t^2} = S\frac{\partial^2 y}{\partial z^2}, \tag{32}$$

where S stands for the tension in the string and m is the mass per unit length (see, for example, Joos, 1959). The wave velocity is then

$$V = (S/m)^{\frac{1}{2}}. \tag{33}$$

If we wish to apply these results to the oscillations of the magnetic lines of force, S is to be replaced by the tension per unit area, which, according to 3.3.1 (14), is

$$S = B^2/4\pi\mu, \tag{34}$$

and m by the fluid mass per unit volume, i.e.

$$m = \rho, \tag{35}$$

and the velocity comes out to be

$$V = B/(4\pi\mu\rho)^{\frac{1}{2}} \tag{36}$$

in agreement with (10).

According to what has just been said it is possible to regard the hydromagnetic waves in a conducting fluid as oscillations of the magnetic lines of force 'materialized' into strings into which the conducting fluid has been 'glued'. This concept gives in many cases a surprisingly good survey of the phenomena. On the other hand, it must be used with great care in *plasmas*, where it may lead to erroneous conclusions (cf. § 5.4).

3.4.4. *Hydromagnetic waves as a special case of electromagnetic waves*

It is of some interest that the hydromagnetic waves can be considered as an extreme case of electromagnetic waves in spite of the fact that their velocity (cf. Table 3.2 on page 103) is often many powers of ten lower than the velocity of light.

Consider a homogeneous magnetic field and an electromagnetic wave which travels in the direction of the field. The velocity v of the wave is given by

$$v = c(\epsilon\mu)^{-\frac{1}{2}}, \tag{37}$$

where c is the velocity of light, ϵ the permittivity, and μ the permeability, which we shall here assume to be unity. The electric field E of the waves causes a displacement current i,

$$i = \frac{\epsilon}{4\pi}\frac{dE}{dt}. \tag{38}$$

In a vacuum we have $\epsilon = 1$. When matter is present this current is supplemented by a current in the matter. Anticipating the results of § 5.3 we consider a *plasma* and restrict ourselves to the simple case where the equivalent circuit of Fig. 5.6 (*b*) applies, cf. § 5.3.2. The values of the components are given by 5.3.2 (22) to (24). For frequencies much smaller than the gyro frequency the inductance L can be neglected. If we also assume that the conductivity is very high, the resistance R is unimportant, and only the two condensers

$$C_0 = \frac{1}{4\pi}$$

and

$$C = \frac{\rho c^2}{B^2}$$

need to be considered. When an electric field E is applied, the displacement current $i_1 = C_0\,dE/dt$ flows through the first condenser and the conduction current $i_c = C\,dE/dt$ through the second. The resulting current is then

$$i = \frac{1}{4\pi}\left(1+\frac{4\pi\rho c^2}{B^2}\right)\frac{dE}{dt}. \tag{39}$$

Comparing this with (38) we find that the actual current corresponds to an *equivalent permittivity* given by

$$\epsilon_{\text{equiv}} = 1+4\pi\rho c^2/B^2. \tag{40}$$

Introducing this value of ϵ into (37) and remembering that $\mu = 1$, we obtain

$$v = (V^{-2}+c^{-2})^{-\frac{1}{2}}, \tag{41}$$

where

$$V = B/(4\pi\rho)^{\frac{1}{2}}. \tag{42}$$

We recognize V as the hydromagnetic velocity met with before. It follows directly from (41) that when

$$V \ll c \tag{43}$$

the speed of propagation (37) of the electromagnetic wave reduces to the hydromagnetic velocity V.

The transition between hydromagnetic waves and electromagnetic waves has been treated by Rydbeck (1948), Åström (1951), and others. A recent survey has been given by Åström (1961); see also Fünfer and Lehner (1962).

For a different derivation of the equivalent permittivity of a plasma see, for example, Fünfer and Lehner (1962), p. 7. The applicability of the equivalent permittivity has recently been discussed by Lehnert; see § 5.3.2.

3.4.5. *Reflection and transmission of hydromagnetic waves*

When a hydromagnetic wave encounters a boundary between two fluids with different values of the hydromagnetic velocity (3.4.1 (10)), the wave is partly reflected and partly transmitted. This phenomenon has been studied by Walén (1944, 1946), Lundquist (1952), Ferraro (1954) and others (see, for example, Ferraro and Plumpton, 1961).

Here we shall limit ourselves to the simple dissipation-free waves ($\sigma = \infty$) studied in § 3.4.1 and to the case where the boundary between the fluids is perpendicular to the undisturbed magnetic field $\mathbf{B}_0$. Let $z = 0$ be the boundary surface between the two fluids, and let the hydromagnetic velocities be V_1 (in the region $z < 0$) and V_2 (in the region $z > 0$). Let the magnetic field of the *incoming wave* be given by 3.4.1 (11)

$$b_y = A \sin \omega(t - z/V_1) \quad (z < 0). \tag{44}$$

Then we can write the magnetic field of the *transmitted wave* in the form

$$b'_y = A' \sin \omega(t - z/V_2) \quad (z > 0), \tag{45}$$

and that of the *reflected wave*

$$b''_y = A'' \sin \omega(t + z/V_1) \quad (z < 0). \tag{46}$$

According to 3.4 (8) the corresponding electric fields are

$$E_x = A \frac{V_1}{c} \sin \omega(t - z/V_1) \quad (z < 0), \tag{47}$$

$$E'_x = A' \frac{V_2}{c} \sin \omega(t - z/V_2) \quad (z > 0), \tag{48}$$

$$E''_x = -A'' \frac{V_1}{c} \sin \omega(t + z/V_1) \quad (z < 0). \tag{49}$$

When $\mu = 1$ the boundary conditions are

$$b_y + b''_y = b'_y \tag{50}$$

and

$$E_x + E''_x = E'_x. \tag{51}$$

As 3.4 (7) shows, the continuity of E_x implies also continuity of the velocity v_y when $\sigma = \infty$. Equations (50) and (51) give the relations

$$A+A'' = A' \tag{52}$$

and

$$AV_1 - A''V_1 = A'V_2. \tag{53}$$

Consequently

$$A' = \frac{2V_1}{V_1+V_2}A, \tag{54}$$

$$A'' = \frac{V_1-V_2}{V_1+V_2}A. \tag{55}$$

Since the velocities V_1 and V_2 are

$$V_1 = B_0/(4\pi\rho_1)^{\frac{1}{2}} \tag{56}$$

and

$$V_2 = B_0/(4\pi\rho_2)^{\frac{1}{2}}, \tag{57}$$

the amplitude relations (54) and (55) are

$$A' = \frac{2\rho_2^{\frac{1}{2}}}{\rho_2^{\frac{1}{2}}+\rho_1^{\frac{1}{2}}}A, \tag{58}$$

$$A'' = \frac{\rho_2^{\frac{1}{2}}-\rho_1^{\frac{1}{2}}}{\rho_2^{\frac{1}{2}}+\rho_1^{\frac{1}{2}}}A. \tag{59}$$

At a reflection against a rigid conducting boundary the reflected wave has a phase-shift (relative to the incident wave) of 180° in the velocity but no phase-shift in the induced magnetic field. (This is seen by putting $\rho_2 = \infty$ in (59) and using (44), (46), and 3.4 (5); cf. Lundquist, 1952.) At a reflection against a free surface there is instead a 180° phase-shift in the induced magnetic field and no phase-shift in the velocity.

Next we consider the case where ρ and hence V varies continuously along a magnetic field line. If the variation is slow enough (that is, $|\text{grad}\,\rho|/\rho \ll 1/\lambda$, where λ is the wavelength), no appreciable reflection occurs. (This is analogous to what is known from the theory of light or sound waves.)

In this case we can find the variation of b_y and v_y in the following simple way. We approximate the continuous density distribution by one where the density jumps by the amount $\Delta\rho = (d\rho/dz)\Delta z$ at a large number of parallel planes with the spacing Δz. At such a plane the amplitude of the transmitted wave changes by the amount ΔA. We find from (58) that ΔA is given by

$$\frac{\Delta A}{A} = \frac{2(\rho+\Delta\rho)^{\frac{1}{2}}}{(\rho+\Delta\rho)^{\frac{1}{2}}+\rho^{\frac{1}{2}}} - 1 = \frac{\Delta\rho}{4\rho}.$$

Hence $$A = \text{const}\,\rho^{\frac{1}{4}}. \tag{60}$$

As A is the amplitude of b_y and v_y is proportional to $b_y/\rho^{\frac{1}{2}}$ (as shown by 3.4.1 (12)), we have

$$b_y = \text{const}\,\rho^{\frac{1}{4}}, \tag{61}$$

$$v_y = \text{const}\,\rho^{-\frac{1}{4}}. \tag{62}$$

This result can also be found from the condition of conservation of energy, because no appreciable energy is lost by reflection when the condition $|\text{grad}\,\rho|/\rho \ll 1/\lambda$ is satisfied.

3.5. Waves of arbitrary form in incompressible fluid

So far we have discussed some special cases in order to become familiar with the waves. We shall now enter into a more general discussion, which was first given by Walén (1944).

As before, we neglect the displacement current. We also assume that the permeability μ is constant. From 3.3 (2) and 3.3 (5) we eliminate $\mathbf{E}$ and introduce 3.3 (1). Because of 3.3 (3), the identity

$$\text{curl}(\text{curl}\,\mathbf{B}) = -\Delta\mathbf{B}+\text{grad}(\text{div}\,\mathbf{B})$$

reduces to $\text{curl}(\text{curl}\,\mathbf{B}) = -\Delta\mathbf{B}$, and we obtain

$$\text{curl}(\mathbf{v}\times\mathbf{B})+\frac{c^2}{4\pi\mu\sigma}\Delta\mathbf{B}-\frac{\partial\mathbf{B}}{\partial t} = 0. \tag{1}$$

Introducing 3.3 (1) into 3.3 (6) we find

$$\frac{\partial\mathbf{v}}{\partial t}+(\mathbf{v}\,\text{grad})\mathbf{v}+\frac{1}{4\pi\mu\rho}\mathbf{B}\times\text{curl}\,\mathbf{B} = \mathbf{G}-\frac{1}{\rho}\text{grad}\,p. \tag{2}$$

The magnetic field $$\mathbf{B} = \mathbf{B}_0+\mathbf{b} \tag{3}$$

consists of the primary field $\mathbf{B}_0$, which is given and supposed to derive from currents outside the fluid so that

$$\text{curl}\,\mathbf{B}_0 = 0, \tag{4}$$

and the induced field $\mathbf{b}$, which is produced by the currents caused by the disturbance. Our problem is to find $\mathbf{v}$ and $\mathbf{b}$ from (1) and (2).

The general solution encounters mathematical difficulties. *For the case of an incompressible fluid with constant density ρ in a homogeneous magnetic field B_0* a solution has been given by Walén (1944). In this case we have

$$\text{div}\,\mathbf{v} = 0, \tag{5}$$

$$\text{grad}\,B_0 = 0. \tag{6}$$

Then the identities

$$\begin{aligned}
\operatorname{curl}(\mathbf{v}\times\mathbf{B}) &= (\mathbf{B}\operatorname{grad})\mathbf{v}-(\mathbf{v}\operatorname{grad})\mathbf{B}-\mathbf{B}\operatorname{div}\mathbf{v}+\mathbf{v}\operatorname{div}\mathbf{B},\\
(\mathbf{v}\operatorname{grad})\mathbf{v} &= \tfrac{1}{2}\operatorname{grad}v^2-\mathbf{v}\times\operatorname{curl}\mathbf{v},\\
\mathbf{B}\times\operatorname{curl}\mathbf{B} &= \tfrac{1}{2}\operatorname{grad}B^2-(\mathbf{B}\operatorname{grad})\mathbf{B}
\end{aligned}$$

can be written

$$\operatorname{curl}(\mathbf{v}\times\mathbf{B}) = \{(\mathbf{B}_0+\mathbf{b})\operatorname{grad}\}\mathbf{v}-(\mathbf{v}\operatorname{grad})\mathbf{B}, \tag{7}$$

$$(\mathbf{v}\operatorname{grad})\mathbf{v} = \operatorname{grad}\tfrac{1}{2}v^2-\mathbf{v}\times\operatorname{curl}\mathbf{v}, \tag{8}$$

$$\begin{aligned}
\mathbf{B}\times\operatorname{curl}\mathbf{B} &= \mathbf{B}_0\times\operatorname{curl}\mathbf{b}+\mathbf{b}\times\operatorname{curl}\mathbf{b}\\
&= \operatorname{grad}(\mathbf{B}_0\mathbf{b})-(\mathbf{B}_0\operatorname{grad})\mathbf{b}+\mathbf{b}\times\operatorname{curl}\mathbf{b}.
\end{aligned} \tag{9}$$

Putting

$$\mathbf{G} = -\operatorname{grad}U, \tag{10}$$

we obtain from (1)

$$(\mathbf{B}_0\operatorname{grad})\mathbf{v}-\frac{\partial\mathbf{b}}{\partial t} = -\frac{c^2}{4\pi\mu\sigma}\Delta\mathbf{b}+(\mathbf{v}\operatorname{grad})\mathbf{b}-(\mathbf{b}\operatorname{grad})\mathbf{v}, \tag{11}$$

and from (2)

$$\frac{\partial\mathbf{v}}{\partial t}+(\mathbf{v}\operatorname{grad})\mathbf{v}+\frac{1}{4\pi\mu\rho}\mathbf{B}_0\times\operatorname{curl}\mathbf{b}+\frac{1}{4\pi\mu\rho}\mathbf{b}\times\operatorname{curl}\mathbf{b}+\operatorname{grad}\left(U+\frac{p}{\rho}\right) = 0 \tag{12}$$

or

$$\frac{1}{4\pi\mu\rho}(\mathbf{B}_0\operatorname{grad})\mathbf{b}-\frac{\partial\mathbf{v}}{\partial t} = \frac{1}{4\pi\mu\rho}\mathbf{b}\times\operatorname{curl}\mathbf{b}-\mathbf{v}\times\operatorname{curl}\mathbf{v}+ \\ +\operatorname{grad}\left[U+\frac{1}{\rho}\left(p+\frac{\rho v^2}{2}\right)+\frac{1}{4\pi\mu\rho}(\mathbf{B}_0\mathbf{b})\right]. \tag{13}$$

3.5.1. *Infinite conductivity*

If $\sigma = \infty$ an exact solution can be found for (11) and (13). We put

$$\mathbf{v} = \mathbf{b}/(4\pi\mu\rho)^{\frac{1}{2}}. \tag{14}$$

As

$$\tfrac{1}{2}\rho v^2+\frac{1}{4\pi\mu}(\mathbf{B}_0\mathbf{b}) = \frac{1}{8\pi\mu}\{(\mathbf{B}_0+\mathbf{b})^2-B_0^2\}, \tag{15}$$

the grad term on the right side in (13) vanishes, if

$$p+\rho U+\frac{1}{8\pi\mu}(\mathbf{B}_0+\mathbf{b})^2 = \text{const}, \tag{16}$$

which means that the sum of the hydrostatic pressure p and the magnetostatic pressure $B^2/8\pi\mu$ equals $-\rho U$.

Then (11) and (13) are both reduced to

$$(\mathbf{B}_0\operatorname{grad})\mathbf{b} = (4\pi\mu\rho)^{\frac{1}{2}}\frac{\partial\mathbf{b}}{\partial t}. \tag{17}$$

This equation shows that *the state of motion is displaced with the velocity*

$$\mathbf{V} = -\mathbf{B}_0/(4\pi\mu\rho)^{\frac{1}{2}}. \tag{18}$$

Equations (11) and (13) are also satisfied by

$$\mathbf{v} = -\mathbf{b}/(4\pi\mu\rho)^{\frac{1}{2}}, \tag{19}$$

the result being that the velocity of propagation becomes

$$\mathbf{V} = \mathbf{B}_0/(4\pi\mu\rho)^{\frac{1}{2}}. \tag{20}$$

Hence, if the mass velocity $\mathbf{v}$ is antiparallel to the magnetic disturbance field $\mathbf{b}$, the wave velocity $\mathbf{V}$ is parallel to the given magnetic field $\mathbf{B}_0$.

In the calculations no second-order terms have been neglected. Consequently the result holds even if $b > B_0$. However, when waves of finite amplitude with opposite propagation velocities meet, complicated non-linear phenomena occur.

3.5.2. *Note on the effect of a Coriolis force*

It is interesting to note that hydromagnetic waves in rotating media are affected by the Coriolis force. The discussion above has been limited to non-rotating media in order to avoid too much complication, and we shall only make a few remarks on the effect of the Coriolis force.

As found by Lehnert (1954*b*, 1955*a*) the presence of a Coriolis force modifies the hydromagnetic waves in a characteristic way. A plane polarized wave is split up into two circularly polarized components travelling with different phase velocities. Further, the group velocity is no longer strictly parallel or antiparallel to the magnetic field, and a disturbance suffers distortion during its propagation. The phase velocities of the circularly polarized wave components are given by the expression

$$V_{\mathrm{ph}} = V[(1+\kappa^2)^{\frac{1}{2}} \pm \kappa]\cos\varphi, \tag{21}$$

where

$$V = B/(4\pi\mu\rho)^{\frac{1}{2}}, \tag{22}$$

$$\kappa = \frac{\omega\lambda}{2V}\frac{\cos\psi}{\cos\varphi}. \tag{23}$$

Here φ is the angle between the wave-front normal and the magnetic field, ψ is the angle between the normal and the axis of rotation, λ the wavelength, and ω the angular velocity.

The propagation of a small-amplitude disturbance is determined by the group velocity (cf. § 3.8.1). This has also been calculated by Lehnert (1955*a*) who has shown that the propagation is not exactly parallel to the lines of force. The maximum deviation in the direction of propagation is again given by the parameter κ.

Estimates of the parameter κ, equation (23), show that the effect of the Coriolis force may very well be important even in a slowly rotating star such as the sun.

When the amplitude of the hydromagnetic wave is large, non-linear terms come into play and cause a coupling between the various partial waves that build up the disturbance. This means that the simple results of § 3.5.1, which in a non-rotating fluid are valid for arbitrary amplitudes, become invalidated for large amplitudes by the non-linear effects introduced by a Coriolis force.

3.6. Hydromagnetic whirl rings in incompressible fluid

If a ring in a conducting magnetized fluid is set in motion with the velocity $2v$, it gives rise to hydromagnetic waves. As Walén (1944, 1946) has shown, the result is that the ring is split into two equal rings, each having the hydrodynamic velocity v (cf. Fig. 3.7 (*a*)). One of them migrates with the velocity $+\mathbf{V}$ (parallel to the given magnetic field $\mathbf{B}_0$), the other is displaced with the velocity $-\mathbf{V}$ (antiparallel to $\mathbf{B}_0$).

According to 3.5.1 (14) we have

$$\frac{\rho v^2}{2} = \frac{b^2}{8\pi\mu}, \tag{1}$$

which means that the kinetic energy of the ring equals its magnetostatic energy. As a consequence of this, the centrifugal force due to the curvature of the ring is always compensated by the magnetostatic pull of the lines of force (cf. 3.3.1 (14)).

Another consequence is that the hydrodynamic reduction in pressure, which equals $\frac{1}{2}\rho v^2$, is always compensated by the magnetostatic pressure $b^2/8\pi\mu$.

If the conductivity is infinite, the shape of the rings remains unchanged during the displacement. If the conductivity is finite, but so large that the damping is not very important, the above considerations are approximate. The main effect of the finite conductivity is that the cross-section of the rings increases. This follows the same law as the penetration of a magnetic field into a conductor, a phenomenon well known from the 'skin-effect' The velocity of the ring 'diffuses' into the surroundings. The limitation of the ring becomes more and more diffuse. As an approximate measure of the diffusion we could say that the cross-section S of the hydromagnetic ring increases at the rate

$$\frac{dS}{dt} = \frac{c^2}{2\pi\sigma\mu}, \tag{2}$$

cf. 3.9 (7) and (8).

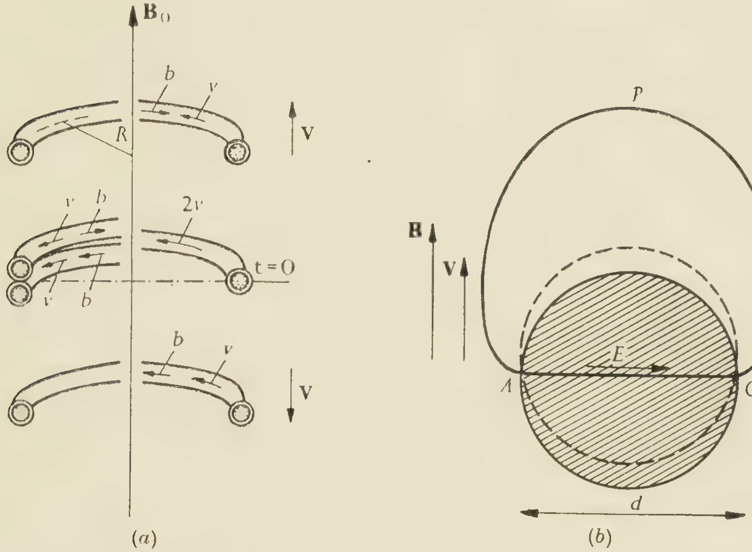


FIG. 3.7. (*a*) An initial hydrodynamic ring is split into two hydromagnetic rings, one migrating in the direction $+\mathbf{B}_0$, the other in the direction of $-\mathbf{B}_0$. (After Walén.) (*b*) The cross-section of a branch of a hydromagnetic ring. The hydrodynamic flow perpendicular to the paper causes a motional induced e.m.f. between A and C, which in the ideal case is exactly compensated by a transformer-induced e.m.f. due to the change of flux in the fixed circuit APC.

3.7. Hydromagnetic oscillations of fluid bodies

In a bounded body of magnetized conducting fluid oscillations can be set up which can be considered as standing hydromagnetic waves with wavelengths comparable to the linear dimension of the body. The frequencies of such oscillations depend on the structure of the body, and for a given structure there exists a set of frequencies corresponding to different modes of oscillation. We can estimate the order of magnitude of the lowest-mode frequency by equating the wavelength $\lambda = 2\pi V/\omega$ to the body's linear dimension l_c. It follows that

$$\omega/2\pi = \frac{V}{l_c} = \frac{B_c}{l_c(4\pi\mu\rho_c)^{\frac{1}{2}}}, \tag{1}$$

where B_c and ρ_c are characteristic values of magnetic field strength and mass density.

The fundamental frequency can also be estimated from energy considerations. Consider a sphere of infinitely conductive fluid in a homogeneous magnetic field $\mathbf{B}_0$.

Following Lundquist (1952) we suppose that the sphere is slightly deformed (see Fig. 3.8) so that it becomes a prolate spheroid with the

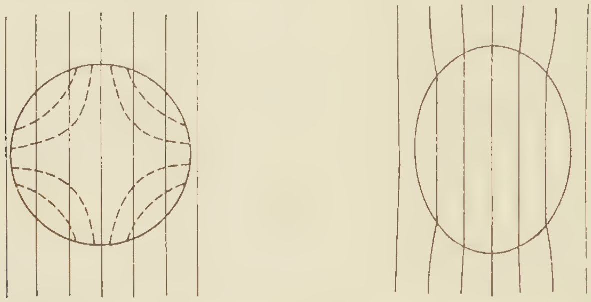


FIG. 3.8. Hydromagnetic oscillations of a sphere. Original field lines and stream lines (dashed curves) are shown in the left figure, deformed state to the right. (After Lundquist 1952.)

half-axes $R(1+\alpha)$ and $R(1-\frac{1}{2}\alpha)$, and then left free to oscillate. The excess magnetic energy at maximum deformation is

$$W_m = \frac{\alpha^2 B_0^2 R^3}{4\mu}. \tag{2}$$

If it is assumed that the oscillation is harmonic and characterized by an angular frequency ω, the kinetic energy is found to be

$$W_{kin} = \tfrac{1}{5}\rho\pi\alpha^2 R^5 \omega^2 \tag{3}$$

at the instant when the body is spherical and the excess magnetic energy vanishes.

Equating W_m and W_{kin} we obtain

$$\omega = \frac{\sqrt{5}}{R}\frac{B_0}{(4\pi\mu\rho)^{\frac{1}{2}}} \tag{4}$$

in agreement with the order-of-magnitude estimate (1).

The first analysis of hydromagnetic oscillations of a sphere was made by Schwarzschild (1949) in an early attempt to explain the behaviour of magnetically variable stars. However, for this application it is necessary to include the gravitational restoring force (Ferraro and Memory, 1952; Cowling, 1952). For a self-gravitating incompressible sphere Dricot and Ledoux (1959) give the result

$$\omega^2 = \tfrac{32}{9}\pi\gamma\rho + \frac{5}{3}\frac{B^2}{\pi\mu\rho R^2},$$

where γ is the gravitational constant. As pointed out by Cowling (1952), the gravitational restoring force is so strong that frequencies low enough to agree with observed periods of magnetic stars can only be obtained for special modes, characterized by a nearly horizontal motion.

The observed variations of magnetic stars are often very irregular and sometimes they involve changes of polarity. It seems doubtful whether they can be explained as due to hydromagnetic oscillations. An alternative explanation, the 'oblique rotor' theory, also encounters difficulties. For reviews see Cowling (1958), Babcock (1958), and Deutsch (1958).

3.8. Hydromagnetic waves in compressible media. Magnetosonic waves

We shall limit ourselves to *small-amplitude*, *plane harmonic waves*, and in the detailed analysis also to the case of *infinite conductivity*.

The assumption of small amplitude allows us to linearize the fundamental equations of § 3.3 by dropping small terms. The equations simplify further when the plane-wave assumption is used. These reductions are performed in § 3.8.1. In § 3.8.2 we derive the dispersion relations by means of which we can discuss the propagation of the waves.

3.8.1. *Linearization and simplification of the fundamental equations*

The total magnetic field

$$\mathbf{B} = \mathbf{B}_0 + \mathbf{b} \tag{1}$$

is a superposition of the unperturbed field $\mathbf{B}_0$ and the induced field $\mathbf{b}$, which is due to the wave motion. The perturbed density is

$$\rho = \rho_0 + \tilde{\rho}, \tag{2}$$

and the perturbed pressure is

$$p = p_0 + \tilde{p}, \tag{3}$$

where ρ_0 and p_0 are constants. The *small-amplitude assumption* implies that $b \ll B_0$, $\tilde{\rho} \ll \rho_0$, and $\tilde{p} \ll p_0$. The quantities $\mathbf{E}$, $\mathbf{v}$, and $\mathbf{i}$ are zero in the unperturbed state.

Neglecting the displacement current as before we obtain from the fundamental equations in § 3.3:

$$\operatorname{curl} \mathbf{b} = \frac{4\pi\mu}{c} \mathbf{i}, \tag{4}$$

$$\operatorname{curl} \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{b}}{\partial t}, \tag{5}$$

$$\operatorname{div} \mathbf{b} = 0, \tag{6}$$

$$\mathbf{E}+(\mathbf{v}/c)\times\mathbf{B}_0 = \mathbf{i}/\sigma, \tag{7}$$

$$\rho_0\frac{\partial\mathbf{v}}{\partial t} = (\mathbf{i}/c)\times\mathbf{B}_0-\operatorname{grad}\tilde{p}, \tag{8}$$

$$-\frac{\partial\tilde{\rho}}{\partial t} = \rho_0\operatorname{div}\mathbf{v}, \tag{9}$$

$$\tilde{p}/p_0 = \gamma\tilde{\rho}/\rho_0. \tag{10}$$

Note that in (8) the term $\rho\mathbf{G}$ does not appear because in the unperturbed state $\rho_0\mathbf{G}-\operatorname{grad}p_0 = 0$.

The assumption of *plane harmonic waves* implies that the variation of each of the fluctuating quantities is of the form

$$f = f_0 e^{j(\omega t-k_x x-k_y y-k_z z)}, \tag{11}$$

where x, y, and z are Cartesian coordinates and $j = \sqrt{(-1)}$, or, in shorter notation,

$$f = f_0 e^{j(\omega t-\mathbf{kr})}, \tag{12}$$

where k_x, k_y, k_z are constants. It is then easily verified that the following simple differentiation rules apply:

$$\frac{\partial}{\partial t} = j\omega, \tag{13}$$

$$\left.\begin{aligned} \operatorname{grad} &= -j\mathbf{k} \\ \operatorname{div} &= -j\mathbf{k}\cdot \\ \operatorname{curl} &= -j\mathbf{k}\times \end{aligned}\right\}. \tag{14}$$

By introducing the plane-wave solution into the linearized fundamental equations we obtain the following set of algebraic equations:

$$-j\mathbf{k}\times\mathbf{b} = (4\pi\mu/c)\mathbf{i}, \tag{15}$$

$$-j\mathbf{k}\times\mathbf{E} = -j\omega\mathbf{b}/c, \tag{16}$$

$$\mathbf{kb} = 0, \tag{17}$$

$$\mathbf{E}+(\mathbf{v}/c)\times\mathbf{B}_0 = \mathbf{i}/\sigma, \tag{18}$$

$$j\omega\rho_0\mathbf{v} = (\mathbf{i}/c)\times\mathbf{B}_0+j\mathbf{k}\tilde{p}, \tag{19}$$

$$-j\omega\tilde{\rho} = \rho_0(-j\mathbf{kv}), \tag{20}$$

$$\tilde{p}/p_0 = \gamma\tilde{\rho}/\rho_0. \tag{21}$$

The expressions (11) and (12) represent a plane wave with wave number k. The wave-number vector $\mathbf{k}$ is perpendicular to the surfaces of constant phase.

The surfaces of constant phase are displaced with the *phase velocity*

$$V_{\mathrm{ph}} = \frac{\omega}{k} = \frac{\omega}{(k_x^2+k_y^2+k_z^2)^{\frac{1}{2}}} \tag{22}$$

or, in vector form,

$$\mathbf{V}_{\mathrm{ph}} = \frac{\omega}{k^2}\mathbf{k}. \tag{22 a}$$

A wave of arbitrary shape can be synthesized by superposition of plane elementary waves of the form (12). The various partial waves of which the 'wave packet' is composed may travel with different phase velocities, and the wave packet itself travels with the *group velocity*, which is given by

$$\mathbf{V}_{\mathrm{g}} = \frac{\partial\omega}{\partial k_x}\hat{\mathbf{x}} + \frac{\partial\omega}{\partial k_y}\hat{\mathbf{y}} + \frac{\partial\omega}{\partial k_z}\hat{\mathbf{z}}, \tag{23}$$

where $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ are the unit vectors in the directions of the x-, y- and z-axes. (See, for example, Madelung, 1950.) The group velocity determines how a given disturbance is propagated.

The calculation of $\mathbf{V}_{\mathrm{ph}}$ and $\mathbf{V}_{\mathrm{g}}$ requires that we have a relation between ω and $\mathbf{k}$. This relation, the *dispersion relation*, will now be calculated from the set of equations (15) to (21).

3.8.2. *The dispersion relation and the modes of propagation*

In order to derive the dispersion relation we start by eliminating from (15) to (21) all variables except $\mathbf{v}$. First $\mathbf{i}$ and p can be eliminated from (19) by means of (15), (20), and (21):

$$j\omega\rho_0\mathbf{v} = (-j/4\pi\mu)(\mathbf{k}\times\mathbf{b})\times\mathbf{B}_0 + \frac{jp_0\gamma}{\omega}(\mathbf{kv})\mathbf{k}. \tag{24}$$

Next we obtain for $\mathbf{b}$ the expression

$$\mathbf{b} = -\frac{1}{\omega}\mathbf{k}\times(\mathbf{v}\times\mathbf{B}_0) + \frac{c^2}{4\pi\mu\sigma j\omega}\mathbf{k}\times(\mathbf{k}\times\mathbf{b}), \tag{25}$$

which follows from (15), (16), and (18). As

$$\mathbf{k}\times(\mathbf{k}\times\mathbf{b}) = \mathbf{k}(\mathbf{kb}) - k^2\mathbf{b},$$

and according to (17)

$$\mathbf{kb} = 0,$$

(25) reduces to

$$\mathbf{b}\left(1 - j\frac{c^2k^2}{4\pi\mu\sigma\omega}\right) = -\frac{1}{\omega}\mathbf{k}\times(\mathbf{v}\times\mathbf{B}_0). \tag{26}$$

Finally, combination of (24) and (26) results in the somewhat lengthy expression

$$j\omega\rho_0\mathbf{v} + \frac{\gamma p_0}{j\omega}(\mathbf{kv})\mathbf{k} = \frac{1}{4\pi\mu j\omega}\left(1 - j\frac{c^2k^2}{4\pi\mu\sigma\omega}\right)^{-1}\mathbf{B}_0\times\{\mathbf{k}\times[\mathbf{k}\times(\mathbf{v}\times\mathbf{B}_0)]\}.$$

By twice using the vector identity

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \mathbf{b}(\mathbf{ac})-\mathbf{c}(\mathbf{ab})$$

we can rewrite it in the form (Lüst, 1959)

$$\left\{\frac{(\mathbf{B}_0\mathbf{k})^2}{4\pi\mu}-F\omega^2\rho_0\right\}\mathbf{v}+\left\{\left(\frac{\mathbf{B}_0^2}{4\pi\mu}+\gamma p_0\,F\right)\mathbf{k}-\frac{(\mathbf{B}_0\mathbf{k})}{4\pi\mu}\mathbf{B}_0\right\}(\mathbf{kv})-$$
$$-\frac{(\mathbf{B}_0\mathbf{k})}{4\pi\mu}(\mathbf{B}_0\mathbf{v})\mathbf{k} = 0, \quad (27)$$

where the factor
$$F = 1-j\frac{c^2k^2}{4\pi\mu\sigma\omega} \quad (28)$$

reduces to unity when $\sigma = \infty$. The following discussion will be restricted to the case of *infinite conductivity* so that we have $F = 1$ in (27).

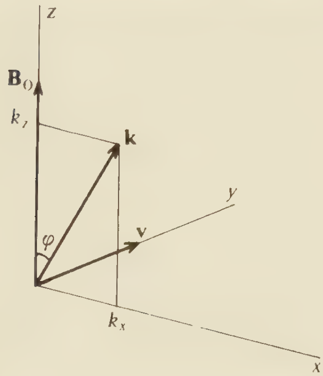


FIG. 3.9. Directions of the vectors $\mathbf{B}_0$, $\mathbf{k}$, and $\mathbf{v}$ in the case of transverse waves.

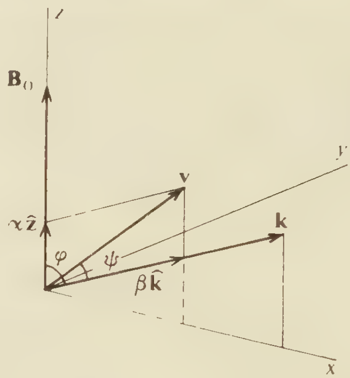


FIG. 3.10. Directions of the vectors $\mathbf{B}_0$, $\mathbf{k}$, and $\mathbf{v}$ and definitions of the angles ϕ and ψ in the case where the velocity is coplanar with $\mathbf{B}_0$ and $\mathbf{k}$.

For different modes of oscillation we obtain from (27) different dispersion relations. It is convenient to consider separately the oscillation mode where $\mathbf{v}$ is perpendicular to $\mathbf{k}$ and $\mathbf{B}_0$ (Fig. 3.9) and those where $\mathbf{v}$ is coplanar with $\mathbf{k}$ and $\mathbf{B}_0$ (Fig. 3.10).

Case 1. $\mathbf{v}$ *perpendicular to* $\mathbf{k}$ *and* $\mathbf{B}_0$ (Fig. 3.9)

(Transverse waves with material motion perpendicular to $\mathbf{B}_0$.)

Since in this case $\mathbf{kv} = 0$ and $\mathbf{B}_0\mathbf{v} = 0$, the dispersion relation (27), with $F = 1$, simplifies to

$$\omega = (\mathbf{B}_0\mathbf{k})/(4\pi\mu\rho_0)^{\frac{1}{2}}. \quad (29)$$

As $\mathbf{B}_0\mathbf{k} = B_0 k\cos\varphi$ (see Fig. 3.9) the phase velocity is, according to (22),

$$V_{\text{ph}} = \frac{\omega}{k} = V\cos\varphi, \quad (30)$$

where
$$V = B_0/(4\pi\mu\rho_0)^{\frac{1}{2}} \tag{31}$$
is the hydromagnetic velocity.

Calculating the group velocity by means of (23) we obtain
$$\mathbf{V}_g = V\hat{\mathbf{z}}, \tag{32}$$
which means that a disturbance travels parallel to the magnetic field $\mathbf{B}_0$ and at a speed equal to the ordinary hydromagnetic velocity. It is actually an ordinary hydromagnetic wave of the kind studied in § 3.4.1. This result is expected, because $\mathbf{v}$ is perpendicular to $\mathbf{k}$, so that, according to (20), the density fluctuation vanishes (in the linear approximation), and the compressibility of the fluid does not play any role. (Note that the pressure changes derived in § 3.4.1 are of second order in the amplitude A.)

Case 2. $\mathbf{v}$ *coplanar with* $\mathbf{B}_0$ *and* $\mathbf{k}$ (Fig. 3.10)

In this case the waves are generally neither purely transverse nor purely longitudinal. Essentially following Lüst (1959) we resolve the velocity vector into components parallel to $\mathbf{B}_0$ and $\mathbf{k}$:
$$\mathbf{v} = \alpha\hat{\mathbf{z}}+\beta\hat{\mathbf{k}}, \tag{33}$$
where $\hat{\mathbf{z}}$ is the unit vector in the direction of the positive z-axis and $\hat{\mathbf{k}}$ the unit vector parallel to the vector $\mathbf{k}$. Since
$$\mathbf{B}_0\mathbf{k} = B_0 k\cos\varphi, \tag{34}$$
$$\mathbf{B}_0\mathbf{v} = \alpha B_0+\beta B_0\cos\varphi, \tag{35}$$
and
$$\mathbf{k}\mathbf{v} = \alpha k\cos\varphi+\beta k, \tag{36}$$
equation (27), with $F = 1$, becomes
$$\left(\frac{\mathbf{B}_0^2}{4\pi\mu}k^2\cos^2\varphi-\omega^2\rho_0\right)(\alpha\hat{\mathbf{z}}+\beta\hat{\mathbf{k}})+ \\ +\left\{\left(\frac{\mathbf{B}_0^2}{4\pi\mu}+\gamma p_0\right)k^2\hat{\mathbf{k}}-\frac{\mathbf{B}_0^2}{4\pi\mu}k^2\cos\varphi\,\hat{\mathbf{z}}\right\}(\alpha\cos\varphi+\beta)- \\ -\frac{\mathbf{B}_0^2}{4\pi\mu}k^2\cos\varphi(\alpha+\beta\cos\varphi)\hat{\mathbf{k}} = 0. \tag{37}$$
Introducing the velocity of sound,
$$c_s = (\gamma p_0/\rho_0)^{\frac{1}{2}} \tag{38}$$
(see, for example, Landau and Lifshitz, 1959, p. 245), and the magneto-hydrodynamic velocity
$$V = B_0/(4\pi\mu\rho_0)^{\frac{1}{2}}, \tag{39}$$

we can write the components of (37), divided by ρ_0, as follows:

$$\begin{cases} \alpha\omega^2+\beta V^2k^2\cos\varphi = 0, & (40) \\ \alpha c_s^2 k^2\cos\varphi+\beta\{(V^2+c_s^2)k^2-\omega^2\} = 0. & (41) \end{cases}$$

This system of equations determines the coefficients α and β and hence the angle ψ between the material velocity and the propagation vector. As the system of equations is linear and homogeneous, the condition for its having a non-zero solution is that its determinant is zero:

$$\begin{vmatrix} \omega^2 & V^2k^2\cos\varphi \\ c_s^2 k^2\cos\varphi & (V^2+c_s^2)k^2-\omega^2 \end{vmatrix} = 0, \tag{42}$$

so that
$$\omega^4-\omega^2k^2(V^2+c_s^2)+c_s^2V^2k^4\cos^2\varphi = 0. \tag{43}$$

Equation (43) is the required dispersion relation.

In terms of the phase velocity, (22), the dispersion relation can alternatively be written

$$V_{\text{ph}}^4-(V^2+c_s^2)V_{\text{ph}}^2+c_s^2V^2\cos^2\varphi = 0. \tag{44}$$

Since the discriminant of this equation is

$$D = (V^2+c_s^2)^2-4c_s^2V^2\cos^2\varphi = (V^2-c_s^2)^2+4c_s^2V^2\sin^2\varphi,$$

there are always two real and non-negative roots for V_{ph}^2, corresponding to two different modes of wave propagation.

We denote the phase velocities of the two modes by $V_1 = V_1(\varphi)$ and $V_2 = V_2(\varphi)$. They are plotted as functions of φ in the polar diagrams in Fig. 3.11. As the diagrams show, one of the modes is characterized by the fact that the phase velocity is equal to V, the hydromagnetic velocity, when $\varphi = 0$. We may call this mode a *modified hydromagnetic wave*, and denote its phase velocity by $V_1 = V_1(\varphi)$. The other mode, whose phase velocity is equal to the velocity of sound, c_s, when $\varphi = 0$, will be called a *modified sound wave*. Its phase velocity is denoted by $V_2 = V_2(\varphi)$.

Next we calculate the angle ψ between the velocity vector $\mathbf{v}$ and the wave-number vector $\mathbf{k}$. When φ is given, the value of ψ is determined by the ratio α/β, see Fig. 3.10.

From (40) it follows that

$$\frac{\alpha}{\beta} = -\frac{V^2}{V_{\text{ph}}^2}\cos\varphi, \tag{45}$$

and straightforward geometrical considerations give

$$\tan\psi = \frac{\sin\varphi\cos\varphi}{\cos^2\varphi-V_{\text{ph}}^2/V^2}, \tag{46}$$

where V_{ph} is in its turn determined by (44).

As the values of ψ differ from 0 and $\frac{1}{2}\pi$ (except when φ also is 0 or $\frac{1}{2}\pi$) the modified sound waves and the modified hydromagnetic waves are generally neither longitudinal ($\psi = 0$) nor transverse ($\psi = \frac{1}{2}\pi$).

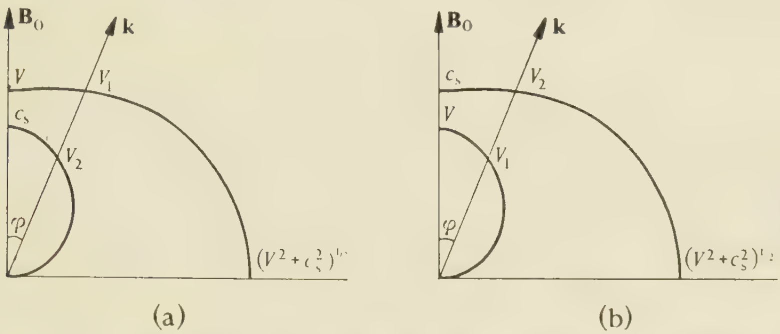


FIG. 3.11. Polar diagram of the phase velocities of the modified hydromagnetic waves (V_1) and the modified sound waves (V_2). (*a*) The case $V > c_s$. (*b*) The case $V < c_s$.

Table 3.1 summarizes the values taken by the phase velocity and the angle ψ in the special cases $\varphi = 0$ and $\varphi = \frac{1}{2}\pi$.

