Definition

A k-*colouring* of a graph $G(V, E)$ is a function $\phi : V \to \{1, \ldots, k\}$ with the property that

$$(u, v) \in E \Rightarrow \phi(u) \neq \phi(v).$$

That is, ϕ assigns distinct values to adjacent vertices.

Definition

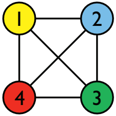
If $G(V, E)$ has a k-colouring then it is said to be k-*colourable*.

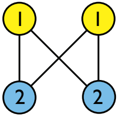
Definition

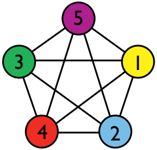
The *chromatic number* $\chi(G)$ of a graph G is the smallest number k for which G is k-colourable.

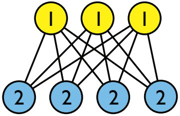
Examples of graphs and their chromatic numbers include

- K_n, the complete graph on n vertices is clearly n-colourable, but not $(n-1)$ colourable: thus $\chi(K_n) = n$.
- $K_{m,n}$, the complete bipartite graph on groups of m and n vertices, is 2-colourable, but not 1-colourable, so $\chi(K_{m,n}) = 2$.









It is generally a hard problem to compute the chromatic number of a graph, but the following results help for small graphs.

Lemma ($\chi(G)$ for bipartite graphs)

A graph G has chromatic number $\chi(G) = 2$ if and only if it is bipartite.

Lemma (Colouring subgraphs)

If H is a subgraph of G and G is k-colourable, then so is H.

Lemma (Chromatic number of subgraphs)

If H is a subgraph of G then $\chi(H) \leq \chi(G)$.

Algorithm (Greedy colouring)

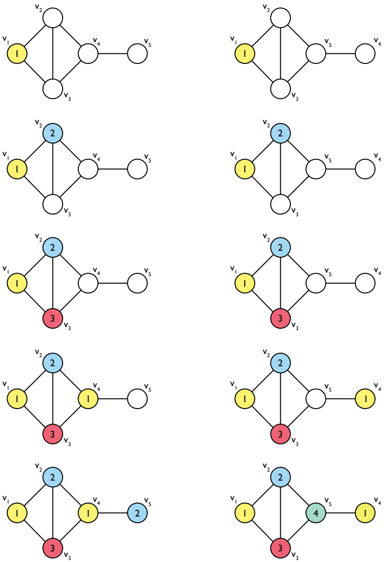
Given a graph $G(V, E)$ with vertex set $V = \{v_1, \ldots, v_n\}$ and adjacency lists A_{v_j}, construct a function $c : V \to \mathbb{Z}^+$ such that if the edge $e = (v_i, v_j) \in E$, then $c(v_i) \neq c(v_j)$.

1. *Set* $c(v_j) \leftarrow 0$ *for all* $1 \leq j \leq n$.
2. $c(v_1) \leftarrow 1$.
3. *For* $2 \leq j \leq n$ $\{$
4. *Choose a colour* $k > 0$ *for vertex* v_j *that differs from its neighbours' colours*
   $c(v_j) \leftarrow \min\left(k \in \mathbb{Z} \,|\, k > 0 \text{ and } k \neq c(w) \;\; \forall w \in A_{v_j}\right)$
5. $\}$ *End of loop over vertices* v_j.

Greedy Colouring Provides Upper Bounds

The two columns at right show the result of applying the greedy colouring algorithm to two versions of the same graph.

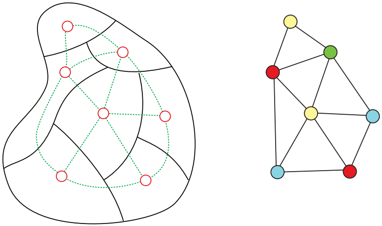
- The only difference between the two columns is that that roles of v_4 and v_5 have been interchanged.
- In the leftmost column we get a three-colouring, which is minimal for this graph, while in the rightmost column we get a four colouring.
- Generally speaking, greedy colouring provides only an upper bound on $\chi(G)$.



Theorem (The Four Colour Theorem)

Every planar graph G has $\chi(G) \leq 4$. That is, a planar graph can always be coloured with four or fewer colours.

The Four Colour Theorem engaged the interest of such famous mathematicians as Hamilton, De Morgan, Cayley and Birkhoff. It also prompted a famous incorrect proof that survived from 1879 to 1890. The first correct proof, produced by Kenneth Appel and Wolfgang Haken in 1977, is also the first and best-known example of a major problem resolved by means of a computer-aided proof.



The Four Colour Theorem was originally conjectured in the 1850's by Francis Guthrie and it arises from an observation about cartography. Guthrie's brother Frederick was thinking about colouring a map in such a way that it was easy to tell the different countries apart. He wanted to choose colours so that adjacent countries—those that share a segment of border—receive different colours and he observed that he never needed more than four colours to accomplish this.