Automated Theorem Proving

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Slides partially based on material by Alexander Fuchs, Harald Ganzinger, John Slaney, Viorica Sofronie-Stockermans and Uwe Waldmann

Purpose of This Lecture

Overview of Automated Theorem Proving (ATP)

- Emphasis on automated proof methods for **first-order logic**
- More “breadth” than “depth”

Standard techniques covered

- Normal forms of formulas
- Herbrand interpretations
- Resolution calculus, unification
- Instance-based methods
- Model computation
- Theory reasoning: Satisfiability Modulo Theories

Part 1: What is Automated Theorem Proving?

First-Order Theorem Proving in Relation to ...

...Calculation: Compute function value at given point:

Problem: $2^2 = ?$ $3^2 = ?$ $4^2 = ?$

"Easy" (often polynomial)

...Constraint Solving: Given:

- Problem: $x^2 = a$ where $x \in [1 \ldots b]$
  (x variable, a, b parameters)
- Instance: $a = 16$, $b = 10$

Find values for variables such that problem instance is satisfied

"Difficult" (often exponential, but restriction to **finite** domains)

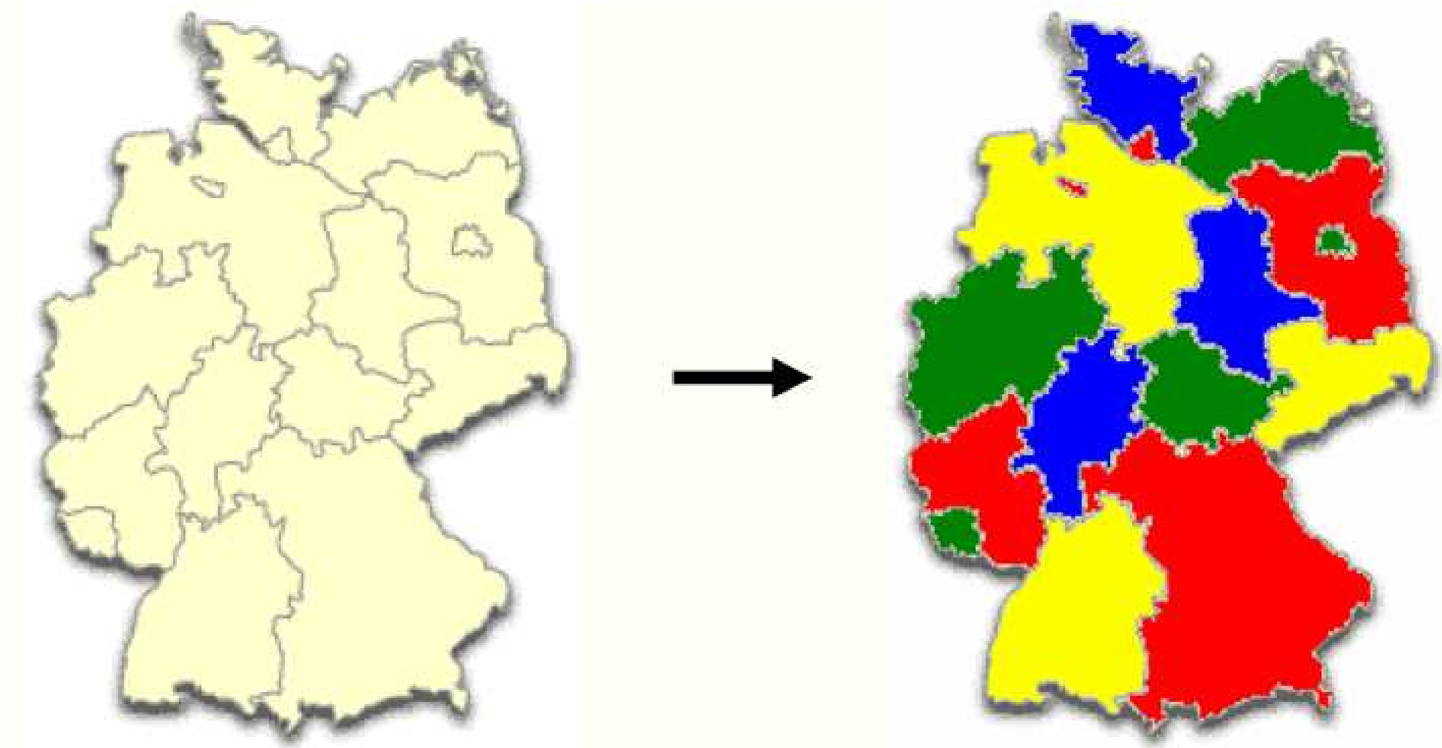
First-Order Theorem Proving: Given:

Problem: $\exists x \, (x^2 = a \wedge x \in [1 \ldots b])$

Is it satisfiable? unsatisfiable? valid?

"Very difficult" (often undecidable)

Logical Analysis Example: Three Coloring Problem

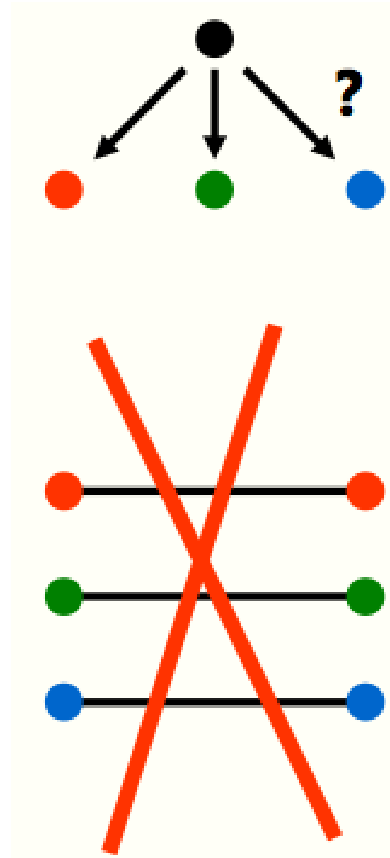


Problem: Given a map. Can it be colored using only three colors, where neighbouring countries are colored differently?

Three Coloring Problem - Graph Theory Abstraction

Problem Instance

Problem Specification



The Rôle of Theorem Proving?