TABLE 3.1

The phase velocity and the angle ψ in the special cases $\varphi = 0$ and $\varphi = \frac{1}{2}\pi$ (cf. Fig. 3.10)

		$c_s < V$		$V < c_s$	
		$\varphi = 0$	$\varphi = \frac{1}{2}\pi$	$\varphi = 0$	$\varphi = \frac{1}{2}\pi$
Modified sound wave	Phase velocity	c_s	0	c_s	$(V^2+c_s^2)^{\frac{1}{2}}$
	Angle ψ	0	$\frac{1}{2}\pi$	0	0
Modified hydro-magnetic wave	Phase velocity	V	$(V^2+c_s^2)^{\frac{1}{2}}$	V	0
	Angle ψ	$\frac{1}{2}\pi$	0	$\frac{1}{2}\pi$	$\frac{1}{2}\pi$

From the table we find that when $\varphi = 0$ the modified hydromagnetic wave is transverse ($\psi = \frac{1}{2}\pi$), and propagates parallel to **B** (because $\varphi = 0$) at a speed that is equal to V. In other words, it has degenerated into an ordinary hydromagnetic wave. Similarly, when $\varphi = 0$, the modified sound wave is longitudinal ($\psi = 0$) and propagates parallel to **B** at the velocity of sound; it is an ordinary sound wave. The table also show that there exist longitudinal waves ($\psi = 0$) propagating with the velocity $(V^2+c_s^2)^{\frac{1}{2}}$ perpendicular to the magnetic field. They are the *magneto-sonic waves* (in the restricted sense). It is not difficult to show by means of (23) and (43) that for these waves not only the phase velocity but also the group velocity is directed perpendicular to **B** (and has the value $(V^2+c_s^2)^{\frac{1}{2}}$) so that the waves are capable of transmitting a disturbance across the magnetic field.

3.9. Criteria for applicability of magneto-hydrodynamics

Equation 3.4 (25) gives the distance z_0, which a weakly damped hydromagnetic wave travels before its amplitude is decreased by a factor e^{-1}. This distance is very much larger than the wavelength if

$$\frac{z_0}{\lambda} = \frac{\mu^{\frac{1}{2}}\sigma B_0 \lambda}{\pi^{\frac{3}{2}}\rho^{\frac{1}{2}}c^2} \gg 1. \tag{1}$$

As the existence of pronounced hydromagnetic waves requires that the damping is small, the inequality (1) is a criterion for the applicability of magneto-hydrodynamics to wave motion.

The criterion (1) is valid also for waves in compressible media. This is seen from 3.8.2 (27) and (28) by putting $k = 2\pi/\lambda$, $\omega/k = B_0/\sqrt{(4\pi\mu\rho)}$ in (28) and noticing that the damping is weak or strong depending on whether the factor F has a small or large imaginary part.

Consequently, disregarding unimportant numerical factors, we can take the inequality

$$\frac{\mu^{\frac{1}{2}}\sigma B_0 \lambda}{\rho^{\frac{1}{2}}c^2} \gg 1 \tag{2}$$

as a criterion for the existence of pronounced hydromagnetic waves. The dimensionless number

$$L = \frac{\mu^{\frac{1}{2}}\sigma B_0 \lambda}{\rho^{\frac{1}{2}}c^2} \tag{3}$$

was first introduced by Lundquist (1952).

We now turn from the special case of waves to motion in general.

One important aspect of the coupling between the material motion and the magnetic field can be studied by means of 3.5 (1):

$$\operatorname{curl}(\mathbf{v}\times\mathbf{B}) + \frac{c^2}{4\pi\mu\sigma}\Delta\mathbf{B} - \frac{\partial\mathbf{B}}{\partial t} = 0, \tag{4}$$

which is obtained by eliminating the electric field from the fundamental equations 3.3 (1), 3.3 (2), and 3.3 (5). In order to discuss the relative importance of the terms in this equation we use the estimates

$$|\mathrm{curl}\,(\mathbf{v}\times\mathbf{B})| = \frac{v_c B_c}{l_c} \tag{5a}$$

and

$$|\Delta\mathbf{B}| = \frac{B_c}{l_c^2}, \tag{5b}$$

where v_c, B_c, and l_c stand for characteristic values of velocity, magnetic field, and dimension of the phenomenon to be studied. This method of estimating terms in differential equations by replacing spatial differentiations by multiplications by inverse characteristic lengths is often useful and commonly practised. But it is obviously a very crude method and care must be taken in its application to specific problems.

From (4), (5a), and (5b) we conclude that for small values of the *magnetic Reynolds number*

$$R_m = 4\pi\mu\sigma l_c v_c/c^2 \tag{6}$$

the first term in (4) is negligible compared with the second (Lehnert, 1952*a*). Then (4) reduces to the diffusion-type equation

$$\Delta\mathbf{B} = \frac{1}{D}\frac{\partial\mathbf{B}}{\partial t}, \tag{7}$$

where the diffusion coefficient is given by

$$D = \frac{c^2}{4\pi\mu\sigma}. \tag{8}$$

This means that the magnetic field is not appreciably influenced by the material motion.

If, on the other hand, R_m is large compared with unity, the second term of (4) can instead be neglected and, from the resulting equation

$$\mathrm{curl}(\mathbf{v}\times\mathbf{B}) - \frac{\partial\mathbf{B}}{\partial t} = 0, \tag{9}$$

it follows that the magnetic flux through any contour following the material motion remains constant. This is so because the rate of change of the flux ϕ through the moving contour (cf. Fig. 3.12) is

$$\frac{d\phi}{dt} = \iint \frac{\partial\mathbf{B}}{\partial t}\,d\mathbf{S} + \oint (\mathbf{v}\times d\mathbf{s})\mathbf{B} = \iint \left[\frac{\partial\mathbf{B}}{\partial t} - \mathrm{curl}(\mathbf{v}\times\mathbf{B})\right] d\mathbf{S} = 0. \tag{10}$$

Thus the condition

$$R_m = 4\pi\mu\sigma l_c v_c/c^2 \gg 1 \tag{11}$$

characterizes situations where magnetic lines of force are carried by the moving fluid and can be considered as 'frozen-in'. However, it is not a criterion for applicability of magneto-hydrodynamics. For example, it may be satisfied in a highly conducting fluid with so weak a magnetic field that it does not influence the motion. Such a fluid behaves according to the laws of ordinary fluid dynamics. For magneto-hydrodynamics to be applicable there must be magnetic field of sufficient strength present in the fluid. Let this field be $\mathbf{B}_0$ and let B_c be a typical value of its strength. A material velocity $\mathbf{v}$ produces an induced magnetic field $\mathbf{b}$, so that the total field is $\mathbf{B}_0+\mathbf{b}$. The condition for the middle term in (4) to be negligible is then

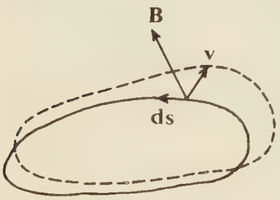


FIG. 3.12. Illustration of the change of magnetic flux through a moving contour.

$$v_c B_c/l_c \gg c^2 b_c/(4\pi\mu\sigma l_c^2). \tag{12 a}$$

Then the first and last terms of (4) are equal, which means

$$v_c B_c/l_c = b_c/t_c. \tag{12 b}$$

For a hydromagnetic phenomenon the magnetic force term in the equation of motion, 3.3 (6), is important, which means that its value is comparable to that of the inertia term. Thus

$$\rho v_c/t_c = b_c B_c/(4\pi\mu l_c) \tag{12 c}$$

(because $\mathbf{i}$ is associated with the induced field $\mathbf{b}$). From (12 b) and (12 c) it follows that

$$b_c = v_c\sqrt{(4\pi\mu\rho)}. \tag{12 d}$$

Combining equations (12 a) and (12 d) we arrive again at the Lundquist criterion

$$L = \frac{\mu^{\frac{1}{2}}\sigma B_c l_c}{\rho^{\frac{1}{2}}c^2} \gg 1, \tag{13}$$

which now contains l_c instead of λ.

Table 3.2, which has been compiled by Lehnert (1958), shows typical values of L in laboratory experiments and cosmical plasmas. Generally the condition (13) is very well satisfied in cosmical plasmas (cf. also Lehnert, 1959).

Viscous damping

Since all results in this chapter are derived for inviscid fluids, we also give the criterion for the viscosity to be negligible. For a viscous incompressible fluid the equation of motion, 3.3 (6), reads

$$\rho\frac{d\mathbf{v}}{dt} = \rho\mathbf{G}+(\mathbf{i}/c)\times\mathbf{B}-\operatorname{grad}p+ +\rho\nu\operatorname{div}\operatorname{grad}\mathbf{v}, \tag{14}$$

where $\rho\nu$ is the dynamic viscosity.

As before, we replace space differentiations by multiplications by $1/l_c$, and time differentiation by multiplication by v_c/l_c, where v_c and l_c

TABLE 3.2

(After Lehnert, 1958.)

Characteristic quantities for magneto-hydrodynamic experiments and for cosmical plasmas

	l_c	B_c	ρ_c	σ_c	V	L	L_1
Laboratory experiments:	(cm)	(gauss)	(g/cm^3)	(e.s.u.)	(cm/sec)		
Mercury	10	10^4	13·50	$9{\cdot}20\times10^{15}$	$7{\cdot}66\times10^2$	1	$6{\cdot}8\times10^6$
Sodium	10	10^4	0·93	$9{\cdot}37\times10^{16}$	$2{\cdot}92\times10^3$	38	$4{\cdot}6\times10^6$
Ionized gas (hydrogen)	10	10^3	10^{-10}	$4{\cdot}8\times10^{14}$	3×10^7	2×10^3	4×10^3
Cosmical plasmas:							
Earth's interior	2×10^8	10 (?)	10	7×10^{15}	1 (?)	2×10^4 (?)	2×10^{10} (?)
Sunspots	10^9	2×10^3	10^{-4}	4×10^{14}	6×10^4	10^9	2×10^{11}
Solar granulation	10^8	10^2	10^{-7}	7×10^{13}	9×10^4	10^7	10^8
Magnetic variable stars	10^{12}	10^4	1 (?)	7×10^{15}	3×10^3 (?)	3×10^{11} (?)	10^{18} (?)
Interstellar space (more condensed regions)	10^{22}	10^{-5} (?)	10^{-24} (?)	7×10^{12} (?)	3×10^6 (?)	3×10^{21} (?)	3×10^7 (?)
Interplanetary space	10^{13}	10^{-4}	10^{-23}	7×10^{14} (?)	9×10^6	10^{15}	1
Solar corona	10^{11}	1 (?)	10^{-18} (?)	7×10^{15} (?)	3×10^8 (?)	3×10^{15} (?)	3×10^2 (?)

l_c = linear dimension.
B_c = magnetic field strength.
ρ_c = density.
σ_c = conductivity.

V = magneto-hydrodynamic velocity.
$L = B_c\, l_c(\sigma/c^2)\mu^{\frac{1}{2}}\rho^{-\frac{1}{2}}$.
$L_1 = B_c\, l_c\, \nu^{-1}\mu^{\frac{1}{2}}\rho^{-\frac{1}{2}}$.

are a characteristic velocity and a characteristic length. Then the order of magnitude of the ratio between the inertia term and the viscous terms in (14) is found to be given by the *Reynolds' number*

$$R = \frac{v_c l_c}{\nu}. \tag{15}$$

The viscous damping is negligible if

$$R = \frac{v_c l_c}{\nu} \gg 1. \tag{16}$$

This condition is exactly analogous to (11) and may be considered as the condition for the vortex lines to be 'frozen-in'. By considerations corresponding to those leading to (13) one arrives at the inequality

$$L_1 = \frac{B_c l_c}{\nu \mu^{\frac{1}{2}} \rho_c^{\frac{1}{2}}} \gg 1 \tag{17}$$

(Lehnert, 1958), which is analogous to the Lundquist criterion. As Table 3.2 shows, (17) is in most cases well satisfied in cosmical plasmas.

Thermal conduction

It may be mentioned that the condition for the isothermal surfaces to move with the fluid (when compression work is negligible) is found to be

$$\frac{v_c l_c}{\kappa} \gg 1, \tag{18}$$

where κ is the thermometric conductivity (which is connected with the thermal conductivity k by the relation $\kappa = k/\rho c_v$, where c_v is the heat capacity at constant volume). For hydromagnetic phenomena this means

$$L_2 = \frac{B_c l_c}{\kappa \mu^{\frac{1}{2}} \rho_c^{\frac{1}{2}}} \gg 1 \tag{19}$$

(Baños, 1958; Lehnert, 1958).

Plasma phenomena

It should be noted that the fundamental equations of magneto-hydrodynamics (§ 3.3) rest on the assumption that the conducting medium can be considered as a fluid. This is an important limitation, for if the medium is a plasma it is sometimes necessary to use a microscopic description in which the motion of the constituent particles is taken into account. Examples of plasma phenomena invalidating a hydromagnetic description are ambipolar diffusion, electron runaway, and generation of microwaves. Of special importance is the fact that in plasmas with sufficiently low density the picture of 'frozen-in' lines of force need not always be valid, see § 5.4.

3.10. Experimental observation of hydromagnetic waves

Because the dimensions available in the laboratory are small, it is difficult to satisfy the condition 3.9 (13) in experiments with electrically conducting liquids, cf. Table 3.2. Nevertheless the first attempts at experimental demonstration of hydromagnetic waves were made with mercury (Lundquist, 1949*a*, *b*) and liquid sodium (Lehnert, 1954*a*).

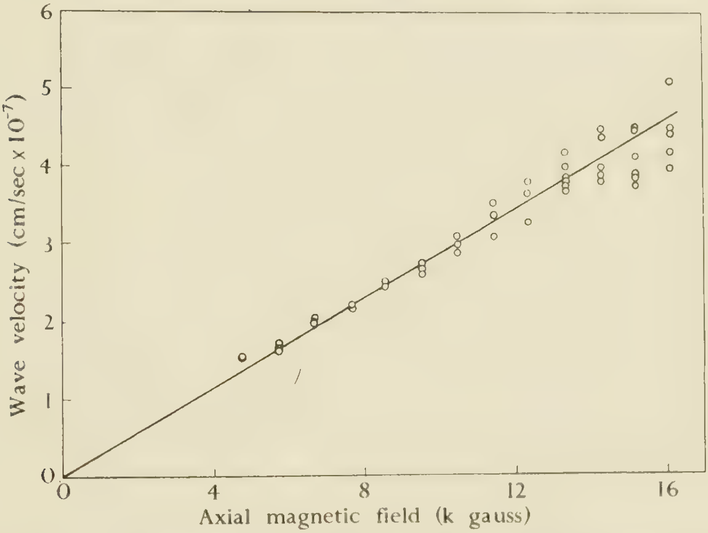


FIG. 3.13. Hydromagnetic wave velocity (V) as a function of the magnetic field strength (B_0). The small circles represent the measured values and the solid line indicates the proportionality between V and B_0. (After Wilcox, De Silva, Cooper, and Boley, 1961.)

In a plasma it is possible to obtain a large value of the Lundquist number $L = B_c l_c(\sigma/c^2)\mu^{\frac{1}{2}}\rho^{-\frac{1}{2}}$, because the density ρ is small. Hydromagnetic waves in plasma have been produced and studied by several workers (Allen *et al.*, 1959; Jephcott, 1959; Wilcox *et al.*, 1960, 1961, 1962; Nagao and Sato, 1960). Some results obtained by the Berkeley group are shown in Figs. 3.13 and 3.14. These results refer to a torsional wave mode in a hydrogen plasma at a temperature of 10 000° K, when the number density of ions is 5×10^{15} cm^{-3}. Fig. 3.13 shows the measured wave velocity as a function of the strength of the steady axial magnetic field. Fig. 3.14 shows the reflection of a hydromagnetic pulse (*a*) from a conducting end plate and (*b*) from a neutral-gas surface. In the former case the electric field of the wave is reversed but the magnetic field of the wave is not; in the latter case the magnetic field but not the electric field is reversed. This is in agreement with theory, cf. § 3.4.5.

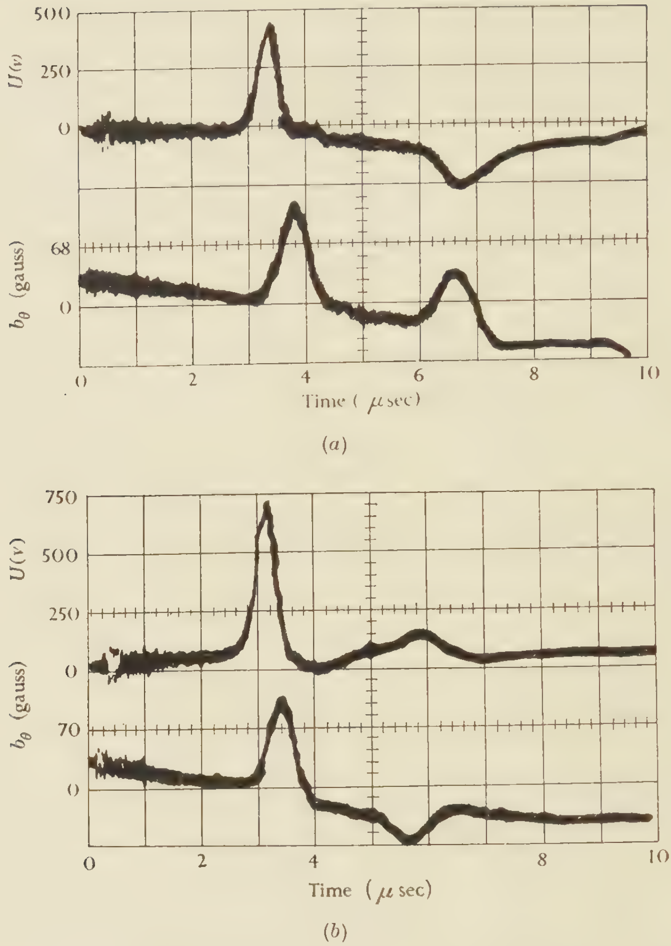


FIG. 3.14. Oscillograms showing the reflection of hydromagnetic waves. Upper traces show the induced electric voltage, U, of the wave, lower traces the induced (azimuthal) magnetic field, b_θ. The first pulse is the induced wave and the second pulse is the reflected wave. The delay between them corresponds to the time required for the waves to go to the end of the tube and back again. (*a*) Reflection against rigid conducting end-plate. (*b*) Reflection against plasma–neutral-gas interface. (After Wilcox, De Silva, Cooper, and Boley, 1961.)

OTHER APPLICATIONS OF MAGNETO-HYDRODYNAMICS

3.11. Steady motion of conducting fluids in the presence of a magnetic field

The motion of a conducting liquid in the presence of a magnetic field was studied by Ampère, who made experiments with mercury. However, due to the low conductivity, the hydromagnetic effects were small. Magneto-hydrodynamics, which might conceivably have been discovered as a consequence of these experiments, was left unexplored for another hundred years.

The problem of steady flow of mercury in pipes across a magnetic field was investigated both theoretically and experimentally by Hartmann and Lazarus (1937). Further investigations of similar problems are due to Shercliff (1953) and Murgatroyd (1955).

Both theoretical and experimental investigations of the behaviour of conducting fluids in magnetic fields have been carried out by Lehnert (1952, 1957, 1958). One of his experiments illustrates in a striking way the anisotropy introduced in a conducting fluid by the presence of a magnetic field (§ 3.11.1).

The presence of a magnetic field has a decisive influence on the onset of turbulence and thermal convection. Although some aspects of these phenomena are important in cosmic physics they are beyond the scope of the present monograph, and readers interested in these topics are referred to Chandrasekhar's book on *Hydrodynamic and Hydromagnetic Stability* (Chandrasekhar, 1961).

The steady rotation of conducting fluid masses in the presence of magnetic fields is a problem with great astrophysical interest. In § 3.11.2 we shall derive the theorem of isorotation or, as it is generally called after its discoverer, *Ferraro's theorem.*

3.11.1. *Anisotropy introduced by a magnetic field. Experiments*

The behaviour of a conducting liquid is very different in the absence and in the presence of a magnetic field. Some of the differences can easily be demonstrated experimentally (Lehnert, 1952*a*). For example, mercury has a water-like consistency in the absence of magnetic fields, and by stirring and shaking one can produce small-scale irregular motions and small fast surface ripples. In the presence of a strong magnetic field the mercury has instead a syrup-like consistency. Stirring produces only large whirls, and such whirls all have their axes parallel to the magnetic field. The surface tends to remain smooth. The tendency of the whirls to be aligned parallel to the magnetic field is one manifestation of the anisotropy produced by the field. In fact there is a tendency for all motions to become uniform along the magnetic field, or, in other words, a tendency towards two-dimensional motion. This is illustrated very clearly in another experiment (Lehnert, 1955*b*). The apparatus is shown in Fig. 3.15 (*a*). In a shallow copper vessel filled with mercury and inserted in a vertical magnetic field, a ringshaped part of the bottom was rotated. It was found that, when the magnetic field was strong, the motion of the mercury followed very closely the motion of the copper below, Fig. 3.15 (*b*) and (*c*).

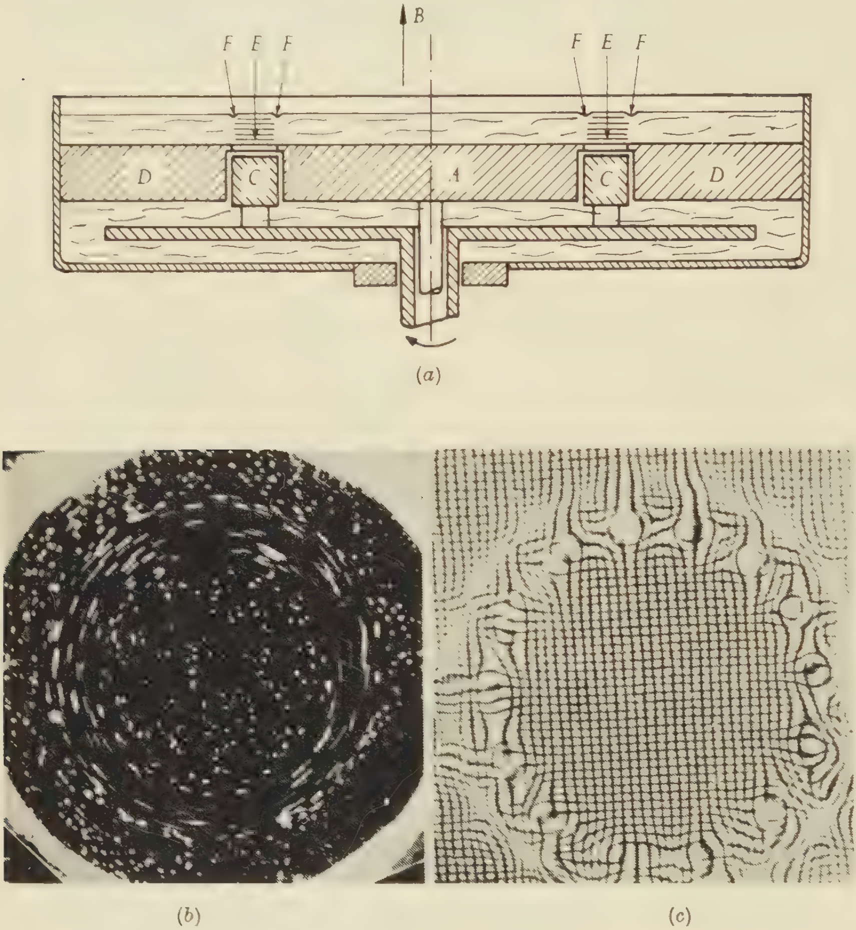


Fig. 3.15. (*a*) Apparatus for studying the motion of mercury above rotating copper disks in the presence of an external magnetic field. (*b*) The mercury surface seen from above when $B = 4300$ gauss. The mean motion has been indicated with grains of sand. (*c*) The reflection of a wire grid shows the deformation of the surface during the motion. A number of stationary whirls are produced on both sides of the moving ring of mercury. (After Lehnert, 1955*b*.)

The tendency of the motion to be constant along the magnetic field is quite similar to a phenomenon in ordinary hydrodynamics. According to the Taylor–Proudman theorem the motion in a rotating fluid tends to be uniform along lines parallel to the axis of rotation (see, for example, Chandrasekhar, 1961, p. 83). This is one of several analogies between a magnetic field and a vorticity field. (A rigid rotation corresponds to a homogeneous vorticity field.)

3.11.2. *Ferraro's law of isorotation*

In the experiments mentioned above the conductivity was rather low. In astrophysical situations the conductivity can often be considered as infinite. When that is true, the tendency of the motion to be constant along a line of force can be expressed very precisely in the form of a general theorem.

According to the fundamental equations in § 3.3 we have

$$\operatorname{curl}\mathbf{B} = \frac{4\pi\mu}{c}\mathbf{i}, \tag{1}$$

$$\operatorname{curl}\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}, \tag{2}$$

$$\operatorname{div}\mathbf{B} = 0, \tag{3}$$

$$\mathbf{i}/\sigma = \mathbf{E}+(\mathbf{v}/c)\times\mathbf{B}. \tag{4}$$

As in a *steady state* $\partial/\partial t = 0$, it follows from (1) to (4) that

$$\operatorname{curl}(\mathbf{v}\times\mathbf{B}) = \frac{c^2}{4\pi\mu}\operatorname{curl}\left(\frac{1}{\sigma}\operatorname{curl}\mathbf{B}\right). \tag{5}$$

If the *conductivity is infinite*, or more precisely if

$$\frac{4\pi\mu\sigma}{c^2}l_c v_c \gg 1, \tag{6}$$

where l_c and v_c are a characteristic length and a characteristic velocity, cf. § 3.9, the right-hand side of (5) vanishes, so that

$$\operatorname{curl}(\mathbf{v}\times\mathbf{B}) = 0. \tag{7}$$

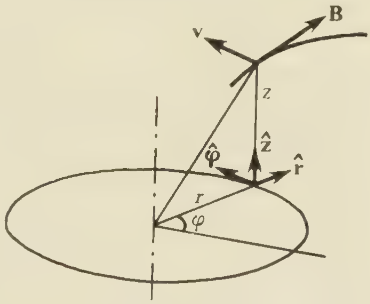


FIG. 3.16. Coordinate system used in the discussion of Ferraro's theorem.

Choose a cylindrical coordinate system (coordinates r, φ, z and unit vectors $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\varphi}}$, $\hat{\mathbf{z}}$, see Fig. 3.16) with the z-axis coinciding with the axis of rotation of the fluid body. The fact that the motion is assumed to be a pure rotation—but not necessarily rigid rotation—implies that

$$\mathbf{v} = \omega r\hat{\boldsymbol{\varphi}}, \tag{8}$$

where ω is a function of the coordinates r and z. (Dependence of ω on φ is excluded by the condition of steady state.) The magnetic field $\mathbf{B}$

may have components in all three coordinate directions:

$$\mathbf{B} = B_r\hat{\mathbf{r}} + B_\varphi\hat{\boldsymbol{\varphi}} + B_z\hat{\mathbf{z}}, \tag{9}$$

but we require *rotational symmetry*, which means

$$\frac{\partial B_r}{\partial \varphi} = \frac{\partial B_\varphi}{\partial \varphi} = \frac{\partial B_z}{\partial \varphi} = 0. \tag{10}$$

From Fig. 3.16 it is seen that

$$\mathbf{v} \times \mathbf{B} = \omega r(B_z\hat{\mathbf{r}} - B_r\hat{\mathbf{z}}). \tag{11}$$

Forming the curl of (11) using the condition (10) we obtain

$$\operatorname{curl}(\mathbf{v}\times\mathbf{B}) = \hat{\boldsymbol{\varphi}}\left\{\frac{\partial}{\partial z}(\omega r B_z) + \frac{\partial}{\partial r}(\omega r B_r)\right\}.$$

Since $\partial\omega/\partial\varphi = 0$, this can be rewritten

$$\operatorname{curl}(\mathbf{v}\times\mathbf{B}) = \hat{\boldsymbol{\varphi}}\left\{r\omega\left(\frac{1}{r}\frac{\partial}{\partial r}(rB_r) + \frac{1}{r}\frac{\partial B_\varphi}{\partial \varphi} + \frac{\partial B_z}{\partial z}\right) + r\left(B_r\frac{\partial \omega}{\partial r} + B_\varphi\frac{1}{r}\frac{\partial \omega}{\partial \varphi} + B_z\frac{\partial \omega}{\partial z}\right)\right\},$$

or, in shorter notation,

$$\operatorname{curl}(\mathbf{v}\times\mathbf{B}) = \hat{\boldsymbol{\varphi}}\{r\omega \operatorname{div}\mathbf{B} + r(\mathbf{B}\operatorname{grad})\omega\}. \tag{12}$$

From (3) and (7) it then follows that

$$(\mathbf{B}\operatorname{grad})\omega = 0, \tag{13}$$

which means that the derivative of ω in the direction of the magnetic field is zero. In other words: *in the steady state the angular velocity is constant along magnetic field lines.*

The law expressed by (13) is called the *law of isorotation* or *Ferraro's theorem.*

A large class of stars are known to have strong magnetic fields, and weaker magnetic fields are present also in our sun (and possibly in most other stars too). The importance of the above theorem in the physics of stellar interiors is therefore obvious. However, it is also important in connexion with the motion of interstellar matter within the reach of stellar magnetic fields. As the assumption of a high conductivity is essential, the theorem is directly applicable only if the density is so high that we can speak of a conductivity ('medium' or 'high' density plasmas, cf. § 5.1). It may be applicable also to 'low' density plasmas but this should not be taken for granted without detailed discussion. In the case of the sun it is possible that a certain inner region of interplanetary space is brought to co-rotate with the sun (Lüst and Schlüter, 1955). This co-rotation may have detectable effects on the variations of cosmic-ray intensity (see, for example, Dattner and Venkatesan, 1959; Sandström, 1956). The co-rotation of plasma outside the sun may have been a circumstance of decisive importance in the formation of the planetary system (Alfvén, 1954, 1962; Alfvén and Wilcox, 1962).

The establishing of isorotation between a magnetized star and the surrounding plasma requires a transfer of angular momentum between the central body and the plasma. The transfer of angular momentum can be effected by the magnetic field, and in this way a rotating star may lose a large part of its angular momentum. This is probably the explanation why some stars, including our sun, have a slow rotation. According to Ter Haar (1949) only an electromagnetic process is capable of producing sufficient braking. Lüst and Schlüter (1955) have analysed the transport of angular momentum by magnetic fields. They estimate that a nearly complete braking of the rotation of a star would be possible in some million years.

Simulation of the co-rotation by scale model experiments is difficult because the dimensions and conductivities available in the laboratory are small. Although correctly scaled model experiments of the cosmical plasma cannot be made (cf. also § 4.2.2), a qualitative demonstration of the co-rotation phenomenon in conducting liquid can be made in the laboratory (Fälthammar, 1961).

3.12. Magneto-hydrostatics

In a static equilibrium of an infinitely conducting medium we have $\mathbf{v} \equiv 0$, $\partial/\partial t = 0$, $1/\sigma = 0$, and so the fundamental equations of § 3.3 reduce to

$$\operatorname{curl} \mathbf{B} = \frac{4\pi\mu}{c}\mathbf{i}, \tag{1}$$

$$\operatorname{div} \mathbf{B} = 0, \tag{2}$$

$$\rho\mathbf{G} + (\mathbf{i}/c)\times\mathbf{B} - \operatorname{grad} p = 0, \tag{3}$$

where $\rho\mathbf{G}$ represents the non-magnetic forces.

Eliminating $\mathbf{i}$ from (1) and (3) we obtain the *magneto-hydrostatic equation*:

$$\rho\mathbf{G} - \operatorname{grad} p + \frac{1}{4\pi\mu}(\operatorname{curl} \mathbf{B})\times\mathbf{B} = 0. \tag{4}$$

Alternatively we may rewrite the magnetic force term as in § 3.3.1

and obtain

$$\rho \mathbf{G} - \operatorname{grad} p - \operatorname{grad} \frac{B^2}{8\pi\mu} + \frac{(\mathbf{B}\operatorname{grad})\mathbf{B}}{4\pi\mu} = 0. \tag{5}$$

We can distinguish between two kinds of magneto-hydrostatic equilibria:

(1) The magnetic forces are balanced by non-magnetic forces of some kind (gravitation, gas-kinetic pressure, etc.).

(2) The magnetic forces and the resultant of the non-magnetic forces vanish separately. The magnetic field is 'force-free' (Lundquist, 1950; Lüst and Schlüter, 1954).

Case (1) may be realized in certain cosmical structures. Discussions of equilibrium problems in sunspots, prominences, and galactic spiral arms are found in Cowling's book *Magneto-hydrodynamics* (1957).

Case (2) may also be very important in cosmical physics. We may expect force-free equilibria in cases where gravitation and gas-kinetic pressure represent energy densities that are much smaller than the energy density of the magnetic field. Furthermore, in plasmas that are so thin that the collision frequency is much smaller than the gyro frequency, the electric current tends to flow parallel to the magnetic field so that the magnetic force vanishes (§ 5.5.3).

The present discussion will be limited to some remarks about Case (2), force-free magnetic fields. Such fields are further discussed in § 5.5.3. For other types of magneto-hydrostatic equilibria see surveys by Cowling (1957, p. 20) or Dungey (1958, p. 43).

3.12.1. *Force-free magnetic fields*

The vanishing of the magnetic force implies that

$$(\operatorname{curl} \mathbf{B}) \times \mathbf{B} = 0. \tag{6}$$

(We assume throughout that $\mu = \text{const.}$)

The condition (6) is satisfied if

$$\operatorname{curl} \mathbf{B} = \alpha \mathbf{B}, \tag{7}$$

where α is a scalar function, which is arbitrary except that it must meet the requirement

$$(\operatorname{grad} \alpha)\mathbf{B} = 0. \tag{8}$$

This requirement follows, if we take the divergence of (7) and observe that $\operatorname{div} \mathbf{B} = 0$.

As an example we may consider a cylindrically symmetric case with $\alpha = \text{const}$, which has been analysed by Lundquist (1950). The components of (7) are

$$-\frac{\partial B_z}{\partial r} = \alpha B_\varphi, \tag{9}$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rB_\varphi) = \alpha B_z. \tag{10}$$

Elimination of B_φ gives the equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial B_z}{\partial r}\right) + \alpha^2 B_z = 0, \tag{11}$$

the solution of which is the zero-order Bessel function

$$B_z = AJ_0(\alpha r), \tag{12}$$

where A is an arbitrary constant. Then B_φ is calculated from (9) to be

$$B_\varphi = AJ_1(\alpha r). \tag{13}$$

The structure of the magnetic field is as shown in Fig. 3.17.

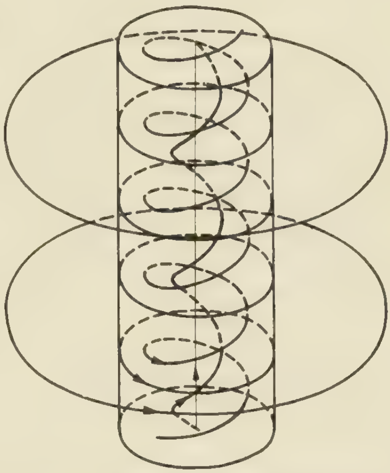


Fig. 3.17. Force-free magnetic field in an infinite cylinder. (After Lundquist 1950.)

In connexion with an application to cosmical plasmas we shall later (§ 5.5.3) discuss a particular kind of force-free field, which does not correspond to a constant α, but which has a certain general similarity to the field given by (12) and (13).

The general rotationally symmetric solution of (7) with constant α has been given by Lüst and Schlüter (1954). As shown by Chandrasekhar (Chandrasekhar, 1956; Chandrasekhar and Kendall, 1957*a*) it can be expressed in terms of Gegenbauer's functions. It is also shown that, given any toroidal† magnetic field, it is possible to find a poloidal† field such that the superposition is force-free, and conversely, if a poloidal field is given, there exists a toroidal field to make the superposition force-free. The general solution without the restriction of constant α has been given by Schlüter (1957) for cylindrical symmetry (also $\partial/\partial z \equiv 0$).

One may ask about the possibility of a force-free magnetic field

† In a cylindrical coordinate system a magnetic field which is everywhere perpendicular to planes through the axis (meridional planes) is called *toroidal* and a magnetic field whose field lines lie everywhere in meridional planes is called *poloidal*.

remaining force-free during decay due to finite electrical conductivity. This problem has been analysed by Lundquist (1950) who has shown that a necessary condition is that α (defined by (7)) is constant.

The decay of a magneto-hydrostatic field obeys the diffusion equation

$$\operatorname{curl}\operatorname{curl}\mathbf{B}+\frac{4\pi\mu\sigma}{c^2}\frac{\partial\mathbf{B}}{\partial t}=0, \tag{14}$$

which follows from 3.3 (1), 3.3 (2), and 3.3 (5) if $\partial\mathbf{D}/\partial t = 0$. Using the condition (7) and the vector identity $\operatorname{curl}(\alpha\mathbf{B}) = \alpha\operatorname{curl}\mathbf{B}+(\operatorname{grad}\alpha)\times\mathbf{B}$ we find for a force-free field

$$\frac{\partial\mathbf{B}}{\partial t} = -\frac{c^2}{4\pi\mu\sigma}\{\alpha^2\mathbf{B}+(\operatorname{grad}\alpha)\times\mathbf{B}\}. \tag{15}$$

From (15) we can immediately draw the conclusion: If the magnetic field is to decay without distortion of its shape, α must be constant. This is so because the condition of no distortion means that $\partial\mathbf{B}/\partial t$ has no component perpendicular to $\mathbf{B}$, and then (15) implies that $(\operatorname{grad}\alpha)\times\mathbf{B}$, which is perpendicular to $\mathbf{B}$, must vanish. This is possible only if $\operatorname{grad}\alpha$ has no component perpendicular to $\mathbf{B}$, and since, by (8), it can have no component parallel to $\mathbf{B}$ either, it follows that

$$\alpha = \text{const.} \tag{16}$$

As (15) then reduces to

$$\frac{\partial\mathbf{B}}{\partial t} = -\frac{c^2\alpha^2}{4\pi\mu\sigma}\mathbf{B},$$

the decay is exponential with a time constant proportional to σ/α^2.

According to Chandrasekhar and Woltjer (1958) the force-free fields with constant α are among the field configurations that have minimum ohmic loss at given magnetic energy. For a closed system of magneto-fluid with infinite conductivity (magnetic flux conserved) Woltjer (1958) found the force-free fields with constant α to be configurations of minimum magnetic energy.

3.13. Hydromagnetic stability

Magneto-hydrostatic equilibria may be classified as stable or unstable depending on how they respond to small perturbations. If any possible small perturbation induces restoring forces that bring the system back to equilibrium, the system is stable. If, on the other hand, there exists *at least one* perturbation such that the system, when left to itself, is removed more and more from the equilibrium configuration by the forces provoked by the perturbation, the system is unstable.

As a very simple illustration of an unstable configuration we may consider a dense fluid superposed on a less dense fluid, the two fluids being separated by a horizontal boundary, and gravity—or an inertia field—pointing downward. A perturbation that ripples the boundary will grow in amplitude. This is the *Taylor instability*. The physical reason for this is, of course, that more potential energy is released by displacing some of the heavy fluid downwards than is absorbed by displacing a corresponding amount of light fluid upward. If surface tension acts at the interface it has a stabilizing effect, particularly for small ripples, because these correspond to an increased surface energy. As a consequence ripples with wavelengths smaller than a certain critical value λ_c are stabilized. Next we may ask about the effect of a homogeneous magnetic field $\mathbf{B}$. If the field is parallel to the interface, it has, of course, no effect on ripples extending uniformly along $\mathbf{B}$, since such perturbations do not perturb the magnetic field (we assume here that the fluids are incompressible). On the other hand, ripples that extend uniformly in a direction perpendicular to the magnetic field cause a bending of the field lines, and, if the fluid is electrically conducting, the tension along the magnetic field lines (cf. § 3.3.1) opposes the deformation. Analysis shows that for this kind of deformation the magnetic field has a stabilizing effect equivalent to that of a surface tension given by

$$T = \frac{\lambda}{\pi} \frac{B^2}{4\pi\mu}$$

(see, for example, Chandrasekhar, 1961, p. 428), where λ is the wavelength of the ripple. (Finally, a magnetic field perpendicular to the interface influences the growth rate of perturbations with small wavelengths but does not achieve stabilization for any wavelength.)

A conducting fluid can also be supported against gravitation by a magnetic field. In this case the supporting force is due to currents at the boundary interacting with the magnetic field. The boundary is again unstable (Kruskal and Schwarzschild, 1954). The instability is generally referred to as the *Kruskal–Schwarzschild instability*, and it is analogous to the Taylor instability.

Another simple example is provided by the 'pinch' configuration, that is, a plasma column with an axial current and an associated azimuthal magnetic field. The magnetic force, which is directed radially inwards, is balanced by a pressure gradient. We assume for simplicity that the fluid is incompressible and has infinite conductivity. If a local constriction occurs, the current density and the azimuthal magnetic field are locally increased. The 'pinching' magnetic force $(\mathbf{i}/c)\times\mathbf{B}$ is therefore larger at the constriction. As a consequence the constriction develops further (the fluid being pushed aside axially to build expansions on either side). A series of constrictions deform the column into a shape exhibiting bulges separated by constrictions, hence the name 'sausage instability'. A bending of the column into a 'kink', increases the magnetic force on the concave side and the deformation is amplified. This is the 'kink instability'.

In the mathematical analysis of stability a standard approach is the *normal-mode method*, which makes use of the possibility of decomposing an arbitrary deformation into a spectrum of normal modes.

For example in cylindrical geometry one analyses deformations where the first-order perturbations of all quantities vary as

$$f(r)e^{j\omega t}e^{j(kz+m\varphi)}. \tag{1}$$

Then from the linearized hydromagnetic equations and the appropriate boundary conditions a dispersion relation is derived, which ascribes to each pair of values of k and m a value for ω. The dispersion relation contains the required information on the stability properties of the system: values of k and m for which ω has a value with negative imaginary part represent perturbations for which the system is unstable.

The various instabilities of a cylindrical column are often classified according to what value of m they correspond to in the normal-mode representation. According to (1) the perturbed boundary surface has, at a given time a shape that is given by (the real part of)

$$R = R_0 + \text{const}\, e^{j(kz+m\varphi)}. \tag{2}$$

We see that in modes with $m = 0$, this is a surface of rotation with a radius varying with z. The 'sausage instability' mentioned above is an $m = 0$ instability. In modes with $m = 1$ the surface is perturbed into a helical shape (the cross-section remaining circular). The same is true for $m = -1$ except that the helix is wound in the opposite sense. The pitch of the helices is $2\pi/k$. By a superposition of $m = 1$ and $m = -1$ perturbations, a periodic 'kink' deformation is obtained. For higher values of m also the cross-section is deformed.

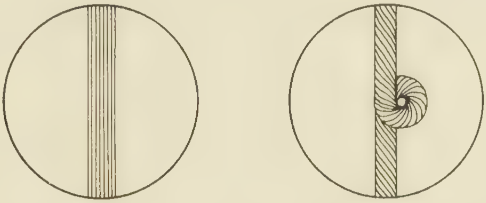
The hydromagnetic stability theory has important applications to cosmical physics. Since it is also important in connexion with magnetically confined plasmas, it has experienced a fast development due to the efforts of thermonuclear research. A comprehensive treatise of stability theory has recently been given by Chandrasekhar (1961), and the particular problems of stellar stability have been treated by Ledoux

(1958). Here we shall only discuss some results regarding the stability of twisted magnetic fields. These results are needed for the application in § 3.14.

Model experiments can sometimes offer an important possibility of studying hydromagnetic stability. To illustrate this we devote § 3.13.2 to a brief presentation of such experiments.

3.13.1. *Stability of twisted magnetic fields*

An originally homogeneous magnetic field in a conducting fluid may become twisted by motions of the fluid. From the analogy between



(a) (b)

Fig. 3.18. Illustration of loop formation in a twisted magnetic field. (*a*) Flux tube before twisting. (*b*) Flux tube after formation of a loop.

magnetic field lines and elastic strings one can qualitatively expect that sufficient twisting should make the configuration unstable for loop formation. As such loop formation may be of importance for maintenance of cosmic magnetic fields (§ 3.14.3) the stability of twisted magnetic fields is of some interest.