Three Coloring Problem - Formalization

Every node has at least one color

$$\forall N\ (\mathrm{red}(N) \lor \mathrm{green}(N) \lor \mathrm{blue}(N))$$

Every node has at most one color

$$\begin{aligned}\forall N\ ((&\mathrm{red}(N) \rightarrow \neg \mathrm{green}(N)) \land \\ (&\mathrm{red}(N) \rightarrow \neg \mathrm{blue}(N)) \land \\ (&\mathrm{blue}(N) \rightarrow \neg \mathrm{green}(N)))\end{aligned}$$

Adjacent nodes have different color

$$\begin{aligned}\forall M, N\ (\mathrm{edge}(M, N) \rightarrow (&\neg(\mathrm{red}(M) \land \mathrm{red}(N)) \land \\ &\neg(\mathrm{green}(M) \land \mathrm{green}(N)) \land \\ &\neg(\mathrm{blue}(M) \land \mathrm{blue}(N))))\end{aligned}$$

Three Coloring Problem - Solving Problem Instances ...

... with a constraint solver:

Let constraint solver find value(s) for variable(s) such that problem instance is satisfied

Here:	Variables:	Colors of nodes in graph
	Values:	Red, green or blue
	Problem instance:	Specific graph to be colored

... with a theorem prover

Let the theorem prover prove that the three coloring formula (see previous slide) $+$ specific graph (as a formula) is satisfiable

- To solve problem instances a constraint solver is usually much more efficient than a theorem prover (e.g. use a SAT solver)
- Theorem provers are not even guaranteed to terminate, in general

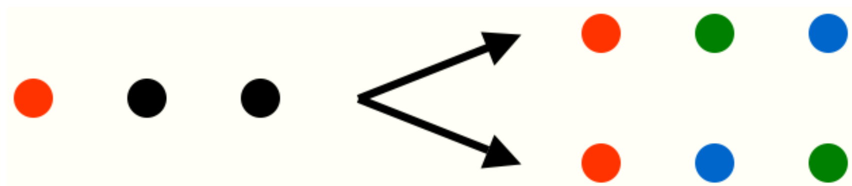
$\boxed{\textbf{Other tasks where theorem proving is more appropriate?}}$

Three Coloring Problem: The Rôle of Theorem Proving

Functional dependency

- Blue coloring depends functionally on the red and green coloring

- Blue coloring does not functionally depend on the red coloring



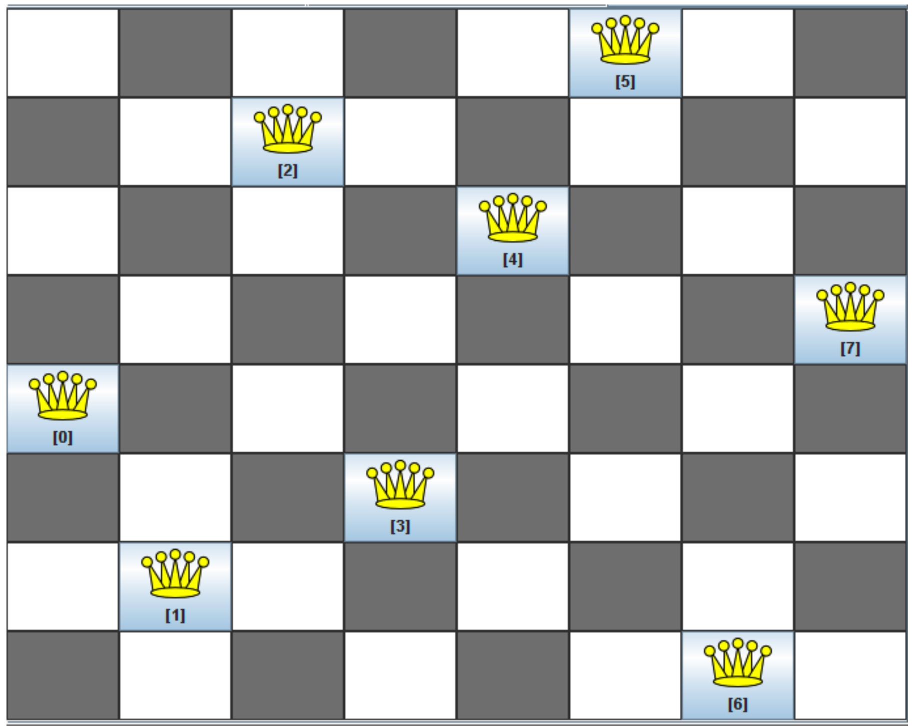
Theorem proving: Prove a formula is valid. Here:

> Is "the blue coloring is functionally dependent on the red/red and green coloring" (as a formula) valid, i.e. holds for all possible graphs?

I.e. analysis wrt. **all** instances ⇒ theorem proving is adequate

Theorem Prover Demo

Another Example: Proving Symmetries



The n-queens problem has “variable symmetries”: mirroring preserves solutions.

Can add a constraint “queen on first row must be left of queen on last row”.

Theorem Prover Demo

Part 2: Methods in Automated Theorem Proving

How to Build a (First-Order) Theorem Prover

1. Fix an **input language** for formulas
2. Fix a **semantics** to define what the formulas mean
   Will be always "classical" here
3. Determine the desired **services** from the theorem prover
   (The questions we would like the prover be able to answer)
4. Design a **calculus** for the logic and the services
   Calculus: high-level description of the "logical analysis" algorithm
   This includes redundancy criteria for formulas and inferences
5. Prove the calculus is **correct** (sound and complete) wrt. the logic and the services, if possible
6. Design a **proof procedure** for the calculus
7. Implement the proof procedure (research topic of its own)

Go through the red issues in the rest of this talk

How to Build a (First-Order) Theorem Prover

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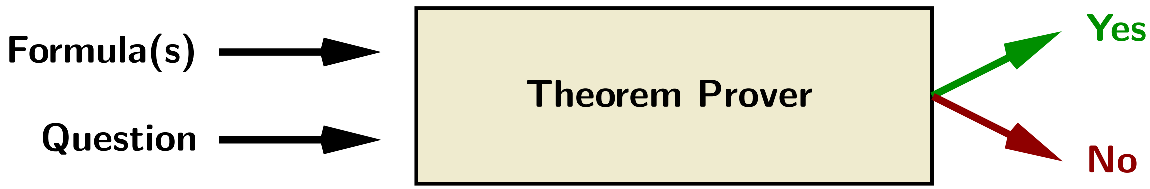
Languages and Services — Propositional SAT



Formula: Propositional logic formula ϕ

Question: Is ϕ satisfiable?

(Minimal model? Maximal consistent subsets?)

Theorem Prover: Based on BDD, **DPLL**, or stochastic local search

Issue: the formula ϕ can be **BIG**

DPLL as a Semantic Tree Method

(1) $A \vee B$ (2) $C \vee \neg A$ (3) $D \vee \neg C \vee \neg A$ (4) $\neg D \vee \neg B$

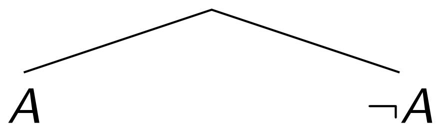
$\langle$empty tree$\rangle$

$$\{\} \not\models A \vee B$$
$$\{\} \models C \vee \neg A$$
$$\{\} \models D \vee \neg C \vee \neg A$$
$$\{\} \models \neg D \vee \neg B$$

- A Branch stands for an interpretation
- **Purpose of splitting:** satisfy a clause that is currently falsified
- Close branch if some clause is plainly falsified by it ($\star$)

DPLL as a Semantic Tree Method

(1) $A \vee B$ (2) $C \vee \neg A$ (3) $D \vee \neg C \vee \neg A$ (4) $\neg D \vee \neg B$



$$\{A\} \models A \vee B$$
$$\{A\} \not\models C \vee \neg A$$
$$\{A\} \models D \vee \neg C \vee \neg A$$
$$\{A\} \models \neg D \vee \neg B$$

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DPLL as a Semantic Tree Method

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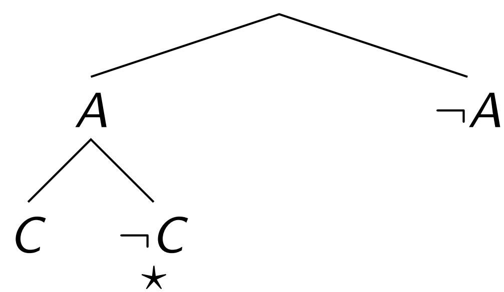

$$\{A, C\} \models A \vee B$$
$$\{A, C\} \models C \vee \neg A$$
$$\{A, C\} \not\models D \vee \neg C \vee \neg A$$
$$\{A, C\} \models \neg D \vee \neg B$$

- A Branch stands for an interpretation
- **Purpose of splitting:** satisfy a clause that is currently falsified
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DPLL as a Semantic Tree Method

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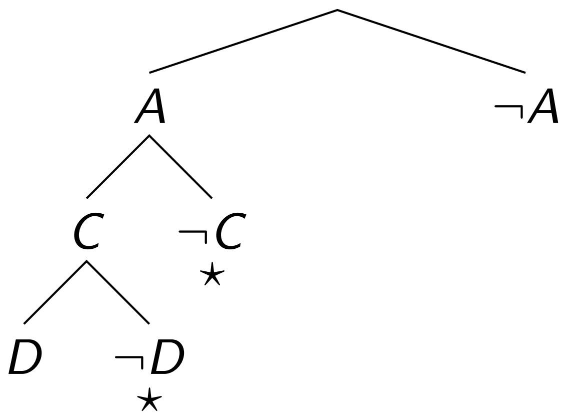


$\{A, C, D\} \models A \vee B$

$\{A, C, D\} \models C \vee \neg A$

$\{A, C, D\} \models D \vee \neg C \vee \neg A$

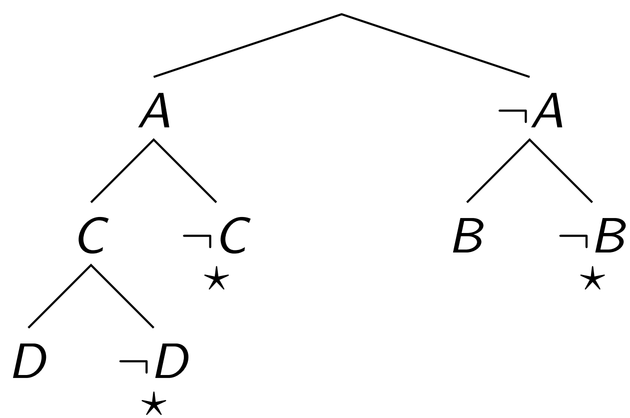
$\{A, C, D\} \models \neg D \vee \neg B$

Model $\{A, C, D\}$ found.

- A Branch stands for an interpretation
- **Purpose of splitting:** satisfy a clause that is currently falsified
- Close branch if some clause is plainly falsified by it ($\star$)

DPLL as a Semantic Tree Method

(1) $A \lor B$ (2) $C \lor \neg A$ (3) $D \lor \neg C \lor \neg A$ (4) $\neg D \lor \neg B$



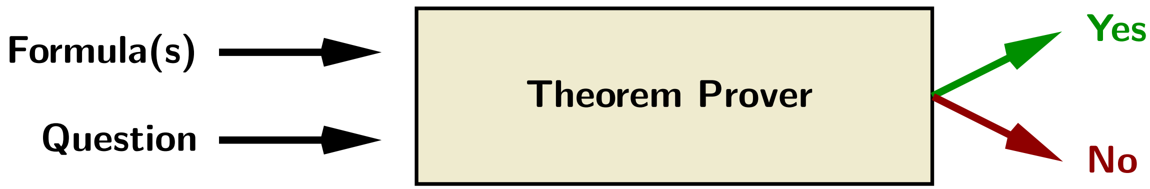
$$\{B\} \models A \lor B$$
$$\{B\} \models C \lor \neg A$$
$$\{B\} \models D \lor \neg C \lor \neg A$$
$$\{B\} \models \neg D \lor \neg B$$

Model $\{B\}$ found.

- A Branch stands for an interpretation
- **Purpose of splitting:** satisfy a clause that is currently falsified
- Close branch if some clause is plainly falsified by it ($\star$)

DPLL is the basis of most efficient SAT solvers today

Languages and Services — Description Logics



Formula: Description Logic TBox + ABox (restricted FOL)

TBox: Terminology

ABox: Assertions

$$\text{Professor} \sqcap \exists\, \text{supervises} \,.\, \text{Student} \sqsubseteq \text{BusyPerson}$$

$$p : \text{Professor} \qquad (p, s) : \text{supervises}$$

Question: Is TBox + ABox satisfiable?