An order-of-magnitude estimate of the condition for instability for loop formation can be obtained by the following simple argument (Alfvén, 1950). Consider a cylindrical flux tube (Fig. 3.18 (*a*)) with length l and radius $R \ll l$. If it is twisted at one end through an angle φ, there is created an azimuthal magnetic field of strength

$$B_\varphi \doteq \frac{r\varphi}{l} B_z, \tag{3}$$

where B_z is the axial field strength. The total magnetic energy of the tube of force is thus increased to the value

$$W_M = \frac{l}{8\pi\mu} \int\limits_0^R (B_\varphi^2 + B_z^2) 2\pi r\, dr = \frac{B_z^2 R^2}{8\mu}\left(l + \frac{R^2\varphi^2}{2l}\right). \tag{4}$$

If the flux tube is deformed by loop formation as shown in Fig. 3.18 (*b*), the length is increased by at least $\Delta l = 2\pi R$. At the same time the twist changes by the amount $\Delta\varphi = -2\pi$. As, in the incompressible case, which we consider, the volume $\pi R^2 l$ remains unchanged, the radius changes by $\Delta R = -R\Delta l/2l = -\pi R^2/l$. As the flux $\pi R^2 B_z$ remains constant, the magnetic field changes by $\Delta B_z = 2\pi B_z R/l$. If now the changes in l, R, and B_z are introduced into (4), one finds that the forming of the loop implies a decrease of magnetic energy if

$$\varphi > (\sqrt{5}-1)l/R. \tag{5}$$

This is the approximate criterion for instability.

Quantitative analyses of the problem have been performed by Lundquist (1951) and by Dungey and Loughhead (1954).

In a cylindrical coordinate system with coordinates r, φ, and z (and unit vectors $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\varphi}}$, and $\hat{\mathbf{z}}$) we can write the twisted magnetic field

$$\mathbf{B} = B_\varphi(r)\hat{\boldsymbol{\varphi}} + B_z(r)\hat{\mathbf{z}}. \tag{6}$$

The lines of force lie on cylindrical surfaces and form the angle

$$\psi = \arctan(B_\varphi/B_z) \tag{7}$$

with the axis, and the pitch is

$$p = 2\pi r B_z/B_\varphi. \tag{8}$$

The magnetic force is radial and in equilibrium it is balanced by a radial pressure gradient.

For a perturbation characterized by a deformation which, in Cartesian coordinates, x, y, z, has the form

$$\boldsymbol{\xi} = A\cos bx \sin az\,\hat{\mathbf{x}} + C\sin bx \cos az\,\hat{\mathbf{z}} \tag{9}$$

with $C/A = -b/a$ (incompressibility), Lundquist calculated the total change in magnetic energy and found the following condition for negative energy change (and thus for instability):

$$\int_0^R B_\varphi^2 r\,dr > 2\int_0^R B_z^2\, r\,dr. \tag{10}$$

(The energy method has later been developed to a high degree of refinement: see Bernstein, Frieman, Kruskal, and Kulsrud, 1958.)

As a particular kind of perturbation, namely (9), was used, (10) is a *sufficient* condition for instability, but need not be a necessary condition.

Equation (10) implies that when both axial and azimuthal fields are present, the configuration becomes unstable when the average energy density of the azimuthal field is twice the average energy density of the axial field. As mentioned above, the pinch with its purely azimuthal magnetic field is unstable.

By means of the normal-mode method Dungey and Loughhead (1954) have found that a cylinder with radius R and a uniformly twisted magnetic field (constant pitch p) is unstable if

$$p < \tfrac{1}{2}R. \tag{11}$$

3.13.2. *Model experiments on instabilities*

Although instabilities of various kinds appear in hot plasmas, it is generally difficult to observe in detail the process of instability growth. This can sometimes be more readily done in conducting liquids.

Dattner, Lehnert, and Lundquist (1958), and Dattner (1962) have made an experiment on instabilities in a current-carrying mercury jet. In this experiment a mercury jet was made to fall vertically from a circular hole in a metallic vessel down to another conducting vessel. Along this approximately cylindrical column was passed an electric current, variable up to 400 amp. The experiment was performed both with and without an externally applied axial magnetic field.

Fig. 3.19 shows pictures of the mercury column in two typical cases. As the downward motion of the liquid is a free fall, the distance from the top of the picture represents a non-uniform time coordinate.

In the case of no axial magnetic field, Fig. 3.19 (*a*), the deformation is essentially axisymmetric with periodic constrictions and expansions growing in amplitude. This is an instability of the 'sausage' type (cf. above).

When there is a superposed axial magnetic field, Fig. 3.19 (*b*), the column is generally deformed into a spiral. The mechanism of this phenomenon is as follows: Any slight 'kink' deformation is amplified by the perturbed magnetic forces. The current along the deformed column gets a radial component at both ends of the kink, and the radial current component interacts with the axial magnetic field. The force $(\mathbf{i}/c)\times\mathbf{B}$ is in the positive azimuthal direction at one end of the kink and in the negative at the other end. As a consequence the initial kink-like deformation is drawn out into the spiral shape found in the picture. The sense of the spiral is such that the axial field of the deformed current channel adds to the externally applied axial field.

A difference between the mercury jet and a hot plasma column is the conductivity, which may make the lines of force 'frozen-in' in the latter case but not in the former. However, in the case with no axial magnetic field the stability properties are very similar for a mercury jet and for an infinitely conducting plasma column. Even the growth rates agree within a factor of 2 (Murty, 1960, 1961). On the other hand, when an axial magnetic field is present, a high or low conductivity makes a considerable difference.

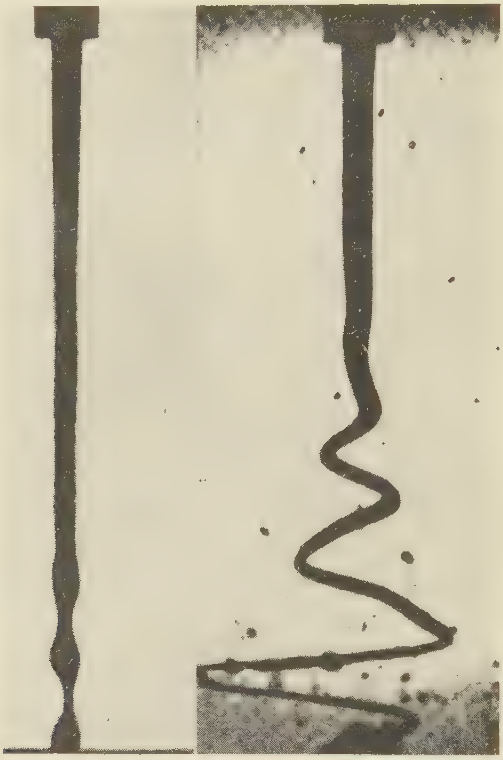


Fig. 3.19. Pictures of falling mercury jet carrying electric current. (*a*) In the absence of an axial magnetic field, sausage instabilities develop. (*b*) When an axial magnetic field (300 gauss) is present, the column is deformed into a spiral. (After Dattner, Lehnert, and Lundquist, 1958.)

Colgate, Furth, and Halliday (1960) have used sodium to perform model experiments on plasma instability. Pieces of solid sodium were introduced into a mirror machine and made subject to a strong pulsed magnetic field. The magnetic forces were strong enough to deform the sodium into characteristic shapes, exhibiting deformations corresponding to theoretically expected modes of instability.

3.14. Mechanisms for generating cosmic magnetic fields

An important problem in magneto-hydrodynamics is the creation of cosmic magnetic fields. As mentioned in Chapter 1, magnetic fields occur in widely different cosmical bodies (earth's interior, sun, magnetic stars, interplanetary and interstellar matter). Although only the 'magnetic' stars and the sun have fields strong enough to be observed from the earth by present-day techniques, it seems likely that most stars possess magnetic fields.

It is possible that essentially different generating processes work in different kinds of cosmic bodies. However, on the other hand, the existence of magnetic fields under a large variety of conditions suggests that possibly a single very general and fundamental process may be responsible for their generation.

Although some early work was done by Larmor, it is essentially through Cowling's theorem (§ 3.14.1) that the basic problem has been made clear. According to Cowling a magnetic field cannot be generated by a symmetric steady-state process. Therefore, one approach to the problem has been through non-symmetric steady-state processes (*self-exciting dynamo*), which will be briefly surveyed in § 3.14.2. See also reviews by Cowling (1955, 1957), Inglis (1955), and Elsasser (1955, 1956). An alternative approach, based on non-steady processes, *kink-instability model,* is reviewed in § 3.14.3–5.

3.14.1. *Cowling's theorem*

Dynamo theory is an inherently complicated subject. A complete theory would involve the solution of the full set of hydromagnetic equations taking into account both the magnetic effects of the material motions and the reaction of the magnetic forces upon these motions. The equations must not be linearized, because the dynamo action is essentially non-linear in character. Furthermore, the geometrical configuration of the dynamo can never be very simple, because Cowling's theorem (1934) states that *no stationary and axially symmetric dynamo is possible.* The proof of Cowling's theorem is, in broad outline, as follows: In any axially symmetric magnetic field of physical interest there exists one or more circular 'neutral lines', where the magnetic field vanishes. In a stationary state there is no electrostatic field along such a line (because that would imply a non-vanishing curl of **E** and so a time-varying **B**). The induced electric field $(\mathbf{v}/c)\times\mathbf{B}$ vanishes on the neutral line because **B** does. Thus there can be no electromotive force along the neutral line, and therefore the current density in the stationary state vanishes (the conductivity being finite). On the other hand, curl **B** does not vanish on the neutral line, provided it represents a first-order zero of the magnetic field. (Close to the neutral line the magnetic lines of force form loops around the neutral line.) By Maxwell's equations the non-vanishing curl **B** and the vanishing current density are in contradiction and this disproves the existence of a rotationally symmetric steady-state dynamo. (This result is valid also in the case of higher-order zeros of **B** on the neutral line.)

3.14.2. *The self-exciting dynamo*

As a consequence of the great complication which is thus inherent in the dynamo problem, most investigations so far are based on a kinematic approach. In this approach one tries to find a suitably chosen velocity distribution capable of dynamo maintenance of a magnetic field. The creation of the material motions and the influence of the magnetic field on them is not subject to simultaneous quantitative analysis, but the general plausibility of the motions from a dynamical point of view can be checked in a qualitative way.

By mathematical methods introduced by Elsasser (1946) the problem of finding velocity distributions capable of dynamo maintenance of magnetic fields can be formulated in terms of an eigenvalue problem. The existence of suitable velocity fields then depends on the existence of real eigenvalues to an infinite system of equations (see, for example, Cowling, 1957). Bullard (1949) and Bullard and Gellman (1954) have made plausible the existence of a magneto-hydrodynamic dynamo by analysing numerically a particular velocity field in a sphere of conducting fluid and finding real eigenvalues. The velocity field considered consisted of a non-uniform rotation of the sphere and simultaneous convection. From the dynamical point of view one may expect this kind of motion in the interior of the earth. If the earth's core is heated sufficiently by radioactive decay, thermal convection could occur, and the conservation of angular momentum of the rising material may cause differential rotation. Herzenberg (1958) has shown that two rotating fluid spheres embedded in a conducting fluid are capable of acting as a dynamo. Although it is difficult to see how such a configuration could be created in a cosmical body, the analysis has the advantage of being free from convergence difficulties. This is important for the question of existence of magneto-hydrodynamic dynamos (cf. Cowling, 1959).

We shall avoid going into the mathematics of dynamo theory and only discuss some features from a physical point of view. Consider a spherical cosmic body. We can distinguish between two phases in the maintenance of a magnetic field in such a body. The first is the generation of a toroidal flux from an existing poloidal magnetic field

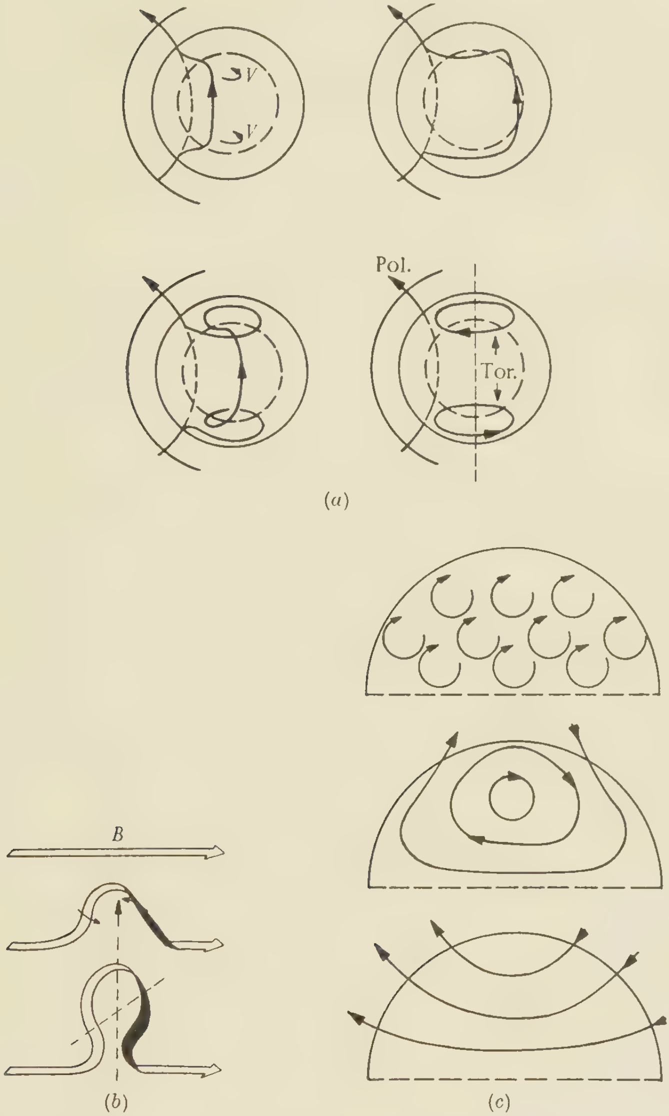


FIG. 3.20. (*a*) In the presence of a poloidal magnetic field a non-uniform rotation creates a toroidal field component. (*b*) Meridional loops created by local lifting and twisting. (*c*) Coalescence of loops to form a general poloidal field. (After Elsasser, 1955.)

and the second is the feedback from the toroidal to the poloidal field.

The creation of a toroidal flux is in principle very simple. As realized already by Larmor, it can be effected by non-uniform rotation, which drags out the poloidal field so that a toroidal component is produced. Fig. 3.20 (*a*) illustrates this. The crucial point is the feedback from the toroidal to the poloidal field. This could conceivably be effected by local lifting of flux tubes and twisting by the action of the Coriolis force so that meridional loops are formed and then coalesce (Parker, 1955), Fig. 3.20 (*b*) and (*c*). (Fig. 3.20 is from the review by Elsasser, 1955.)

It should be noted that toroidal fields of the kind postulated in dynamo theories may exist in the earth's core without being observed at the surface, because only the poloidal magnetic field can penetrate the insulating crust. (Toroidal fields at the surface would require currents penetrating the surface.) Elsasser (1956) has discussed the possibility that the sunspots are whirls associated with the sun's general magnetic field, the bipolar spots resulting from upsurges of toroidal flux tubes from beneath the surface. This concept is also part of Babcock's (1961) picture of the solar magnetic field and its solar-cycle variations. According to Babcock the strong magnetic fields, of which the sunspots bear evidence, are only a surface phenomenon. On the other hand, the very regular features of the sunspot progression cycle seems to be difficult to explain without assuming a strong poloidal field throughout the solar body (Alfvén, 1943; Walén, 1944 *a*, *b*, 1946, 1948).

The interaction of *turbulent* motions and irregular ('turbulent') magnetic fields has also been considered as a possibility for amplifying already existing magnetic fields. Such fields are, of course, not steady but may have some steady average characteristics. There are certain indications that in magneto-hydrodynamic turbulence there should be a tendency towards equipartition at least in some parts of the spectrum. The theoretical problems involved here are very complicated and far from solved. A discussion is found in Cowling's review of dynamo theories (Cowling, 1955).

3.14.3. *The kink-instability model*

The patterns of motion postulated in the *steady* self-exciting dynamos are rather complicated, and it is not easy to see how they could be produced. And even if such a mechanism may work in the earth's interior, it seems unlikely that it could also be responsible for the magnetic fields of stars and interstellar matter.

Under such conditions it is desirable to study also *non-stationary* processes. This has been done earlier by Alfvén (1950), Lundquist (1951), Dungey and Loughhead (1954), Inglis (1955), and recently by

Alfvén (1961). These authors have discussed the possibility of magnetic amplification by a *kink-instability process.*

The basic idea is that motions in a conducting fluid body change a certain initial magnetic field in such a way that it becomes unstable and, by a convulsion, goes back to its initial shape but with increased strength. This may happen, for example, if, in an initial poloidal field, motions produce a twist of the magnetic field lines, by which the total magnetic energy is increased. When the twist exceeds a certain limit, which has been calculated by Lundquist and Dungey-Loughhead (see § 3.13.1), it becomes unstable and produces a 'kink' with a closed magnetic flux in such a direction as to amplify the initial poloidal flux.

Since the first papers on magnetic amplification were written, the knowledge of hydromagnetic instabilities has increased very much as a consequence of thermonuclear research. However, the theoretical investigations of hydromagnetic instabilities are generally limited to the linear initial phase of instability growth. As the flux amplification depends on the phenomena that take place in the late stages of development of the instability, a quantitative theory is still out of reach. On the other hand, the technical facilities now available have made an experimental approach possible. Thus in an experiment intended for studying magnetized plasma rings Lindberg (Lindberg *et al.*, 1960, 1961) has discovered flux amplification of a kind that brings an important new element into the discussion of the origin of cosmic magnetic fields.

3.14.4. *The plasma-ring experiment*

In the experiments by Lindberg *et al.* a coaxial plasma gun produces a plasma ring, which, at the exit end of the gun, encounters a radial magnetic field, Fig. 3.21. The plasma ring that emerges from the gun has therefore both a trapped toroidal field originally due to the currents in the gun and a poloidal field that has been captured by the ring at the passage through the radial field at the exit of the gun. The magnetized plasma ring then continues its axial motion in a large drift tube, which is much wider than the gun. The essential features of the magnetic field are represented by the schematic drawing in Fig. 3.21.

It was expected that, at any time, the flux of the poloidal field would, at most, equal the initial flux. However, it was found that under some circumstances the measured poloidal flux exceeded the initial flux, sometimes by a factor of 5 or even more. As the toroidal magnetic field in the experiment is much stronger than the poloidal field, the phenomenon could be interpreted as an equalization of the fields, so that part of the toroidal field energy is transformed into poloidal field energy.

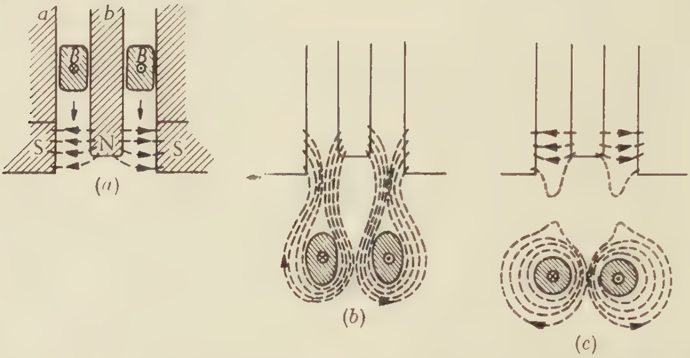


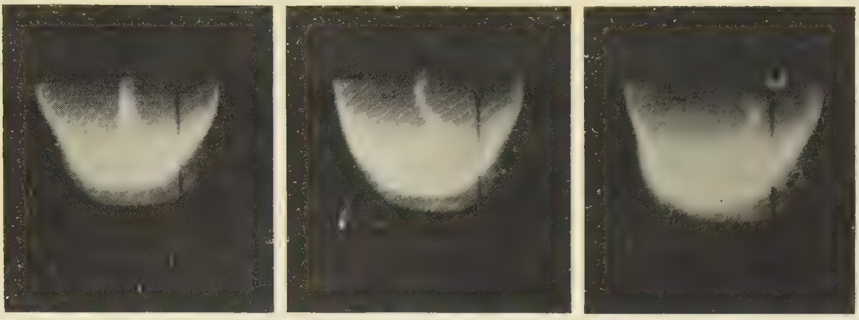
FIG. 3.21. Geometry of the plasma-ring experiment. (*a*) The plasma-ring before leaving the gun. (*b*) The plasma-ring pulling out the lines of force of the static magnetic field. (*c*) Plasma-ring with captured poloidal field.

Fig. 3.22 (*a*) shows Kerr-cell pictures of the plasma taken at various times after ignition. The successive deformation of the central column is clearly visible. Pictures taken simultaneously from different directions show that the deformed shape is helical. In all cases the sense of the helix is such that the discharge current flowing in the helix produces a poloidal field reinforcing the original poloidal field.

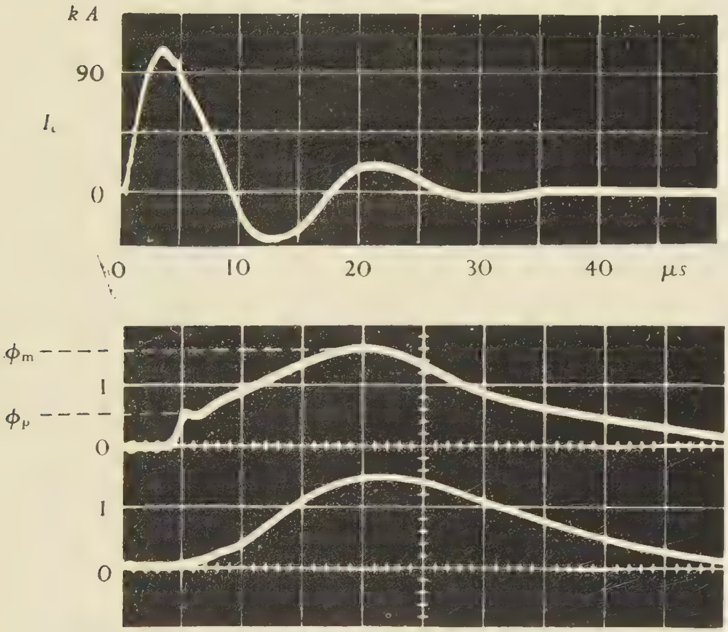
The helical distortion of the central column occurred under certain conditions, and it was under the same conditions that the flux amplification occurred. Typical oscillograms of the poloidal magnetic flux measured at two different distances from the gun are shown in Fig. 3.22 (*b*). At the smaller distance from the gun, the flux first rises to a plateau ϕ_p, which is approximately equal to the static flux, and then slowly increases to a maximum ϕ_m, which it reaches after the primary current has passed zero. In the picture the ratio ϕ_m/ϕ_p is about four, but even larger values have been observed. At the greater distance from the gun the plateau is smeared out and the flux grows continuously to its maximum value. At later times the plasma expands to the glass wall, where it is cooled, and the flux dissipates. A different, but possibly related kind of flux amplification has been discovered by Bickerton (1958).

3.14.5. *Field amplification by instability*

We can now outline a mechanism in which the process described above contributes to the production of general magnetic fields of celestial bodies.



(a)



(b)

FIG. 3.22. (a) Plasma column turning into a helix. Pictures are taken between 4·0 and 4·6 μsec after ignition. (After Lindberg and Jacobsen, 1961.) (b) Oscillograms showing discharge current and flux measured by two loops at 15 and 30 cm distance from the gun. Static flux $\phi_s = 5{\cdot}5 . 10^4$ gauss cm². The flux unit on the vertical axis is 10^5 gauss cm². (After Lindberg and Jacobsen, 1961.)

Consider a rotating fluid sphere which may represent the earth's fluid interior, the sun, an interstellar cloud, or perhaps a galaxy. Let us assume that it is initially magnetized with a poloidal magnetic field. Then the first step in the process is the creation of a toroidal field. As pointed out above, this is easily achieved by differential rotation, and Fig. 3.20 (*a*) may be taken as an illustration of this step. We only note that the twist produced by the differential rotation is propagated along the magnetic lines of force with the magneto-hydrodynamic velocity

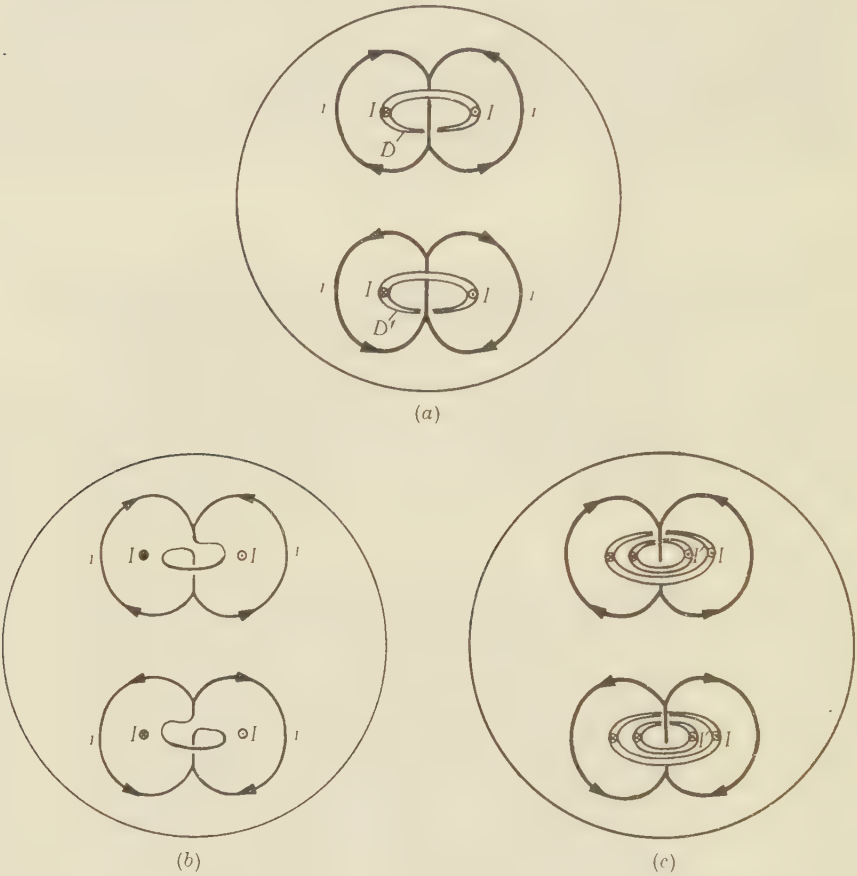


FIG. 3.23. Field amplification by kink instability. (*a*) Current system. The original poloidal field derives from the ring currents I. The toroidal field in the rings D and D is associated with the meridional currents i. (*b*) At the axis the currents i become unstable for kinks, and two helices are formed, both in such a direction that they tend to increase the poloidal field. (*c*) The helical currents are converted into an axial current and a ring current I', which amplifies the initial current I. The configuration is the same as in (*a*), but with an amplified poloidal field.

$V = B/(4\pi\rho\mu)^{\frac{1}{2}}$, so that if the differential rotation is too slow, the lines of force will have time to straighten themselves out. For a considerable toroidal field to be produced, the differential velocity should be at least comparable to V.

Next comes the feedback from the toroidal to the poloidal field. Before this feedback has taken place the current configuration is as shown (schematically) in Fig. 3.23 (*a*). The original poloidal magnetic field corresponds to a toroidal current system I, and the toroidal magnetic field to a poloidal current system i. The current systems i and I correspond precisely to the current systems in Lindberg's experiment. We should therefore expect that under certain conditions the axial columns of the current system i are unstable, become deformed into helices and produce a poloidal flux as in the experiment. This is illustrated in Fig. 3.23 (*b*) and (*c*). The latter figure shows a later stage of development, where the helical loops have transformed into toroidal current rings. They are in such a direction as to reinforce the original ring current.

As regards the condition for the onset of the instability the investigations of Lundquist (1951) and Dungey–Loughhead (1954) show that it should be expected when the average toroidal and poloidal magnetic field strengths satisfy the condition

$$B_{\text{tor}} > \alpha B_{\text{pol}},$$

where the factor α should be of the order of magnitude unity. In terms of the differential velocity v, this would imply

$$v > \alpha V,$$

where V is again the magneto-hydrodynamic velocity.

The pattern of motion we have postulated is very simple, and it should be considered only as a special example of a much more general type of motion.

REFERENCES

Alfvén, H., 1943: 'On Sunspots and the Solar Cycle', *Ark. f. mat. astr. o. fys.* **29 A,** no. 12.

—— 1950: 'Discussion of the Origin of the Terrestrial and Solar Magnetic Fields', *Tellus*, **2,** 74.

—— 1954: *On the Origin of the Solar System*, Clarendon Press, Oxford.

—— 1961: 'On the Origin of Cosmic Magnetic Fields', *Astrophys. J.* **133,** 1049.

—— 1962: 'On the Mass Distribution in the Solar System', *Astrophys. J.* **136,** 1005.

—— and Wilcox, J., 1962: 'On the Origin of the Satellites and the Planets', ibid. **136,** 1016.

Allen, T. K., Baker, W. R., Pyle, R. V., and Wilcox, J. M., 1959: 'Experimental Generation of Plasma Alfvén Waves', *Phys. Rev. Letters*, **2,** 383.

Åström, E., 1951: 'On Waves in an Ionized Gas', *Ark. f. fys.* **2,** 443.

—— 1961: 'On the Electromagnetic Behaviour of a Plasma', ibid. **19,** 163.

Babcock, H. W., 1958: 'Magnetic Fields of the A-type Stars', *Astrophys. J.* **128,** 228.

BABCOCK, H. W., 1961: 'Topology of the Sun's Magnetic Field and the 22-year Cycle', *Astrophys. J.* **133**, 572.

BAÑOS, A., 1958: 'Magneto-hydrodynamic Waves in Compressible Fluids with Finite Viscosity and Heat Conductivity', *Electrical Phenomena in Cosmical Physics*, International Astronomical Union Symposium no. 6, Stockholm, B. Lehnert, ed., Cambridge University Press, p. 15.

BERNSTEIN, I. B., FRIEMAN, E. A., KRUSKAL, M. D., and KULSRUD, R. M., 1958: 'An Energy Principle for Hydromagnetic Stability Problems', *Proc. Roy. Soc.* A, **244**, 17.

BICKERTON, R. J., 1958: 'The Amplification of a Magnetic Field by a High Current Discharge', *Proc. Phys. Soc.* **72**, 618.

BULLARD, E. C., 1949: 'The Magnetic Field within the Earth', *Proc. Roy. Soc.* A, **197**, 433.

—— and GELLMAN, H., 1954: 'Homogeneous Dynamos and Terrestrial Magnetism', *Phil. Trans. Roy. Soc.* **247**, 213.

CHANDRASEKHAR, S., 1956: *Proc. Nat. Acad. Sci.* (*Washington*), **42**, 1.

—— and KENDALL, P. C., 1957*a*: 'On Force-free Magnetic Fields', *Astrophys. J.* **126**, 457.

—— and WOLTJER, L., 1958: 'On Force-free Magnetic Fields', *Proc. Nat. Acad. Sci.* (*Washington*), **44**, 285.

—— 1961: *Hydrodynamic and Hydromagnetic Stability*, Clarendon Press, Oxford.

CHEW, G. F., GOLDBERGER, M. L., and LOW, F. E., 1959: 'The Boltzmann Equation and the One-fluid Hydromagnetic Equations in the Absence of Particle Collisions', *Proc. Roy. Soc.* A, **236**, 112.

COLGATE, S. A., FURTH, H. P., and HALLIDAY, F. O., 1960: 'Hydromagnetic Equilibrium Experiments with Liquid and Solid Sodium', *Rev. Mod. Phys.* **32**, 744.

COWLING, T. G., 1934: 'The Magnetic Field of Sunspots', *M.N.R.A.S.* **94**, 39.

—— 1952: 'The Oscillation Theory of Magnetic Variable Stars', ibid. **112**, 527.

—— 1955: 'Dynamo Theories of Cosmic Magnetic Fields', *Vistas in Astronomy*, **1**, 313.

—— 1957: *Magneto-hydrodynamics*, Interscience, New York.

—— 1958: 'Theoretical Problems of Stellar Magnetism', *Electrical Phenomena in Cosmical Physics*, International Astronomical Union Symposium no. 6, Stockholm, B. Lehnert, ed., Cambridge University Press, p. 169.

—— 1959: 'Magnetic Field of the Earth', *Nature*, **183**, 937.

DATTNER, A., LEHNERT, B., and LUNDQUIST, S., 1958: 'Liquid Conductor Model of Instabilities in a Pinched Discharge', *Proc. 2nd UN Int. Conference on the Peaceful Uses of Atomic Energy, Geneva*, **31**, Pergamon Press, p. 325.

—— and VENKATESAN, D., 1959: 'Anisotropies in Cosmic Radiation', *Tellus*, **11**, 239.

—— 1962: 'Current-induced Instabilities of a Mercury Jet', *Ark. f. fys.* **21**, 71.

DEUTSCH, A. J., 1958: 'Magnetic Fields of Stars', *Encyclopedia of Physics*, **51**, Springer-Verlag, Berlin, p. 689.

DRICOT, G., and LEDOUX, P., 1959: 'Remarque sur la théorie des oscillations d'une masse fluide incompressible en présence d'un champ magnétique', *Bull. Soc. Roy. Sci.* (*Liège*), **28**, 115.

DUNGEY, J. W., 1958: *Cosmic Electrodynamics*, Cambridge University Press.

—— and LOUGHHEAD, R. E., 1954: 'Twisted Magnetic Fields in Conducting Fluids', *Aust. J. Phys.* **7**, 5.

ELSASSER, W. M., 1946: 'Induction Effects in Terrestrial Magnetism', *Phys. Rev.* **69,** 106.

—— 1950: 'The Hydromagnetic Equations', ibid. **79,** 183.

—— 1955: 'Hydromagnetism I, A Review', *Amer. J. Phys.* **23,** 590.

—— 1956: 'Hydromagnetic Dynamo Theory', *Rev. Mod. Phys.* **28,** 135.

FÄLTHAMMAR, C.-G., 1961: 'Experiments on Magneto-hydrodynamic Co-rotation in Mercury', *Ark. f. fys.* **19,** 109.

FERRARO, V. C. A., 1937: 'Non-uniform Rotation of the Sun and its Magnetic Field', *M.N.R.A.S.* **97,** 458.

—— and MEMORY, D. J., 1952: 'Oscillations of a Star in its own Magnetic Field: An Illustrative Problem', ibid. **112,** 361.

—— 1954: 'On the Reflection and Refraction of Alfvén Waves', *Astrophys. J.* **119,** 293.

—— and PLUMPTON, C., 1961: *Magneto-fluid Mechanics,* Clarendon Press, Oxford.

FÜNFER, E., and LEHNER, G., 1962: 'Plasmaphysik', *Ergebnisse der exakten Naturwissenschaften,* **34,** 1. Springer-Verlag, Berlin.

TER HAAR, D., 1949: 'Stellar Rotation and Age', *Astrophys. J.* **110,** 321.

HARTMANN, J., and LAZARUS, F., 1937: 'Hg-Dynamics', *Kgl. Danske Videnskab. Selskab, Mat.-fys. Medd.* **15,** nos. 6 and 7.

HERLOFSON, N., 1950: 'Magneto-hydrodynamic Waves in a Compressible Fluid Conductor', *Nature,* **165,** 1020.

HERZENBERG, A., 1958: 'Geomagnetic Dynamos', *Phil. Trans.* A, **250,** 543; and *Ann. Geophys.* **14,** 522.

DE HOFFMAN, F., and TELLER, E., 1950: 'Magneto-hydrodynamic Shocks', *Phys. Rev.* **80,** 692.

VAN DE HULST, H. C., 1951: *Problems of Cosmical Aerodynamics,* Proc. Symposium on Motion of Gaseous Masses of Cosmical Dimensions, Central Air Documents Office, Dayton, Ohio.

INGLIS, D. R., 1955: 'Theories of the Earth's Magnetism', *Rev. Mod. Phys.* **27,** 212.

JEPHCOTT, D. F., 1959: 'Alfvén Waves in a Gas Discharge', *Nature,* **183,** 1652.

JOOS, G., 1959: *Lehrbuch der theoretischen Physik,* Leipzig 10th edition (English translation: Blackie, London).

KRUSKAL, M. D., and SCHWARZSCHILD, M., 1954: 'Some Instabilities of a Completely Ionized Plasma', *Proc. Roy. Soc.* A, **223,** 348.

LANDAU, L. D., and LIFSHITZ, E. N., 1959: *Fluid Mechanics,* Pergamon Press.

LEDOUX, P., 1958: 'Stellar Stability', *Encyclopedia of Physics,* **51,** Springer-Verlag, Berlin, p. 605.

LEHNERT, B., 1952*a*: 'On the Behaviour of an Electrically Conductive Liquid in a Magnetic Field', *Ark. f. fys.* **5,** 69.

—— 1952*b*: 'Experiments on Non-laminar Flow of Mercury in presence of a Magnetic Field', *Tellus,* **4,** 63.

—— 1954*a*: 'Magneto-hydrodynamic Waves in Liquid Sodium', *Phys. Rev.* **94,** 815.

—— 1954*b*: 'Magneto-hydrodynamic Waves under the Action of the Coriolis Force', *Astrophys. J.* **119,** 647.

—— 1955*a*: 'Magneto-hydrodynamic Waves under the Action of the Coriolis Force. II', ibid. **121,** 481.

LEHNERT, B., 1955*b*: 'An Instability of Laminar Flow of Mercury caused by an External Magnetic Field', *Proc. Roy. Soc.* A, **233,** 299.

—— 1957: 'An Experiment on Axisymmetric Flow of Liquid Sodium in a Magnetic Field', *Ark. f. fys.* **13,** 109.

—— 1958: 'Magneto-hydrodynamic Experiments', *Electro-magnetic Phenomena in Cosmical Physics*, International Astronomical Union, Symposium no. 6, B. Lehnert ed., Cambridge University Press, p. 50.

—— 1959: 'Plasma Physics on Cosmical and Laboratory Scale', *Nuovo Cimento* (X), Suppl. **13,** 1.

LINDBERG, L., WITALIS, E., and JACOBSEN, C., 1960: 'Experiments with Plasma Rings', *Nature*, **185,** 452.

—— and JACOBSEN, C., 1961: 'On the Amplification of the Poloidal Magnetic Flux in a Plasma', *Astrophys. J.* **133,** 1043.

LUNDQUIST, S., 1949*a*: 'Experimental Demonstration of Magneto-hydrodynamic Waves', *Nature* **164,** 145.

—— 1949*b*: 'Experimental Investigations of Magneto-hydrodynamic Waves', *Phys. Rev.* **76,** 1805.

—— 1950: 'Magneto-hydrostatic Fields', *Ark. f. fys.* **2,** 361.

—— 1951: 'On the Stability of Magneto-hydrostatic Fields', *Phys. Rev.* **83,** 307.

—— 1952: 'Studies in Magneto-hydrodynamics, *Ark. f. fys.* **5,** 297.

LÜST, R., and SCHLÜTER, A., 1954: 'Kraftfreie Magnetfelder', *Z. Astrophys.* **34,** 263.

—— —— 1955: 'Drehimpulstransport durch Magnetfelder und die Abbremsung schnell rotierender Sterne', ibid. **38,** 190.

—— 1959: 'Über die Ausbreitung von Wellen in einem Plasma', *Fortschritte der Physik*, **7,** 503.

MADELUNG, E., 1950: *Die mathematischen Hilfsmittel des Physikers*, 4th ed., Springer-Verlag, Berlin, p. 189.

MURGATROYD, W., 1955: 'Experiments on Magneto-hydrodynamic Channel Flow', *Phil. Mag.* **44,** 1348.

MURTY, G. S., 1960: 'Instability of a Conducting Fluid Cylinder due to Axial Current', *Ark. f. fys.* **18,** 241.

—— 1961: 'Instability of a Conducting Fluid Cylinder in the presence of an Axial Current, a Longitudinal Magnetic Field and a Coaxial Conducting Cylinder', ibid. **19,** 483.

NAGAO, S., and SATO, T., 1960: 'Some Experiments on Plasma Alfvén Waves', *J. Phys. Soc. Japan*, **15,** 735.

PARKER, E. N., 1955: 'Hydromagnetic Dynamo Models', *Astrophys. J.* **122,** 293.

RYDBECK, O., 1948: 'On the Propagation of Waves in Inhomogeneous Media', *Trans. Chalmer's University of Technology*, Gothenburg, Sweden, **74.**

SANDSTRÖM, A. E., 1956: 'Experimental Evidence of the Directional Components of Cosmic Rays', *Tellus*, **8,** 18.

SCHLÜTER, A., 1957: 'Kraftfreie Magnetfelder II', *Z. Naturforsch.* **12a,** 855.

SHERCLIFF, J. A., 1953: 'Steady Motion of Conducting Fluids in Pipes under Transverse Magnetic Fields', *Proc. Camb. Phil. Soc.* **49,** 136.

SCHWARZSCHILD, M., 1949: 'Magnetic Oscillations of a Highly Idealized Star', *Ann. d'Astrophys.* **12,** 148.

STRATTON, J. A., 1941: *Electromagnetic Theory*, McGraw-Hill, New York.

WALÉN, C., 1944*a*: 'On the Theory of Sunspots', *Ark. f. mat. astr. o. fys.* **30 A,** no. 15.

—— 1944*b*: 'On the Theory of Sunspots', ibid. **31 B,** no. 3.

—— 1946: 'On the Distribution of the Solar General Magnetic Field', ibid. **33 A,** no. 18.

—— 1948: *On the Vibratory Rotation of the Sun*, Henrik Lindståhls Bokhandel, Stockholm.

WILCOX, J. M., BAKER, W. R., BOLEY, F. J., COOPER, W. S., DE SILVA, A. W., and SPILLMAN, G. R., 1962: 'Devices for generating Highly-ionized Hydrogen Plasma', *J. Nuclear Energy*, Part C, *Plasma Physics*, **4,** 307.

—— BOLEY, F. I., and DE SILVA, A. W., 1960: 'Experimental Study of Alfvén Wave Propagation', *Phys. Fluids*, **3,** 15.

—— DE SILVA, A. W., and COOPER, W. S., 1961: 'Experiments on Alfvén-wave Propagation', ibid. **4,** 1506.

—— —— —— and BOLEY, F. I., 1961: 'Experiments on Alfvén-wave Propagation', *Radiation and Waves in Plasmas*, M. Mitchner, ed., Stanford University Press, p. 138.

WOLTJER, L., 1958: 'A Theorem on Force-free Magnetic Fields', *Proc. Nat. Acad. Sci. (Washington)*, **44,** 489.

—— 1958: 'The Stability of Force-free Magnetic Fields', *Astrophys. J.* **128,** 384.

ZEMANSKY, M. W., 1957: *Heat and Thermodynamics*, McGraw-Hill, New York.

4

PLASMA

4.1. Introduction

The term 'plasma' was introduced by Langmuir for the state in the positive column of electric discharges in gases. It is nowadays used as a synonym for 'ionized gas'.† The properties of plasmas are of paramount interest in cosmical physics because most of the matter in the universe is in the plasma state. In the interiors of stars the gas is almost completely ionized. In the photosphere of the sun (and other stars) the degree of ionization is not very high, but above the photosphere, in the chromosphere and the corona, the ionization is high again (almost 100 per cent). Vast regions of interstellar space around the stars, particularly the hot stars of early spectral type, are highly ionized (HII-regions), whereas far away from the stars the degree of ionization is low (HI-regions). In the sun and in interplanetary space, probably also in interstellar and intergalactic space, the plasma is penetrated by magnetic fields. This is probably the case in all stars, in any case in the 'magnetic stars'. As a consequence the astrophysicist's interest in plasma physics is mainly concentrated on *magnetic plasmas*.

Whereas the treatment of the magnetic plasma will be reserved for Chapter 5, the present chapter will be devoted to the general properties of a plasma. In both chapters we shall especially stress the important difference between a plasma and the idealized 'fluid' which has been treated in Chapter 3.

Most of our knowledge about plasmas has been gained by a century of research on 'electric discharges in gases' and a decade of intense efforts in the field of 'thermonuclear research'.

The study of *electric discharges in gases* (see, for example, von Engel and Steenbeck, 1932, 1934; and Loeb, 1955) has been concentrated on plasmas with a small degree of ionization (usually much less than 1 per cent). From an experimental point of view such plasmas are often produced by electric currents of the order of one ampere or less, but much work has also been devoted to currents of the order of kiloamperes.

† The electron gas in a solid is also often called a plasma.

Investigations of spectra emitted from electric discharges in gases have been of fundamental importance for clarifying the structure of the atom. Although most spectroscopists have regarded the discharges only as light sources there has also been much work devoted to the properties of the plasma itself. When a plasma is produced by an electric current, the heating and ionization is produced by an electric field, and when the term 'plasma' was introduced by Langmuir the presence of an electric field was essential by definition.

A very important contribution to the theory of ionized gases was made by Chapman and Cowling (1939), who deduced from kinetic theory some of the fundamental properties of ionized gases. However, many of their results have not yet been verified experimentally. Moreover, experiments have revealed a multitude of 'plasma' phenomena which are not expected from their theory.

In principle it should be possible to predict all the properties of an electric discharge by calculating the motions and interactions of the particles and quanta in it. In practice, however, it is necessary to use a number of simplifying assumptions in order to reduce the complexity of the mathematical procedure. If these simplifications are not chosen properly, misleading results may come out of the calculations. Many times the experiments have revealed theoretically unexpected behaviour in gas discharges. Hence it has been possible to build up the theory of electric discharges only through an intimate contact with experiments, and this co-operation between theory and experiment is still essential for the further progress of plasma physics.

Thermonuclear research (see, for example, Rose and Clark, 1961) started in several countries as secret projects aiming at the construction of fusion reactors in which a plasma of deuterium and tritium 'burns' at a temperature of the order of 10^8 °K. This requires that the plasma be kept in a 'magnetic bottle', i.e. prevented by a magnetic field from reaching the walls of the vessel in which it is enclosed. Therefore the interest has been focused on the properties of a *magnetic plasma*. Although much of the work so far has been concerned with the unexpected behaviour of large machines, thermonuclear research has induced fast progress also in the fundamental physics of magnetic plasmas, both theoretically and experimentally.

The experiences in the thermonuclear field again show clearly the necessity of close contact between theory and experiment. It therefore appears desirable to obtain some experimental evidence also on the behaviour of cosmical plasmas. The possibility of direct model experiments in which all important characteristics are preserved is often excluded for reasons which we shall discuss in § 4.2.2. However, in spite of this limitation, experiments may in some cases provide valuable information on particular aspects of cosmical plasmas. In this connexion the spectacular advances in experimental technique made in thermonuclear research are important.

Owing to the development of *space research*, artificial satellites and space probes have recently become available as powerful tools for exploration of the cosmical plasma. By means of them it has now become possible to observe the exospheric and interplanetary plasma *in situ* and even to perform experiments on a planetary scale ('Project Argus', Christofilos, 1959; see also Porter, 1959). Theoretical investigations based on space probe measurements will certainly be one of the most powerful approaches to cosmical electrodynamics.

4.2. General considerations

4.2.1. *Survey of electric discharges*

Electric discharges in the laboratory are usually divided into two groups: *non-sustained discharges*, which depend upon an 'external' ionizer to produce at least an essential part of the ions and electrons which carry the current, and *self-sustained discharges*, where the ionization is mainly produced by the discharge itself. *Ceteris paribus* the second group is characterized by higher current densities than the first. This is due to the fact that in the laboratory we have at our disposal only very weak ionizers. In cosmic physics, where the 'external ionizer' may be a high temperature which ionizes the matter more or less completely, non-sustained discharges may carry very large currents.

The domain of the self-sustained discharges is very extensive, including Townsend discharges, glow discharges, and arcs. Moreover, there are several special forms such as the spark, which is essentially a short-lived arc. In most of the discharges between electrodes we can discern three different regions:

1. The cathode region, where the electrons (which carry the main part of the current) are produced by emission from the cathode and by ionization of the gas.

2. The anode region (which is rather unimportant), associated with the passing of the current between the discharge and the anode.

3. The plasma which extends from the region of the cathode mechanism to that of the anode mechanism. The properties of the plasma can

be regarded as characteristic for a gaseous conductor in the absence of disturbances from electrodes.

An important distinction between the different types of discharges lies in the cathode mechanism. In the *Townsend* and *the glow discharge* the emission takes place from a cold cathode: in *the arc* the cathode is hot enough to give thermionic emission (or it emits abundantly for some other reason).

In cosmic physics the cold cathode mechanisms are of little interest. If we can speak of electrodes these usually consist of ionized gaseous layers of higher density than the discharge space. Such layers can give off electrons abundantly, so that the cathode mechanism is most similar to that of an arc discharge.

The properties of the plasma are not immediately connected with the cathode mechanism, so in principle the plasma could have the same properties for different types of self-sustained, and even for non-sustained, discharges. The state of the plasma depends upon the current density, and this is usually increasing when we go from non-sustained to Townsend and further to glow and to arc discharges. Although in principle the same phenomena occur in all plasmas, the properties of an arc plasma are, because of the high current density, different from that of a glow discharge, and still more different from that of a non-sustained discharge.

In a discharge tube the plasma may fill almost the whole tube out to the walls, from which it is separated by a thin sheath. This is usually the case if the pressure as well as the discharge current is low. However, at high pressures (e.g. atmospheric pressure) the plasma is usually *contracted* to a thin column, the diameter of which is determined by the discharge itself. Also at very large currents a contraction takes place due to electromagnetic forces (pinch effect), cf. § 5.5.2.

4.2.2. *Similarity transformations*

In the theory of gaseous discharges certain 'similarity laws' have proved very valuable (see Cobine, 1941, p. 209; or von Engel and Steenbeck, **2**, 1934, p. 95). When the linear scale is changed by a factor γ the general character of the phenomena remains unchanged if at the same time we change other quantities according to Table 4.1.

Proportionality between length and time is required by Maxwell's equations. The most characteristic features of a discharge depend upon the interactions between atoms, electrons, and quanta. As these interactions depend in a very complicated way upon the energies involved,

TABLE 4.1

Similarity transformation applicable to gaseous discharges

Length, time, inductance, capacity	vary as γ^{+1}
Particle energy, velocity, potential, current, resistance	vary as γ^{0}
Electric and magnetic field, conductivity, gaseous density	vary as γ^{-1}
Current density, space charge density	vary as γ^{-2}

we must leave all energies, and hence the electrostatic potential (which determines the kinetic energy of a charged particle) unchanged. If we change the linear dimensions l by a factor γ, the electric field E must be changed by γ^{-1} in order to leave the potential $V \sim lE$ unchanged. Because of Maxwell's equations we must change D, H, and B in the same way as E. The current density i which is equivalent to the displacement current $\partial D/\partial t$ must be changed by the factor γ^{-2}, which means that the total current $I = il^2$ remains unchanged. The conductivity $\sigma\,(= i/E)$ changes as γ^{-1}, the inductance L, which equals $V(dI/dt)$ and the capacity C $(\sim l)$ change as γ. Further, the mean free path, which is of fundamental importance in gaseous discharges, must vary as the linear dimension, so that the energy acquired in one mean free path is unchanged. Hence the density ρ of the gas, which is inversely proportional to the mean free path, must be changed as γ^{-1}.

In the theory of gaseous discharges the above transformation has proved to be very useful for a general survey, but it must be used with some care, because it refers to some fundamental phenomena only, and many secondary phenomena, which in special cases become important, do not obey the transformation. For example, the number of charged particles per unit volume is proportional to i and hence varies as γ^{-2}, whereas the number of molecules is proportional to ρ and hence varies as γ^{-1}. Hence the degree of ionization is not invariant, as we should like it to be, but varies as γ^{-1}. Further, as the force $f\,(= iB/c)$, which acts upon unit volume traversed by a current i in the presence of a magnetic field B, is proportional to γ^{-3}, but the density is proportional to γ^{-1}, the acceleration becomes porportional to γ^{-2}, and not, as it ought to be because of its dimension lt^{-2}, to γ^{-1}. One of the consequences of this is that hydromagnetic waves (Chapter 3) do not obey the transformation.

It must be observed that the transformation does not affect atomic quantities. For example, atomic dimensions, wavelength of emitted light, and lifetime of metastable states will remain unchanged.

If we want to apply the results obtained in a laboratory apparatus with the linear extension of 10 cm to cosmic phenomena, we have to increase the scale by a factor of 10^8–10^9 with regard to the conditions around the earth, a factor of 10^7–10^{10} for the sun, 10^{12} for interplanetary space, and 10^{21}–10^{22} for the galaxy. Perhaps it is of more interest to go the other way, i.e. to transform the cosmic phenomena down to laboratory scale, because this gives us some hint concerning the general type of the phenomena. It shows what quantities are the most important ones, and indicates to what extent it is possible to make scale-model experiments illustrating cosmic phenomena.

Table 4.2 shows how the similarity transformation may be applied to some important domains of cosmic physics.

The table shows some features of interest. The first is that most densities are to be considered as very high. Except in the close vicinity of the earth there is no analogy to high-vacuum phenomena. The laboratory analogy of cosmic space is not the vacuum in a tank but a highly ionized gas of a very high density.

Still more striking than the high densities are the very strong magnetic fields in the cosmos. In fact they are so strong that at present our laboratory resources do not suffice to produce field strengths large enough for model experiments.

The powerful magnetic fields have two important consequences. The first is that the motion of charged particles is usually of a different type from what we are familiar with in the laboratory. The radius of curvature is very small and the particles move in the direction of the magnetic field or 'drift' perpendicular to it. This type of motion has been studied in Chapter 2.

The other consequence is that strong electric fields are easily produced by any motion across the magnetic field (see § 1.3). To give an example, in a reduced magnetic field of 10^6 gauss, a velocity of 3.10^5 cm/sec causes an electric field of $E = 10$ e.s.u. $= 3000$ V/cm, and in a field of 10^{10} gauss the same velocity gives 30.10^6 V/cm. Thus also the electric fields in the cosmos, when reduced to laboratory scale, are often very strong.

Finally, the time-scale transformation in Table 4.2 is of interest. Solar flares, coronal arcs, and also the initial phase of a magnetic storm should be regarded as very short-lived phenomena. In fact their equivalent duration is of the order of the ignition time of an electric discharge. This means that transient phenomena are very important in cosmical physics.

TABLE 4.2

Region	*Characteristic dimension (cm)*			*Density (particles/cm³)*		*Magnetic field (gauss)*		*Characteristic times*	
	Actual	*Reduced*	*Scale factor*	*Actual*	*Reduced*	*Actual*	*Reduced*	*Actual*	*Reduced*
Ionosphere (F_1-layer)	10^6–10^7	10	$\gamma = 10^{-5}$–10^{-6}	10^{10}	10^{15}–10^{16}	0·5	5×10^4– 5×10^5	*Period of giant pulsation* 100 sec	 0·1–1 msec
Exosphere . .	10^9	10	$\gamma = 10^{-8}$	10^5 10	10^{13} 10^9	0·5–5×10^{-4}	5×10^7– 5×10^4	*One day* 10^5 sec	 1 msec
Interplanetary space	10^{13}	10	γ 10^{-12}	1–10	10^{12} 10^{13}	10^{-4}	10^8	*One solar rotation* 2×10^6 sec	 2 μsec
Interstellar space .	3×10^{22}	10	$\gamma = 3\times10^{-22}$	1	3×10^{21}	10^{-6}–10^{-5}	3×10^{15}– 3×10^{16}	*Period of galactic rotation* 3×10^8 years	 3 μsec
Intergalactic space .	$> 3\times10^{27}$	10	$\gamma < 3\times10^{-27}$	10^{-4} (?)	$> 3\times10^{22}$	10^{-7} (?)	$> 3\times10^{19}$	*Age of universe* 10^{10} years	 $< 10^{-3}$ μsec
Solar chromosphere .	10^8	10	$\gamma = 10^{-7}$	10^{11}–10^{14}	10^{18}–10^{21}	10^3–1	10^{10}–10^7	*Life of solar flare* 10^3 sec *Life of prominence* 10^5 sec	 100 μsec 10 msec
Solar corona . .	10^{10}–10^{11}	10	$\gamma = 10^{-9}$–10^{-10}	10^8–10^6	10^{17}–10^{16}	10^2–10^{-1}	10^{11}–10^9	*Life of coronal arc* 10^3 sec *Solar cycle* 22 years	 10^{-1}–1 μsec 70–700 msec

The interplanetary density varies very much in time. The larger density value given in the table is applicable during solar corpuscular emission.

The figures of density and magnetic field in intergalactic space are only speculative.

In the data for the chromosphere and corona the larger value of magnetic field strength refers to regions above sunspots.

4.2.3. *General properties of a plasma*

A laboratory plasma consists of neutral molecules (monatomic or polyatomic), electrons, positive (and in many cases also negative) ions, and also quanta, emitted from the excited atoms. In laboratory discharges the degree of ionization is generally small, but in some devices used for thermonuclear research a very high degree of ionization is obtained. In cosmic physics the ionization may be more or less complete except in planetary atmospheres and in interstellar HI-regions.

The electrons, ions, and molecules collide mutually. In a typical plasma only a very small fraction of the electrons have velocities so large that they can ionize or excite the molecules. Hence most collisions between electrons and molecules are elastic. Due to the large difference in mass between electrons and other particles, the exchange of energy is small at such a collision. In fact an electron transmits only a fraction of the order of m_e/M (m_e = electronic, M = molecular mass) of its kinetic energy when colliding with a heavy particle (see, for example, Landau and Lifshitz, 1960, p. 44). Hence if the mean energy of the electrons is different from that of the molecules, several thousand collisions are required before the energies are equalized (m_e/M being $\leqslant 1/1840$). On the other hand, the ions and molecules have masses of the same order, so that at collisions the energy exchange is of the same order as the total kinetic energy. A difference in mean energy is rapidly smoothed out. Cf. § 4.3.2.

In a typical plasma the velocity distribution of the molecules is, at least to a first approximation, Maxwellian, as in an ordinary gas. We call its temperature T_M. The ions and electrons are affected by the electric field, which gives them a systematic velocity parallel or antiparallel to the field. This velocity is often small compared with the random velocity. When the electric field is strong, the distribution of the random velocity will differ from the Maxwellian distribution, particularly in the case of a highly ionized plasma. However, in many cases even the ions and electrons have a nearly Maxwellian distribution. Thus we can often speak of an electronic gas having a certain 'electronic temperature' T_e which is defined by the condition that $\frac{3}{2}kT_e$ (k = Boltzmann's constant) shall equal the average energy due to the random velocity of the electrons. In the same way we can often speak of an 'ionic temperature' T_i.

The systematic motion in the electric field causes a heating of the electronic gas as well as of the ionic gas. Owing to the slow energy exchange between the electrons and the other constituents, the electronic gas may reach a temperature which is one or two (or even three) powers of 10 above that of the molecular gas. On the other hand, the thermal contact between the ionic and the molecular gas is generally good enough to ensure that no big difference between the ionic and molecular temperature is established.

The average energy of the electrons is usually much lower than the ionization or excitation energies of the molecules. Only that small part of the electrons which, due to the Maxwellian distribution, have energies several times the average energy are able to ionize or excite.

The behaviour of a plasma is very complicated because there are so many different interactions between electrons, more or less excited or ionized atoms or molecules, and quanta. In principle it is possible to treat the phenomena by exact statistical methods. In practice, however, most theories must be approximate because of the complexity of the problems. For several purposes we reach sufficient accuracy without using the more elaborate methods of statistical mechanics.

The Debye distance

The average positive and negative space charge densities in a plasma are very nearly equal. As illustrated in § 1.4 the local value of their ratio cannot differ appreciably from unity over large regions because enormous electric potentials would be created. Appreciable differences of positive and negative charge density are possible only in sufficiently small regions, such as for example the sheath that separates a plasma from a surrounding wall. An order of magnitude estimate of the size of such regions can be made in the following way (Spitzer, 1956). The potential energy per particle in the electric field created by uncompensated space charge should not exceed the thermal energy per particle. The potential field V is governed by the Poisson equation

$$\Delta V = -4\pi(e_i n_i + e_e n_e), \tag{1}$$

where e_e, e_i and n_e, n_i are the charges and number densities of electrons and positive ions. If for simplicity we consider a plane slab with thickness d from which all positive particles have been removed, the potential difference created is

$$V = -4\pi n_e e_e \frac{d^2}{2} \tag{2}$$

(provided the electric field strength vanishes at one of the surfaces of the slab). For an electron the difference of potential energy between the two sides of the slab is $4\pi n_e e_e^2(d^2/2)$. The mean kinetic energy (of thermal motion) due to the velocity component perpendicular to the slab is $\frac{1}{2}kT_e$. The potential and kinetic energies are equal if

$$4\pi n_e e_e^2 \frac{d^2}{2} = \tfrac{1}{2}kT_e \tag{3}$$

and then the thickness of the slab is

$$d = \left(\frac{kT_e}{4\pi n_e e_e^2}\right)^{\frac{1}{2}} = \lambda_D. \tag{4}$$

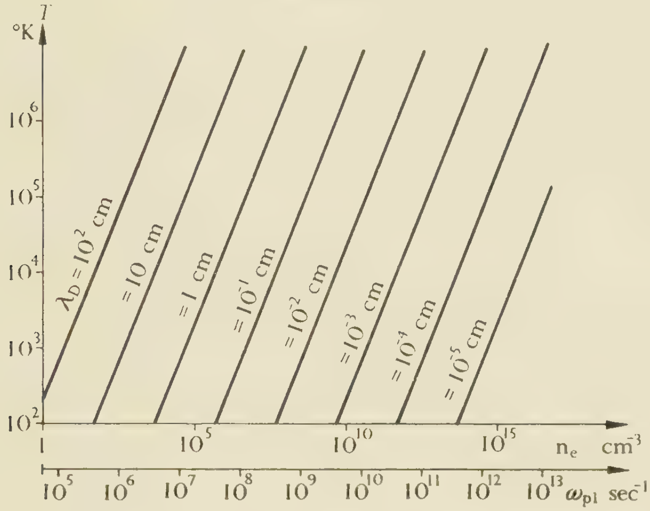


FIG. 4.1. Diagram of Debye distance λ_D, plasma frequency ω_{pl}.

The quantity λ_D defined by (4) is called the '*Debye* (*shielding*) *distance*' It was first introduced by Debye, who found that in an electrolyte the potential from a point charge has the form

$$V = \text{const}\,\frac{e^{-r/\lambda_D}}{r}. \tag{5}$$

The exponential factor describes the electrostatic shielding of the point charge by the rearrangement of surrounding charged particles. By this shielding the field (5) from the point charge is essentially limited to a sphere of radius λ_D; the 'Debye sphere'. Analogous considerations are valid in a plasma in thermal equilibrium, and similar results, among them equation (5), hold. Thus in a thermal plasma the effective depth of penetration of an electrostatic field is of the order of λ_D.

In a plasma which is far from thermal equilibrium (as is often the case with a 'low density' plasma, cf. § 5.1) the Debye distance is not always very well defined.

The value of the Debye distance for various values of electron density n_e and electron temperature T_e are shown in Fig. 4.1.

The plasma frequency

Another characteristic quantity of a plasma is the so-called *plasma frequency*, which is actually closely related to the Debye distance. Consider an infinite slab of plasma with thickness d (see Fig. 4.2 (a)) and let the electrons be displaced a small distance $\xi \ll d$ relative to the ions. A surface charge density $\pm n_e e_e \xi$ is built up at each surface.

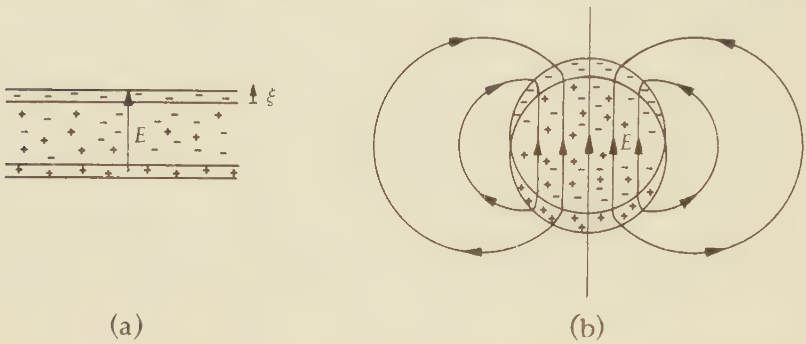


Fig. 4.2. Examples of collective oscillation modes of the electrons in plasma. (From Herlofson, 1951.) (a) Oscillating slab. (b) Oscillating cylinder.

This produces an electric field strength $4\pi n_e e_e \xi$ which extends throughout the slab and tends to pull back the electrons to their equilibrium position. The equation of motion is

$$m_e n_e d \frac{d^2\xi}{dt^2} = -e_e n_e d \,.\, 4\pi n_e e_e \xi \tag{6}$$

and represents an oscillation with the angular frequency

$$\omega_{pl} = \left(\frac{4\pi n_e e_e^2}{m_e}\right)^{\frac{1}{2}}. \tag{7}$$

The quantity ω_{pl} defined by equation (7) is called the *plasma frequency*.

Of course, this simple case represents only one of many possible modes of oscillation. Fig. 4.2 (b) shows one of the oscillation modes of a cylindrical column. In that case the angular frequency is $\omega_{pl}/\sqrt{2}$ (Herlofson, 1951).

The plasma frequency is characteristic of the rate at which the electrostatic restoring forces in a plasma eliminate deviations from neutrality, and it enters in a fundamental way in the theory of wave propagation in a plasma.

As regards the relation of plasma frequency and Debye distance, a comparison between (4) and (7) shows that $1/\omega_{pl}$ is just the time required for a particle to move a distance equal to λ_D if it travels with the velocity $(kT_e/m_e)^{\frac{1}{2}}$.

The numerical value of ω_{pl} for given n_e can be found in Fig. 4.1.

4.3. Collision processes in a plasma

When an electric field is applied to an ionized gas, all charged particles are accelerated. Positive and negative particles move in opposite directions. By collisions between charged particles and the neutral gas as well as between charged particles of opposite sign the average velocities soon reach stationary values. Cf., however, § 4.3.4.

The interaction between a charged and a neutral particle is rather different from that between two charged particles. In the former case the forces have a short range and therefore are appreciable only when the particles are at a distance comparable to the size of the particles. Except in stellar interiors this distance is much smaller than the separation of the particles. As a consequence any given particle interacts with others only during brief time intervals at the collisions, and spends most of its time in free flight. This is the background of the 'free-path method', which is very convenient for approximate study of the behaviour of gases. Another consequence of short-range forces is that when two particles collide the probability of a third particle being sufficiently close to have any influence is very small, and hence only *binary* collisions need to be considered.

The force between two charged particles decreases only as the inverse square of the distance. The average number of particles at a distance less than r is proportional to r^3 and so increases with distance faster than the force from each of them diminishes. A consequence of this is that any given particle is more strongly influenced by the continual interaction with numerous distant particles than by the relatively rare interactions with particles that it happens to pass close by.

Since the long-range interactions require a slightly different approach to the analysis, it is convenient to discuss separately the properties of *weakly ionized gases*, where the collisions of the charged particles with the neutral gas molecules are the most important, and those of *highly ionized gases*, where collisions between charged particles play a dominant role. It must then be observed that due to the large effective cross-section for collisions between charged particles, such collisions can be dominant even at a relatively low degree of ionization. Thus, *as far as collision processes are concerned, plasmas with degrees of ionization larger than* 1 *per cent are to be considered as highly ionized.*

In the case of weakly ionized plasmas (§ 4.3.1) we shall use the mean-free-path method to calculate the mobility of the charged particles and from this the conductivity.

The dynamics of Coulomb interactions are treated in § 4.3.2 and the friction produced by such interactions is calculated. The results are then used in § 4.3.3 for the calculation of the conductivity in a highly ionized plasma and in § 4.3.4 for discussion of 'runaway electrons'.

As long as we are dealing with time-independent currents in non-magnetic plasmas the current density vector $\mathbf{i}$ has the same direction as the electric-field vector $\mathbf{E}$, with which it is associated. In magnetic plasmas the situation is often much more complicated (§§ 5.1 to 5.3).

There are very few experimental measurements of the conductivity of plasmas because the quantity is difficult to measure and often physically irrelevant.

4.3.1. *Electric current in a weakly ionized plasma*

Suppose that the number density of one kind of charged particles with charge e_k and mass m_k is n_k. An electric field gives to these particles an average 'drift' velocity $\mathbf{u}_k$:

$$\mathbf{u}_k = b_k \mathbf{E}. \tag{1}$$

For weak fields b_k is a constant called the *mobility*. When the mobilities of all the constituents of a gas are known, its *conductivity* σ can easily be computed. In fact, the current density $\mathbf{i}$ produced by the field is given by

$$\mathbf{i} = \sum_k n_k e_k \mathbf{u}_k = \mathbf{E} \sum_k n_k e_k b_k, \tag{2}$$

where the summation includes all kinds of charged particles. As the conductivity σ is defined by

$$\mathbf{i} = \sigma \mathbf{E} \tag{3}$$

we have

$$\sigma = \sum_k n_k e_k b_k. \tag{4}$$

For negative particles e_k as well as b_k is negative.

Using exact statistical methods Chapman and Cowling have calculated the conductivity of ionized gases (see Chapman and Cowling, 1939). Their calculations are mathematically complicated, so for a survey it is preferable to use the simpler 'free-path method' which more easily demonstrates the physical process. As the cited authors have shown, this gives, in general, formulae which are sufficiently accurate in view of the fact that the final results anyhow depend on uncertain parameters, such as atomic collisional cross-sections.

In the free-path method it is assumed that the molecules (including ions and electrons) of a gas make instantaneous collisions with each other, but move freely between the collisions. In a rigorous treatment according to this method the statistical distribution of molecular velocities, free paths, etc., is taken into consideration, but in a less accurate variant of the method only the average values of the quantities are used. The results differ in general by less than a factor of 2, an accuracy which is enough for many applications.

Let v_k be the mean thermal velocity, λ_k the mean free path, and τ_k the mean interval between two collisions which a particle of kind k makes with other particles:

$$\tau_k = \lambda_k/v_k. \tag{5}$$

During the time τ_k the particle falls freely in the electric field a distance

$$D_k = \frac{e_k E}{2m_k}\tau_k^2. \tag{6}$$

Hence its average velocity in the direction of the electric field is

$$u_k = \frac{D_k}{\tau_k} = \frac{e_k E}{2m_k}\tau_k = \frac{e_k \lambda_k}{2m_k v_k}E. \tag{7}$$

This gives the mobility

$$b_k = u_k/E = \frac{\gamma e_k \lambda_k}{m_k v_k} \tag{8}$$

with $\gamma = \frac{1}{2}$. A more rigorous treatment, taking account of the statistical distribution of velocities and free paths, gives the same formula but with $\gamma = 1$. Different authors gives values of γ between these (see Cobine, 1941, p. 33). For electrons we may put $\gamma = 0{\cdot}85$.

For low values of E, v_k and hence b_k are independent of E, but when E is greater than a certain value, they vary as we shall now see.

When the particle drifts in the electric field its kinetic energy W_k, which is

$$W_k = \tfrac{1}{2}m_k v_k^2, \tag{9}$$

increases at the rate

$$dW_k/dt = e_k E u_k. \tag{10}$$

Let κ_k be the fraction of its kinetic energy which it loses on an average when colliding with a molecule. A stationary state is reached when the increase in energy due to the drift in the electric field equals the energy losses at the $1/\tau_k$ collisions per second:

$$\frac{dW_k}{dt} = \frac{\kappa_k W_k}{\tau_k}. \tag{11}$$

Combining (10) and (11), and introducing (5), (8), and (9), we obtain

$$\tfrac{1}{2}\gamma e_k^2 \lambda_k^2 E^2 = \kappa_k W_k^2 \tag{12}$$

or

$$W_k = (\gamma/2\kappa_k)^{\frac{1}{2}} e_k \lambda_k E. \tag{13}$$

The value of the collision loss ratio κ_k depends upon the character of the collision. If it is inelastic κ_k may be as high as 1, the total kinetic energy of the colliding particle being transformed into excitation or ionization energy.

Usually most collisions are elastic, and in this case κ_k can be computed from a formula given by Cravath (1930):

$$\kappa_k = \gamma_1 \frac{m_k}{M}\left(1-\frac{W_M}{W_k}\right), \tag{14}$$

where M is the molecular mass, W_M the average kinetic energy of a molecule, and γ_1 is a constant of order unity:

$$\gamma_1 = \frac{8}{3}\left(\frac{M}{m_k+M}\right)^2 \tag{15}$$

Introducing (14) into (12) we find

$$W_k = \tfrac{1}{2}W_M + \{\tfrac{1}{4}W_M^2 + \gamma_2 M(e_k \lambda_k E)^2/m_k\}^{\frac{1}{2}}, \tag{16}$$

where

$$\gamma_2 = \tfrac{1}{2}\gamma/\gamma_1. \tag{17}$$

γ_2 is a numerical constant. If the charged particles are ions ($k \to i$) so that $M/m_i \approx 1$, the average energy W_i of these is about the same as the average molecular energy W_M unless E is very large. If the charged particles are electrons ($k \to e$) the ratio $M/m_e \geq 1840$, so that even relatively weak fields make W_e considerably larger than W_M. This is in agreement with what was stated in § 4.2: the ionic temperature T_i is usually approximately equal to the gas temperature, whereas the electronic temperature T_e easily becomes much higher. As we have

$$W_M = \tfrac{3}{2}kT_M \tag{18}$$

and, if the charged particles are electrons,

$$W_e = \tfrac{3}{2}kT_e, \tag{19}$$

we can write for (16):

$$T_e = \tfrac{1}{2}T_M + \{\tfrac{1}{4}T_M^2 + \gamma_2 M(\epsilon\lambda_e E)^2/m_e\}^{\frac{1}{2}}, \tag{20}$$

where ϵ is the conversion factor between electron volts and temperature

$$\epsilon = 2|e_e|/3k = 2{\cdot}32\times10^6 \text{ e.s.u.} = 7700 \text{ degrees/eV}. \tag{21}$$

The electronic temperature becomes considerably different from the gas temperature when $\lambda_e E$ increases beyond the value given by

$$\lambda_e E = \frac{1}{\epsilon}\left(\frac{m_e}{M}\right)^{\frac{1}{2}} T_M. \tag{22}$$

If $M = 2000 m_e$, this occurs for $T_M = 300°$, when the product of the mean free path and the electric field, $\lambda_e E$, equals 0·001 V, and for $T_M = 6000°$ when $\lambda_e E = 0{\cdot}02$ V. In laboratory discharges electron temperatures of 20 000–50 000° are frequently measured.

As long as $T_e \approx T_M$, the drift is proportional to the electric field E. According to (8) it is given by:

$$u_e = \gamma \frac{e_e}{m_e} \frac{\lambda_e}{v_e} E, \tag{23}$$

where $\gamma \approx 1$, and m_e means the electronic mass; v_e is given by

$$\tfrac{1}{2} m_e v_e^2 = \tfrac{3}{2} k T_e \approx \tfrac{3}{2} k T_M. \tag{24}$$

If instead $T_e \gg T_M$, we have approximately (according to (20))

$$T_e = \epsilon(\gamma_2 M/m_e)^{\frac{1}{2}} \lambda_e E. \tag{25}$$

From (9), (19), and (21) we find the thermal velocity

$$v_e = (|e_e|/m_e)^{\frac{1}{2}} (4\gamma_2 M/m_e)^{\frac{1}{4}} (\lambda_e E)^{\frac{1}{2}}. \tag{26}$$

Introducing (26) into (8) we find that the drift velocity is

$$u_e = (|e_e|/m_e)^{\frac{1}{2}} (\tfrac{1}{2}\gamma^3\gamma_1)^{\frac{1}{4}} (m_e/M)^{\frac{1}{4}} (\lambda_e E)^{\frac{1}{2}} \approx (m_e/M)^{\frac{1}{2}} v_e. \tag{27}$$

Thus u_e is proportional to the square root of the electric field.

When the drift velocities of all charged particles are known, the current density i is found from (2).

As we have seen above, for values of the electric field E so small that the charged particle temperatures approximately equal the gas temperature, u_i and u_e are proportional to E. In this case the conductivity is given by

$$\sigma = \gamma\left(\frac{e_i^2 n_i \lambda_i}{m_i v_i} + \frac{e_e^2 n_e \lambda_e}{m_e v_e}\right). \tag{28}$$

As $\lambda_e > \lambda_i$, $m_e v_e \ll m_i v_i$, and usually $n_i \approx n_e$ (see §§ 1.4 and 4.2.3), the electronic conductivity is usually much higher than the ionic conductivity, so that the first term can be neglected:

$$\sigma = \gamma \frac{e_e^2}{m_e} \frac{n_e \lambda_e}{v_e} = \gamma \frac{e_e^2}{m_e} n_e \tau_e. \tag{29}$$

($\tau_e = \lambda_e/v_e$ = time between two collisions.) If the electron temperature is higher than the gas temperature, i according to (27) is proportional to $E^{\frac{1}{2}}$. If we still want to use (3) we must put σ proportional to $E^{-\frac{1}{2}}$. Thus the conductivity is independent of E for small values of E, but as soon as E increases beyond the value given by (22), the conductivity begins to decrease.

4.3.2. *Coulomb collisions in a plasma*

The binary collision between particles interacting by inverse-square forces is treated in most textbooks on dynamics (see, for example, Landau and Lifshitz, 1960, p. 53). For two particles with charges e_1 and e_2 separated by a distance r the strength of the electrostatic force is

$$F = \frac{e_1 e_2}{r^2}. \tag{30}$$

In the centre of mass system each of the two colliding particles is deflected through an angle χ given by

$$\tan \tfrac{1}{2}\chi = \frac{e_1 e_2}{mv^2 l}, \tag{31}$$

where m is the reduced mass,

$$m = \frac{m_1 m_2}{m_1 + m_2}, \tag{32}$$

v is the asymptotic relative velocity, and l is the impact parameter (the closest distance to which the particles would approach each other if they were not deflected by mutual forces). In the limit of small deflexion angles the formula (31) reduces to

$$\chi = \left(\frac{e_1 e_2}{l}\right) \bigg/ \left(\frac{mv^2}{2}\right). \tag{33}$$

The inverse-square force characterizes both Coulomb interaction and gravitational attraction. As a consequence some analogous scattering problems arise in plasma physics and in stellar dynamics, and certain results can be taken over from one field to the other. For instance the work by Chandrasekhar (1942, 1943 *a*, *b*) on relaxation times of a stellar system is very important in plasma physics.

To analyse the collision phenomena in a plasma it is convenient to consider the behaviour of a selected particle, the *test particle* moving through a background of *field particles* (Chandrasekhar, 1942, 1943 *a*, *b*). For simplicity we consider first the case where the field particles are heavy ($m_2 \gg m_1 \approx m$) and immobile. (This applies approximately to an electron moving among ions.)

In order to suffer a deflexion through an angle of 90° or more the test particle must approach a field particle so closely that

$$l \leqslant l_c = \frac{e_1 e_2}{m_1 v_1^2}. \tag{34}$$

The probable number per second of such collisions, which we shall call *close collisions*, is $n_2 v_1 \pi l_c^2$.

In other words the average collision frequency for close collisions is

$$\nu_{\text{close}} = 1/\tau_{\text{close}} = n_2 v_1 \pi l_c^2 = \frac{\pi n_2 e_1^2 e_2^2}{m_1^2 v_1^3}, \tag{35}$$

where the subscript 1 refers to the test particle and the subscript 2 to the field particles.

As the deflexion angle decreases only slowly with the impact parameter, interaction of the test particle with distant field particles is very important. Although each of these *distant collisions* does not cause any considerable deflexion, their number is so large that their cumulative effect influences the test particle even more than the relatively rare close collisions can do. (The corresponding phenomenon in stellar dynamics was discussed by Jeans in 1929.)

To estimate the effect of distant collisions we make the following simple considerations. (For a more detailed discussion see Spitzer, 1956.) Each distant collision causes a slight angular deviation given by (33) and a corresponding change in the momentum vector of the test particle. The momentum component $p_\perp$ normal to the original path changes by the amount

$$\delta p_\perp = \chi p = \chi m v_1 = \frac{2e_1 e_2}{v_1 l}, \tag{36}$$

if, for simplicity, we assume $m_2 \gg m_1 \approx m$, $v_2 \ll v_1$.

If the distant collisions occur at random, the average rate of change of $p_\perp^2$ is

$$\frac{d}{dt} p_\perp^2 = \int \left(\frac{2e_1 e_2}{v_1 l}\right)^2 v_1 n 2\pi l \, dl. \tag{37}$$

This integral is to be taken between the parameter values l_c (defined by (34)) and λ_D (given by 4.2.3 (4)), and so we obtain

$$\frac{d}{dt} p_\perp^2 = \frac{e_1^2 e_2^2 n_2}{v_1} 8\pi \ln \Lambda, \tag{38}$$

where

$$\ln \Lambda = \ln(\lambda_D / l_c), \tag{39}$$

and

$$\lambda_D = \left(\frac{kT_e}{4\pi n_e e_e^2}\right)^{\frac{1}{2}}. \tag{40}$$

The argument for choosing the upper limit of the integral equal to the Debye distance λ_D is that the test particle is electrically shielded from field particles at distances larger than λ_D. Detailed analysis has shown that this choice of the cut-off parameter is generally a good approximation (see Pines and Bohm, 1952; Henon, 1956; von Ecker and Voslamber, 1960; cf. also Thompson and Hubbard, 1960; Hubbard, 1961; Aono, 1961, 1962).

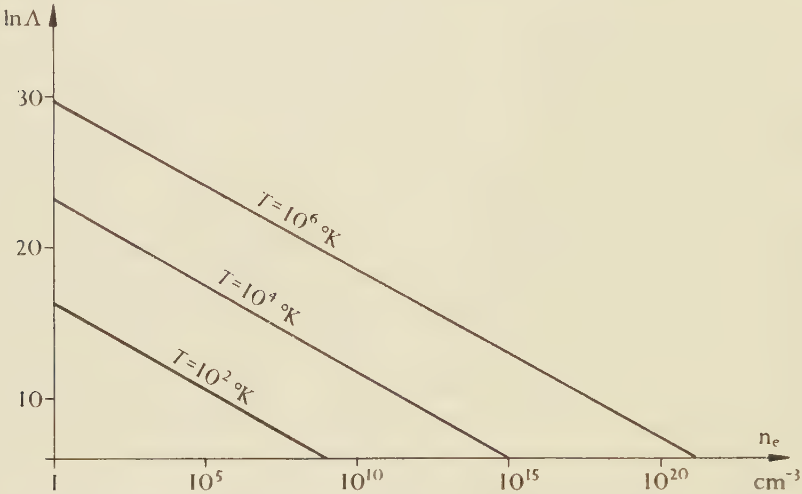


FIG. 4.3. Diagram of the Coulomb logarithm. (From a table by Spitzer, 1956.)

The factor $\ln \Lambda$ is the so-called *Coulomb logarithm.* Its value for hydrogen plasma at various densities and temperatures is shown in Fig. 4.3.

After a certain time τ the perpendicular component of momentum has grown to a value equal to the initial momentum $m_1 v_1$. At this time, which is given by

$$\tau = (m_1 v_1)^2 \Big/ \frac{dp_\perp^2}{dt} = \frac{m_1^2 v_1^3}{e_1^2 e_2^2 n_2 \, 8\pi \ln \Lambda}, \tag{41}$$

the accumulated deflexion will on the average be comparable to $\frac{1}{2}\pi$, which means that a large-angle deflexion has accumulated from many distant collisions. The corresponding equivalent collision frequency for distant collisions, $1/\tau$, is larger than the close-collision frequency (35) by a factor $8 \ln \Lambda$, which is generally much larger than unity (typically about 100, cf. Fig. 4.3). Thus the effect of close collisions is negligible compared with that of distant collisions.

The deflexion time τ for electrons colliding with protons is one of the various *relaxation times*, which characterize changes in the velocity distribution of a plasma. The accurate calculation of relaxation times requires that the motions of the field particles are taken into account. The calculations are lengthy and will not be given here, but the final results of Chandrasekhar (1942, 1943*a*, *b*) and Spitzer (1940, 1956) will be cited.

Consider an electron gas with number density n_e and temperature T_e and let the test particles have the mean thermal velocity, namely $(3kT_e/m_e)^{\frac{1}{2}}$. For such test particles the deflexion time (in the sense used above but with the motion of the field particles taken into account) is

$$\tau_{ee} = \frac{m_e^2(3kT_e/m_e)^{\frac{3}{2}}}{e_e^4 n_e . 0{\cdot}714 . 8\pi \ln \Lambda}. \tag{42}$$

It is called the *self-collision time of electrons* and characterizes the rate at which the interactions among the electrons themselves extinguish anisotropies in the velocity distribution. It also characterizes the rate of thermalization of energy among the electrons.

Similarly the *self-collision time of ions* with mass m_i and charge e_i at a temperature T_i and number density n_i is

$$\tau_{ii} = \frac{m_i^2(3kT_i/m_i)^{\frac{3}{2}}}{e_i^4 n_i . 0{\cdot}714 . 8\pi \ln \Lambda}. \tag{43}$$

For ions of molecular weight A and $e_i = -Ze_e$ the self-collision time is larger than that for electrons (of the same temperature and number density) by a factor

$$\tau_{ii}/\tau_{ee} \approx 43 A^{\frac{1}{2}} Z^{-3} \tag{44}$$

since $e_i n_i = -e_e n_e$.

When an electron gas and an ion gas of different temperatures interact, thermal energy flows from the hotter to the cooler gas. If both electrons and ions have nearly Maxwellian velocity distributions with temperatures T_e and T_i respectively, the temperatures approach each other according to the equation

$$-\frac{dT_e}{dt} = \frac{T_e - T_i}{\tau_{eq}}, \tag{45}$$

where τ_{eq} is the *equipartition time*

$$\tau_{eq} = \frac{m_e m_i[3kT_e/m_e + 3kT_i/m_i]^{\frac{3}{2}}}{e_e^2 e_i^2 n_i (6\pi)^{\frac{1}{2}} 8 \ln \Lambda} \tag{46}$$

(Spitzer, 1940, 1956). In case the temperatures T_e and T_i are equal, comparison with (43) shows that

$$\tau_{eq} = 0{\cdot}517 \frac{e_i^2}{e_e^2}\left(\frac{m_i}{m_e}\right)^{\frac{1}{2}} \tau_{ii}, \tag{47}$$

so that the equipartition time is generally much larger than the self-collision time for ions. For example in a plasma of electrons and protons at equal temperatures we have

$$\tau_{\mathrm{eq}} \approx 22\tau_{\mathrm{ii}} \approx 950\tau_{\mathrm{ee}}. \tag{48}$$

This illustrates the fact that equipartition of energy is approached so slowly that in many cases both the electron gas and the ion gas can maintain, separately, nearly Maxwellian distributions during the process.

Finally, we need the relaxation time characterizing the slowing down of a charged particle (mass m_1, charge e_1, and speed v) by collisions with field particles (mass m_2, charge e_2, temperature T_2, and number density n_2). This is the *slowing-down time* τ_{s}.

As the average rate of retardation is v/τ_{s}, the effective retarding force acting on the test particle is

$$F = -m_1 v/\tau_{\mathrm{s}}. \tag{49}$$

The quantity v/τ_{s} has been calculated by Chandrasekhar (1942, 1943 *a*, *b*), who has called this quantity the *coefficient of dynamical friction*. From his results we have

$$F = -\frac{4\pi e_1^2 e_2^2(1+m_1/m_2)n_2 \ln\Lambda}{m_1 v_T^2} f(v/v_T), \tag{50}$$

where

$$v_T = (2kT_2/m_2)^{\frac{1}{2}} \tag{51}$$

and

$$f(v/v_T) = \frac{2}{\sqrt{\pi}} \frac{\left(\int\limits_0^{v/v_T} e^{-x^2}dx - (v/v_T)e^{-v^2/v_T^2}\right)}{(v/v_T)^2}. \tag{52}$$

The function $f(v/v_T)$ is represented graphically in Fig. 4.4. In the limits $v/v_T \ll 1$ and $v/v_T \gg 1$, (52) reduces to

$$f(v/v_T) = \frac{4}{3\sqrt{\pi}}\frac{v}{v_T} \quad (v/v_T \ll 1), \tag{53}$$

$$f(v/v_T) = \left(\frac{v}{v_T}\right)^{-2} \quad (v/v_T \gg 1). \tag{54}$$

4.3.3. *Current in a highly ionized plasma in a weak electric field*

In an electric field the positive ions and electrons drift in opposite directions. Provided the electric field is weak enough, the relative drift velocity is much smaller than the thermal velocity of the electrons, and its value is determined by the requirement that in equilibrium the friction of the electrons against the ions just compensates the accelerating force from the electric field.