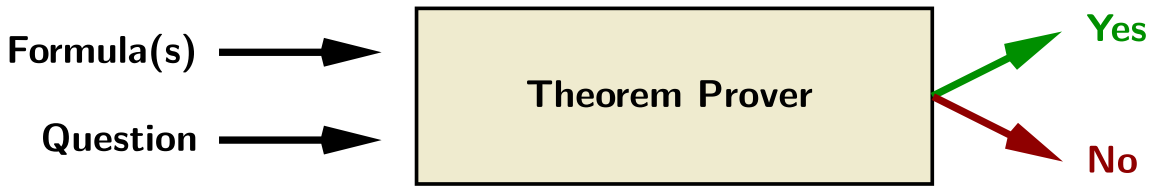
(Does C subsume D?, Concept hierarchy?)

Theorem Prover: Tableaux algorithms (predominantly)

Issue: Push expressivity of DLs while preserving decidability

See overview lecture by Maurice Pagnucco on “Knowledge Representation and Reasoning”

Languages and Services — Satisfiability Modulo Theories (SM



Formula: Usually **variable-free** first-order logic formula ϕ

Equality $\doteq$, combination of theories, free symbols

Question: Is ϕ valid? (satisfiable? entailed by another formula?)

$$\models_{\mathbb{N} \cup \mathbb{L}} \forall l \; (c = 5 \rightarrow \mathrm{car}(\mathrm{cons}(3 + c, l)) \doteq 8)$$

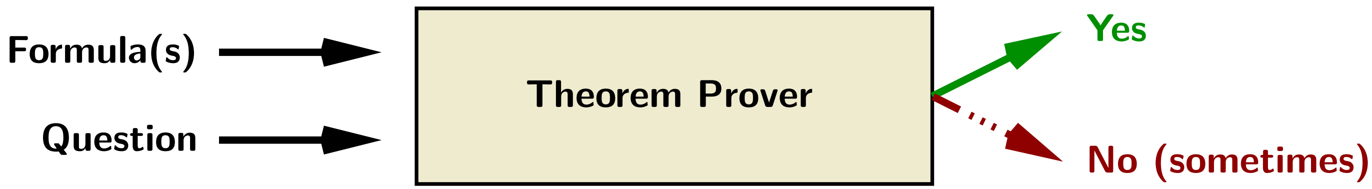
Theorem Prover: DPLL(T), translation into SAT, first-order provers

Issue: essentially undecidable for non-variable free fragment

$$P(0) \wedge (\forall x \; P(x) \rightarrow P(x+1)) \models_{\mathbb{N}} \forall x \; P(x)$$

Design a “good” prover anyways (ongoing research)

Languages and Services — “Full” First-Order Logic



Formula: First-order logic formula ϕ (e.g. the three-coloring spec above)
Usually with equality $\doteq$

Question: Is ϕ formula valid? (satisfiable?, entailed by another formula?)

Theorem Prover: Superposition (Resolution), Instance-based methods

Issues

- Efficient treatment of equality
- Decision procedure for sub-languages or useful reductions?
  Can do e.g. DL reasoning? Model checking? Logic programming?
- Built-in inference rules for arrays, lists, arithmetics (still open research)

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Semantics

"The function f is continuous", expressed in (first-order) predicate logic:

$$\forall\varepsilon(0 < \varepsilon \rightarrow \forall a \exists \delta(0 < \delta \wedge \forall x(|x - a| < \delta \rightarrow |f(x) - f(a)| < \varepsilon)))$$

Underlying Language

Variables ε, a, δ, x

Function symbols 0, $|\_|$, $\_ - \_$, $f(\_)$

Terms are well-formed expressions over variables and function symbols

Predicate symbols $\_ < \_$, $\_ = \_$

Atoms are applications of predicate symbols to terms

Boolean connectives $\wedge$, $\vee$, $\rightarrow$, $\neg$

Quantifiers $\forall$, $\exists$

The function symbols and predicate symbols comprise a signature Σ

Semantics

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$$\forall \varepsilon (0 < \varepsilon \rightarrow \forall a \exists \delta (0 < \delta \wedge \forall x (|x - a| < \delta \rightarrow |f(x) - f(a)| < \varepsilon)))$$

"Meaning" of Language Elements – Σ-Algebras

Universe (aka Domain): Set U

Variables $\mapsto$ values in U (mapping is called "assignment")

Function symbols $\mapsto$ (total) functions over U

Predicate symbols $\mapsto$ relations over U

Boolean connectives $\mapsto$ the usual boolean functions

Quantifiers $\mapsto$ "for all ... holds", "there is a ..., such that"

Terms $\mapsto$ values in U

Formulas $\mapsto$ Boolean (Truth-) values

Semantics - Σ-Algebra Example

Let Σ_{PA} be the standard signature of Peano Arithmetic
The standard interpretation $\mathbb{N}$ for Peano Arithmetic then is:

$$
\begin{array}{rcl}
U_{\mathbb{N}} & = & \{0, 1, 2, \ldots\} \\
0_{\mathbb{N}} & = & 0 \\
s_{\mathbb{N}} & : & n \mapsto n + 1 \\
+_{\mathbb{N}} & : & (n, m) \mapsto n + m \\
*_{\mathbb{N}} & : & (n, m) \mapsto n * m \\
\leq_{\mathbb{N}} & = & \{(n, m) \mid n \text{ less than or equal to } m\} \\
<_{\mathbb{N}} & = & \{(n, m) \mid n \text{ less than } m\}
\end{array}
$$

Note that $\mathbb{N}$ is just one out of **many possible** Σ_{PA}-interpretations