To estimate the drift velocity v let us assume that the electrons have a Maxwellian velocity distribution in a coordinate system following the drift motion ('displaced Maxwellian distribution'). In this system the ions move through a stationary electron gas of temperature T_e. In so doing each ion experiences a friction force F given by 4.3.2 (50), if the ions are considered as test particles and the electrons as field particles. The friction force on the ions in a unit volume is $F_i = n_i F$.

An equally strong but opposite force acts on a unit volume of the electron gas.

Neglecting m_e compared with m_i we find from (50) and (53)

$$F_e = -F_i = \frac{16\sqrt{\pi}}{3} \frac{e_e^2 e_i^2 n_i n_e \ln\Lambda}{m_e v_T^2} \frac{v}{v_T}. \tag{55}$$

On the other hand, the electric accelerating force is

$$F'_e = e_e n_e E = -e_i n_i E. \tag{56}$$

The condition
$$F_e + F'_e = 0 \tag{57}$$

determines the equilibrium value of v and hence the current density which becomes

$$i = -e_e n_e v = \frac{3}{4\sqrt{(2\pi)}\ln\Lambda} \frac{(kT_e)^{\frac{3}{2}}}{m_e^{\frac{1}{2}}|e_e|e_i} E. \tag{58}$$

From this expression we can get the conductivity σ by dividing by E. However, the conductivity value so obtained is only approximate, because of the simplifying assumption made regarding the velocity distribution.

Derivation of an accurate value of σ requires a more detailed examination of the velocity distribution of the electrons. Calculations have been made by Chapman (1928), Cowling (1945), and others (see Chapman and Cowling, 1939). Taking into account also electron-electron collisions Spitzer and Härm (1953) have obtained the value

$$\sigma = \gamma \frac{2^{\frac{5}{2}}}{\pi^{\frac{3}{2}}} \frac{(kT_e)^{\frac{3}{2}}}{m_e^{\frac{1}{2}}|e_e|e_i \ln\Lambda}, \tag{59}$$

where γ is a factor between $\frac{1}{2}$ and 1 which depends on the ionic charge e_i (Spitzer, 1956).

For singly charged ions ($e_i = -e_e = |e|$)

$$\gamma = 0{\cdot}582, \tag{60}$$

and when e_i increases, γ approaches 1.

Numerical values of the conductivity are found in § 5.2.3.3 (pp. 178–80). Values of the mean free path of electrons are found in § 5.1.4 (pp. 167–8).

4.3.4. *Highly ionized plasma in a strong electric field. Runaway electrons*

The friction force on an electron gas drifting through a background of cold ions was calculated in the preceding section, formula 4.3.3 (55). In the weak-field limit, where $v \ll v_T$, an expression for the conductivity

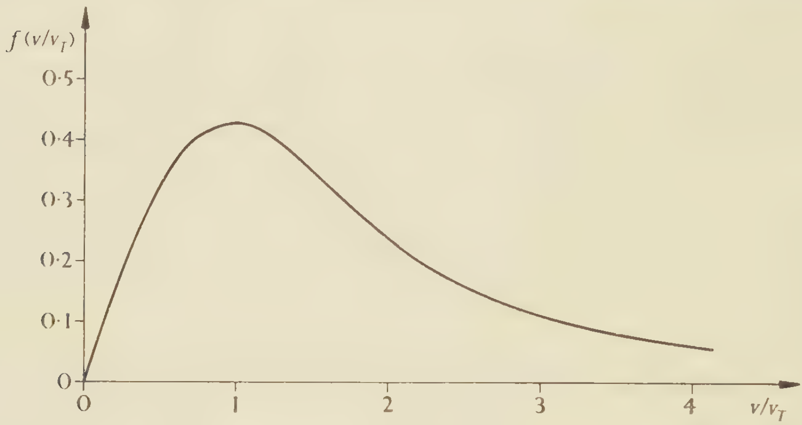


FIG. 4.4. Graph of the function $f(v/v_T)$, which is proportional to the friction force given by 4.3.2 (50).

was obtained. Let us now abandon the restriction $v \ll v_T$. If the ions are singly charged, $e_i = -e_e = |e|$, the force per unit volume is

$$F_e = n_e |e| E_c f(v/v_T), \tag{61}$$

where
$$v_T = \sqrt{\left(\frac{2kT_e}{m_e}\right)} \tag{62}$$

and
$$E_c = \frac{4\pi |e|^3 n_i \ln \Lambda}{m_e v_T^2}. \tag{63}$$

The function f is given by 4.3.2 (52) and plotted in Fig. 4.4. As the diagram shows, the function has an absolute maximum $f_{max} = 0{\cdot}427$. This means that when

$$E > E_c f_{max}, \tag{64}$$

the drag force is always less than the accelerating electric force, and no stationary state can exist. As the presence of the electric field causes ohmic heating, the electron temperature increases so that E_c decreases. Consequently the condition (64) may become satisfied after some time even if it is not satisfied at the instant when the electric field is applied.

This lack of equilibrium in a highly ionized plasma subject to a strong electric field was predicted by Giovanelli (1949) and has been investigated in detail by Dreicer (1959, 1960), who obtained the results (61) to (64) and (66) to (69) using the Focker–Planck formalism.

In a strong electric field the electron velocity distribution can differ much from a Maxwellian distribution. This is the case both for weakly and highly ionized plasma. Equations (61) to (64), which are based on the assumption of a displaced Maxwellian distribution, are therefore only approximate.

Even when the 'strong-field' condition (64) is far from satisfied there are some very fast electrons (in the 'tail of the distribution') that lose less momentum by friction against the ions than they gain from the electric field. Since they also collide comparatively infrequently with other electrons, they are decoupled from the main body of the electron population and able to run away independently ('weak-field' runaway). Such a particle experiences a total friction force F_{tot} (from other electrons and from ions, which we here assume to be singly charged). Using 4.3.2 (50) and (54) we find

$$F_{\mathrm{tot}} = 3|e|E_{\mathrm{c}}(v/v_T)^{-2}, \tag{65}$$

if the assumption is made that the ions are stationary and the main population of electrons have a Maxwellian distribution. The equation of motion of the fast electron is then

$$m_{\mathrm{e}}\frac{d\mathbf{v}}{dt} = -|e|\mathbf{E}-3|e|E_{\mathrm{c}}\left(\frac{v_T}{v}\right)^2\frac{\mathbf{v}}{v}. \tag{66}$$

If a coordinate system with the z-axis antiparallel to the electric field is used, the x- and y-components of (66) can be written

$$\frac{1}{v_x}\frac{dv_x}{dt} = \frac{1}{v_y}\frac{dv_y}{dt} = -\frac{3|e|E_{\mathrm{c}}}{m_{\mathrm{e}}}\frac{v_T^2}{(v_x^2+v_y^2+v_z^2)^{\frac{3}{2}}}, \tag{67}$$

and multiplying (66) by $\mathbf{v}$ we further obtain

$$\frac{d}{dt}(v_x^2+v_y^2+v_z^2) = \frac{2|e|E}{m_{\mathrm{e}}}\left[v_z-\frac{3v_T^2E_{\mathrm{c}}/E)}{(v_x^2+v_y^2+v_z^2)^{\frac{1}{2}}}\right]. \tag{68}$$

According to (67) the components v_x and v_y can only decrease. Therefore, if at any instant the right-hand side of (68) is positive, $v_x^2+v_y^2+v_z^2$ increases due to increasing v_z, and the right hand of (68) becomes even more positive, causing continued acceleration. The region of velocity space where runaway takes place is therefore given by the inequality

$$v_z(v_x^2+v_y^2+v_z^2)^{\frac{1}{2}} > 3v_T^2(E_{\mathrm{c}}/E). \tag{69}$$

Experimentally the runaway effect has been found in certain thermonuclear machines, where the runaway electrons finally strike the walls and produce X-rays (see, for example, Bernstein *et al.*, 1958).

4.4. Ambipolar diffusion

As soon as the number density n_k of a certain kind of particle varies in space, a diffusion occurs. The flow of particles is given by $-D_k \operatorname{grad} n_k$ (see, for example, Cobine, 1941, p. 53). The *diffusion coefficient* D_k is, in the approximate theory,

$$D_k \approx \frac{kT_k \tau_k}{m_k} \approx \frac{kT_k}{n_k e_k^2} \sigma_k, \tag{1}$$

where T_k, τ_k and σ_k are temperature, collision time, and conductivity (cf. 4.3.1 (4), (8), and (28)). The diffusion of charged particles corresponds to an electric current

$$\mathbf{i}_{k\mathrm{D}} = e_k(-D_k \operatorname{grad} n_k), \tag{2}$$

which adds to the current $\sigma_k \mathbf{E}$ produced by the electric field $\mathbf{E}$. Thus the resultant current carried by the electrons is

$$\mathbf{i}_\mathrm{e} = \sigma_\mathrm{e} \mathbf{E} - e_\mathrm{e} D_\mathrm{e} \operatorname{grad} n_\mathrm{e} \tag{3}$$

and that carried by the ions

$$\mathbf{i}_\mathrm{i} = \sigma_\mathrm{i} \mathbf{E} - e_\mathrm{i} D_\mathrm{i} \operatorname{grad} n_\mathrm{i}. \tag{4}$$

Suppose that, in the absence of an electric field, a gas is ionized inside a certain volume, where the number densities of electrons and ions are n_e and n_i. Initially we have $e_\mathrm{e} n_\mathrm{e} + e_\mathrm{i} n_\mathrm{i} = 0$. At the border of the ionized region electrons and ions diffuse outwards. Because of the higher mobility of the electrons, these diffuse more rapidly, so that after some time there is immediately outside the border a region containing a surplus of electrons, whereas immediately inside the border there is a surplus of ions. Hence an electric field is produced which impedes the outward motion of the electrons and accelerates that of the ions. As in the cases of interest in cosmic physics, the relative difference between positive and negative space charge density can never be large (see §§ 1.4 and 4.2.3), the electric field soon attains such a value that the diffusion of the electrons proceeds at a rate equal to that of the ions. This type of diffusion is called *ambipolar* diffusion. It is a very important phenomenon in ordinary gaseous discharges (see von Engel and Steenbeck, **1**, 1932, p. 197; Cobine, 1941, p. 48; or Allis, 1956, p. 397).

For quantitative analysis of the ambipolar diffusion we start from (3) and (4). Assuming the ions to be singly ionized we put $e_\mathrm{i} = -e_\mathrm{e} = |e|$, $n = n_\mathrm{e} = n_\mathrm{i}$, and $\mathbf{i}_\mathrm{i} + \mathbf{i}_\mathrm{e} = 0$. Then the electric field $\mathbf{E}$ produced by the ambipolar diffusion of electrons and ions is

$$\mathbf{E} = -\frac{|e|(D_\mathrm{e} - D_\mathrm{i})}{\sigma_\mathrm{e} + \sigma_\mathrm{i}} \operatorname{grad} n. \tag{5}$$

Introducing this into (3) and (4) we get

$$\mathbf{i}_{\mathrm{i}} = -\mathbf{i}_{\mathrm{e}} = -\frac{\sigma_{\mathrm{e}} D_{\mathrm{i}} + \sigma_{\mathrm{i}} D_{\mathrm{e}}}{\sigma_{\mathrm{e}} + \sigma_{\mathrm{i}}} |e| \operatorname{grad} n. \tag{6}$$

According to (1) this can also be written

$$\mathbf{i}_{\mathrm{i}} = -\mathbf{i}_{\mathrm{e}} = -\frac{T_{\mathrm{e}} + T_{\mathrm{i}}}{T_{\mathrm{i}}/D_{\mathrm{i}} + T_{\mathrm{e}}/D_{\mathrm{e}}} |e| \operatorname{grad} n. \tag{7}$$

If $\sigma_{\mathrm{i}} \ll \sigma_{\mathrm{e}}$, which normally is the case (§ 4.3.1) and which, according to (1), implies $D_{\mathrm{i}}/T_{\mathrm{i}} \ll D_{\mathrm{e}}/T_{\mathrm{e}}$, the expression reduces to

$$\mathbf{i}_{\mathrm{i}} = -\mathbf{i}_{\mathrm{e}} = -\left(1 + \frac{T_{\mathrm{e}}}{T_{\mathrm{i}}}\right) D_{\mathrm{i}} |e| \operatorname{grad} n. \tag{8}$$

REFERENCES

Allis, W. P., 1956: 'Motions of Ions and Electrons', *Encyclopedia of Physics*, **21**, Springer-Verlag, Berlin.

Aono, O., 1961, 1962: 'Fluctuations in a Plasma. I. Ion-electron Temperature Relaxation. II. The Numerical Factor in the Coulomb Logarithm'. *J. Phys. Soc. Japan*, **16**, 2264; **17**, 853.

Bernstein, W., Chen, F. F., Heald, M. A., and Kranz, A. Z., 1958: '"Runaway" Electrons and Cooperative Phenomena in B-1 Stellarator Discharges', *Phys. Fluids*, **1**, 430.

Chandrasekhar, S., 1942: *Principles of Stellar Dynamics*, University of Chicago Press.

—— 1943*a*: 'Dynamical Friction', *Astrophys. J.* **97**, 225.

—— 1943*b*: 'Stochastic Problems in Physics and Astronomy', *Rev. Mod. Phys.* **15**, 1.

Chapman, S., 1928: 'The Electrical Conductivity of Stellar Matter', *M.N.R.A.S.* **89**, 54.

—— and Cowling, T. G., 1939: *The Mathematial Theory of Non-uniform Gases*, Cambridge University Press, 2nd ed., 1958.

Christofilos, N. C., 1959: 'The Argus Experiment', *J. Geophys. Res.* **64**, 869.

Cobine, J. D., 1941: *Gaseous Conductors*, McGraw-Hill, New York.

Cowling, T. G., 1945: 'Electrical Conductivity of an Ionized Gas in a Magnetic Field, with Applications to the Solar Atmosphere and the Ionosphere', *Proc. Roy. Soc.* A, **183**, 453.

Cravath, A. M., 1930: 'The Rate at which Ions lose Energy in Elastic Collisions', *Phys. Rev.* **36**, 248.

Dreicer, H., 1959: 'Electron and Ion Runaway in a Fully Ionized Gas I', ibid. **115**, 238.

—— 1960: 'Electron and Ion Runaway in a Fully Ionized Gas II', ibid. **117**, 329.

Von Ecker, G., and Voslamber, D., 1960: 'Zum Cut-off Parameter der Impulsstreukoeffizienten von Systemen langreichweitiger Wechselwirkung', *Z. Naturforsch.* **15a**, 1107.

von Engel, A., and Steenbeck, M., 1932: *Elektrische Gasentladungen*, **1**, Springer-Verlag, Berlin. 1934: ibid. **2**, Springer-Verlag, Berlin. See also von Engel, 1955: *Ionized Gases*, Clarendon Press, Oxford.

Giovanelli, R. G., 1949: 'Electron Energies in Highly Ionized Gases', *Phil. Mag.* **40**, 206.

Henon, M., 1956: 'Un calcul amélioré des perturbations des vitesses stellaires', *Ann. d'Astrophys.* **21**, 186.

Herlofson, N., 1951: 'Plasma Resonance in Ionospheric Irregularities', *Ark. f. fys.* **3**, 247.

Hubbard, J., 1961: 'The Friction and Diffusion Coefficients in Fokker-Planck Equation for Plasmas', Parts I and II, *Proc. Roy. Soc.* A, **260**, 114; A, **261**, 371.

Jeans, J., 1929: *Astronomy and Cosmogony*, Cambridge University Press.

Landau, L. D., and Lifshitz, E. M., 1960: *Mechanics*, Pergamon Press.

Loeb, L. B., 1955: 'Basic Processes of Gaseous Electronics', University of California Press.

Massey, H. S. W., and Burhop, E. H. S., 1952: *Electronic and Ionic Impact Phenomena*, Clarendon Press, Oxford.

Pines, D., and Bohm, D., 1952: 'A Collective Description of Electron Interactions. II. Collective vs. Individual Particle Aspects of the Interactions', *Phys. Rev.* **85**, 338.

Porter, R. W. (ed.), 1959: 'Symposium on the Scientific Effects of Artificially Introduced Radiations at High Altitudes', *J. Geophys. Res.* **64**, 865.

Rose, D. J., and Clark, M., 1961: *Plasmas and Controlled Fusion*, M.I.T. Press.

Spitzer, L., Jr., 1940: 'The Stability of Isolated Clusters', *M.N.R.A.S.* **100**, 396.

—— 1954: 'Behaviour of Matter in Space', *Astrophys. J.* **120**, 1.

—— 1956: *Physics of Fully Ionized Gases*, Interscience, New York. Second edition, 1962.

—— and Härm, R., 1953: 'Transport Phenomena in a Completely Ionized Gas', *Phys. Rev.* **89**, 977.

Thompson, W. B., and Hubbard, J., 1960: 'Long-range Forces and the Diffusion Coefficients of a Plasma', *Rev. Mod. Phys.* **32**, 714.

5

MAGNETIC PLASMA

5.1. Introduction. Classification of plasmas

If there is a magnetic field present in a plasma, a number of interesting phenomena are produced. The properties of a magnetic plasma depend on many parameters, such as its density, temperature, degree of ionization and chemical composition, and further on the strength and homogeneity of the magnetic field. It is convenient to divide plasmas into three classes, which for the sake of simplicity will be referred to as 'high-density', 'medium-density', and 'low-density plasmas'. Very roughly we can say that a *high-density* plasma obeys reasonably well the laws which in Chapter 3 have been derived for liquids and compressible media. In a *medium-density* plasma the magnetic field has introduced a pronounced anisotropy and in several respects the plasma becomes quite different from a magnetized fluid. Finally, a *low-density 'collision-less' plasma* forms a sort of transition to the case of single charges moving in high vacuum (as treated in Chapter 2), although still the condition of quasi-neutrality is obeyed. All three classes are important in cosmical physics.

For this classification of a plasma three parameters are suitable: (1) The *mean free path* λ of the electrons, (2) the *Larmor radius* ρ of the electrons, and (3) the *characteristic length* l_c of the region we consider. For example, l_c may be the scale-height of an atmosphere or the distance an electron can travel before it is reflected by a magnetic mirror.

We shall now describe the properties of typical plasmas belonging to the different classes. Of course, in reality there are a great variety of transitions between them.

5.1.1. *High-density plasmas, $\lambda \ll \rho$*

If the velocity of a charged particle is v with a component $v_\perp$ perpendicular to $\mathbf{B}$, the (angular) gyro frequency is $\omega = v_\perp/\rho$ and the collision time $\tau = \lambda/v$. If $v_\perp$ is of the same order of magnitude as v the condition $\lambda \ll \rho$ implies $\omega\tau \ll 1$.

Between consecutive collisions an electron moves in almost straight lines without being affected very much by the magnetic field. This has the consequence that the plasma is isotropic but for the anisotropic macroscopic force $(\mathbf{i}/c)\times\mathbf{B}$ which a magnetic field always introduces in a conducting medium. A manifestation of this anisotropy is the anisotropic propagation of hydromagnetic waves (§§ 3.1–10). However, the diffusion and the electric conductivity are isotropic.

In Chapter 4 we have treated a number of typical plasma phenomena (e.g. ambipolar diffusion). Such phenomena are, of course, important also in magnetic plasmas. Hence even a high-density plasma differs in many respects from the ideal fluid treated in Chapter 3.

5.1.2. *Medium-density plasmas*, $\rho \ll \lambda \ll l_c$

The motion parallel to **B** obeys the same laws as in high-density plasmas. At high or medium densities the parallel motion is a random walk. If an electric field $E_\|$ is applied parallel to **B**, a systematic drift is superimposed on the random motion. Hence a current i is produced, which, under the conditions discussed in § 4.3, is proportional to $E_\|$ so that we have

$$i_\| = \sigma_\| E_\|, \tag{1}$$

where the parallel conductivity $\sigma_\|$ is a constant given by 4.3.3 (59) for a highly ionized plasma.

Projected on a plane perpendicular to **B** an electron moves in a circle. Between collisions the electron can move freely a distance λ parallel to **B**, but perpendicular to **B** it can be elongated only a distance ρ from the guiding centre. After having described a number of turns it collides, and its guiding centre is displaced a distance which typically is of the order ρ. In addition, the electron may drift, for example under the action of an electric field perpendicular to **B**. The result is that, for example, diffusion and electric conductivity are highly anisotropic.

5.1.3. *Low-density plasmas*, $l_c \ll \lambda$

As long as we consider regions which are much larger than ρ (as we usually do in cosmical physics), the condition $\lambda \gg l_c$ implies $\lambda \gg \rho$. Hence in a plane perpendicular to **B** the particles in a low-density plasma behave essentially as in a medium-density plasma.

The difference between a medium-density plasma and a low-density plasma refers to the motion parallel to **B**. In the low-density case the mean free path λ is much larger than the linear dimension l_c of the region we consider. Suppose that there is only an electric force parallel to **B**. Then an individual electron obeys the equation of motion

$$m_e \frac{dv_\|}{dt} = e_e E_\| \tag{2}$$

and the corresponding current density $i_{\|}$ is given by

$$\frac{di_{\|}}{dt} = \frac{n_e e_e^2 E_{\|}}{m_e}. \tag{3}$$

This shows that in the low-density case there is no relation of the form (1) between $i_{\|}$ and $E_{\|}$, and the conductivity $\sigma_{\|}$ defined by (1) has no meaning. Even if $E_{\|} = 0$ we may have $i_{\|} \neq 0$, and for $E_{\|} \neq 0$ the current at a given instant may be zero or even antiparallel to $E_{\|}$.

It is interesting to study the case when the magnetic field is inhomogeneous in such a way that the guiding centres of the particles oscillate along a field line between two mirror points at a distance l_c. Suppose that the plasma consists of electrons and one kind of positive ions, and that their magnetic moments are μ_e and μ_i. In the presence of an electric field $E_{\|}$ parallel to **B** they are acted upon by the average forces (see 2.3.6 (61))

$$f_{e\|} = -\mu_e \frac{dB}{ds} + e_e E_{\|} \tag{4}$$

and

$$f_{i\|} = -\mu_i \frac{dB}{ds} + e_i E_{\|} \tag{5}$$

parallel to the magnetic field.

If under the influence of this force a particle oscillates with the variable velocity $v_{k\|}$ between the points s_1 and s_2, it spends the time

$$dt = ds/v_{k\|} \tag{6}$$

on the line element ds, and its half-period is

$$\tau_k = \int_{s_1}^{s_2} ds/v_{k\|}. \tag{7}$$

Hence it gives rise to an average space charge

$$dq_k = \frac{e_k\,ds}{\tau_k v_{k\|}} \tag{8}$$

on the line element ds. (The subscript k is used to distinguish between different kinds of particles.) Let N_i positive ions and N_e electrons oscillate simultaneously between the same mirror points. The space charge at each line element is $N_i\,dq_i + N_e\,dq_e$, and if N_e is so large that we have a plasma (which must be quasi-neutral), we must have $N_i\,dq_i + N_e\,dq_e = 0$ and also $N_i e_i + N_e e_e = 0$. According to (8) this means that $\tau_i v_{i\|} = \tau_e v_{e\|}$, or, if we introduce $\alpha = \tau_i/\tau_e$,

$$v_{e\|} = \alpha v_{i\|}. \tag{9}$$

Here α is a constant which can have any value (because electrons and ions need not have equal energies).

Equation (9) and the equations of motion

$$m_e \frac{dv_{e\parallel}}{dt} = f_{e\parallel}, \qquad m_i \frac{dv_{i\parallel}}{dt} = f_{i\parallel}$$

give
$$f_{e\parallel} = m_e v_{e\parallel} \frac{dv_{e\parallel}}{ds} = \frac{\alpha^2 m_e}{m_i} m_i v_{i\parallel} \frac{dv_{i\parallel}}{ds} = \frac{\alpha^2 m_e}{m_i} f_{i\parallel}. \tag{10}$$

If we assume $e_i = -e_e = |e|$, we obtain by introducing (4) and (5) into (10):

$$-|e|E_\parallel - \mu_e \frac{dB}{ds} = \frac{\alpha^2 m_e}{m_i}\left(|e|E_\parallel - \mu_i \frac{dB}{ds}\right), \tag{11}$$

or
$$E_\parallel = -K\frac{dB}{ds}, \tag{12}$$

with
$$K = \frac{1}{|e|}\,\frac{\mu_e/m_e - \alpha^2\mu_i/m_i}{1/m_e + \alpha^2/m_i}. \tag{13}$$

Introducing (9) and $\mu = W_\perp/B$ we find the following expression for the *invariant* K:

$$K = \frac{W_{i\parallel}W_{e\perp} - W_{e\parallel}W_{i\perp}}{|e|B(W_{i\parallel}+W_{e\parallel})}. \tag{14}$$

We have used the notations $W_\parallel = \frac{1}{2}mv_\parallel^2$ and $W_\perp = \frac{1}{2}mv_\perp^2$ with the indices e and i referring to electrons and positive ions.

Equation (12) shows that in a low-density plasma *the electric field parallel to the magnetic field* is zero only when the magnetic field is homogeneous ($dB/ds = 0$) or the relation

$$\frac{W_{i\parallel}}{W_{e\parallel}} = \frac{W_{i\perp}}{W_{e\perp}} \tag{15}$$

is satisfied. This relation means that the helices of the ions and of the electrons have the same pitch angle. If (15) is satisfied, so that $K = 0$, both kinds of particles oscillate with the same amplitude in the absence of an electric field. However, if $K \neq 0$, so that $W_{i\parallel}/W_{i\perp} \neq W_{e\parallel}/W_{e\perp}$, electrons and ions would oscillate with different amplitude, if there were no electric field. As the quasi-neutrality of the plasma requires that $N_i\,dq_i + N_e\,dq_e = 0$ everywhere, particles of both kinds must oscillate with equal amplitude. This can only be achieved by setting up an electric field.

We integrate (12) between two points A and C where the magnetic field strengths are B_A and B_C. The voltage difference between C and A is

$$V = V_C - V_A = K(B_C - B_A). \tag{16}$$

We introduce
$$\gamma = B_C/B_A. \tag{17}$$

Then (16) becomes

$$|e|V = (\gamma-1)\left(\frac{W_{i\parallel}W_{e\perp}-W_{e\parallel}W_{i\perp}}{W_{i\parallel}+W_{e\parallel}}\right)_A, \tag{18}$$

where the expression within the brackets refers to the point A.

The resultant voltage depends on how the particle population is injected, because this influences the values of $W_{i\parallel}$ etc. in (18). As an illustration we consider the following simple model. (The actual injection mechanism in the magnetosphere is not yet known.) A line of force of the geomagnetic dipole field intersects the ionosphere at C (Fig. 5.1). Outside the ionosphere there is a low-density plasma

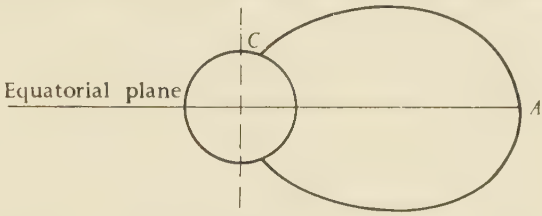


FIG. 5.1. Geomagnetic field line in the magnetosphere.

which is produced by injection of hot plasma in the equatorial plane (at A in Fig. 5.1) and evaporation of low-energy particles from the ionosphere. For the sake of simplicity we assume that in the equatorial plane only particles of one sign, say electrons, are injected. At C, where the field strength is B_C ($B_C > B_A$), there is a source emitting ions with negligible energy. The emission takes place as soon as there is a voltage difference between the plasma at C and the source. In this way the voltage of the plasma at C is fixed by the voltage of the source. The injection of electrons at A lowers the voltage at this point, and as a consequence positive particles are emitted from the source at C, until (18) is satisfied.

As the positive ions have been accelerated by the voltage V before they reach A, we have $W_{i\parallel} = |e|V$, and $W_{i\perp} = 0$. Hence we obtain from (18)

$$|e|V = (\gamma-1)\frac{|e|VW_{e\perp}}{|e|V+W_{e\parallel}}, \tag{19}$$

which besides the solution $V = 0$ also gives

$$|e|V = (\gamma-1)W_{e\perp}-W_{e\parallel}. \tag{20}$$

If in a medium-density plasma we have two similar sources on the same field line, we obtain a conduction current carrying electrons from the source A to C and ions from C to A. This current tends to annihilate any voltage difference between A and C. As in cosmical physics the conductivity is usually large, the current would thus prevent any large voltage difference between A and C. Contrary to this, in a low-density plasma the voltage difference V given by (20) is produced before any appreciable current can flow. If we inject electrons at A they cannot reach C unless their energy is increased by V to such an extent as to displace the mirror point to C. Only if the voltage difference exceeds

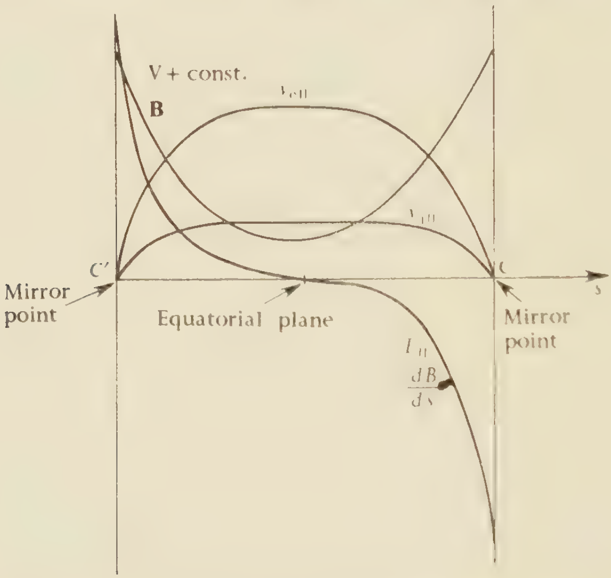


FIG. 5.2. Illustration of the simple model of low-density plasma confined between magnetic mirrors (s is a length coordinate following the magnetic lines of force). All particles have the same mirror points A and C. The electric field $E_{\|}$ parallel to **B** is drawn to such a scale that it coincides with dB/ds (cf. 5.1 (12)). The electric potential V and the magnetic field strength B are brought to coincide by addition of a constant to V.

the value given by (20) can electrons reach C so that a current is produced. Further, as the ions emitted from C oscillate along the field line without collision, they cannot carry any average current when a stationary state is reached.

In our model a magnetic field line, which intersects the equatorial plane (field strength B_A) at 5 or 10 earth radii, reaches the ionosphere at a point where the field strength (B_C) is so much larger that κ is 100 or 1000. This means that under our simplifying assumptions there may be a voltage difference between the equatorial plane and the ionosphere which exceeds the volt energy of particles injected at A by a very large factor.

In case the collision frequency in the plasma is finite, the oscillating particles emitted from C will lose their energy and accumulate at A, whereas the particles emitted from A will have a chance of being scattered into the 'loss cone' so that they move along the field lines to C. In this way a current is caused so that the voltage V given by (20) is eliminated. For finite collision frequencies the voltage V is large only under the condition that the 'lifetime' of the plasma is shorter than the collision time. By 'lifetime' we mean the time before the plasma has drifted away from the region we consider, and new plasma has been injected.

5.1.4. *Magnetic plasmas in cosmical physics*

The border between a high-density and a medium-density plasma is given by

$$\omega\tau = 1.$$

As $\omega = |e|B/(m_e c)$, the border depends on the magnetic field. The critical collision frequency $1/\tau = \omega$ for different magnetic fields is given in Table 5.1.

TABLE 5.1

Magnetic field B	0·4	40	4000 gauss
Critical collision frequency	$0{\cdot}7\times10^{7}$	$0{\cdot}7\times10^{9}$	$0{\cdot}7\times10^{11}$ sec^{-1}

At the solar surface magnetic field strengths up to 4000 gauss are observed in sunspots, but there are also regions with field strengths of the order of 1 gauss. Collision frequencies of the orders of magnitude given in the table are characteristic for the upper photosphere. Hence the interior of the sun (and probably stars in general), including most of the photosphere, should be counted as high-density magnetic plasmas, whereas the chromosphere and the corona are characterized by medium-density plasmas. In the ionosphere ($B \approx 0{\cdot}4$ gauss) the critical collision frequency is $0{\cdot}7\times10^{7}$ sec^{-1} which is reached at about 70 km height. At heights greater than this the ionosphere is a medium-density plasma which is highly anisotropic.

The limit between a medium- and a low-density plasma depends on the mean free path λ. The mean free path of an electron in a fully ionized plasma can be written $\lambda = v_e\tau$, where τ is the deflexion time given in 4.3.2 (41). Using for v_e the mean thermal velocity $(3kT_e/m_e)^{\frac{1}{2}}$, we find for an electron-proton plasma

$$\lambda_e = \frac{1{\cdot}3\times10^{5}}{\ln\Lambda}\frac{T_e^2}{n_e}. \qquad (21)$$

The values of the Coulomb logarithm $\ln \Lambda$ varies only slowly with temperature and density (Fig. 4.3). For order of magnitude estimates we can put

$$\lambda \approx 10^4 \frac{T_e^2}{n_e}.$$

Table 5.2 shows the order of magnitude of λ for different values of n and T.

TABLE 5.2

Mean free path of electrons in a completely ionized hydrogen plasma

Temperature T =	10^2	10^3	10^4	10^5	10^6 *degrees*
Density n = 1	10^8	10^{10}	10^{12}	10^{14}	10^{16} cm
(cm^{-3}) 10^2	10^6	10^8	10^{10}	10^{12}	10^{14}
10^4	10^4	10^6	10^8	10^{10}	10^{12}
10^6	10^2	10^4	10^6	10^8	10^{10}
10^8	1	10^2	10^4	10^6	10^8

At a solar distance $R = 5R_{\text{sun}} = 3{\cdot}5 \times 10^{11}$ cm, the density of the corona is estimated to be about $n = 10^4$ cm^{-3} and the temperature 3×10^5 degrees. This gives $\lambda = 10^{11}$, so if we identify l_c with the solar distance we have $l_c \approx 3{\cdot}5\lambda$.

In interplanetary space near the orbit of the earth (solar distance $1{\cdot}5 \times 10^{13}$ cm) the density is of the order $n \approx 1$ cm^{-3}. The temperature is not very well known but may be 10^5 degrees or more. This gives $\lambda > 10^{14}$ so that $\lambda \gg l_c$.

In interstellar space the average density is estimated to $n \approx 1$ cm^{-3} and the temperature is in general so low that according to Table 5.2 we have $\lambda \ll l_c$ (l_c being of the order 10^{18} or 10^{20} cm). The same holds in general for interstellar gas clouds and also for intergalactic space.

Let us consider the magnetosphere of the earth, in particular a magnetic field line which intersects the equatorial plane at, say, five times the earth's radius, i.e. at 3×10^9 cm. The density in the equatorial plane is probably of the order 10^2 cm^{-3} or less. If we put $l_c = 3 \times 10^9$ we find that $\lambda \approx l_c$ if the temperature is about 10^4 degrees. If the temperature is lower, the magnetosphere should be counted as a medium-density region, whereas if the temperature is higher it is a low-density region. No reliable temperature measurements seem to exist and it can perhaps not be excluded that during undisturbed conditions the temperature may be low. However, during magnetic storms the magnetosphere is invaded by a presumably hot plasma (perhaps 10^6 degrees) ejected from the sun, and this is still more heated by magnetic compression when it penetrates into the magnetosphere. Therefore, at least during magnetic storms the magnetosphere is likely to be very hot and consequently a low-density region.

5.1.5. *Survey of the properties of magnetic plasmas*

A survey of the properties of plasmas at different densities is given in Table 5.3 (p. 170). For comparison the case of single charges moving in perfect vacuum is also included.

Stellar interiors are high-density plasmas.

Medium-density plasmas are found in the solar chromosphere and corona, in interstellar and intergalactic space and also in gas nebulae. To this should be added the ionosphere above 70 km.

At least during magnetically disturbed conditions the magnetosphere of the earth is probably a low-density region. Interplanetary space may be a low-density region or an intermediate case. One of the consequences of this is that we cannot exclude the existence of considerable electric fields parallel to the magnetic field lines in those parts of the magnetosphere and interplanetary space where the magnetic field is inhomogeneous.

The distinction between the medium-density case and the low-density case is very important. In cosmical physics a medium-density plasma has often so high conductivity that we can put $\sigma = \infty$. In contrast to this it is more appropriate to put $\sigma = 0$ or to leave the conductivity undefined in a low-density plasma.

There is a whole literature based on the assumption that $\sigma = \infty$ and $E_{\parallel} = 0$ in low-density plasmas. Several of the conclusions based on this assumption are probably erroneous and should be reconsidered.

The difference between the vacuum case and a low-density plasma is that in the former case the quasi-neutrality condition $e_i n_i + e_c n_c \approx 0$ need not to be satisfied. If the characteristic length l_c is smaller than the Debye distance, we have to do with high-vacuum phenomena.

5.2. Theory of magnetic plasmas

5.2.1. *The microscopic and macroscopic descriptions of plasmas*

The behaviour of a plasma can be analysed from a microscopic or from a macroscopic point of view (cf., e.g., Grad, 1961).

In the *microscopic* approach the starting-point is the motion of the individual particle. However, the detailed motion of the charged particle in the presence of a magnetic field is complicated. It is, therefore, advantageous to study the motion of the centre of gyration. The velocity of the latter is given by 2.3.4 (39) to (43), where we have to

TABLE 5.3

Survey of characteristic properties of plasmas and of single charges in high vacuum

	High-density plasma	*Medium-density plasma*	*Low-density plasma*	*Single charges in high vacuum*
Criterion	$\lambda \ll \rho$	$\rho \ll \lambda \ll l_c$	$l_c \ll \lambda$	$l_c \ll \lambda_D$
Diffusion	Isotropic	Anisotropic	Anisotropic and small	No diffusion
Conductivity	Isotropic	Anisotropic	Not defined	Not defined
Electric field parallel to **B** in completely ionized gas	Small	Small	Any value	Any value
Particle motion in plane perpendicular to **B**	Almost straight path between collisions	Circle between collisions	Circle	Circle
Path of guiding centre parallel to **B**	Straight path between collisions	Straight path between collisions	Oscillations (e.g. between mirror points)	Oscillations (e.g. between mirror points)
Debye distance λ_D	$\lambda_D \ll l_c$	$\lambda_D \ll l_c$	$\lambda_D \ll l_c$	$\lambda_D \gg l_c$

include in **f** an equivalent force representing the effect of collisions between particles.

In the *macroscopic* approach one considers instead the average velocity **v** of all the particles (of each kind), contained in a volume element, which is large compared to the Larmor radius and the mean distance between particles, but still so small that it can be considered as a nearly homogeneous region. If the volume element under consideration has linear dimensions much larger than the mean free path λ, the particles contained in the volume element at a given instant remain in it during its motion with the velocity **v**. In this case each component of the plasma can be considered as a fluid (electron fluid, ion fluids, and neutral-molecule fluids), the motion of which is described by a macroscopic velocity.

When the mean free path is larger than the linear dimension of any volume element of interest, the fluid description is no longer suitable. The average velocity of the particles present in a given volume element at a given time is still a well-defined but not a very useful quantity.

As the condition $\lambda \ll l_c$ is satisfied in high- and medium-density plasmas, these can often be described by fluid models. On the other hand, such models are not suitable for low-density plasmas, where, by definiton, $\lambda \gg l_c$.

5.2.2. *Relation between drift velocity and macroscopic velocity*

The macroscopic velocity **v** is defined as the average velocity of the *particles* present in a given volume element. The average velocity $\mathbf{u}_g$ of all the *centres of gyration* present in the same volume element may very well be different from **v**. This is so because some of the particles in the volume element have their centres of gyration outside it and vice versa. As an illustration we shall calculate the quantitative relation between **v** and $\mathbf{u}_g$, assuming straight magnetic field lines.

We shall here limit the discussion to the case where quantities such as magnetic field strength, density, pressure, etc., vary only slowly, so that

$$\rho \ll l_c, \tag{1}$$

where ρ is the Larmor radius and l_c the characteristic length of the variation. From this assumption it follows directly that the number densities of particles and of gyration centres are approximately equal,

$$n \approx n_g. \tag{2}$$

As we shall see, the velocities **v** and $\mathbf{u}_g$ can be essentially different in spite of (1), cf. Spitzer (1952).

For calculating $\mathbf{v}-\mathbf{u}_g$ we use a coordinate system with the z-axis parallel to $\mathbf{B}$. Let $\Delta V = \Delta x\, \Delta y\, \Delta z$ be a volume element with linear dimensions much larger than the Larmor radius ρ. The total time-averaged momentum perpendicular to $\mathbf{B}$ of the particles in the interior is, apart from smaller terms, $m(n\, dxdydz)\mathbf{u}_{g\perp}$. Let us next calculate the momentum of the particles crossing the boundary 1 (see Fig. 5.3). We denote by $\mathbf{w}_\perp$ the instantaneous velocity of an individual particle. According to the definition of macroscopic velocity a 'boundary particle' contributes to the macroscopic velocity of the volume element only during that part of the gyration where the particle is inside the boundary.

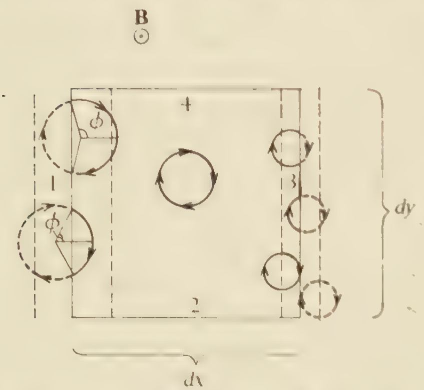


FIG. 5.3. Illustrating the difference between macroscopic velocity and average gyration-centre velocity.

Therefore its effective velocity is given by

$$\bar{w}_\perp = \frac{1}{2\pi}\int_{-\phi}^{\phi} w_\perp \cos\varphi \, d\varphi = \frac{\sin\phi}{\pi} w_\perp. \tag{3}$$

(As the gyrations are not complete, the effective velocity $\bar{w}_\perp$ is of the same order of magnitude as $w_\perp$, and so $\bar{w}_\perp$ can be much larger than the drift velocity. This is why the boundary particles, which are few in number, play any role at all.) The number of particles with ϕ-values in the range ϕ, $\phi+d\phi$ is $n\, dydz\, \rho \sin\phi\, d\phi$. According to (3) each contributes a momentum $(mw_\perp/\pi)\sin\phi$. Integration over ϕ gives

$$\int_0^\pi \frac{mw_\perp \sin\phi}{\pi} n\, dydz\, \rho\sin\phi\, d\phi = \frac{w_\perp}{2} mn\rho\, dydz. \tag{4}$$

Corresponding expressions are valid at the other boundaries. Thus the total perpendicular momentum $nm\mathbf{v}_\perp\, dxdydz$ is given by the expression

$$\begin{aligned}
\pm nm\mathbf{v}_\perp\, dxdydz = {} & \pm mn\mathbf{u}_{g\perp}\, dxdydz - (\tfrac{1}{2}w_\perp\, mn\rho\, dydz)_x\, \hat{\mathbf{y}} + \\
& + (\tfrac{1}{2}w_\perp\, mn\rho\, dydz)_{(x+dx)}\hat{\mathbf{y}} + (\tfrac{1}{2}w_\perp\, mn\rho\, dxdz)_y\, \hat{\mathbf{x}} - \\
& - (\tfrac{1}{2}w_\perp\, mn\rho\, dxdz)_{(y+dy)}\hat{\mathbf{x}} \\
= {} & dxdydz\left\{\pm mn\mathbf{u}_{g\perp} + \tfrac{1}{2}m\frac{\partial}{\partial x}(n\rho w_\perp)\hat{\mathbf{y}} - \tfrac{1}{2}m\frac{\partial}{\partial y}(n\rho w_\perp)\hat{\mathbf{x}}\right\}.
\end{aligned} \tag{5}$$

where $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are the unit vectors in the x- and y-directions, and the upper signs apply to positive particles. Since, according to 2.2.1 (8),

$$\rho = \frac{mw_\perp c}{|e|B}, \tag{6}$$

we can rewrite (5) as

$$\mathbf{v}_\perp = \mathbf{u}_{g\perp} + \frac{c}{enB}\mathbf{B}\times\mathrm{grad}\left(\frac{nmw_\perp^2}{2B}\right). \tag{7}$$

In terms of the pressure perpendicular to the lines of force (cf. § 5.9.3), which is

$$p_\perp = \frac{nmw_\perp^2}{2}, \tag{8}$$

we have

$$\mathbf{v}_\perp - \mathbf{u}_{g\perp} = \frac{c}{enB}\mathbf{B}\times\mathrm{grad}(p_\perp/B). \tag{9}$$

If $w_\perp$ is different for different particles, as is the case in a plasma, $w_\perp^2$ in (7) and (8) stands for the average over the velocity distribution, and (9) remains unchanged.

For the motion along the lines of force the formula

$$\frac{d\mathbf{v}_\parallel}{dt} - \frac{d\mathbf{u}_{g\parallel}}{dt} = -\frac{\mathbf{B}}{nm}\mathrm{grad}(p_\parallel/B) \tag{10}$$

applies, where $p_\parallel = nmw_\parallel^2$ (cf. Spitzer, 1952, and Lehnert, 1963*b*).

In the general case it can be shown (see, e.g., Lehnert, 1963*b*) that

$$\mathbf{v} - \mathbf{u}_g = -\frac{c}{en}\,\mathrm{curl}\left(\frac{nmw_\perp^2}{2B}\frac{\mathbf{B}}{B}\right) = \frac{c}{en}\,\mathrm{curl}\left[n\mu\left(-\frac{\mathbf{B}}{B}\right)\right], \tag{10 a}$$

where μ is the orbital magnetic moment given by 2.2.1 (13). Note that the quantity $\mathbf{M} = n\mu(-\mathbf{B}/B)$, which enters into (10 a), represents the volume density of magnetic moment due to the circling motion of the charged particles.

5.2.3. *The macroscopic equations*

5.2.3.1. *The three-fluid model*

In this section we derive the general equations relating current density, velocity, and electric and magnetic fields in a three-component plasma consisting of *electrons*, one kind of *ions*, and one kind of *neutral molecules*. The analysis is based on the *macroscopic equations of motion* for each kind of particle. These equations, 5.2.3.1 (21) to (23), can be rigorously derived from the Boltzmann equation (cf. § 5.9), but here we will only make them plausible by starting from the drift equations of § 2.3.4 and including the effects of gyration of the particles (by means of 5.2.2 (9) and (10)) and of collisions.

The velocity distribution is supposed to be almost isotropic. This assumption limits the applicability of our results. Especially at low densities the velocity distribution is often anisotropic.

We start from 2.3 (39) to (43) for the drift motion of a single particle with mass m_k and charge e_k,

$$\mathbf{u}_{gk} = -\frac{c}{e_k B^2}\mathbf{B}\times\left\{e_k\mathbf{E}+\mathbf{f}_k-\mu_k \operatorname{grad} B-m_k\frac{d\mathbf{u}_{gk}}{dt}\right\}, \tag{11}$$

$$0 = \left(e_k\mathbf{E}+\mathbf{f}_k-\mu_k \operatorname{grad} B-m_k\frac{d\mathbf{u}_{gk}}{dt}\right)_{\parallel}, \tag{12}$$

where
$$\mu_k = \frac{m_k w_{k\perp}^2}{2B} \tag{13}$$

and $\mathbf{f}_k$ is the total mechanical force acting on the particle (including, for example, the gravitation force $m_k\mathbf{g}$). The subscripts $\perp$ and $\parallel$ are used to distinguish components perpendicular and parallel to $\mathbf{B}$. The magnetic moment μ_k is different for different individual particles, but under the assumption of isotropy its average value over the velocity distribution is

$$\langle\mu_i\rangle = \frac{\langle m_i w_{i\perp}^2\rangle}{2B} = \frac{kT_i}{B} = \frac{p_i}{n_i B} \tag{14 a}$$

for ions, and
$$\langle\mu_e\rangle = \frac{\langle m_e w_{e\perp}^2\rangle}{2B} = \frac{kT_e}{B} = \frac{p_e}{n_e B} \tag{14 b}$$

for electrons. T_i and T_e are the ion and electron temperatures and p_i and p_e the corresponding pressures. (The assumed isotropy implies that the pressure tensor reduces to a scalar pressure, cf. § 5.9.)

Thus the average *gyration-centre velocity* perpendicular to $\mathbf{B}$ is for the electrons

$$\mathbf{u}_{ge} = -\frac{c}{e_e B^2}\mathbf{B}\times\left\{e_e\mathbf{E}+\mathbf{f}_e-\frac{p_e}{n_e B}\operatorname{grad} B-m_e\frac{d\mathbf{u}_{ge}}{dt}\right\} \tag{15}$$

and for the ions

$$\mathbf{u}_{gi} = -\frac{c}{e_i B^2}\mathbf{B}\times\left\{e_i\mathbf{E}+\mathbf{f}_i-\frac{p_i}{n_i B}\operatorname{grad} B-m_i\frac{d\mathbf{u}_{gi}}{dt}\right\}. \tag{16}$$

The terms $\mathbf{f}_e$ and $\mathbf{f}_i$ represent the average values of the mechanical forces. In the following we shall assume the ions to be singly ionized so that $e_i = -e_e = |e|$, and use the condition of quasi-neutrality,

$$n_i e_i + n_e e_e = 0 \tag{17}$$

so that
$$n_i = n_e = n. \tag{17 a}$$

(Cf. §§ 1.4 and 4.2.3.)

Introducing 5.2.2 (9) and (10), and observing that

$$\operatorname{grad}(p/B) = \frac{1}{B}\operatorname{grad} p - \frac{p}{B^2}\operatorname{grad} B,$$

we can calculate the *macroscopic velocity of the electron gas.* The result is, apart from smaller terms,

$$\mathbf{v}_{e\perp} = \frac{c}{|e|B^2}\mathbf{B}\times\left\{-m_e\frac{d\mathbf{v}_e}{dt} - |e|\mathbf{E}+\mathbf{f}_e - \frac{1}{n}\operatorname{grad} p_e\right\}, \tag{18a}$$

$$0 = \left(-m_e\frac{d\mathbf{v}_e}{dt} - |e|\mathbf{E}+\mathbf{f}_e - \frac{1}{n}\operatorname{grad} p_e\right)_{\|}. \tag{18b}$$

Similarly we find for the *ion gas*

$$\mathbf{v}_{i\perp} = \frac{c}{|e|B^2}\mathbf{B}\times\left\{m_i\frac{d\mathbf{v}_i}{dt} - |e|\mathbf{E}-\mathbf{f}_i + \frac{1}{n}\operatorname{grad} p_i\right\}, \tag{19a}$$

$$0 = \left(m_i\frac{d\mathbf{v}_i}{dt} - |e|\mathbf{E}-\mathbf{f}_i + \frac{1}{n}\operatorname{grad} p_i\right)_{\|}. \tag{19b}$$

It is of interest to observe that no term contains grad B. Thus a magnetic gradient cannot directly cause any macroscopic velocity and hence no current and no $\mathbf{i}\times\mathbf{B}$ force. This result, however, depends on our assumption that the velocity distribution is isotropic. Consequently, it does not hold in boundary layers, where the velocity distribution is anisotropic. (This explains why it is possible to confine a plasma in a magnetic field, e.g. in a mirror field.)

In a collision between an electron and a neutral molecule, the momentum loss of the electron is on the average $m_e(\mathbf{v}_e-\mathbf{v}_n)$. If the frequency of such collisions is $1/\tau_{en}$, the average force per electron from these collisions is $-m_e(\mathbf{v}_e-\mathbf{v}_n)/\tau_{en}$. Similar considerations apply to electron–ion collisions, provided we use the 'effective' collision time τ_{ei}, taking into account distant collisions (see § 4.3.2). Hence, if we include also gravitation, we can write the mechanical forces on electrons and ions

$$\mathbf{f}_e = -\frac{m_e(\mathbf{v}_e-\mathbf{v}_n)}{\tau_{en}} - \frac{m_e(\mathbf{v}_e-\mathbf{v}_i)}{\tau_{ei}} + m_e\mathbf{g} \tag{20a}$$

and

$$\mathbf{f}_i = -\frac{m_i(\mathbf{v}_i-\mathbf{v}_n)}{\tau_{in}} + \frac{m_e(\mathbf{v}_e-\mathbf{v}_i)}{\tau_{ei}} + m_i\mathbf{g}, \tag{20b}$$

where τ_{en} and τ_{in} are the collision times for electrons and ions with neutrals. Note that τ_{in} is the 'effective' collision time for ions with neutrals (taking into account the fact that not the whole momentum is lost at each real collision).

Multiplying (18a) and (19a) vectorially by $n|e|\mathbf{B}/c$ and rearranging, we can rewrite (18) and (19) as

$$nm_e\frac{d\mathbf{v}_e}{dt} = -n|e|\{\mathbf{E}+(\mathbf{v}_e/c)\times\mathbf{B}\}-\operatorname{grad}p_e+n\mathbf{f}_e, \tag{21}$$

$$nm_i\frac{d\mathbf{v}_i}{dt} = n|e|\{\mathbf{E}+(\mathbf{v}_i/c)\times\mathbf{B}\}-\operatorname{grad}p_i+n\mathbf{f}_i, \tag{22}$$

where $\mathbf{f}_e$ and $\mathbf{f}_i$ are given by (20a) and (20b). Further, if the neutral molecules have the mass m_n and the number density n_n, we have

$$n_n m_n\frac{d\mathbf{v}_n}{dt} = n_n m_n\mathbf{g}-\operatorname{grad}p_n+\frac{nm_e}{\tau_{en}}(\mathbf{v}_e-\mathbf{v}_n)+\frac{nm_i}{\tau_{in}}(\mathbf{v}_i-\mathbf{v}_n), \tag{23}$$

where p_n is the neutral-gas pressure.

Equations (21), (22), and (23) are the *equations of motion in the three-fluid model* for the case of singly charged ions (cf. for example, Lehnert, 1959).

5.2.3.2. *The equations of motion and the generalized Ohm's law*

When the velocities of electrons and ions are nearly equal, it is often convenient to rewrite (21) and (22) in terms of the *current density* $\mathbf{i}$ and the velocity $\mathbf{v}_c$ of the *charged-particle component* of the plasma.

We introduce the following symbols referring to the charged-particle component of the plasma:

mass density: $$\rho_c = n(m_i+m_e); \tag{24}$$

momentum per unit volume:

$$\rho_c\mathbf{v}_c = n(m_i\mathbf{v}_i+m_e\mathbf{v}_e); \tag{25}$$

current density: $$\mathbf{i} = |e|n(\mathbf{v}_i-\mathbf{v}_e). \tag{26}$$

We can express $\mathbf{v}_i$ and $\mathbf{v}_e$ in terms of $\mathbf{v}_c$ and $\mathbf{i}$ as follows:

$$\mathbf{v}_i = \mathbf{v}_c+\frac{m_e}{|e|n(m_i+m_e)}\mathbf{i} \approx \mathbf{v}_c+\frac{m_e}{m_i}\frac{\mathbf{i}}{|e|n}, \tag{27}$$

$$\mathbf{v}_e = \mathbf{v}_c-\frac{m_i}{|e|n(m_i+m_e)}\mathbf{i} \approx \mathbf{v}_c-\frac{\mathbf{i}}{|e|n}. \tag{28}$$

The equations of motion

The sum of the mechanical forces per unit volume of the charged component of the plasma is, according to (20a) and (20b),

$$n(\mathbf{f}_i+\mathbf{f}_e) = n\mathbf{v}_n\left(\frac{m_e}{\tau_{en}}+\frac{m_i}{\tau_{in}}\right)-\frac{nm_e\mathbf{v}_e}{\tau_{en}}-\frac{nm_i\mathbf{v}_i}{\tau_{in}}+\rho_c\mathbf{g}. \tag{29}$$

Using (27) and (28) we can rewrite this as

$$n(\mathbf{f}_\mathrm{i}+\mathbf{f}_\mathrm{e}) = n\left(\frac{m_\mathrm{e}}{\tau_\mathrm{en}}+\frac{m_\mathrm{i}}{\tau_\mathrm{in}}\right)(\mathbf{v}_\mathrm{n}-\mathbf{v}_\mathrm{c})+\frac{m_\mathrm{e}}{|e|}\left(\frac{1}{\tau_\mathrm{en}}-\frac{1}{\tau_\mathrm{in}}\right)\mathbf{i}+\rho_\mathrm{c}\mathbf{g}. \tag{30}$$

As $m_\mathrm{e} \ll m_\mathrm{i}$, it is a good approximation to write

$$\rho_\mathrm{c} = nm_\mathrm{i}, \tag{31}$$

and consequently

$$n(\mathbf{f}_\mathrm{i}+\mathbf{f}_\mathrm{e}) = -\frac{\rho_\mathrm{c}}{\tau_\mathrm{in}}\left(1+\frac{m_\mathrm{e}\tau_\mathrm{in}}{m_\mathrm{i}\tau_\mathrm{en}}\right)(\mathbf{v}_\mathrm{c}-\mathbf{v}_\mathrm{n})+\frac{\rho_\mathrm{c}}{\tau_\mathrm{in}}\frac{m_\mathrm{e}}{m_\mathrm{i}}\left(\frac{\tau_\mathrm{in}}{\tau_\mathrm{en}}-1\right)\frac{\mathbf{i}}{|e|n}+\rho_\mathrm{c}\mathbf{g}. \tag{32}$$

Adding (21) and (22) and using (31) and (32) we obtain

$$\rho_\mathrm{c}\frac{d\mathbf{v}_\mathrm{c}}{dt} = (\mathbf{i}/c)\times\mathbf{B}-\mathrm{grad}(p_\mathrm{e}+p_\mathrm{i})+$$
$$+\rho_\mathrm{c}\mathbf{g}-\frac{\rho_\mathrm{c}}{\tau_\mathrm{in}}\left[\left(1+\frac{m_\mathrm{e}\tau_\mathrm{in}}{m_\mathrm{i}\tau_\mathrm{en}}\right)(\mathbf{v}_\mathrm{c}-\mathbf{v}_\mathrm{n})-\frac{m_\mathrm{e}}{m_\mathrm{i}}\left(\frac{\tau_\mathrm{in}}{\tau_\mathrm{en}}-1\right)\frac{\mathbf{i}}{|e|n}\right]. \tag{33}$$

This is the *equation of motion for the charged component of the plasma.*

The neutral-gas velocity $\mathbf{v}_\mathrm{n}$ which enters in (33) is governed by (23).

When also $\mathbf{v}_\mathrm{n}$ is nearly equal to $\mathbf{v}_\mathrm{c}$, we obtain from (23) and (33) the *equation of motion for the gas as a whole*:

$$\rho\frac{d\mathbf{v}}{dt} = (\mathbf{i}/c)\times\mathbf{B}-\mathrm{grad}\,p+\rho\mathbf{g}, \tag{34}$$

where

$$\rho = \rho_\mathrm{e}+\rho_\mathrm{i}+\rho_\mathrm{n}, \tag{35}$$

$$\rho\mathbf{v} = \rho_\mathrm{e}\mathbf{v}_\mathrm{e}+\rho_\mathrm{i}\mathbf{v}_\mathrm{i}+\rho_\mathrm{n}\mathbf{v}_\mathrm{n}, \tag{36}$$

$$p = p_\mathrm{e}+p_\mathrm{i}+p_\mathrm{n}. \tag{37}$$

The generalized Ohm's law

In order to calculate the current density we multiply (21) by $1/(n|e|)$ and (22) by $m_\mathrm{e}/(m_\mathrm{i}n|e|)$ and subtract (21) from (22). We use (27) and (28) and neglect $m_\mathrm{e}/m_\mathrm{i}$ in comparison with unity. If we also linearize the equations by leaving out terms of second order in $\mathbf{v}$ and $\mathbf{i}$, we obtain after rearranging:

$$\frac{\mathbf{i}}{|e|nc}\times\mathbf{B}+\frac{m_\mathrm{e}}{e^2n}\frac{\partial\mathbf{i}}{\partial t} = \mathbf{E}+(\mathbf{v}_\mathrm{c}/c)\times\mathbf{B}+\frac{1}{|e|n}\left(\mathrm{grad}\,p_\mathrm{e}-\frac{m_\mathrm{e}}{m_\mathrm{i}}\mathrm{grad}\,p_\mathrm{i}\right)-\mathbf{P}, \tag{38}$$

where the quantity $\mathbf{P}$, which represents the friction between the different fluids, is given by

$$\mathbf{P} = \frac{1}{|e|}\left(\mathbf{f}_\mathrm{e}-\frac{m_\mathrm{e}}{m_\mathrm{i}}\mathbf{f}_\mathrm{i}\right) = \frac{m_\mathrm{e}}{|e|}\left[(\mathbf{v}_\mathrm{i}-\mathbf{v}_\mathrm{e})\frac{1}{\tau_\mathrm{ei}}+\frac{\mathbf{v}_\mathrm{i}}{\tau_\mathrm{in}}-\frac{\mathbf{v}_\mathrm{e}}{\tau_\mathrm{en}}-\mathbf{v}_\mathrm{n}\left(\frac{1}{\tau_\mathrm{in}}-\frac{1}{\tau_\mathrm{en}}\right)\right]$$
$$= \frac{m_\mathrm{e}}{|e|}\left[(\mathbf{v}_\mathrm{i}-\mathbf{v}_\mathrm{e})\left(\frac{1}{\tau_\mathrm{ei}}+\frac{1}{\tau_\mathrm{en}}\right)+(\mathbf{v}_\mathrm{i}-\mathbf{v}_\mathrm{n})\left(\frac{1}{\tau_\mathrm{in}}-\frac{1}{\tau_\mathrm{en}}\right)\right]$$
$$= \eta\mathbf{i}-\beta(\mathbf{v}_\mathrm{c}-\mathbf{v}_\mathrm{n}). \tag{39}$$

Here we have introduced the notation

$$\beta = \frac{m_e}{|e|}\left(\frac{1}{\tau_{en}} - \frac{1}{\tau_{in}}\right), \tag{40}$$

$$\eta = \frac{m_e}{e^2 n}\left(\frac{1}{\tau_{ei}} + \frac{1}{\tau_{en}} + \frac{m_e}{m_i}\frac{1}{\tau_{in}}\right) \tag{41}$$

(cf. Lehnert, 1959). Putting

$$\frac{1}{\tau_e} = \frac{1}{\tau_{ei}} + \frac{1}{\tau_{en}} + \frac{m_e}{m_i}\frac{1}{\tau_{in}}, \tag{42}$$

we have

$$\eta = \frac{m_e}{e^2 n}\frac{1}{\tau_e}, \tag{43}$$

where $1/\tau_e$, as defined by (42), is the effective collision frequency for electrons.

We can now write (38) in the form

$$\eta \mathbf{i} + \frac{1}{|e|n}(\mathbf{i}/c)\times\mathbf{B} + \frac{m_e}{e^2 n}\frac{\partial \mathbf{i}}{\partial t} = \mathbf{E}_{\text{eff}}, \tag{44}$$

or

$$\eta\left(\mathbf{i} + \omega_e \tau_e \mathbf{i} \times \frac{\mathbf{B}}{B} + \tau_e \frac{\partial \mathbf{i}}{\partial t}\right) = \mathbf{E}_{\text{eff}}, \tag{45}$$

with

$$\mathbf{E}_{\text{eff}} = \mathbf{E} + (\mathbf{v}_c/c)\times\mathbf{B} + \frac{1}{|e|n}\left(\operatorname{grad} p_e - \frac{m_e}{m_i}\operatorname{grad} p_i\right) + \beta(\mathbf{v}_c - \mathbf{v}_n) \tag{46}$$

and

$$\omega_e = \frac{|e|B}{m_e c}. \tag{47}$$

Equation (44) is one form of the '*generalized Ohm's law*'.

5.2.3.3. *Numerical values of the conductivity*

The expression for the conductivity σ and the resistivity η

$$\sigma = \frac{1}{\eta} = \frac{e^2 n \tau_e}{m_e} \tag{48}$$

contains, apart from atomic constants, only two variables, the number density of electrons n and the collision time for electrons τ_e. The former is a well-defined quantity, but the latter, τ_e, requires some comments.

Consider first a *weakly ionized plasma* where only collisions with molecules are important. If n_n is the neutral-molecule number density and S_n the collision cross-section that a molecule presents to an electron, the mean free path of the electron is

$$\lambda_e = 1/n_n S_n. \tag{49}$$

If the electron moves at a velocity v_e, the collision time is

$$\tau_e = \lambda_e / v_e. \tag{50}$$

Introducing (49) and (50) into (48) we find

$$\sigma = \frac{e^2}{m_e} \frac{1}{S_n} \frac{1}{v_e} \frac{n_e}{n_n}. \tag{51}$$

Although the mean thermal speed of the electrons is

$$v_e = (3kT_e/m_e)^{\frac{1}{2}}, \tag{52}$$

many electrons move faster or slower than that. Since S_n and hence λ_e depends on v_e, τ_e is different for different electrons. Therefore the distribution of electron velocities v_e, and hence of collision times τ_e, must be taken into account in accurate calculations.

The value of S_n depends on the velocity of the impinging electron and is in many cases a complicated function of it. Diagrams and tables of experimental data are found in textbooks (e.g. von Engel and Steenbeck, **1**, 1932, 168; Cobine, 1941, p. 29; Massey and Burhop, 1952, p. 8). Usually the value of $3{\cdot}6 \times 10^{16} S_n$ is given, which is the sum of cross-sections of all molecules contained in a cubic centimetre of gas at a pressure of 1 mm Hg and a temperature of 0° C. The inverse value gives the mean free path at this pressure and temperature. For electrons below some hundred electron-volts the cross-section S_n of most gases is of the order of magnitude of 10^{-15} cm^2.

In a *completely ionized plasma* the forces have long range, and a moving particle interacts simultaneously with many other particles as discussed in § 4.3.2. The formula (48) is still applicable if we insert for τ_e an 'effective collision time' calculated with distant collisions taken into account. The value to insert in this case is essentially the slowing down time given by 4.3.2 (41). The value so obtained is still approximate to the extent that proper account has not been taken of the actual velocity distribution and of the influence of electron–electron collisions. However, it differs by less than a factor 2 from the more accurate value 4.3.3 (59), which is (with $e_i = -Ze_e = Z|e|$)

$$\sigma = \gamma \frac{2^{\frac{5}{2}}}{\pi^{\frac{3}{2}}} \frac{(kT_e)^{\frac{3}{2}}}{m_e^{\frac{1}{2}} Z e^2 \ln \Lambda} = \frac{\gamma . 2{\cdot}37 . 10^8}{Z \ln \Lambda} T^{\frac{3}{2}} \text{ e.s.u.} \tag{53}$$

(Spitzer and Härm, 1953). The factor γ depends slightly on the ionic charge e_i and takes values between 0·582 (for $Z = 1$) and 1 (for $Z = \infty$). The factor $\ln \Lambda$ (Fig. 4.3) varies only slowly with temperature and density. Fig. 5.4 shows the conductivity of a hydrogen plasma as a function of temperature (and density). It is interesting to compare the cross-section of ions and of molecules. We can write the conductivity in the form

$$\sigma = \frac{e^2 n_e \tau_{e,\mathrm{eff}}}{m_e} = \frac{e^2}{m_e v_e} \frac{1}{S_{\mathrm{eff}}},$$

where v_e is the mean thermal velocity, $v_e = (3kT_e/m_e)^{\frac{1}{2}}$. It then follows that for protons

$$S_{\mathrm{eff}} \approx 0{\cdot}3 \ln \Lambda 10^{-5} T_e^{-2}, \tag{54}$$

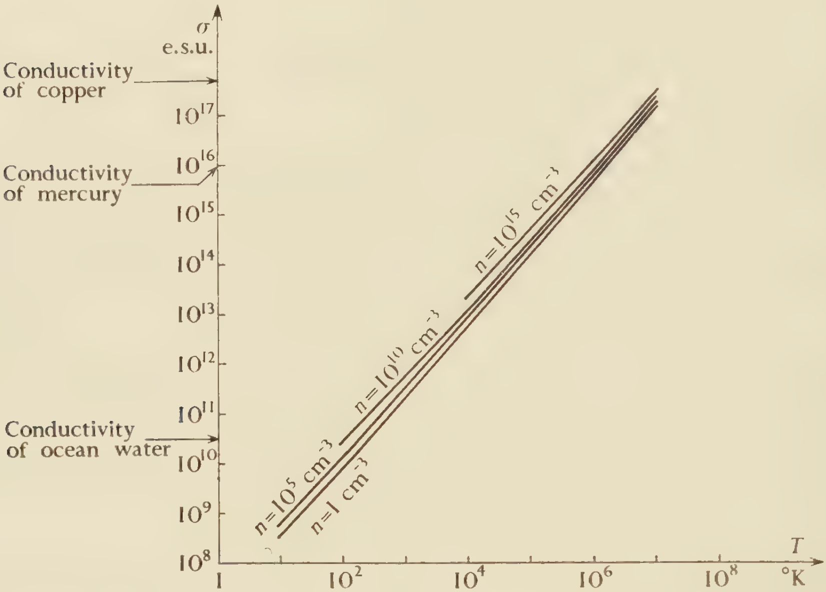


Fig. 5.4. Conductivity of a completely ionized hydrogen plasma.

whereas the cross-section for neutral molecules is of the order of magnitude 10^{-15} cm². For example, at $T_e = 5000°$ K, the effective cross-section of ions is about a thousand times larger than that of neutral molecules. This means that the electron–ion collisions become more important than the electron–molecule collisions already when the degree of ionization is as small as 10^{-3}. This illustrates the fact that, provided the electron temperature is not too high, plasmas with a degree of ionization larger than 1 per cent can often be considered as completely ionized as far as collision processes are concerned.

5.3. The cross-conductivity and the Hall conductivity

From 5.2.3.2 (45) it is seen that, except when $\omega_e \tau_e \ll 1$, an electric field $\mathbf{E}$ and the current density $\mathbf{i}$, which it produces, are in general not parallel. Thus, whereas *high-density* plasmas conduct electricity isotropically, *medium-density* plasmas are strongly anisotropic in this respect.

When $\omega_i \tau_i \gg 1$ the ions can make a significant contribution to the current. Consider for example a thin weakly ionized plasma ($\tau_{ei} \gg \tau_{en}$ and $p_e \approx p_i \approx 0$). It follows from 5.2. (44) and (33) that an electric field with the components $\mathbf{E}_{\|}$ and $\mathbf{E}_{\perp}$ parallel and perpendicular to $\mathbf{B}$ produces a current density with the components

$$\mathbf{i}_{\|} = (\sigma_{\|}^{e}+\sigma_{\|}^{i})\mathbf{E}_{\|}, \tag{1}$$

$$\mathbf{i}_{P} = (\sigma_{P}^{e}+\sigma_{P}^{i})(\mathbf{E}_{\perp}+\mathbf{v}_{n}\times\mathbf{B}), \tag{2}$$

$$\mathbf{i}_{H} = (\sigma_{H}^{e}-\sigma_{H}^{i})(\mathbf{B}/B)\times(\mathbf{E}_{\perp}+\mathbf{v}_{n}\times\mathbf{B}), \tag{3}$$

where

$$\sigma_{\|}^{e} = \frac{e^2 n \tau_{en}}{m_e}, \qquad \sigma_{\|}^{i} = \frac{e^2 n \tau_{in}}{m_i}, \tag{4 a, b}$$

$$\sigma_{P}^{e} = \frac{\sigma_{\|}^{e}}{(1+\omega_e^2 \tau_{en}^2)}, \qquad \sigma_{P}^{i} = \frac{\sigma_{\|}^{i}}{(1+\omega_i^2 \tau_{in}^2)}, \tag{5 a, b}$$

$$\sigma_{H}^{e} = \frac{\sigma_{\|}^{e}\omega_e \tau_{en}}{(1+\omega_e^2 \tau_{en}^2)}, \qquad \sigma_{H}^{i} = \frac{\sigma_{\|}^{i}\omega_i \tau_{in}}{(1+\omega_i^2 \tau_{in}^2)}. \tag{6 a, b}$$

(Cf. also Chapman and Cowling, 1939; Cowling, 1945; Baker and Martyn, 1953; and Chapman, 1956.) The 'Hall current' $\mathbf{i}_H$ is produced by the drift motion, perpendicular to $\mathbf{B}$, of electrons and ions, see Fig. 5.5.

The quantity $\sigma_P = \sigma_P^e+\sigma_P^i$ is called the *cross-conductivity* or '*Pedersen conductivity*', and $\sigma_H = \sigma_H^e-\sigma_H^i$ is called the '*Hall conductivity*'.

For deriving (1) to (6), or corresponding more general formulae, one can also start from 5.2.3.1 (21) to (23). calculate separately the electron current $\mathbf{i}_e = -n|e|\mathbf{v}_e$ and the ion current $\mathbf{i}_i = n|e|\mathbf{v}_i$, and then form their sum $\mathbf{i} = \mathbf{i}_e+\mathbf{i}_i$. This analysis will not be carried through here, but for later application we write down the relations

$$\mathbf{i}_{eH} = \omega_e \tau_e(\mathbf{B}/B)\times\mathbf{i}_{eP}, \tag{7}$$

$$\mathbf{i}_{iH} = -\omega_i \tau_i(\mathbf{B}/B)\times\mathbf{i}_{iP}, \tag{8}$$

which can be seen directly from 5.2.3.1 (21) and (22).

In *high-density plasmas* we have $\omega_e \tau_e \ll 1$ and the conductivity is *isotropic*. If in particular the plasma is fully ionized, the friction term

$\beta(\mathbf{v}_e - \mathbf{v}_n)$ in 5.2.3 (46) vanishes, and if in addition the pressures p_e and p_i are negligible, the generalized Ohm's law reduces to the form

$$\mathbf{i} = \sigma\{\mathbf{E} + (\mathbf{v}/c) \times \mathbf{B}\}. \tag{9}$$

This is the simple equation that is valid for ideal magneto-fluids (cf. § 3.3). However, even for a high-density plasma there are many cases when the generalized Ohm's law cannot be reduced to the simple form (9). For example the friction term $-\beta(\mathbf{v}_e - \mathbf{v}_n)$ in 5.2.3 (46) can sometimes be very important and represent an additional loss mechanism in addition to the ordinary ohmic heating. (For surveys of the interaction between the charged-particle plasma and the neutral gas see, for example, Cowling, 1957, and Lehnert, 1959.)

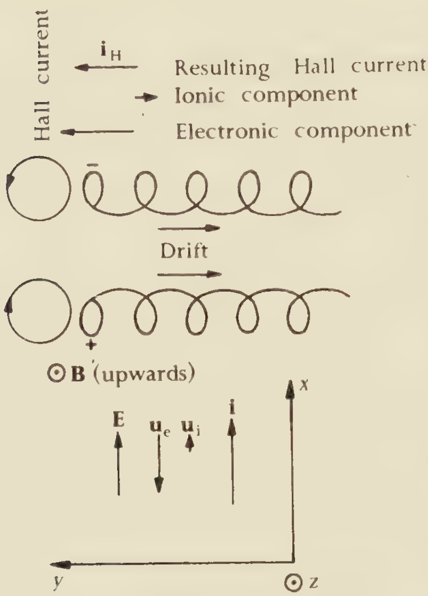


FIG. 5.5. In an electric field **E** positive ions drift with the velocity $\mathbf{u}_i$ and electrons with the velocity $\mathbf{u}_e$, producing a current **i**. In the presence of a magnetic field **B** (upwards through the paper) the particles also drift perpendicular to **E** and **B** thus causing a Hall current $\mathbf{i}_H$.

Another simple special case, which will be the basis of the discussion in § 5.3.3, is the following. Consider a three-component plasma consisting of electrons, singly charged ions, and neutral molecules. In the generalized Ohm's law we neglect the $\partial\mathbf{i}/\partial t$ term because we consider a (quasi-) stationary state and the terms grad p_e and grad p_i (which is justified, for example, in a cold plasma). We further assume that the neutral gas is at rest and that the friction between ion-gas and neutral-gas is sufficient to keep the former also nearly at rest. This situation can be realized provided

$$\omega_i \tau_i \ll 1 \tag{10}$$

(even if at the same time $\omega_e \tau_e \gg 1$). If (10) is valid, and if τ_i is mainly determined by ion–molecule collisions, we find from 4.3.1 (7) that the ion-gas velocity is of the order of magnitude

$$v_i \approx \frac{|e|\tau_{in}}{2m_i} E. \tag{11}$$

As $\mathbf{v}_e \approx \mathbf{v}_i$, we find that the friction term in 5.2.3 (46) is of the magnitude

$$|\beta(\mathbf{v}_e - \mathbf{v}_n)| \approx \frac{m_e}{|e|}\left|\frac{1}{\tau_{en}} - \frac{1}{\tau_{in}}\right|\frac{|e|\tau_{in}}{2m_i}E \approx \frac{m_e}{m_i}E \ll E. \tag{12}$$

Further the term $(\mathbf{v}_e/c)\times\mathbf{B}$ in 5.2.3 (46) can be estimated to be

$$|(\mathbf{v}_e/c)\times\mathbf{B}| \approx \frac{|e|\tau_{in}EB}{2m_i c} = \tfrac{1}{2}\omega_i\tau_i E \ll E. \tag{13}$$

Thus, under the conditions we study, the ion-neutral friction is sufficient to keep the term $(\mathbf{v}_e/c)\times\mathbf{B}$ small without the friction term $\beta(\mathbf{v}_e - \mathbf{v}_n)$ being of importance in the generalized Ohm's law, which now reduces to

$$\left(\mathbf{i} + \omega_e\tau_e\mathbf{i}\times\frac{\mathbf{B}}{B}\right) = \mathbf{E}/\eta = \sigma\mathbf{E}. \tag{14}$$

5.3.1. *Effect of inhibiting the Hall current*

We shall next discuss what happens if the Hall current is prevented from flowing. This could be done by inserting electrically insulating planes parallel to $\mathbf{B}$ and $\mathbf{E}$. Then an electric field $\mathbf{E}_1$ is produced perpendicular to these planes. Putting the current perpendicular to the planes equal to zero we obtain directly from (14)

$$\mathbf{i} + 0 = \sigma\mathbf{E}, \tag{15}$$

so that the current density is

$$\mathbf{i} = (1/\eta)\mathbf{E} = \sigma\mathbf{E} \tag{16}$$

as it would be if there were no magnetic field. Hence *the stopping of the Hall current replaces the cross-conductivity by the ordinary conductivity*. At the same time an electric polarization field is produced. According to 5.2.3.2 (45) it is given by

$$\mathbf{E}_1 = \eta\omega_e\tau_e\mathbf{i}\times\frac{\mathbf{B}}{B} = \omega_e\tau_e\mathbf{E}\times\frac{\mathbf{B}}{B}. \tag{17}$$

The result (16) is only approximate, because the details of the velocity distribution have not been taken into account. According to Spitzer (1952) a more accurate result is

$$\mathbf{i} = (0{\cdot}56/\eta)\mathbf{E} = 0{\cdot}56\sigma\mathbf{E}. \tag{18}$$

5.3.2. *Equivalent circuits*

It is sometimes useful to represent the electrical behaviour of a plasma by an equivalent circuit. Generally the equivalent circuits are complicated (see, for example, Lehnert, 1962*a*), but in certain cases the equivalent circuit can have a simple structure. We shall briefly discuss such a case.

Consider first an electrically conducting liquid, with mass density ρ and conductivity σ, in a homogeneous magnetic field $\mathbf{B}$. An electric field $\mathbf{E}$ applied perpendicular to $\mathbf{B}$ produces a current density $\mathbf{i} = \sigma\mathbf{E}$. The force $(\mathbf{i}/c)\times\mathbf{B}$ accelerates the medium in a direction perpendicular to $\mathbf{E}$ and $\mathbf{B}$. The electric field in a coordinate system moving with the plasma is

$$\mathbf{E}' = \mathbf{E}+(\mathbf{v}/c)\times\mathbf{B}. \tag{19}$$

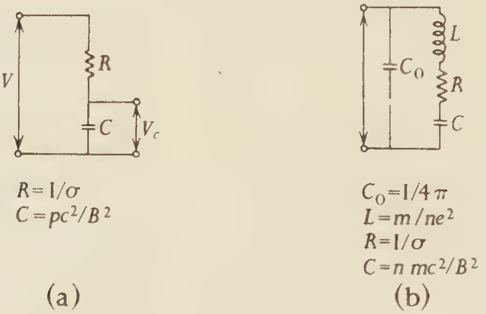


FIG. 5.6. Equivalent circuits of conductors in a magnetic field. (*a*) Conducting liquid. R represents the specific resistance; the energy of condenser C represents the kinetic energy. (*b*) Charged-particle gas. L represents the inertia of the charged particles. C_0 represents the capacity in vacuum.

The acceleration goes on until

$$\mathbf{v} = c\frac{\mathbf{E}\times\mathbf{B}}{B^2}, \tag{20}$$

so that

$$\mathbf{E}' = 0, \tag{21}$$

and the current density also vanishes. The equivalent circuit for a unit cube of the liquid is shown in Fig. 5.6 (*a*). From what has been said we conclude that the following correspondence holds:

$$\left.\begin{aligned} I &\to i \\ V &\to E \\ V_c &\to vB/c \\ R &\to 1/\sigma \\ C &\to \rho c^2/B^2 \end{aligned}\right\}. \tag{22}$$

The charging of the capacitor corresponds to the acceleration of the liquid by the current density i. The magnitude of the capacitor is chosen so as to make its electrostatic energy $\frac{1}{2}CV_c^2$ equal to the kinetic energy $\frac{1}{2}\rho v^2$ of the liquid.

When the medium is a gas consisting of charged particles the influence of the inertia of the particles (ion mass $= m$, number density $= n$) is represented by a series inductance L,

$$L \to \frac{m}{ne^2}. \tag{23}$$

The resonant frequency of this inductance and the capacity C,

$$C \to \frac{\rho c^2}{B^2} = \frac{nmc^2}{B^2}, \tag{24}$$

is the gyro frequency $\omega = (LC)^{-\frac{1}{2}} = eB/mc$. In order to account even for the displacement current $i = (1/4\pi)\,dE/dt$ we must introduce a condenser $C_0 = 1/4\pi$. The circuit is seen in Fig. 5.6 (b).

5.3.3. *On currents perpendicular to the magnetic field in an inhomogeneous plasma*

Equation (5 a) indicates a great reduction in the conductivity perpendicular to a magnetic field as soon as $\omega_c \tau_c \gg 1$. However, this result rests on the idealized assumption of a homogeneous plasma. In inhomogeneous plasmas (which usually occur in cosmical physics) the situation may be very different as we shall now show by a simple example.

Consider a plasma for which 5.3 (14) holds. Let the geometrical configuration be as shown in Fig. 5.7 (a). In a narrow strip A which is parallel to the x-axis of a Cartesian coordinate system and has the breadth $\beta\,\Delta y$ the conductivity is σ_0. In an adjacent strip B with the breadth Δy the conductivity has the smaller value σ_0/α (e.g. due to lower degree of ionization). Owing to the Hall current there are produced electric fields in the y-direction. We denote the electric field strength in the first strip by E_y and that in the second strip by E'_y. We assume

$$\beta\,\Delta y\,E_y + \Delta y\,E'_y = 0. \tag{25}$$

The condition (25) makes it possible to draw conclusions from our model in cases where, due to different geometry, the Hall current is closed over a finite path, as, for example, in Fig. 5.7 (b).

As continuity considerations require that E_x as well as i_y is the same in both regions, we have (if for brevity we write $\omega_e \tau_e = \gamma$)

$$i_{xA} + \gamma i_y = \sigma_0 E_x, \tag{26}$$

$$i_y - \gamma i_{xA} = \sigma_0 E_y, \tag{27}$$

$$i_{xB} + \gamma i_y = \sigma_0 E_x/\alpha, \tag{28}$$

$$i_y - \gamma i_{xB} = \sigma_0 E'_y/\alpha = -(\beta/\alpha)\sigma_0 E_y, \tag{29}$$

where i_{xA} and i_{xB} are the current densities in the first and the second strip.

If we put

$$i_{xA} = \sigma_A E_x, \tag{30}$$

$$i_{xB} = \sigma_B E_x, \tag{31}$$

$$i_y = \sigma_C E_x, \tag{32}$$

we find from (26) to (29)

$$\frac{\sigma_A}{\sigma_0} = 1 - \frac{1+\beta}{\lambda+\beta}\,\frac{\gamma^2}{1+\gamma^2}, \tag{33}$$

$$\frac{\sigma_B}{\sigma_0} = \frac{1}{\lambda} - \frac{1+\beta}{\lambda+\beta}\,\frac{\gamma^2}{1+\gamma^2}, \tag{34}$$

$$\frac{\sigma_C}{\sigma_0} = \frac{1+\beta}{\lambda+\beta}\,\frac{\gamma}{1+\gamma^2}. \tag{35}$$

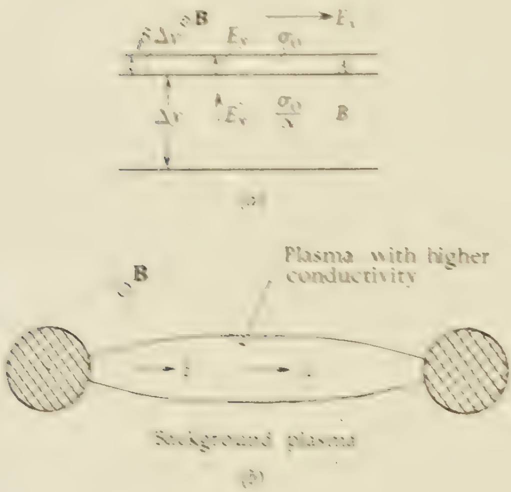


FIG. 5.7. (a) Geometrical configuration of the simplified model of an inhomogeneous plasma. (b) Sketch of a configuration to which one may wish to apply the results from the simple model.

If the plasma is homogeneous ($\lambda = 1$) we find

$$\sigma_A = \sigma_B = \sigma_0/(1+\gamma^2)$$

and

$$\sigma_C = \sigma_0 \frac{\gamma}{1+\gamma^2}$$

in agreement with 5.3 (5a) and (6a). If $\lambda = \infty$, the Hall current is prevented because the region B is an insulator. In this case we obtain $\sigma_A = \sigma_0$ as expected.

From (33) to (35) it is possible to estimate how sensitive the cross-conductivity is to small inhomogeneities. Suppose that with

$$\gamma = \omega_e \tau_e = 10 \quad \text{and} \quad \beta = 0{\cdot}1$$

the conductivity varies by 20 per cent, so that $\alpha = 1{\cdot}2$. Then we obtain from (33)

$$\frac{\sigma_A}{\sigma_0} = 0{\cdot}16,$$

whereas $\alpha = 1$ gives $\sigma_A/\sigma_0 = 0{\cdot}01$. Thus in this case the variation of the conductivity by as little as 20 per cent changes the effective cross-conductivity in the strip A by more than one order of magnitude.