Semantics - Σ-Algebra Example

Evaluation of terms and formulas

Under the interpretation $\mathbb{N}$ and the assignment $\beta : x \mapsto 1, y \mapsto 3$ we obtain

$$
\begin{array}{lcl}
(\mathbb{N}, \beta)(s(x) + s(0)) & = & 3 \\
(\mathbb{N}, \beta)(x + y \doteq s(y)) & = & \mathit{True} \\
(\mathbb{N}, \beta)(\forall z\ z \leq y) & = & \mathit{False} \\
(\mathbb{N}, \beta)(\forall x \exists y\ x < y) & = & \mathit{True} \\
\mathbb{N}(\forall x \exists y\ x < y) & = & \mathit{True} \qquad \text{(Short notation when } \beta \text{ irrelevant)}
\end{array}
$$

Important Basic Notion: Model

If ϕ is a closed formula, then, instead of $I(\phi) = \mathit{True}$ one writes

$$I \models \phi \qquad (\text{“}I \text{ is a model of } \phi\text{”})$$

E.g. $\mathbb{N} \models \forall x \exists y\ x < y$

Standard reasoning services can now be expressed semantically

Services Semantically

E.g. "entailment":

$$\text{Axioms over } \mathbb{R} \wedge \text{continuous}(f) \wedge \text{continuous}(g) \models \text{continuous}(f+g) \text{ ?}$$

Services

Model(I,ϕ): $I \models \phi$? (Is I a model for ϕ?)

Validity(ϕ): $\models \phi$? ($I \models \phi$ for every interpretation?)

Satisfiability(ϕ): ϕ satisfiable? ($I \models \phi$ for some interpretation?)

Entailment(ϕ,ψ): $\phi \models \psi$? (does ϕ entail ψ?, i.e.
for every interpretation I: if $I \models \phi$ then $I \models \psi$?)

Solve(I,ϕ): find an assignment β such that $I, \beta \models \phi$

Solve(ϕ): find an interpretation and assignment β such that $I, \beta \models \phi$

Additional complication: fix interpretation of some symbols (as in $\mathbb{N}$ above)

What if theorem prover's native service is only "Is ϕ unsatisfiable?" ?

Semantics - Reduction to Unsatisfiability

- Suppose we want to prove an entailment $\phi \models \psi$
- Equivalently, prove $\models \phi \rightarrow \psi$, i.e. that $\phi \rightarrow \psi$ is valid
- Equivalently, prove that $\neg(\phi \rightarrow \psi)$ is not satisfiable (unsatisfiable)
- Equivalently, prove that $\phi \wedge \neg\psi$ is unsatisfiable

Basis for (predominant) refutational theorem proving

Dual problem, much harder: to disprove an entailment $\phi \models \psi$ find a model of $\phi \wedge \neg\psi$

One motivation for (finite) model generation procedures

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Calculus - Normal Forms

Most first-order theorem provers take formulas in **clause normal form**

Why Normal Forms?

- Reduction of logical concepts (operators, quantifiers)
- Reduction of syntactical structure (nesting of subformulas)
- Can be exploited for efficient data structures and control

Translation into Clause Normal Form

Formula → **Prenex normal form** → **Skolem normal form** → **Clause normal form** → **Clausal Theorem Prover**

Prop: the given formula and its clause normal form are equi-satisfiable

Prenex Normal Form

Prenex formulas have the form

$$Q_1 x_1 \ldots Q_n x_n \; F,$$

where F is quantifier-free and $Q_i \in \{\forall, \exists\}$

Computing prenex normal form by the rewrite relation $\Rightarrow_P$:

$$\begin{array}{rcll}
(F \leftrightarrow G) & \Rightarrow_P & (F \rightarrow G) \wedge (G \rightarrow F) & \\
\neg Q x F & \Rightarrow_P & \overline{Q} x \neg F & (\neg Q) \\
(Q x F \;\rho\; G) & \Rightarrow_P & Q y (F[y/x] \;\rho\; G), \;\; y \text{ fresh}, \;\; \rho \in \{\wedge, \vee\} & \\
(Q x F \rightarrow G) & \Rightarrow_P & \overline{Q} y (F[y/x] \rightarrow G), \;\; y \text{ fresh} & \\
(F \;\rho\; Q x G) & \Rightarrow_P & Q y (F \;\rho\; G[y/x]), \;\; y \text{ fresh}, \;\; \rho \in \{\wedge, \vee, \rightarrow\} &
\end{array}$$

Here $\overline{Q}$ denotes the quantifier **dual** to Q, i.e., $\overline{\forall} = \exists$ and $\overline{\exists} = \forall$.

In the Example

$$\forall\varepsilon(0<\varepsilon\rightarrow\forall a\exists\delta(0<\delta\wedge\forall x(|x-a|<\delta\rightarrow|f(x)-f(a)|<\varepsilon)))$$

$$\Rightarrow_P$$

$$\forall\varepsilon\forall a(0<\varepsilon\rightarrow\exists\delta(0<\delta\wedge\forall x(|x-a|<\delta\rightarrow|f(x)-f(a)|<\varepsilon)))$$

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$$\Rightarrow_P$$

$$\forall\varepsilon\forall a\exists\delta\forall x(0<\varepsilon\rightarrow(0<\delta\wedge(|x-a|<\delta\rightarrow|f(x)-f(a)|<\varepsilon)))$$

Skolem Normal Form

Formula $\longrightarrow$ **Prenex normal form** $\longrightarrow$ **Skolem normal form** $\longrightarrow$ **Clause normal form** $\longrightarrow$ **Clausal Theorem Prover**

Intuition: replacement of $\exists y$ by a concrete choice function computing y from all the arguments y depends on.

Transformation $\Rightarrow_S$

$$\forall x_1, \ldots, x_n \exists y \; F \quad \Rightarrow_S \quad \forall x_1, \ldots, x_n \; F[f(x_1, \ldots, x_n)/y]$$

where f/n is a new function symbol (**Skolem function**).