Fig. 5.8 (a) and (b) show how σ_A, σ_B, and σ_C vary with γ for $\alpha = 2$ and $\alpha = 10$. Fig. 5.8 (c) shows σ_A, σ_B, and σ_C as functions of the

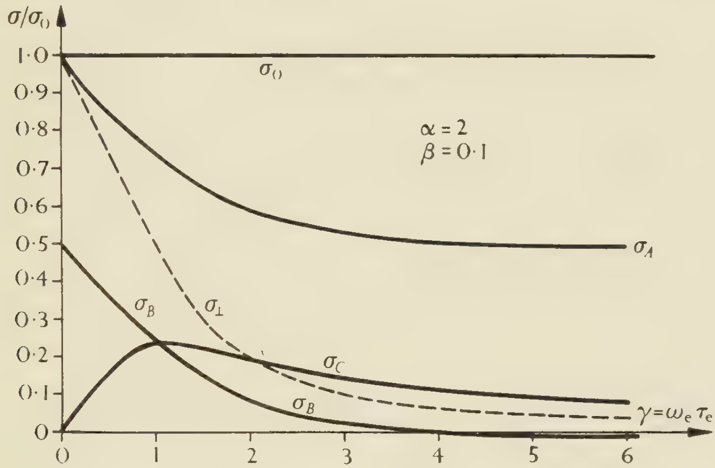


FIG. 5.8 (a)

conductivity ratio α when $\gamma = 3$. In all three cases we have put $\beta = 0{\cdot}1$. This corresponds to narrow strips with increased conductivity. We see from the diagrams that the magnetic field does not cause a large reduction of the current density in the strip with the higher conductivity. For $\alpha = 2$ the reduction never exceeds about 50 per cent and for $\alpha = 10$ it is only about 10 per cent. On the other hand, in the strip with the smaller conductivity the current goes down very rapidly when γ increases. (It even reverses, and for large γ the conductivity σ_B reaches the asymptotic value $-(\beta/\alpha)\sigma_A$.) The Hall current density is reduced by a factor α, and in the above-mentioned cases it is always far below the conduction current density i_{xA}.

If instead the strip with higher conductivity is *parallel to the y-axis*

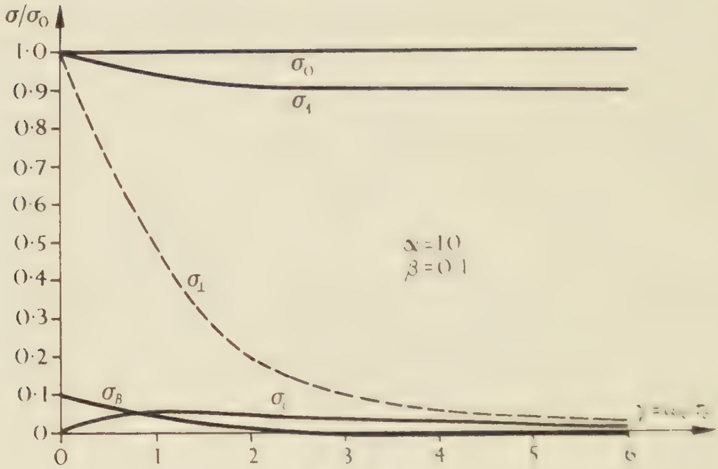


FIG. 5.8 (b)

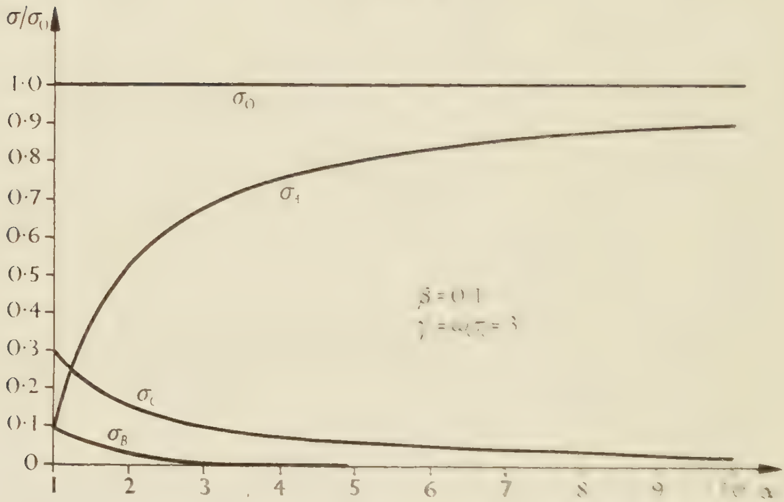


FIG. 5.8 (c)

FIG. 5.8. The effective conductivities σ_A, σ_B, and σ_C (*a*) as functions of γ when $\alpha = 2$ and $\beta = 0{\cdot}1$, (*b*) as functions of γ when $\alpha = 10$ and $\beta = 0{\cdot}1$, (*c*) as functions of α when $\beta = 0{\cdot}1$ and $\gamma = 3$.

we have $i_{xA} = i_{xB}$; $E_{yA} = E_{yB} = 0$, from which follows $i_{yA} = i_{yB}$; $E_{xA} = E_{xB}/\alpha$. Hence the only effect of the inhomogeneity is a reduction of the electric field by a factor α.

Formation of an electrojet

Our results indicate that the usual theories of plasma currents perpendicular to a magnetic field very easily break down when applied to a non-homogeneous plasma. In the cases we have treated, the main effect *of a magnetic field is to concentrate most of the current to the region of highest conductivity.* Inside this region the magnetic field does not reduce the conductivity very much, but outside it, the current is small. Cf. also § 5.5.1.

5.4. Frozen-in lines of force

5.4.1. *Macroscopic model*

A magnetic field line (line of force) is defined as a line which at every point has the same direction as the magnetic field.

If a solid body with infinite electric conductivity is magnetized, the magnetic flux through the body is constant, because it follows from Ohm's law ($E = i/\sigma$) that $E \equiv 0$ and hence $\partial \mathbf{B}/\partial t \equiv 0$.

We may extend this picture to a fluid with infinite conductivity, if we consider the fluid to consist of a very large number of very small bodies. The magnetic field is frozen-in in all these bodies, and when the fluid moves, the magnetic field lines move with it as if the field lines were frozen-in in every mass element. (An alternative proof of this fact has already been given in § 3.9.) One may express this by saying that the field lines are 'materialized' by the fluid, which is 'glued' on them. (As we have seen in §§ 3.4 and 3.9, this picture is natural in the study of hydromagnetic waves.)

Let us as illustrations consider a few simple cases.

(1) Suppose that an infinitely conducting fluid is situated in a magnetic field, which is homogeneous at a certain instant $t = 0$. We introduce a Cartesian coordinate system, with the z-axis parallel to $\mathbf{B}$, and assume that a pillar of the fluid moves in the x-direction. Say, for simplicity, that the velocity is given by

$$v = \begin{cases} v_0 & (|z| < a, \quad |y| < d) \\ \dfrac{b-|z|}{b-a} v_0 & (a < |z| < b, \quad |y| < d) \\ 0 & (|z| > b, \quad |y| < d), \end{cases}$$

see Fig. 5.9. If a certain line of force $ABCDEF$ is a straight line at the instant $t = 0$, it is deformed to the shape $ABC_1 D_1 EF$ at a later instant, Δt. The deformation of the field lines is due to a superimposed magnetic field $\mathbf{b}$, which is produced by induced (surface) currents $\mathbf{i}$, see Fig. 5.9. The shape of the lines of force is the same as if they were elastic strings moving with the fluid (cf. Lundquist, 1952).

(2) Suppose next that, in the same geometrical configuration, we let the two regions $a < z < b$ be insulating slabs, whereas everywhere else we have an infinitely conducting, inviscid fluid moving with the velocity

$$v = \begin{cases} v_0 & (|z| < a, \quad |y| < d) \\ 0 & (|z| > b, \quad |y| < d), \end{cases}$$

see Fig. 5.10. In this case the insulating slabs prevent the establishing of current systems like those in Fig. 5.9. (This is still true if instead of the slabs we have insulating planes.) Therefore there is no induced magnetic field. The magnetic field remains homogeneous, and the magnetic lines of force remain straight lines. We can describe this situation

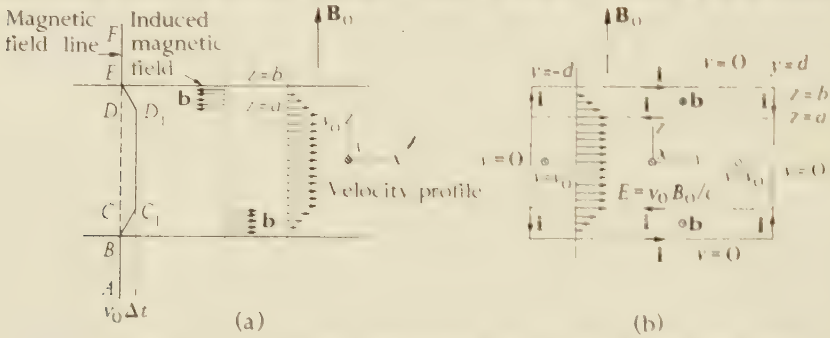


FIG. 5.9. When in an infinitely conducting fluid a pillar is set into motion perpendicular to **B**, the magnetic lines of force are deformed due to a superposed magnetic field **b** from induced electric currents. The configuration is shown projected on the zx plane (a) and on the yz plane (b).

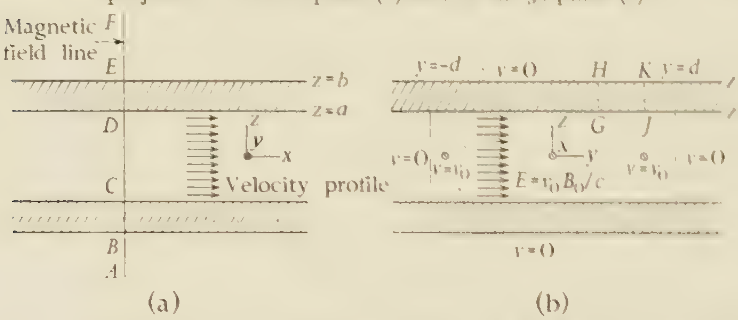


FIG. 5.10. When the moving pillar is separated from the fluid on either side by insulating slabs (or planes), no induced currents can flow, and the magnetic lines of force remain straight.

in two ways. We may say that the medium in the pillar moves across the lines of force under the influence of an electric field **E**. Alternatively we may say that the lines of force move with the pillar, but such a statement is possible only if we add that the insulating slabs 'cut the lines of force' in the sense that the lines of force lose their identity when they enter an insulating region (or pass through an insulating plane).

In a coordinate system at rest there is a potential difference between the points G and J but not between H and K. Therefore the potential difference V_{GH} must be different from V_{JK}, and so there exists in the insulator an *electric field* with a non-vanishing component *parallel to* **B**.

(3) If the conductivity is everywhere finite and isotropic, we have a case intermediate between (1) and (2). The magnetic lines of force are carried with the moving liquid at a reduced speed. They 'slip' through the medium (cf. § 3.9).

(4) If the conductivity is everywhere zero, there is no coupling between the material motion and the magnetic field. The lines of force remain undisturbed by any motion of the matter.

5.4.2. *Microscopic model. Low-density plasma*

Consider a plasma from the microscopic point of view. The individual particles drift according to the formulae given in Chapter 2. If the drifts are due to a magnetic gradient or non-electric forces, positive and negative particles drift in opposite directions. Under such circumstances the concept of moving lines of force is not very useful. We therefore limit the discussion to the case where non-electric forces are absent. In the presence of an electric field **E** both positive and negative particles drift perpendicular to **B** with the velocity $\mathbf{u}_\perp = c(\mathbf{E}\times\mathbf{B})/B^2$.

We can rewrite this as

$$\mathbf{E}_\perp + (\mathbf{u}/c)\times\mathbf{B} = 0. \tag{1}$$

As long as

$$\mathbf{E}_\parallel = 0, \tag{2}$$

(1) implies

$$\mathbf{E} + (\mathbf{u}/c)\times\mathbf{B} = 0. \tag{3}$$

Then we can directly apply the proof given in § 3.9, and it follows that the magnetic flux through a contour moving with the velocity $\mathbf{u}_\perp$ is conserved.

If there exist electric fields parallel to **B**, (3) does not follow from (1), the proof in § 3.9 does not apply, and the magnetic lines of force need not be frozen-in.

As shown in § 5.1.3, a non-vanishing $\mathbf{E}_\parallel$ is possible in low-density plasmas. If $\mathbf{E}_\parallel$ is different along different lines of force, one can have a situation analogous to that shown in Fig. 5.10. Hence, *in low-density plasmas the concept of frozen-in lines of force is questionable.*

The concept of frozen-in lines of force may be useful in solar physics, where we have to do with high- and medium-density plasmas (cf. § 5.1.4), but may be grossly misleading if applied to the magnetosphere of the earth. To plasma in interplanetary space it should be applied with some care.

5.5. Production of plasma inhomogeneities

Most theoretical investigations of cosmical plasmas have been devoted to the study of homogeneous plasmas. However, observations indicate that most cosmical plasmas are strongly inhomogeneous. In the ionosphere a small-scale structure is often observed, most pronounced in connexion with aurorae. Auroral rays are often very thin, and the degree of ionization and hence the conductivity may vary by two or three powers of ten within a few kilometres or less. Also the magnetosphere has probably a filamentary structure as indicated by the study of whistler propagation. Similarly, the solar atmosphere has a ray structure. On good eclipse photographs the ray structure of the corona is often very pronounced, and radio observations indicate that the rays continue at least out to 10 or 20 solar radii ('supercorona'). Closer to the surface of the sun we observe prominences, which very often have a threadlike structure. The chromosphere is often regarded as a filamentary network of miniature prominences. In gas nebulae filamentary structures are often conspicuous.

Hence medium-density plasmas (and perhaps also low-density plasmas) seem very often to be strongly inhomogeneous, exhibiting a filamentary structure which often may be parallel to the magnetic field. Therefore it is of basic importance to discuss mechanisms which are capable of producing a filamentary structure. A survey of such mechanisms is given in § 5.5.1.

5.5.1. *Constriction of discharges*

A discharge may fill all the space between the electrodes or it may be confined to a narrow channel. The former is the case for a glow discharge at low pressure. Examples of the latter case are arcs, sparks, flashes of lightning and the 'pinched discharges' studied in thermonuclear research.

In laboratory discharges, especially at a low degree of ionization, the constriction is often connected with a 'falling characteristic', by which we mean that the electric field necessary to maintain the discharge is a decreasing function of the current density. If the total discharge current is given, the electric field strength becomes smaller when the current concentrates to a small channel than when it fills the whole space. A discharge often adjusts itself so that the electric field strength becomes a minimum.

The constriction of a discharge is a phenomenon which is influenced by several different factors (see von Engel and Steenbeck, **2**, 1934, 138; and Cobine, 1941, p. 317).

A *thermal constriction mechanism* is operative in arcs.

The arc plasma does not obey the similarity laws of § 4.2.2. Instead it obeys rather complicated laws, the essence of which is that the heat produced by the electric current shall cover the thermal losses, mainly through convection and conduction, to the cold surroundings (see Cobine, p. 317). If we change a discharge according to the similarity laws of § 4.2.2, the conditions for thermal constriction become less favourable the lower the pressure (and hence the larger the dimensions). This is mainly due to the fact that the power Ei developed per unit volume varies as γ^{-3}, and hence becomes small when we go to larger dimensions and lower pressures. It is a general experience that at atmospheric pressure discharges are usually constricted, whereas at pressure below, say, 1 mm Hg, constriction becomes a more rare phenomenon. (It should be observed that even at low pressure constriction may occur if certain gases (e.g. CO_2) are present. This constriction has nothing to do with the thermal effect discussed above.)

The mechanisms mentioned above operate independently of the presence of a magnetic field. However, the *magnetic effects* may give rise to constrictions of different types.

Currents *perpendicular to* **B** may be constricted. Examples of such phenomena in the ionosphere are the *equatorial electrojet and the auroral electrojet.* According to Singer and others the equatorial electrojet is produced by an interplay between magnetic and gravitational forces. As found in § 5.3.3, a magnetic field in an inhomogeneous plasma may cause a contraction of a current perpendicular to it so that a jet current is produced. This may be the explanation of the auroral electrojet.

The most important constriction mechanism is the *electromagnetic attraction* between parallel currents. A manifestation of this mechanism is the *pinch effect,* which was studied by Bennett long ago (Bennett, 1934), and has received much attention in connexion with thermonuclear research. As we shall see, phenomena of this general type should exist also on a cosmical scale and lead to a bunching of currents and magnetic fields to filaments. This bunching is also accompanied by an accumulation of matter, and it may explain the observational fact that cosmic matter exhibits an abundance of *filamentary structures.*

5.5.2. *The pinch effect. The Bennett relation*

Consider a cylindrical fully ionized plasma column in an axial electric field **E** which produces an axial current, Fig. 5.11. The axial current is associated with an azimuthal magnetic field. The current flowing across its own magnetic field exerts a force $(\mathbf{i}/c)\times\mathbf{B}$, which is directed radially inward, and causes the plasma to be compressed towards the axis (hence the name 'pinch effect'). In equilibrium between the compressing electromagnetic force and the pressure of the plasma we have

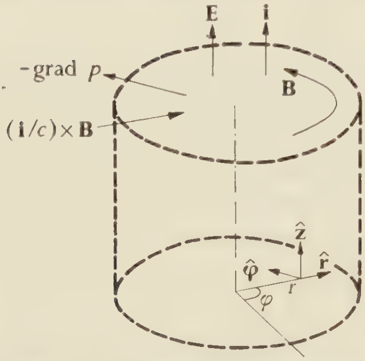


Fig. 5.11. In the stationary pinched discharge the electromagnetic force $(\mathbf{i}/c)\times\mathbf{B}$ (attraction between parallel currents) is compensated by the gradient of the plasma pressure.

$$-\frac{dp}{dr} = -\frac{d}{dr}(p_e+p_i) = \frac{iB}{c}. \tag{1}$$

The axial electric field is perpendicular to the azimuthal magnetic field produced by the axial current, Fig. 5.11. In spite of this the current density is essentially given by

$$i = E/\eta = \sigma E \tag{2}$$

(apart from the correction 5.3.1 (18)) as soon as a stationary state has been reached. This is so because in the stationary state no radial Hall current can flow. Compare the discussion in § 5.3.1.

Combining (1) with the Maxwell equation

$$\frac{1}{r}\frac{d}{dr}(rB) = \frac{4\pi}{c}i \tag{3}$$

we find

$$\int r^2\frac{dp}{dr}\,dr = -\int \frac{rB}{4\pi}\,d(rB). \tag{4}$$

The left-hand side can be integrated by parts:

$$\int\limits_0^R r^2\frac{dp}{dr}\,dr = [r^2p]_{r=R}-[r^2p]_{r=0}-\int\limits_0^R 2pr\,dr, \tag{5}$$

R being the discharge radius. Outside the discharge the pressure is zero, and so the integrated parts vanish. If the temperatures T_e and T_i are constant, we can write the last term of (5) in the form

$$-\int 2pr\,dr = -\frac{1}{\pi}\int nk(T_e+T_i)2\pi r\,dr$$
$$= -\frac{k(T_e+T_i)}{\pi}\int n2\pi r\,dr = -\frac{Nk(T_e+T_i)}{\pi}, \tag{6}$$

where N is the number of electrons per unit length of the column.

Combining (4), (5), and (6) we have

$$2Nk(T_e+T_i) = (RB)^2/4. \tag{7}$$

On the other hand, the total current is, according to (3),

$$I = \int_0^R i2\pi r\,dr = \tfrac{1}{2}c(RB), \tag{8}$$

so that (7) can be written

$$2Nk(T_e+T_i) = (I/c)^2. \tag{9}$$

Equation (9) is the *Bennett relation.*

When the temperature is roughly constant, the same is true for the conductivity and the current density i. Then B is proportional to r, and it follows from (1) that the density has a parabolic distribution,

$$n = n_0 - \text{const}\; r^2. \tag{10}$$

5.5.3. *Filamentary currents in force-free magnetic fields*

In an ordinary pinched discharge the electric current can flow across its own magnetic field because the electromagnetic forces are balanced by the plasma pressure, and the radial motion vanishes. In *cosmical plasmas* the pressure is often negligible and hence the magnetic field force-free. (Lundquist, 1950, cf. § 3.12.1.)

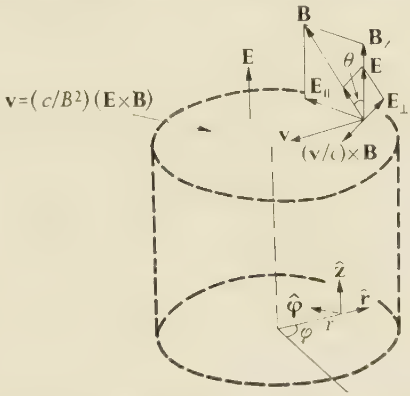


Fig. 5.12. When the pressure is negligible, the plasma acquires a drift velocity **v** such that the electric field in the moving plasma is parallel to **B**. Therefore current flows only along the magnetic lines of force.

Consider a medium-density plasma of the configuration shown in Fig. 5.12. The electric field is homogeneous and directed along the z-axis, and the magnetic field, which may derive partly from currents in the plasma and partly from external sources, has both φ- and z-components.

Under the influence of the electric and magnetic fields both electrons and ions drift with the velocity

$$\mathbf{v} = (c/B^2)(\mathbf{E}\times\mathbf{B}) \tag{11}$$

so that the plasma as a whole moves radially inward.

The electric field in a local coordinate system moving with the plasma is

$$\mathbf{E}' = \mathbf{E} + (\mathbf{v}/c)\times\mathbf{B}. \tag{12}$$

According to the generalized Ohm's law the electron pressure (and, if the plasma were only partly ionized, also the friction against neutral gas) contributes to the 'effective' electric field, but in the present context we assume this contribution to be negligible. It follows from (11) and (12) that

$$\mathbf{E}' = \mathbf{E}+(\mathbf{B}/B)(\mathbf{EB})-\mathbf{E} = \mathbf{E}_{\parallel}, \tag{13}$$

cf. Fig. 5.12. This means that the moving plasma 'feels' only an electric field parallel to **B**, and consequently current flows only along the magnetic field. As the magnetic force per unit volume is $(\mathbf{i}/c)\times\mathbf{B}$, it vanishes in the present situation. The magnetic field is force-free (cf. § 3.12.1). This is a consequence of the assumption that the pressure is negligible (Lundquist, 1950).

The inward drift motion causes an accumulation of matter near the axis. If the accumulation process were to go on for a sufficiently long time, the pressure would cease to be negligible and the situation would finally approach that in the ordinary pinch, which we have already discussed. Here we shall analyse the force-free state which is of interest in the cosmical applications.

Since the vectors **i** and **B** are parallel, we have

$$\frac{i_\varphi}{i_z} = \frac{B_\varphi}{B_z} = \tan\theta = K, \tag{14}$$

where θ is the angle between the magnetic field and the z-direction. θ and K are functions of the radius r. In terms of the electric field and the conductivity, which we assume constant, the current density and its components can be written as follows:

$$i = (i_\varphi^2+i_z^2)^{\frac{1}{2}} = \sigma E\cos\theta = i_0\cos\theta, \tag{15}$$

$$i_z = i_0\cos^2\theta, \tag{16}$$

$$i_\varphi = i_0\cos\theta\sin\theta. \tag{17}$$

On the other hand, the components of the Maxwell equation

$$\operatorname{curl}\mathbf{B} = \frac{4\pi}{c}\mathbf{i} \tag{18}$$

can be integrated over r to give

$$B_\varphi = \frac{4\pi}{c}\frac{1}{r}\int_0^r i_z r\,dr \tag{19}$$

and

$$B_z = B_0-\frac{4\pi}{c}\int_0^r i_\varphi\,dr. \tag{20}$$

B_0 is the field strength at the axis.

If (14), (16), and (17) are combined with (19) and (20), the result is

$$\frac{4\pi i_0}{Kr}\int_0^r \frac{r\,dr}{1+K^2}+4\pi i_0\int_0^r \frac{K\,dr}{1+K^2} = B_0 c. \tag{21}$$

It is suitable to introduce the length unit

$$b = \frac{B_0 c}{4\pi i_0} \tag{22}$$

and the dimensionless variable

$$x = r/b. \tag{23}$$

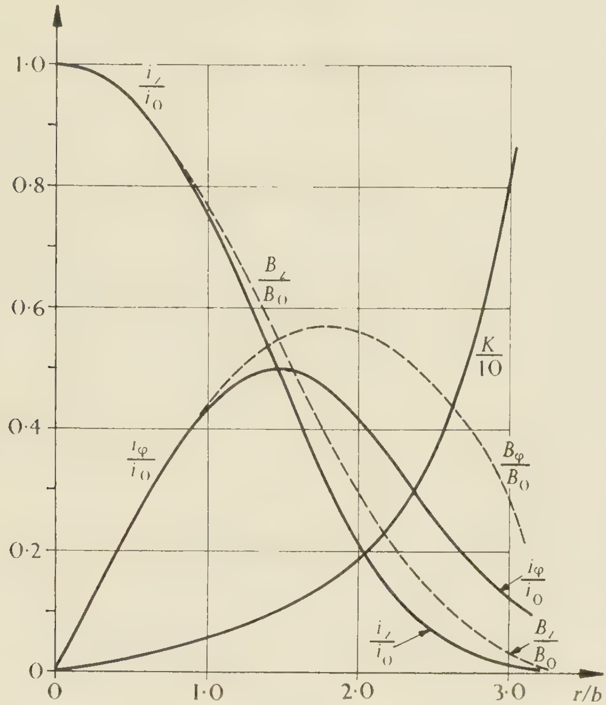


FIG. 5.13. The current density components i_φ and i_z, the magnetic field components B_φ and B_z, and the quantity $K = \tan\theta$ as functions of r/b.

If this is done, (21) becomes

$$\frac{1}{Kx}\int\limits_0^x \frac{x\,dx}{1+K^2} + \int\limits_0^x \frac{K\,dx}{1+K^2} = 1. \tag{24}$$

This integral equation has been solved numerically by Murty (1961). Fig. 5.13 shows K, i_φ, i_z, B_φ, and B_z as functions of r/b.

As the figure shows, the distribution of the axial magnetic field is essentially limited to a cylinder with a radius of the order of magnitude of b. Similarly, the current is 'constricted' to such a cylinder and forms a 'line current'. At radii much larger than b, B_z goes towards zero more rapidly than B_φ, and so the distant magnetic field has almost only a φ-component. (A magnetic field with a finite z-component at infinity is not compatible with a finite value of B_z at the axis.) Fig. 5.14 shows qualitatively the general form of the field-line pattern, which suggests for itself the name of 'magnetic rope'.

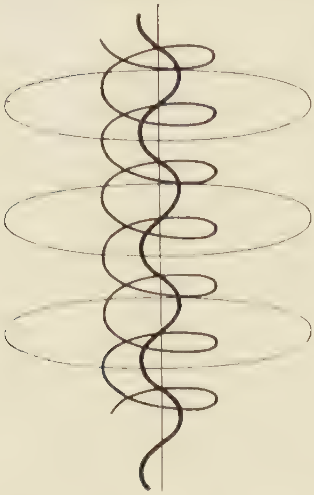


FIG. 5.14. General pattern of magnetic field lines in a 'magnetic rope'.

The total magnetic flux in the z-direction is

$$\phi = \int_0^\infty B_z\, 2\pi r\, dr = \gamma\pi b^2 B_0. \tag{25}$$

The constant γ comes out of the numerical calculations and is approximately 3·1.

Equations (22) and (25) allow us to express B_0 and b in terms of the total flux ϕ and the maximum current density i_0. We find

$$B_0 = \left(\frac{16\pi}{\gamma c^2}\, i_0^2\, \phi\right)^{\frac{1}{3}} \tag{26}$$

and

$$b = \left(\frac{\phi c}{4\pi^2 \gamma i_0}\right)^{\frac{1}{3}}. \tag{27}$$

The total current is given by

$$I = \int_0^\infty i_z\, 2\pi r\, dr = \kappa\pi b^2 i_0 \tag{28}$$

with $\kappa \approx 2{\cdot}6$, and it can be rewritten

$$I = \frac{\kappa}{\pi^{\frac{1}{3}}}\left(\frac{c\phi}{4\gamma}\right)^{\frac{2}{3}} (\sigma E)^{\frac{1}{3}}, \tag{29}$$

which is an unfamiliar relation between electric field strength and current.

In the present treatment we have assumed a constant conductivity, but Murty (1961) has shown that similar results hold for force-free fields under much more general conditions.

As mentioned above the electric and magnetic fields cause an inward drift motion by which matter is accumulated. 'Magnetic ropes' should therefore tend to coincide with material filaments that have a somewhat higher density than the surroundings. The cosmic 'magnetic ropes' are not observable themselves, but the associated filaments of condensed matter can be observable by the radiation they emit and absorb. The constriction mechanism discussed here may therefore explain the filamentary structures that are observed to occur so abundantly in cosmical plasmas (filamentary interstellar clouds, coronal filaments, prominences, etc.). The application to solar physics is of particular interest (Alfvén, 1961*a*, *b*, 1962).

5.6. Diffusion in a magnetic plasma

In order to discuss in a simple way the essentials of diffusion we shall again use the *mean-free-path method*. Although this is not very adequate in fully ionized plasmas, where the particles interact by long-range forces, it still gives approximately correct results if the collision time used is the effective collision time, which includes the effect of distant collisions, see § 4.3.2.

In the time intervals between collisions the motion of the charged particles is a superposition of a rapid gyration and a slow drift across the magnetic field. When the temperature of a certain kind of particle is T_k, the individual particle has on the average a gyration velocity

$$w_{k\perp} = (2kT_k/m_k)^{\frac{1}{2}} \tag{1}$$

and a velocity

$$w_{k\parallel} = (kT_k/m_k)^{\frac{1}{2}} \tag{2}$$

parallel to the lines of force.

5.6.1. *Diffusion parallel to the magnetic field*

Let the magnetic field be directed parallel to the z-axis of a Cartesian coordinate system. The presence of the magnetic field leaves the collision time τ_k essentially unaffected. The mean displacement in the z-direction between two collisions has the value

$$\lambda_{k\parallel} = w_{k\parallel}\tau_k \tag{3}$$

or, according to (2),

$$\lambda_{k\parallel} = \tau_k(kT_k/m_k)^{\frac{1}{2}}. \tag{4}$$

The quantity $\lambda_{k\parallel}$ agrees with the mean displacement in the z-direction in the absence of a magnetic field. Hence, if we denote by $D_{\parallel k}$ the diffusion coefficient for diffusion parallel to **B** and by D_k the diffusion coefficient in the absence of a magnetic field, we have

$$D_{\parallel k} \approx D_k. \tag{5}$$

5.6.2. *Diffusion across the magnetic field*

The diffusion normal to the magnetic field lines is different in high-density plasmas and in medium- or low-density plasmas.

During an average collision interval the particle gyrates an angle of $\omega_k\tau_k$ radians. In *high-density* plasmas $\omega_k\tau_k$ is very small, and the path between collisions is nearly straight. The diffusion is then nearly unaffected by the magnetic field so that

$$D_{\perp k} \approx D_{\parallel k} \approx D_k, \qquad \omega_k \tau_k \ll 1, \tag{6}$$

where $D_{\perp k}$ is the coefficient of diffusion perpendicular to $\mathbf{B}$, and D_k the diffusion coefficient in the absence of a magnetic field.

In *medium-* and *low-density* plasmas $\omega_k \tau_k$ is large. The particle spirals many turns before it collides, and (apart from possible drift motion) its centre of gyration remains fixed to a field line between collisions. When a collision takes place, the centre of gyration is displaced from one field line to another. The displacement is given by 2.2.2 (18).

It is important to notice that collisions between particles of the same kind do not contribute to the diffusion transverse to the magnetic field. This is so because the total momentum $\mathbf{p}_1 + \mathbf{p}_2$ is conserved in the collision. Therefore the momentum changes $\Delta\mathbf{p}_1$ and $\Delta\mathbf{p}_2$ are of equal magnitude but opposite direction. If the particles have the same masses and charges, it follows from 2.2.2 (18) that the gyration centres are also displaced by equal and opposite amounts. As a result the like-particle collisions do not contribute to the transverse diffusion.†

Since the average momentum change in a collision is of the same order of magnitude as the original momentum, the displacement is of the order of magnitude of the Larmor radius

$$\rho_k = \frac{m_k w_{k\perp} c}{|e_k| B} = w_{k\perp} \omega_k^{-1}. \tag{7}$$

Thus we have now a two-dimensional random walk with steps of length ρ_k taken at intervals τ_k. In the absence of a magnetic field the diffusion perpendicular to z is a random walk characterized by steps $w_{k\perp} \tau_k$ taken at intervals τ_k. As the diffusion coefficient is proportional to the square of the step-length (see, for example, Kennard, 1938, p. 286), we have

$$D_{\perp k} \approx D_k (w_{k\perp} \omega_k^{-1} / w_{k\perp} \tau_k)^2 = \frac{D_k}{\omega_k^2 \tau_k^2}, \tag{8}$$

when $\omega_k \tau_k \gg 1$.

The formulae (6) and (8) are limits of the more general expression

$$D_{\perp k} = \frac{D_k}{1 + \omega_k^2 \tau_k^2}, \tag{9}$$

see, for example, Chapman and Cowling, 1939, p. 325, or Lehnert, 1962*b*.

† If higher-order effects are taken into account, a small contribution from like-particle collisions is found (Spitzer, 1956; Simon, 1955; Longmire and Rosenbluth, 1956).

Since $\omega_e \tau_e \gg \omega_i \tau_i$, the magnetic field reduces the transverse diffusion coefficient much more for electrons than for ions, so that in a strong magnetic field we can have

$$D_{\perp e} \ll D_{\perp i}, \tag{10}$$

although at the same time

$$D_{\parallel e} \gg D_{\parallel i}. \tag{11}$$

5.6.3. *Ambipolar diffusion*

As discussed in § 4.4, the different diffusion rates for positive and negative charged particles in a plasma lead to a charge separation and an electric field, which slows down the fast-diffusing particles and accelerates the slow-diffusing ones until both kinds of particles diffuse with the same speed. Formulae were given for the electric fields and currents resulting from this ambipolar diffusion. Analogous phenomena occur in a magnetized plasma, although there one must distinguish between the diffusion in different directions. To diffusion along the magnetic field the formulae of § 4.4 are directly applicable. For the transverse diffusion they are also applicable provided the values used for the diffusion coefficients and conductivities are $D_{\perp k}$, given by (9), and σ_P^k, given by 5.3 (5).

5.6.4. *Diffusion in medium- and low-density plasmas by encounters with magnetic irregularities*

A role much like that of collisions between particles can be played by encounters of charged particles with magnetic irregularities. As a particle passes through a small-scale magnetic irregularity its guiding centre is displaced. This was discussed in § 2.2.2, where a formula for the displacement was given. Although the irregularity taken as an example there extended uniformly along the magnetic field, the irregularities have in practice a limited extent along the field, and so the motion of the particle parallel to the field lines saves it from colliding incessantly with the same irregularity. If irregularities are distributed at random in the magnetic field, the particle will encounter new irregularities at random in the course of its motion along the lines of force. Consequently the individual guiding centres perform a kind of random walk perpendicular to the magnetic field, and for a group of charged particles this means a diffusion across the magnetic field. When the magnetic fields are very tangled, one can expect the motion of the guiding centres to have the character of a diffusion in all three directions of space.

Diffusion by interaction with magnetic irregularities is of particular interest in cosmical physics, where in many cases the collisions between particles are much too rare to produce any diffusion. For example, interplanetary space is permeated by magnetic fields which should contain irregularities of linear dimensions of a small scale compared to the Larmor radii of high-energy charged particles. The motion of these particles has therefore the character of diffusion. (The ordinary collisions are negligible in this context.) This diffusion, together with a simultaneous acceleration of the kind discussed in § 2.7, may explain the observed momentum spectrum of cosmic radiation.

5.6.5. *Anomalous diffusion*

According to the classical theory the diffusion coefficient for transverse diffusion (equation (9)) decreases rapidly with $\omega\tau$ when $\omega\tau \gg 1$, and this means that a plasma situated in a magnetic field should be efficiently prevented from escaping by diffusion transverse to the field lines. However, ordinary diffusion is not the only way for the plasma to escape. If electric fields are set up in the plasma by charge separation, both positive and negative particles obtain a drift velocity $(c/B^2)\mathbf{E}\times\mathbf{B}$ by which they may under certain conditions escape rapidly across the magnetic field.

The existence of a diffusion by random particle drift due to fluctuating electric fields in a plasma was suggested by Bohm (1949). This so-called 'drain diffusion' was to be characterized by a $1/B$-variation of the diffusion coefficient, and according to theoretical work by Spitzer (1960) and Taylor (1962) such mechanisms exist in non-thermal plasmas.

Extensive experiments have shown that under certain circumstances another kind of enhanced diffusion occurs (Lehnert, 1958; for recent reviews of experiments and theory see Lehnert, 1962*b*, and Hoh, 1962). When the enhanced diffusion sets in, the plasma takes on a helical shape, and according to theories by Kadomtsev and Nedospasov (1960) and Hoh and Lehnert (1961) the enhanced diffusion is due to a separation between an ion helix and an electron helix. This produces an azimuthal electric field and a radial $\mathbf{E}\times\mathbf{B}$ drift by which particles escape abundantly. The experimental configuration has very little in common with cosmical situations but the instability mechanism invoked to explain it is of a character that may well be of great interest in cosmical physics. The 'spiral instability' and the closely related 'flute instability' (Rosenbluth and Longmire, 1957; Lehnert, 1962*c*, 1963*a*, *b*) are examples of mechanisms by which a thin plasma can escape freely across the magnetic field without deforming the latter. Related phenomena may be important also in connexion with the behaviour of plasmas in cosmic magnetic fields.

5.7. Magnetic properties of plasmas

5.7.1. *Plasmas in thermodynamic equilibrium*

According to formula 2.2.1 (11) a solitary charged particle moving in a homogeneous magnetic field **B** with a velocity component $w_\perp$ perpendicular to the field has a magnetic moment μ, which has the value

$$\mu = \frac{mw_\perp^2}{2B} = \frac{W_\perp}{B} \tag{1}$$

and is directed opposite to **B**. The same is true for each individual particle in a magnetized plasma as long as the mean free path λ is much larger than the Larmor radius ρ.

Although the charged particles in the plasma have their magnetic moments directed opposite to the magnetic field, it is not legitimate to conclude generally that the plasma as a whole is diamagnetic. For example, it is well known from the classical theory of magnetic properties of metals that no resultant diamagnetism is caused by the conduction electrons in a metal. Bohr (1911) has interpreted this physically by considering the influence of the boundaries of the metal, considered as perfectly reflecting walls, see Fig. 5.15.

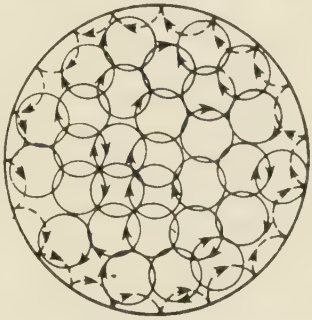


FIG. 5.15. Electron gas enclosed by perfectly reflecting walls.

Similarly, a hypothetical *fully ionized plasma confined by perfectly reflecting walls* has no magnetic effect at all. In terms of Bohr's interpretation the motion of the electrons that are reflected by the walls represents a current, and the magnetic moment of this current cancels exactly the magnetic moment of the electrons spiralling in the interior. This is illustrated in Fig. 5.16. This consideration can easily be verified quantitatively by means of a microscopic description of the plasma, where we consider the gyration-centre of the particle as an *equivalent magnet* carrying both electric charge and magnetic moment. If the cylinder cross-section is S, the magnetic moment, per unit length, of the interior electrons and ions is

$$\begin{aligned} M &= -Sn_e m_e w_{e\perp}^2/2B - Sn_i m_i w_{i\perp}^2/2B \\ &= -Snk(T_e+T_i)/B. \end{aligned} \tag{2}$$

A certain number of electrons, say N_e, strike the wall per unit time and unit length of the circumference, cf. Fig. 5.16. Since they exert the force

p_e on the wall, the reaction force per electron is p_e/N_e and the average drift of their gyration centres is, according to 2.3.4 (39),

$$u_{e\perp} = -\frac{cp_e}{|e|N_e B},$$

which corresponds to a wall current (of equivalent magnets with charge $-|e|$). This wall current is given by

$$I_{ew} = -|e|N_e u_{e\perp} = \frac{cp_e}{B} = \frac{cnkT_e}{B},$$

and consequently it corresponds to a magnetic moment

$$M_{ew} = (S/c)I_{ew} = SnkT_e/B. \tag{3}$$

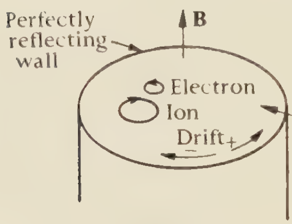


FIG. 5.16

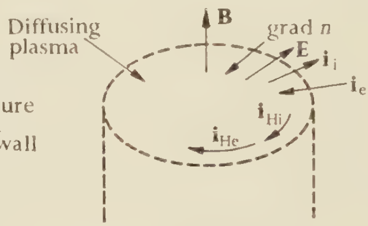


FIG. 5.17

FIG. 5.16. Completely ionized plasma confined by perfectly reflecting wall. The magnetic moment of the particles gyrating in the interior is compensated by the magnetic moment associated with the drift of particles hitting the wall.

FIG. 5.17. During ambipolar diffusion of the plasma a Hall diffusion current flows, which gives a resultant magnetic effect.

Fully analogous considerations apply to the ions, and so the total magnetic moment of the wall current, including both electrons and ions is

$$M_w = Snk(T_e+T_i)/B. \tag{4}$$

This cancels exactly the magnetic moment of the particles in the interior.

From the *macroscopic* point of view there is no magnetic effect in the example just considered, because the macroscopic current density vanishes everywhere. In the interior the electrons and ions are uniformly distributed. The resultant current density is zero at every point, because the distribution of velocities, for each kind of particle, is isotropic. The isotropy, and hence absence of currents, holds even close to the wall, because this is perfectly reflecting (cf. Fig. 5.15). As isotropy is a consequence of thermodynamic equilibrium, it follows quite generally that a plasma in thermodynamic equilibrium has no resultant magnetic effect. This was pointed out by Cowling (1929) in connexion with a discussion on the radial limitation of the solar magnetic field.

On the other hand, as a single particle produces a diamagnetic moment, a plasma consisting of a multitude of particles should also be diamagnetic. This is true when the plasma is confined by a magnetic field instead of material walls. Then there are no guiding-centre currents at the wall to compensate the magnetic moment produced by the particles gyrating in the interior. From the *macroscopic* point of view: the plasma pressure is balanced by a current at the boundary.

5.7.2. *Diffusing, weakly ionized, plasma*

Consider a cylindrical plasma column situated in a magnetic field parallel to the cylinder axis. Let the ions and electrons be present only inside a certain radius and have a number density much smaller than that of neutral molecules (low degree of ionization). The neutral molecules are present also outside the ionized region.

The difference between this situation and that with the completely ionized plasma supported by a perfectly reflecting wall is that the pressure of the ions and electrons is now taken up partly by the magnetic field and partly by collisions with the neutral gas. As before there are two alternative ways to look upon the situation.

(1) The *microscopic* method, where we consider the guiding centres as electrically charged equivalent magnets and take into account both their intrinsic magnetic moments and the magnetic moment of the electric currents associated with their motion.

(2) The *macroscopic* method, where we analyse the macroscopic current, which, as we shall see, can be interpreted as a Hall diffusion current. The existence of a Hall diffusion current is an indication of the lack of thermodynamic equilibrium.

In this section we shall carry through only the macroscopic analysis and we shall use the formulae derived in §§ 5.2 and 5.3.

The density gradients of ions and electrons cause a radial diffusion. We assume that the diffusion is ambipolar, and that the ions are singly ionized so that $e_\mathrm{i} = -e_\mathrm{e} = |e|$, and $n_\mathrm{i} = n_\mathrm{e} = n$. According to § 5.6.3, the radial ionic and electronic current densities are then given by 4.4 (7) if the diffusion coefficients 5.6.2 (9) are inserted. Thus the radial current densities, counted positive in the outward direction, are

$$i_\mathrm{i} = -i_\mathrm{e} = -\frac{(T_\mathrm{e}+T_\mathrm{i})D_{\mathrm{e}\perp}D_{\mathrm{i}\perp}}{T_\mathrm{i}D_{\mathrm{e}\perp}+T_\mathrm{e}D_{\mathrm{i}\perp}}|e|\frac{dn}{dr}. \tag{5}$$

As shown in §§ 5.2 and 5.3, the radial electronic and ionic currents (5) are associated with Hall currents in the azimuthal direction. The strengths of these currents are, according to 5.3 (7) and (8), $-\omega_e \tau_e i_e$ and $\omega_i \tau_i i_i$. The total Hall current, counted positive counterclockwise in Fig. 5.17, is then

$$i_H = -\omega_i \tau_i i_i + \omega_e \tau_e i_e = (\omega_i \tau_i + \omega_e \tau_e) \frac{(T_e + T_i) D_{e\perp} D_{i\perp}}{T_i D_{e\perp} + T_e D_{i\perp}} |e| \frac{dn}{dr}, \tag{6}$$

where

$$D_{e\perp} = \frac{kT_e \tau_e}{m_e} \frac{1}{1+\omega_e^2 \tau_e^2}, \tag{7}$$

$$D_{i\perp} = \frac{kT_i \tau_i}{m_i} \frac{1}{1+\omega_i^2 \tau_i^2}, \tag{8}$$

$$\omega_e = \frac{|e|B}{m_e c}, \tag{9}$$

$$\omega_i = \frac{|e|B}{m_i c}, \tag{10}$$

so that

$$i_H = \frac{ck(T_e + T_i)}{B} f(\omega_e \tau_e, \omega_i \tau_i) \frac{dn}{dr}, \tag{11}$$

where

$$f(\omega_e \tau_e, \omega_i \tau_i) = \frac{\omega_e \tau_e \omega_i \tau_i}{1+\omega_e \tau_e \omega_i \tau_i}. \tag{12}$$

Although the charged-particle density n varies with the radius, the collison times τ_e and τ_i are constant, because in a weakly ionized plasma they are determined by collisions with the neutral gas, which has a nearly constant density. In the following we shall assume that the temperatures T_e and T_i are constant and that the charged-particle pressure $p_e + p_i$ is small compared with the magnetic pressure $B^2/8\pi$. The latter condition implies that the relative variation in B is small so that we can consider ω_e and ω_i as nearly constant. Then (11) can be integrated radially to give the total Hall current

$$I_H = -\frac{cnk(T_e + T_i)}{B} f(\omega_e \tau_e, \omega_i \tau_i). \tag{13}$$

From the Maxwell equation

$$\frac{\partial B}{\partial r} = -\frac{4\pi}{c} i \tag{14}$$

we find that the magnetic field in the centre of the plasma, B_I, is weaker than that outside, B_E, and the difference is

$$B_E - B_I = -\frac{4\pi}{c} I_H = \frac{4\pi}{B} nk(T_e + T_i) f(\omega_e \tau_e, \omega_i \tau_i). \tag{15}$$

At high collision frequencies, $\omega\tau \ll 1$, the difference has the small value

$$B_E - B_I = \frac{4\pi}{B} nk(T_e + T_i)\omega_e \tau_e \omega_i \tau_i, \tag{16}$$

and in the nearly collision-free case $\omega\tau \gg 1$ the difference has its maximum value

$$B_E - B_I = \frac{4\pi}{B} nk(T_e + T_i). \tag{17}$$

The special case (17) is evidently a direct consequence of the fact that when collisions are rare, the whole plasma pressure $nk(T_e + T_i)$ must be taken up by a difference in the magnetic pressure

$$nk(T_e + T_i) = \frac{B_E^2}{8\pi} - \frac{B_I^2}{8\pi} \approx \frac{B_E(B_E - B_I)}{4\pi}. \tag{18}$$

Note that in the macroscopic description we are considering the plasma as a vacuum in which particles move, not as a diamagnetic medium. Therefore $B = H$.

Experimental data concerning the magnetic properties of weakly ionized plasmas are not abundant. The fact that flames are diamagnetic has been known since the time of Faraday and attracted some interest in the nineteenth century, as mentioned in old handbooks (e.g. Graetz, 1915). Rather recently the spin-resonance technique has been used for studying plasmas immersed in magnetic fields (Marshall, Kawcyn, and Goldstein, 1960). The authors find that their results agree with formula (15) within 10 per cent and are also in agreement with older measurements performed by Steenbeck (1936).

5.7.3. *Magnetically confined, completely ionized plasma*

Finally we consider a magnetically confined, completely ionized plasma. Such a plasma is not in thermodynamic equilibrium and collisions between particles cause it to spread from the original confinement volume. However, when the temperature is high, the collisions (which are of Coulomb type) become so rare that they allow the plasma to be confined for long times. The limitation to the confinement time is often set not by collisions but by instabilities.

Let us, therefore, for simplicity assume that no collisions occur. On the other hand, we do not put any restriction on the magnitude of the plasma pressure in relation to the magnetic pressure. Then 5.2.3 (33) reads

$$0 = -\mathrm{grad}(p_e + p_i) + (\mathbf{i}/c) \times \mathbf{B}, \tag{19}$$

and combined with the Maxwell equation

$$\operatorname{curl}\mathbf{B} = \frac{4\pi}{c}\mathbf{i} \tag{20}$$

it gives the *magnetostatic* equilibrium equation (cf. § 3.12)

$$\operatorname{grad}(p_e+p_i) = \frac{1}{4\pi}(\operatorname{curl}\mathbf{B})\times\mathbf{B}. \tag{21}$$

In order to consider a simple configuration similar to that discussed for weakly ionized plasmas we assume cylindrical symmetry and a magnetic field that is everywhere parallel to the cylinder axis. Then (21) takes the simple form

$$\frac{d}{dr}\left(p_e+p_i+\frac{B^2}{8\pi}\right) = 0, \tag{22}$$

which means that the generalized pressure $p_e+p_i+B^2/8\pi$ is constant. If the magnetic field strength is B_I in the interior of the plasma, where the pressure is p_e+p_i, or $nk(T_e+T_i)$, it is related to the external field strength B_E by the equation

$$\frac{B_I^2}{8\pi} = \frac{B_E^2}{8\pi}-(p_e+p_i) = \frac{B_E^2}{8\pi}-nk(T_e+T_i). \tag{23}$$

Thus in this case, too, the magnetic field is weaker inside the plasma than outside.

5.8. Expanding stream of magnetic plasma

It was shown in Chapter 2 that in the presence of an electric field it is possible for low-energy charged particles to move across a magnetic field and also penetrate deep into the geomagnetic dipole field (see Fig. 2.11).

When a multitude of particles are present and form a plasma, an electric field can be produced by polarization of the plasma by a slight separation of the positive and negative charges. The motion of plasma in magnetic fields is of basic importance in the theory of magnetic storms and aurorae, because these phenomena are produced by magnetic plasma streams, which originate at the sun, cross interplanetary space, and approach the geomagnetic dipole field.

We shall devote the present section to discussing in detail some properties of magnetic plasma streams. In order to avoid unnecessary complication we base the discussion on a very simple model (§ 5.8.1), which does not pretend to simulate the geometric structure of the actual solar streams, but which is sufficient for discussing some fundamental phenomena. The emphasis is on the transformation of the electric and magnetic fields and their sources. Even though the stream moves with a velocity much smaller than the velocity of light it is necessary to consider relativistic effects in order to reconcile the difference between the net charge densities in a fixed coordinate system and in one moving with the stream (§ 5.8.1).

Seen from a coordinate system at rest there is an electric field in the plasma stream, and this accelerates or decelerates energetic charged particles (cosmic rays) that pass through the stream. It has sometimes been argued that there would be a contradiction here, because in the moving coordinate system, where the electric field vanishes, no acceleration could take place. This is only an apparent contradiction, which is resolved by a simple application of the theory of relativity as is shown in § 5.8.2.

In § 5.8.3 we discuss briefly the drift motion caused by the magnetic gradient and the inertia force. The role of this drift for the creation of the electric field and the expulsion of the stream is mentioned, but the quantitative analysis of these phenomena, as well as the discussion of the large-scale dynamics of the stream, is beyond the scope of the present monograph.

5.8.1. *Electric and magnetic fields*

In order to define the shape of the plasma-stream model (in a fixed frame of reference) we choose spherical coordinates, radius r, azimuth φ, latitude λ, Fig. 5.18. The corresponding unit vectors are written $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\varphi}}$, $\hat{\boldsymbol{\lambda}}$.

Let the boundaries of the stream be given by

$$\lambda = \pm\lambda_0/2, \tag{1}$$

$$\varphi = \pm\varphi_0/2. \tag{2}$$

Further let the velocity be of the form†

$$\begin{cases} \mathbf{v} = v(r)\hat{\mathbf{r}} & (|\varphi| < \varphi_0/2,\ |\lambda| < \lambda_0/2), \\ \mathbf{v} = 0 & (|\varphi| > \varphi_0/2). \end{cases} \tag{3}$$

Magnetic and electric fields in the fixed coordinate system

We assume that the magnetic field has the structure

$$\begin{cases} \mathbf{B} = B(r)\hat{\boldsymbol{\lambda}} & (|\varphi| < \varphi_0/2,\ |\lambda| < \lambda_0/2), \\ \mathbf{B} = 0 & (|\varphi| > \varphi_0/2). \end{cases} \tag{4}$$

The average motion of a particle in the interior of the stream is approximately given by the motion of its gyration centre. The domi-

† Generalization to cases where the background plasma outside the beam also has a non-vanishing radial velocity is straightforward.

nating term in the expression for the guiding centre velocity perpendicular to the magnetic field is

$$\mathbf{v} = c\frac{\mathbf{E}\times\mathbf{B}}{B^2}, \tag{5}$$

and consequently

$$\mathbf{E}_\perp + \frac{\mathbf{v}}{c}\times\mathbf{B} = 0. \tag{6}$$

In the absence of electric fields parallel to **B** (we have $dB/ds = 0$, cf. § 5.1.3), equation (6) implies conservation of flux through contours moving with the gyration centres (cf. § 5.4.2). It follows that

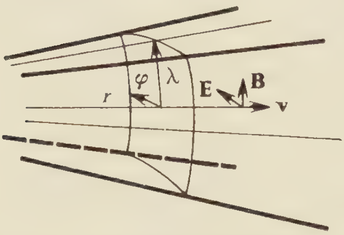


FIG. 5.18. Coordinate system and geometry of the plasma-stream model (φ is the azimuth and λ the latitude).

$$Bn^{-1}r^{-1} = \text{const}, \tag{7}$$

where n is the number density.

Further, the equation of continuity requires

$$nvr^2 = \text{const}, \tag{8}$$

and from (7) and (8) it follows that

$$B = \frac{\text{const}}{vr} = B_0\frac{v_0 r_0}{vr}. \tag{9}$$

Returning to (6) we conclude from (4) and (9) that inside the stream

$$\mathbf{E} = \frac{\text{const}}{r}\hat{\boldsymbol{\varphi}} = E_0\frac{r_0}{r}\hat{\boldsymbol{\varphi}} \quad \begin{cases} |\varphi| < \varphi_0/2 \\ |\lambda| < \lambda_0/2, \end{cases} \tag{10}$$

where

$$E_0 = \frac{v_0 B_0}{c}. \tag{11}$$

In the background plasma on either side of the stream an equation similar to (6) holds, and as in those regions we have assumed $\mathbf{v} = 0$, it follows that

$$\mathbf{E} = 0 \quad (|\varphi| > \varphi_0/2). \tag{12}$$

From (10) and (12) it follows that there are surface charges on the boundaries of the stream, namely

$$q_s = \frac{E_0}{4\pi}\frac{r_0}{r} \quad \text{at} \quad \begin{cases} \varphi = -\varphi_0/2 \\ |\lambda| < \lambda_0/2, \end{cases} \tag{13}$$

$$q_s = -\frac{E_0}{4\pi}\frac{r_0}{r} \quad \text{at} \quad \begin{cases} \varphi = \varphi_0/2 \\ |\lambda| < \lambda_0/2. \end{cases} \tag{14}$$

The planes $\varphi = \pm\varphi_0/2$ are (apart from edge effects) equipotential surfaces with a potential difference

$$V_0 = r\varphi_0 E = \frac{\varphi_0 r_0 v_0 B_0}{c}. \tag{15}$$

Similarly, the sources of the magnetic field consist of radial surface currents i_s with densities

$$i_s = \frac{cB}{4\pi} \quad \text{at} \quad \begin{cases} \varphi = -\varphi_0/2 \\ |\lambda| < \lambda_0/2, \end{cases} \tag{16}$$

$$i_s = -\frac{cB}{4\pi} \quad \text{at} \quad \begin{cases} \varphi = \varphi_0/2 \\ |\lambda| < \lambda_0/2, \end{cases} \tag{17}$$

and an azimuthal volume current density

$$i = \frac{c}{4\pi}\frac{1}{r}\frac{\partial}{\partial r}(rB). \tag{18}$$

In the simplest case, where $v = \text{const}$, it follows from (9) that B is inversely proportional to r so that the total surface current is independent of r and the volume current vanishes.

Magnetic and electric fields in a coordinate system moving with the plasma

In order to consider the plasma stream from a frame of reference moving with it, choose a local Cartesian coordinate system S' with coordinates ξ, η, and ζ, unit vectors $\hat{\boldsymbol{\xi}}$ (parallel to $\hat{\mathbf{r}}$), $\hat{\boldsymbol{\eta}}$ (parallel to $\hat{\boldsymbol{\varphi}}$), and $\hat{\boldsymbol{\zeta}}$ (parallel to $\hat{\boldsymbol{\lambda}}$), and origin O' situated at r', φ', λ'. In the neighbourhood of the origin,

$$(\xi^2+\eta^2+\zeta^2)^{\frac{1}{2}} \ll r', \tag{19}$$

the transformation of coordinates between S and S' has the simple form

$$\xi = r-r', \tag{20}$$

$$\eta = r'(\varphi-\varphi')\cos\lambda', \tag{21}$$

$$\zeta = r'(\lambda-\lambda'). \tag{22}$$

The magnetic field inside the stream is now

$$\mathbf{B}' \approx \mathbf{B} = B\hat{\boldsymbol{\zeta}}, \tag{23}$$

and it has the partial time derivative, cf. (9),

$$\frac{\partial \mathbf{B}}{\partial t} = v\frac{\partial B}{\partial r}\hat{\boldsymbol{\zeta}} = v_0 B_0 \frac{r_0}{r'}\left(\frac{1}{B}\frac{\partial B}{\partial r}\right)_{r'}\hat{\boldsymbol{\zeta}}. \tag{24}$$

The electric field is

$$\mathbf{E}' = \mathbf{E} + \frac{\mathbf{v}(r',\phi',\lambda')\times\mathbf{B}}{c} = \frac{v_0 B_0}{c}\frac{r_0}{r'}\left\{-\frac{\eta}{r'}\hat{\boldsymbol{\xi}} - \frac{\xi}{r'}\left(\frac{1}{rB}\frac{\partial(rB)}{\partial r}\right)_{r'}\hat{\boldsymbol{\eta}}\right\}, \tag{25}$$

where (9) to (11) have been used. In the moving system the electric field vanishes locally in the sense that it is of first order in the small quantities ξ/r' and η/r'. However, it has a zero-order curl,

$$\operatorname{curl}\mathbf{E}' = -\frac{v_0 B_0}{c}\frac{r_0}{r'}\left(\frac{1}{B}\frac{\partial B}{\partial r}\right)_{r'}\hat{\boldsymbol{\zeta}} \tag{26}$$

as expected from the fact that the magnetic field has the zero-order time derivative (24) in the moving system.

In order to discuss the currents and charges at the boundary we place the local system S' at $\varphi = -\varphi_0/2$. Then $\mathbf{E}'$ and $\mathbf{B}'$ are given by (23) and (25) in the region $\eta > 0$ and vanish in the region $\eta < 0$.

The jump in the magnetic field at the boundary reveals a surface current density

$$i_s' = \frac{cB}{4\pi} \tag{27}$$

in agreement with that observed in the fixed system i_s, equation (16). On the other hand, the electric field is continuous and so the surface charge vanishes,

$$q_s' = 0. \tag{28}$$

The difference in surface charge density measured in the two systems is a relativistic effect (see, for example, Becker–Sauter, 1957, **1**, 255 ff.) and in fact the result (28) follows from the transformation formula

$$q_s' = \left(q_s - \frac{vi_s}{c^2}\right)(1-v^2/c^2)^{-\frac{1}{2}} \approx q_s - \frac{vi_s}{c^2} \tag{29}$$

if (9), (11), (13), and (16) are used. The effect is thus due to the existence of surface currents associated with the spatially limited magnetic field. It would not occur if the magnetic field were the same outside and inside. (In the applications both the case with a jump in the magnetic field at the plasma boundary and that without may be of importance in different situations.)

The difference in net surface charge density in the two systems can be given the following simple interpretation in terms of the Lorentz contraction: The surface current i_s consists of positive ions drifting with a velocity u_i (relative to the plasma) and electrons drifting at the velocity u_e. If N_i' and N_e' are the surface densities of positive and negative particles (measured in the plasma system) the surface current density is

$$i_s' = e_i N_i' u_i + e_e N_e' u_e. \tag{30}$$

Consider a certain number, say N, of the electrons taking part in this surface current. When seen in a coordinate system following the drift motion they are distributed over a certain length element l_e^e in the radial direction. When seen from the plasma this length is contracted to

$$l_e' = l_e^e \sqrt{\{1-(u_e/c)^2\}}. \tag{31}$$

As $u_e \ll v \ll c$, the relativistic addition theorem for velocities reduces in sufficient approximation to the classical additive law, and the velocity of the electrons measured in the fixed system is u_e+v. Consequently the length l_e^e has, in the fixed system, the apparent value

$$l_e = l_e^e\sqrt{\{1-(u_e+v)^2/c^2\}}. \tag{32}$$

Thus if N_e' is the number density of electrons at the surface, when measured in the plasma system, it follows from (31) and (32) that the corresponding number density in the fixed system is

$$N_e = N_e'\frac{\sqrt{\{1-(u_e/c)^2\}}}{\sqrt{\{1-(u_e+v)^2/c^2\}}} \approx N_e'\left(1+\frac{v^2}{2c^2}+\frac{v}{c^2}u_e+\ldots\right). \tag{33}$$

Analogously we have for the ions

$$N_i = N_i'\frac{\sqrt{\{1-(u_i/c)^2\}}}{\sqrt{\{1-(u_i+v)^2/c^2\}}} \approx N_i'\left(1+\frac{v^2}{2c^2}+\frac{v}{c^2}u_i+\ldots\right). \tag{34}$$

The surface charge in the fixed system is thus

$$q_s = \left[(N_i'e_i+N_e'e_e)\left(1+\frac{v^2}{2c^2}\right)+(v/c^2)(N_i'e_i u_i+N_e'e_e u_e)\right].$$

If, finally, in the last term of the bracket, we again use (33) and (34), neglecting u_i/v, u_e/v, and v/c compared with unity, we find

$$q_s = [N_i'e_i+N_e'e_e+(v/c^2)(N_i e_i u_i+N_e e_e u_e)]\left(1+\frac{v^2}{c^2}\right) \approx q_s'+(v/c^2)i_s \tag{35}$$

in agreement with (29).

5.8.2. *Acceleration of high-energy charged particles*

When a charged particle of high energy passes the stream in the φ-direction, its energy changes by the amount

$$\Delta W = eV_0 = e\varphi_0 r_0 B_0 v_0/c \tag{36}$$

corresponding to the potential difference (15) between the boundaries (e includes the sign of the charge).

It is of some interest to see *how this acceleration takes place when observed from a moving coordinate system, where the electric field vanishes.* Consider for simplicity an ultra-relativistic charged particle passing a narrow beam, $\varphi_0 \ll 1$, with nearly parallel boundaries. Let its momentum and energy in the fixed system S be p and W and the corresponding quantities in the moving system p' and W'. In the moving system S' the electric field vanishes, and so p' and W' are constants of the motion. The magnetic field $B' = B$ deflects the path by the amount

$$d\alpha = \frac{eB}{cp'\cos\alpha}d\eta, \tag{37}$$

where $d\eta$ is the width of the stream and α is the angle of incidence (see Fig. 5.19).

As the velocity v of the moving coordinate system is slow, $v \ll c$, the relativistic transformation of energy (see, for example, Landau and Lifshitz, 1959, p. 27) simplifies to

$$W = W' + vp' \sin\alpha, \tag{38}$$

where α is the angle between the path and the η-direction. For an ultra-relativistic particle $p' \approx W'/c$, and (38) becomes

$$W = W'\{1+(v/c)\sin\alpha\}. \tag{39}$$

Since the energy is invariant in the moving system, the energy observed in the fixed system changes by the amount

$$dW = W'\frac{v}{c}\cos\alpha\, d\alpha \tag{40}$$

as a consequence of the deflexion $d\alpha$.

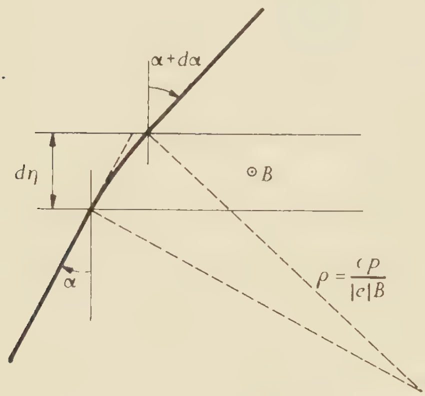


FIG. 5.19. When a high-energy charged particle passes across a magnetized plasma stream, it is deflected by a small angle $d\alpha$ by the magnetic field in the stream.

From (37) and (40) follows, when terms of smaller order in v/c are left out,

$$dW = \frac{eBv}{c}\, d\eta$$

or, since $d\eta = r'\varphi_0$ and $Bv = B_0 v_0 r_0/r'$,

$$dW = e\varphi_0 r_0 B_0 v_0/c = eV_0 \tag{41}$$

in agreement with (36).

5.8.3. *Particle drifts in the azimuthal direction*

Each particle in the stream has a 'thermal' velocity superposed on the mean velocity of the stream. Let the components perpendicular and parallel to $\mathbf{B}$ be $\mathbf{w}'_\perp$ (corresponding to a kinetic energy $W'_\perp$) and $\mathbf{w}'$ (energy $W'_\parallel$). As the particle follows the expanding stream, $w'_\perp$ decreases by deceleration in the decreasing magnetic field. The adiabatic invariance of the quantity

$$\frac{mw'^2_\perp}{2B} = \frac{W'_\perp}{B} = \mu \tag{42}$$

(non-relativistic case) implies a change of $W'_\perp$ at the rate

$$\frac{dW'_\perp}{dt} = \frac{W'_\perp}{B}\frac{dB}{dt} = \frac{W'_\perp}{B}\frac{cE}{B}\frac{dB}{dr}. \tag{43}$$

At the same time the gradient in the magnetic field causes a drift in the φ-direction. This drift is directed opposite to the electric field for a positive particle and parallel to it for a negative particle. Therefore the particle does work against the electric field and tends to increase the latter. An increased electric field corresponds to an increased radial drift velocity and the corresponding inertia force causes an additional drift, which partly cancels that due to the magnetic gradient.

The resultant drift in the φ-direction is, in the case $w'_{\parallel} = 0$, given by

$$u_{\varphi} = -\frac{c\mu}{eB}\left(-\frac{dB}{dr}\right)+\frac{cm}{eB}\frac{dv}{dt}. \tag{44}$$

On multiplication by eE and application of (5), (42), and (43), equation (44) can be written as

$$-\frac{dW'_{\perp}}{dt} = -eEu_{\varphi}+\frac{d}{dt}\left(\frac{mv^2}{2}\right), \tag{45}$$

which shows that the energy loss due to decreasing $W_{\perp}$ goes partly into potential energy and partly into kinetic energy associated with the radial motion.

Phenomena of the same nature as those just described occur also when redistribution of energy takes place between different degrees of freedom, although the relation between $W'_{\perp}$ and B then differs from (42).

The fact that the drift due to radial decrease of the magnetic field transfers energy to the ordered outward motion is important for the ejection of plasma streams from the sun (Alfvén, 1939). The phenomenon can be regarded as a diamagnetic repulsion of the plasma from the region of strong magnetic field and cooling by expansion.

5.9. The distribution function

In plasma physics as well as in the kinetic theory of ordinary gases an accurate mathematical analysis must take into account the distribution of velocities among the particles. This is conveniently done by introducing a *distribution function* for each kind of particle. The distribution function depends on the position coordinates, the velocity components, and the time. It is so defined that the probable number of particles in a given volume element $dxdydz$ (at the point x, y, z) which, at a given instant t, have their velocity components in the intervals (v_x, v_x+dv_x), (v_y, v_y+dv_y), and (v_z, v_z+dv_z) is $f(x,y,z,v_x,v_y,v_z,t)dxdydzdv_x dv_y dv_z$ or, in shorter notation, $f(\mathbf{r},\mathbf{v},t)\,d\tau$. The combined coordinate space formed by the coordinates x, y, z of *configuration space* and the coordinates v_x, v_y, v_z of *velocity space* is six-dimensional, and we denote its 'volume element' by $d\tau$. The distribution function can be interpreted as the number density in this six-dimensional space.

5.9.1. *Equations governing the distribution function*

To derive an equation for the distribution function we consider the particles present in a certain element $d\tau$ at a given instant t. The number of these particles is $f(\mathbf{r}, \mathbf{v}, t)\, d\tau$. At a later time, $t+dt$, the same particles occupy an element $d\tau'$ situated at the position $\mathbf{r}+\mathbf{v}\, dt$ and have acquired the velocity $\mathbf{v}+(\mathbf{F}/m)\, dt$, where $\mathbf{F}$ is the force per particle and m the particle mass. If there were no collisions, all the particles present in the element $d\tau$ at the instant t would be found in $d\tau'$ at the time $t+dt$, so that

$$f(\mathbf{r}+\mathbf{v}\, dt,\, \mathbf{v}+(\mathbf{F}/m)\, dt,\, t+dt)\, d\tau' = f(\mathbf{r}, \mathbf{v}, t)\, d\tau. \tag{1}$$

Now, if the force $\mathbf{F}$ does not depend on $\mathbf{v}$, and also if $\mathbf{F}$ is of the Lorentz-force type $(e/c)\mathbf{v}\times\mathbf{B}$ it can be proved (see, for example, Chapman and Cowling, 1939, p. 322) that the elements $d\tau$ and $d\tau'$ have the same size, so that

$$d\tau = d\tau', \tag{2}$$

and

$$d\tau[f(\mathbf{r}+\mathbf{v}\, dt,\, \mathbf{v}+(\mathbf{F}/m)\, dt,\, t+dt)-f(\mathbf{r}, \mathbf{v}, t)] = 0. \tag{3}$$

If collisions take place, (3) is no longer exactly true, since those particles that undergo collisions in the time interval dt change their velocities abruptly by a large amount and are not present in $d\tau'$ at the time $t+dt$. There are also other particles that are thrown *into* the element considered. The change in particle number is proportional to dt and to $d\tau$ and can be written $(\partial f/\partial t)_{\text{coll}}\, d\tau dt$. Thus in the presence of collisions we have instead of (3) the relation

$$[f(\mathbf{r}+\mathbf{v}\, dt,\, \mathbf{v}+(\mathbf{F}/m)\, dt,\, t+dt)-f(\mathbf{r}, \mathbf{v}, t)]\, d\tau = \left(\frac{\partial f}{\partial t}\right)_{\text{coll}} d\tau dt. \tag{4}$$

In the limit $dt \to 0$ this becomes

$$\frac{\partial f}{\partial t}+v_x\frac{\partial f}{\partial x}+v_y\frac{\partial f}{\partial y}+v_z\frac{\partial f}{\partial z}+\frac{F_x}{m}\frac{\partial f}{\partial v_x}+\frac{F_y}{m}\frac{\partial f}{\partial v_y}+\frac{F_z}{m}\frac{\partial f}{\partial v_z} = \left(\frac{\partial f}{\partial t}\right)_{\text{coll}} \tag{5}$$

or in shorter notation

$$\frac{\partial f}{\partial t}+(\mathbf{v}\,\text{grad})f+\left(\frac{\mathbf{F}}{m}\text{grad}_v\right)f = \left(\frac{\partial f}{\partial t}\right)_{\text{coll}}. \tag{6}$$

Equation (5) is the *Boltzmann equation*, which is of fundamental importance in the kinetic theory of gases. When more than one type of particle is present, as is always the case in ionized gases, separate equations of the type (5) apply to each kind of particle. For example, in an electron-proton plasma we have for the distribution function of electrons:

$$\frac{\partial f_e}{\partial t}+(\mathbf{v}_e\,\mathrm{grad})f_e+\left(\frac{\mathbf{F}_e}{m_e}\mathrm{grad}_v\right)f_e=\left[\left(\frac{\partial f_e}{\partial t}\right)_{\mathrm{coll}}\right]_{\mathrm{ee}}+\left[\left(\frac{\partial f_e}{\partial t}\right)_{\mathrm{coll}}\right]_{\mathrm{ep}}, \quad (7)$$

where the last term refers to electron-proton collisions. A corresponding equation is valid for the protons.

In the case of short-range forces (e.g. in a weakly ionized plasma) the collision term is usually expressed by 'collision integrals' taken over the velocity distribution, and the Boltzmann equation takes the form of an integro-differential equation. In the case of Coulomb forces, which are long-range, most collisions produce very small velocity changes (cf. § 4.3.2). It is then appropriate to make an expansion of the collision integrals in powers of the velocity change, and as a result one obtains the *Fokker–Planck equation* (see, for example, Allis, 1956, p. 429).

In low-density plasmas the mean free paths are very long. The charged particles do not interact much by collisions but instead by the electric fields that are created by the collective motion of the particles. It is then a good first approximation to leave out the collision term in (5) altogether. The *collision-less Boltzmann equation* together with *Maxwell's equations* are generally referred to as the *Vlasov equations* (Vlasov, 1945). They are appropriate for low-density 'collision-less' plasmas.

5.9.2. *Equilibrium distribution*

The true steady state solution of (5) is the Maxwell–Boltzmann distribution, which corresponds to thermodynamical equilibrium. For such an equilibrium to exist the non-magnetic forces must have a potential. We can then write the total force in the form

$$\mathbf{F}=e(\mathbf{v}/c)\times\mathbf{B}-\mathrm{grad}\,\phi. \quad (8)$$

The Maxwell–Boltzmann distribution is then given by

$$f=n_0\left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}}e^{-[\phi+m(v_x^2+v_y^2+v_z^2)/2]/kT}, \quad (9)$$

where n_0 and T are constants representing the density at the equipotential surface $\phi=0$ and the temperature. Note that in equilibrium the temperature is the same everywhere, also in the presence of forces.

In particular we see from (9) that a magnetic field alone does not affect the density distribution in thermal equilibrium. For example, a *magnetically confined plasma is not in thermodynamic equilibrium* although it may be in magnetohydrostatic equilibrium, that is, correspond to a steady-state solution of the magnetohydrostatic equations (cf. § 3.12). Since the approach to thermodynamic equilibrium is often slow, magnetically confined plasmas are of great importance both in the laboratory (thermonuclear machines) and in cosmical physics (e.g. the earth's radiation belts).

5.9.3. *The pressure tensor*

The macroscopic quantities, such as density, mass velocity, etc., are averages (or more precisely moments) over the distribution function. The natural method of deriving the macroscopic equations in § 5.2.3 is therefore to perform, in the equations governing the distribution function, suitable integrations over velocity space. If these calculations are carried through (see, for example, Linhart, 1960, Thompson, 1962, or Lehnert, 1963*b*), they lead to the macroscopic equations of § 5.2.3 in a somewhat more general form, which is useful also in cases where the velocity distribution is anisotropic. In these more general equations the pressure-gradient terms $-\operatorname{grad} p_{\mathrm{e}}$ and $-\operatorname{grad} p_{\mathrm{i}}$ are replaced by tensor divergences $-\operatorname{Div} \boldsymbol{\Psi}_{\mathrm{e}}$ and $-\operatorname{Div} \boldsymbol{\Psi}_{\mathrm{i}}$. Dropping the subscripts e and i we can write these expressions explicitly in the form

$$\operatorname{Div} \boldsymbol{\Psi} = \hat{\mathbf{x}}\left(\frac{\partial}{\partial x}\Psi_{xx}+\frac{\partial}{\partial y}\Psi_{xy}+\frac{\partial}{\partial z}\Psi_{xz}\right)+\hat{\mathbf{y}}\left(\frac{\partial}{\partial x}\Psi_{yx}+\frac{\partial}{\partial y}\Psi_{yy}+\frac{\partial}{\partial z}\Psi_{yz}\right)+ \\ +\hat{\mathbf{z}}\left(\frac{\partial}{\partial x}\Psi_{zx}+\frac{\partial}{\partial y}\Psi_{zy}+\frac{\partial}{\partial z}\Psi_{zz}\right), \quad (10)$$

where $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ denote unit vectors along the three coordinate axes.

The elements, Ψ_{xx}, etc., of *the pressure tensor* $\boldsymbol{\Psi}$ can be expressed as follows in terms of averages over the velocity distribution of the pair-products of the particle-velocity components w_x, w_y, w_z measured in a frame of reference following the average motion:

$$\boldsymbol{\Psi} = \begin{pmatrix} \Psi_{xx} & \Psi_{xy} & \Psi_{xz} \\ \Psi_{yx} & \Psi_{yy} & \Psi_{yz} \\ \Psi_{zx} & \Psi_{zy} & \Psi_{zz} \end{pmatrix}$$

$$= \begin{pmatrix} nm\langle w_x^2\rangle & nm\langle w_x w_y\rangle & nm\langle w_x w_z\rangle \\ nm\langle w_y w_x\rangle & nm\langle w_y^2\rangle & nm\langle w_y w_z\rangle \\ nm\langle w_z w_x\rangle & nm\langle w_z w_y\rangle & nm\langle w_z^2\rangle \end{pmatrix}, \quad (11)$$

where the symbol $\langle\,\rangle$ has been used for averages over the velocity distribution.

In many applications the pressure tensor reduces to a much simpler form than (11). For example in a magnetic plasma, it is often possible to write it in the form

$$\boldsymbol{\Psi} = \begin{pmatrix} p_\perp & 0 & 0 \\ 0 & p_\perp & 0 \\ 0 & 0 & p_\parallel \end{pmatrix} \tag{12}$$

with

$$p_\perp = mn\langle w_x^2\rangle = mn\langle w_y^2\rangle, \tag{13}$$

$$p_\parallel = mn\langle w_z^2\rangle, \tag{14}$$

if one of the coordinate axes (in (12) to (14) the z-axis) is chosen parallel to the magnetic field lines (see, e.g., Thompson, 1962, p. 162). Then the force terms $-\mathrm{Div}\,\boldsymbol{\Psi}_e$ and $-\mathrm{Div}\,\boldsymbol{\Psi}_i$ have the form

$$-\mathrm{Div}\,\boldsymbol{\Psi} = \hat{\mathbf{x}}\left(-\frac{\partial p_\perp}{\partial x}\right) + \hat{\mathbf{y}}\left(-\frac{\partial p_\perp}{\partial y}\right) + \hat{\mathbf{z}}\left(-\frac{\partial p_\parallel}{\partial z}\right). \tag{15}$$

Physically we can interpret this expression by saying that there are different pressures parallel to the magnetic field and perpendicular to it. Such anisotropy may exist, when the coupling (by collisions or by instabilities) between the different translational degrees of freedom is weak and energy is fed preferentially into one of the velocity components $v_\parallel$ or $v_\perp$. Laboratory examples of such situations are provided by thin magnetically compressed plasmas, where $p_\perp$ tends to be large (Post and Perkins, 1961), and plasmas with electron runaway (cf. § 4.3.4) along the magnetic field. In cosmic physics both magnetic fields and magnetic acceleration processes are important (cf. § 2.7) and in low-density regions, where collisions are rare, anisotropic velocity distributions may occur.

Finally, in the simplest case, where the velocity distribution is isotropic, so that

$$p_\perp = p_\parallel = p, \tag{16}$$

and the off-diagonal terms negligible, the pressure tensor reduces to

$$\boldsymbol{\Psi} = \begin{pmatrix} p & 0 & 0 \\ 0 & p & 0 \\ 0 & 0 & p \end{pmatrix}, \tag{17}$$

and this is now true however the coordinate system is oriented. The expression (10) (or (15)) then reduces to

$$-\mathrm{Div}\,\boldsymbol{\Psi} = -\mathrm{grad}\,p$$

as expected.

REFERENCES

ALFVEN, H., 1939: 'A Theory of Magnetic Storms and of the Aurorae', *Kgl. Sv. Vet. Akad. Handl.*, ser. 3, **18**, no. 3.

—— 1959: 'Momentum Spectrum of Cosmic Radiation', *Tellus*, **11**, 106.

—— 1961*a*: 'Filamentary Structures produced by an Electric Current in a Plasma', *Ark. f. fys.* **19**, 375; *see also*

—— 1961*b*: 'Confined Electron Streams in a Plasma', ibid. **20**, 389; and

—— 1955: 'Line Currents in Cosmical Physics', *Proc. Roy. Soc.* A, **233**, 296.

—— 1962: 'On the Filamentary Structure of the Solar Corona', *Proceedings of International Astronomical Union Symposium No.* 16 *at Cloudcroft, Colorado.*

ALLIS, W. P., 1956: 'Motions of Ions and Electrons', *Encyclopedia of Physics*, **21**, Springer-Verlag, Berlin, 383–444.

BAKER, W. G., and MARTYN, D. F., 1953: 'Electric Currents in the Ionosphere. I. The Conductivity', *Phil. Trans. Roy. Soc.* A **246**, 281.

BECKER, R., and SAUTER, F., 1957: *Theorie der Elektrizität*, **1**, Teubner, Stuttgart, 16th edn.

BENNETT, W. H., 1934: 'Magnetically Self-focusing Streams', *Phys. Rev.* **45**, 890.

BOHM, D., 1949: 'Qualitative Description of the Arc Plasma in a Magnetic Field', *The Characteristics of Electrical Discharges in Magnetic Fields*, A. Guthrie and R. K. Wakerling, eds., McGraw-Hill, p. 1.

BOHR, N., 1911: Dissertation, Copenhagen.

CHANDRASEKHAR, S., 1943: 'Stochastic Problems in Physics and Astronomy', *Rev. Mod. Phys.* **15**, 1.

CHAPMAN, S., 1956: 'The Electrical Conductivity of the Ionosphere', *Nuovo Cimento*, Suppl. **4**, 1385.

—— and COWLING, T. G., 1939: *The Mathematical Theory of Non-Uniform Gases*, Cambridge University Press, 2nd edn. 1958.

COBINE, J. D., 1941: *Gaseous Conductors*, McGraw-Hill, New York.

COWLING, T. G., 1929: 'On the Radial Limitation of the Sun's Magnetic Field', *M.N.R.A.S.* **90**, 140.

—— 1945: 'Electrical Conductivity of an Ionized Gas in a Magnetic Field, with Applications to the Solar Atmosphere and the Ionosphere', *Proc. Roy. Soc.* A, **183**, 453.

—— 1957: *Magneto-hydrodynamics*, Interscience, New York.

ENGEL, A. VON, and STEENBECK, M., 1932: *Elektrische Gasentladungen*, **1**, Springer-Verlag, Berlin.

—— —— 1934: ibid. **2**.

GRAD, H., 1961: 'Microscopic and Macroscopic Models in Plasma Physics', *Proceedings of the Symposium on Electromagnetics and Fluid Dynamics of Gaseous Plasma*, J. Fox, ed., Polytechnic Press, Brooklyn, N.Y., 37.

GRAETZ, L., 1915: *Handbuch der Elektrizität und des Magnetismus*, Bd. **4**, 2, 780, Leipzig.

HERDAN, R., and LILEY, B. S., 1960: 'Dynamical Equations and Transport Relations for a Thermal Plasma', *Rev. Mod. Phys.* **32**, 731.

HOH, F. C., 1962: 'Low-temperature Plasma Diffusion in a Magnetic Field' *Rev. Mod. Phys.* **34**, 267.

—— and LEHNERT, B., 1961: 'Screw Instability of a Plasma Column', *Phys. Rev. Letters*, **7**, 75.

KADOMTSEV, B. B., and NEDOSPASOV, A. V., 1960: *J. Nuclear Energy*, Part C, **1**, 230.

KENNARD, E. H., 1938: *Kinetic Theory of Gases*, McGraw-Hill, New York.

LANDAU, L. D., and LIFSHITZ, E. M., 1959: *The Classical Theory of Fields*, Pergamon Press.

—— —— 1960: *Mechanics*, Pergamon Press.

LEHNERT, B., 1958: 'Diffusion Processes in the Positive Column in a Longitudinal Magnetic Field', *Proc. Second U.N. Int. Conf. on the Peaceful Uses of Atomic Energy*, **32**, 349.

—— 1959: 'Plasma Physics on Cosmical and Laboratory Scale', *Nuovo Cimento*, ser. X, Suppl. **13**, 1.

—— 1962*a*: 'Rotating Plasma Devices', *Plasma Physics and Thermonuclear Research*, ed. J. L. Tuck, Progr. Nuclear Energy Series 11, vol. **2**.

—— 1962*b*: 'Diffusion of Ions and Electrons', *Proc. Internat. Conf. on High Magnetic Fields*, MIT Press and Wiley, p. 617.

—— 1962*c*: 'Plasma Stability in an Inhomogeneous Magnetic Field', *Phys. Fluids*, **5**, 432.

—— 1963*a*: 'Screw and Flute Instabilities in a Low Pressure Plasma', *Atomnoya Energiya*, **14**, 82.

—— 1963*b*: *Dynamics of Charged Particles*, North-Holland Publ. Co., Amsterdam.

LINHART, J. G., 1960: *Plasma Physics*, North-Holland Publ. Co., Amsterdam. Second edition, 1961.

LONGMIRE, C. L., and ROSENBLUTH, M. N., 1956: 'Diffusion of Charged Particles across a Magnetic Field', *Phys. Rev.* **103**, 507.

LUNDQUIST, S., 1950: 'Magneto-hydrostatic Fields', *Ark. f. fys.* **2**, 361.

—— 1952: 'Studies in Magneto-hydrodynamics', ibid. **5**, 297.

MARSHALL, T. C., KAWCYN, R. A., and GOLDSTEIN, L., 1960: 'Diamagnetic Moment of Gaseous Plasmas measured by Spin Resonance Techniques', *Nature*, **187**, 584.

MASSEY, H. S. W., and BURHOP, E. H. S., 1952: *Electronic and Ionic Impact Phenomena*, Clarendon Press, Oxford.

MURTY, G. S., 1961: 'Filamentary Structure associated with Force-free Magnetic Fields', *Ark. f. fys.* **21**, 203.

POST, R. F., and PERKINS, W. A., 1961: 'Velocity-space Plasma Instabilities observed in a Mirror Machine', *Phys. Rev. Letters*, **6**, 85.

ROSENBLUTH, M. N., and LONGMIRE, C. L., 1957: 'Stability of Plasmas Confined by Magnetic Fields', *Ann. Phys.* **1**, 120.

SCHLÜTER, A., 1951: 'Dynamik des Plasmas II', *Z. Naturforsch.* **6a**, 73.

SIMON, A., 1955: 'Diffusion of Like Particles across a Magnetic Field', *Phys. Rev.* **100**, 1557.

SPITZER, L., JR., 1952: 'Equations of Motion for an Ideal Plasma', *Astrophys. J.* **116**, 299.

—— 1956: *Physics of Fully Ionized Gases*, Interscience, New York. Second edition, 1962.

—— 1960: 'Particle Diffusion across a Magnetic Field', *Phys. Fluids*, **3**, 659.

STEENBECK, M., 1936: 'Über die magnetischen Eigenschaften des Plasmas von Gasentladungen', *Wissenschaftliche Veröffentlichungen aus den Siemens-Werken*, **15**, 2, 1.

Taylor, J. B., 1962: 'Stochastic Methods in the Theory of Plasma Diffusion', *J. Plasma Physics and Thermonuclear Fusion*, **1962**, Suppl. Part 2, 477.

Thompson, W. B., 1961: 'The Dynamics of High-temperature Plasmas', *Reports on Progress in Physics*, **14**, Inst. of Phys. and the Phys. Soc. (London).

—— 1962: *An Introduction to Plasma Physics*, Pergamon Press.

Vlasov, A., 1945: 'On the Kinetic Theory of an Assembly of Particles with Collective Interaction', *J. Phys.* (*USSR*), **9**, 25.

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