In the Example

$$\forall \varepsilon \forall a \exists \delta \forall x (0 < \varepsilon \rightarrow 0 < \delta \land (|x - a| < \delta \rightarrow |f(x) - f(a)| < \varepsilon))$$

$$\Rightarrow_S$$

$$\forall \varepsilon \forall a \forall x (0 < \varepsilon \rightarrow 0 < d(\varepsilon, a) \land (|x - a| < d(\varepsilon, a) \rightarrow |f(x) - f(a)| < \varepsilon))$$

Clausal Normal Form (Conjunctive Normal Form)

Rules to convert the matrix of the formula in Skolem normal form into a conjunction of disjunctions:

$$
\begin{array}{rcl}
(F \leftrightarrow G) & \Rightarrow_K & (F \rightarrow G) \land (G \rightarrow F) \\
(F \rightarrow G) & \Rightarrow_K & (\neg F \lor G) \\
\neg(F \lor G) & \Rightarrow_K & (\neg F \land \neg G) \\
\neg(F \land G) & \Rightarrow_K & (\neg F \lor \neg G) \\
\neg\neg F & \Rightarrow_K & F \\
(F \land G) \lor H & \Rightarrow_K & (F \lor H) \land (G \lor H) \\
(F \land \top) & \Rightarrow_K & F \\
(F \land \bot) & \Rightarrow_K & \bot \\
(F \lor \top) & \Rightarrow_K & \top \\
(F \lor \bot) & \Rightarrow_K & F
\end{array}
$$

They are to be applied modulo associativity and commutativity of $\land$ and $\lor$

In the Example

$$\forall \varepsilon \forall a \forall x (0 < \varepsilon \rightarrow 0 < d(\varepsilon, a) \wedge (|x - a| < d(\varepsilon, a) \rightarrow |f(x) - f(a)| < \varepsilon))$$

$$\Rightarrow_K$$

$$0 < d(\varepsilon, a) \ \vee \ \neg (0 < \varepsilon)$$

$$\neg (|x - a| < d(\varepsilon, a)) \ \vee \ |f(x) - f(a)| < \varepsilon \ \vee \ \neg (0 < \varepsilon)$$

Note: The universal quantifiers for the variables ε, a and x, as well as the conjunction symbol $\wedge$ between the clauses are not written, for convenience

The Complete Picture

$$
\begin{array}{llll}
F & \Rightarrow^*_P & Q_1 y_1 \ldots Q_n y_n \; G & (G \text{ quantifier-free}) \\[1em]
 & \Rightarrow^*_S & \forall x_1, \ldots, x_m \; H & (m \leq n,\; H \text{ quantifier-free}) \\[1em]
 & \Rightarrow^*_K & \underbrace{\underbrace{\forall x_1, \ldots, x_m}_{\text{leave out}} \bigwedge_{i=1}^{k} \underbrace{\bigvee_{j=1}^{n_i} L_{ij}}_{\text{clauses } C_i}}_{F'} &
\end{array}
$$

$N = \{C_1, \ldots, C_k\}$ is called the **clausal (normal) form** (CNF) of F

Note: the variables in the clauses are implicitly universally quantified

Instead of showing that F is unsatisfiable, the proof problem from now is to show that N is unsatisfiable

Can do better than "searching through all interpretations"

Theorem: N is satisfiable iff it has a Herbrand model

Herbrand Interpretations

A **Herbrand interpretation** (over a given signature Σ) is a Σ-algebra $\mathcal{A}$ such that

- The universe is the set T_Σ of ground terms over Σ
  (a **ground term** is a term without any variables):

$$U_\mathcal{A} = \mathsf{T}_\Sigma$$

- Every function symbol from Σ is "mapped to itself":

$$f_\mathcal{A} : (s_1, \ldots, s_n) \mapsto f(s_1, \ldots, s_n), \text{ where } f \text{ is } n\text{-ary function symbol in } \Sigma$$

Example

- $\Sigma_{Pres} = (\{0/0, s/1, +/2\},\ \{</2, \leq/2\})$
-
$$U_\mathcal{A} = \{0, s(0), s(s(0)), \ldots, 0+0, s(0)+0, \ldots, s(0+0), s(s(0)+0), \ldots\}$$
- $0 \mapsto 0, s(0) \mapsto s(0), s(s(0)) \mapsto s(s(0)), \ldots, 0+0 \mapsto 0+0, \ldots$

Herbrand Interpretations

Only interpretations $p_{\mathcal{A}}$ of predicate symbols $p \in \Sigma$ is undetermined in a Herbrand interpretation

- $p_{\mathcal{A}}$ represented as the set of ground atoms

  $$\{p(s_1, \ldots, s_n) \mid (s_1, \ldots, s_n) \in p_{\mathcal{A}} \text{ where } p \in \Sigma \text{ is } n\text{-ary predicate symbol}\}$$

- Whole interpretation represented as $\bigcup_{p \in \Sigma} p_{\mathcal{A}}$

Example

- $\Sigma_{Pres} = (\{0/0, s/1, +/2\},\ \{</2, \leq/2\})$ (from above)
- $\mathbb{N}$ as Herbrand interpretation over Σ_{Pres}

  $$\begin{aligned} I = \{ \quad & 0 \leq 0,\ 0 \leq s(0),\ 0 \leq s(s(0)),\ \ldots, \\ & 0+0 \leq 0,\ 0+0 \leq s(0),\ \ldots, \\ & \ldots,\ (s(0)+0)+s(0) \leq s(0)+(s(0)+s(0)), \ldots \quad \} \end{aligned}$$

Herbrand's Theorem

Proposition

A Skolem normal form $\forall\phi$ is unsatisfiable iff it has no Herbrand model

Theorem (Skolem-Herbrand-Theorem)

$\forall\phi$ has no Herbrand model iff some finite set of ground instances $\{\phi\gamma_1, \ldots, \phi\gamma_n\}$ is unsatisfiable

Applied to clause logic:

Theorem (Skolem-Herbrand-Theorem)

A set N of Σ-clauses is unsatisfiable iff some finite set of ground instances of clauses from N is unsatisfiable

Leads immediately to theorem prover "Gilmore's Method"

Gilmore's Method - Based on Herbrand's Theorem

Preprocessing:

Given Formula

$$\forall x\, \exists y\; P(y,x) \\ \land \forall z\; \neg P(z,a)$$

$\longrightarrow$

Clause Form

$$P(f(x),x) \\ \neg P(z,a)$$

Outer loop:
Grounding

Inner loop:
Propositional
Method

Gilmore's Method - Based on Herbrand's Theorem

Preprocessing:

Given Formula

$\forall x\ \exists y\ P(y, x)$
$\land \forall z\ \neg P(z, a)$

$\longrightarrow$

Clause Form

$P(f(x), x)$
$\neg P(z, a)$

Outer loop:
Grounding

$P(f(a), a)$
$\neg P(a, a)$

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Method

Gilmore's Method - Based on Herbrand's Theorem

Preprocessing:

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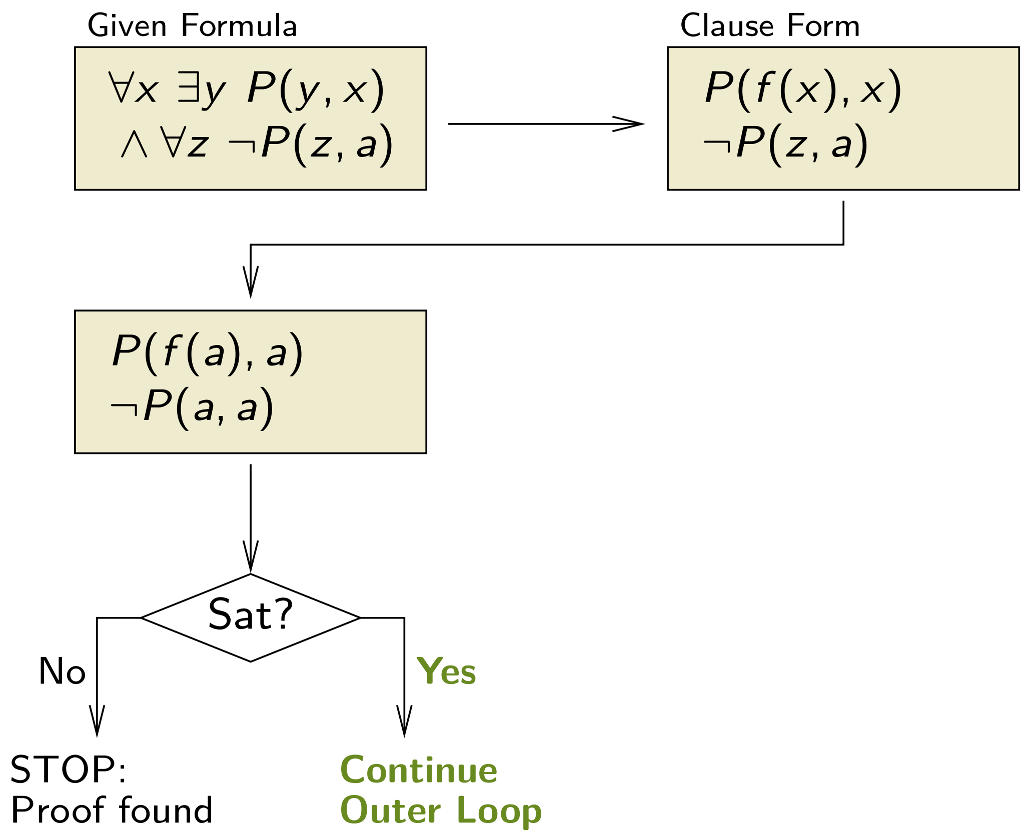


Gilmore's Method - Based on Herbrand's Theorem

Preprocessing:

Given Formula

$$\forall x\ \exists y\ P(y,x) \land \forall z\ \neg P(z,a)$$

$\longrightarrow$

Clause Form

$P(f(x),x)$
$\neg P(z,a)$

$\downarrow$

Outer loop:
Grounding

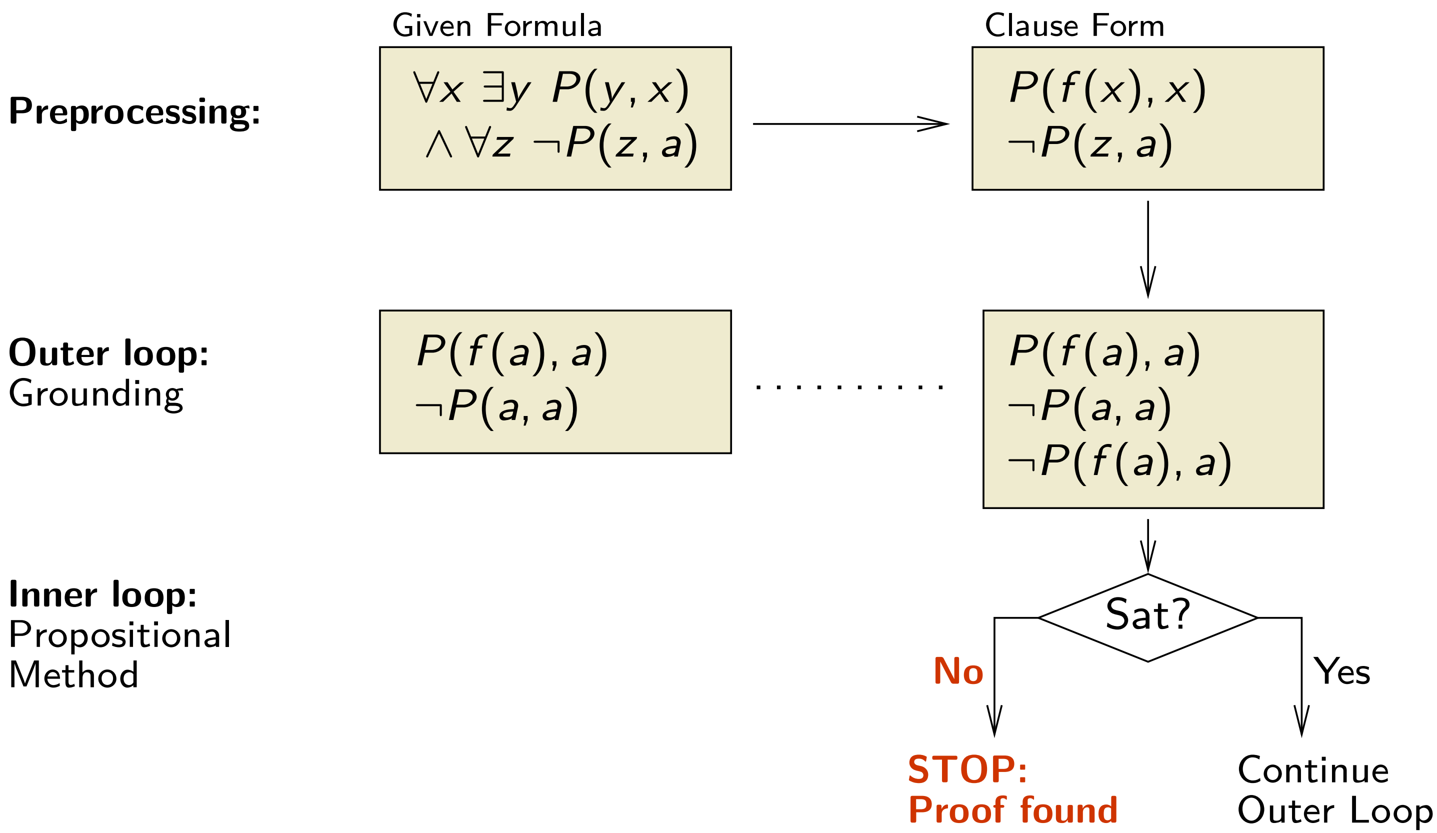
$P(f(a),a)$
$\neg P(a,a)$

. . . .

$P(f(a),a)$
$\neg P(a,a)$
$\neg P(f(a),a)$

Inner loop:
Propositional
Method

Gilmore's Method - Based on Herbrand's Theorem



Calculi for First-Order Logic Theorem Proving

- Gilmore's method reduces proof search in first-order logic to propositional logic unsatisfiability problems
- Main problem is the unguided generation of (very many) ground clauses
- All modern calculi address this problem in one way or another, e.g.
  - **Guidance:** Instance-Based Methods are similar to Gilmore's method but generate ground instances in a guided way
  - **Avoidance:** Resolution calculi need not generate the ground instances at all

    Resolution inferences operate directly on clauses, not on their ground instances

Next: propositional Resolution, lifting, first-order Resolution

The Propositional Resolution Calculus *Res*

Modern versions of the first-order version of the resolution calculus [Robinson 1965] are (still) the most important calculi for FOTP today.

Propositional resolution inference rule:

$$\frac{C \vee A \qquad \neg A \vee D}{C \vee D}$$

Terminology: $C \vee D$: **resolvent**; A: **resolved atom**

Propositional (positive) factorisation inference rule:

$$\frac{C \vee A \vee A}{C \vee A}$$

These are **schematic inference rules**:

C and D – propositional clauses

A – propositional atom

"$\vee$" is considered associative and commutative

Sample Proof

1.	$\neg A \vee \neg A \vee B$	(given)
2.	$A \vee B$	(given)
3.	$\neg C \vee \neg B$	(given)
4.	C	(given)
5.	$\neg A \vee B \vee B$	(Res. 2. into 1.)
6.	$\neg A \vee B$	(Fact. 5.)
7.	$B \vee B$	(Res. 2. into 6.)
8.	B	(Fact. 7.)
9.	$\neg C$	(Res. 8. into 3.)
10.	$\perp$	(Res. 4. into 9.)

Soundness of Propositional Resolution

Proposition

Propositional resolution is sound

Proof:

Let $I \in \Sigma$-Alg. To be shown:

1. for resolution: $I \models C \vee A,\ I \models D \vee \neg A \ \Rightarrow\ I \models C \vee D$
2. for factorization: $I \models C \vee A \vee A \ \Rightarrow\ I \models C \vee A$

Ad (i): Assume premises are valid in I. Two cases need to be considered: (a) A is valid in I, or (b) $\neg A$ is valid in I.

a) $I \models A \Rightarrow I \models D \Rightarrow I \models C \vee D$

b) $I \models \neg A \Rightarrow I \models C \Rightarrow I \models C \vee D$

Ad (ii): even simpler

Completeness of Propositional Resolution

Theorem:

Propositional Resolution is refutationally complete

- That is, if a propositional clause set is unsatisfiable, then Resolution will derive the empty clause $\perp$ eventually
- More precisely: If a clause set is unsatisfiable and closed under the application of the Resolution and Factorization inference rules, then it contains the empty clause $\perp$
- Perhaps easiest proof: semantic tree proof technique (see blackboard)
- This result can be considerably strengthened, some strengthenings come for free from the proof

Propositional resolution is not suitable for first-order clause sets

Lifting Propositional Resolution to First-Order Resolution

Propositional resolution

Clauses	Ground instances
$P(f(x), y)$	$\{P(f(a), a), \ldots, P(f(f(a)), f(f(a))), \ldots\}$
$\neg P(z, z)$	$\{\neg P(a), \ldots, \neg P(f(f(a)), f(f(a))), \ldots\}$

Only common instances of $P(f(x), y)$ and $P(z, z)$ give rise to inference:

$$\frac{P(f(f(a)), f(f(a))) \qquad \neg P(f(f(a)), f(f(a)))}{\perp}$$

Unification

All common instances of $P(f(x), y)$ and $P(z, z)$ are instances of $P(f(x), f(x))$

$P(f(x), f(x))$ is computed deterministically by **unification**

First-order resolution

$$\frac{P(f(x), y) \qquad \neg P(z, z)}{\perp}$$

Justified by existence of $P(f(x), f(x))$

Can represent infinitely many propositional resolution inferences

Substitutions and Unifiers

- A **substitution** σ is a mapping from variables to terms which is the identity almost everywhere

  Example: $\sigma = [y \mapsto f(x),\ z \mapsto f(x)]$

- A substitution can be **applied** to a term or atom t, written as $t\sigma$

  Example, where σ is from above: $P(f(x), y)\sigma = P(f(x), f(x))$

- A substitution γ is a **unifier** of s and t iff $s\gamma = t\gamma$

  Example: $\gamma = [x \mapsto a, y \mapsto f(a),\ z \mapsto f(a)]$ is a unifier of $P(f(x), y)$ and $P(z, z)$

- A unifier σ of s is **most general** iff for every unifier γ of s and t there is a substitution δ such that $\gamma = \sigma \circ \delta$; notation: $\sigma = \mathrm{mgu}(s, t)$

  Example: $\sigma = [y \mapsto f(x),\ z \mapsto f(x)] = \mathrm{mgu}(P(f(x), y), P(z, z))$

  There are (linear) algorithms to compute mgu's or return "fail"

Resolution for First-Order Clauses

$$\frac{C \vee A \qquad D \vee \neg B}{(C \vee D)\sigma} \quad \text{if } \sigma = \mathrm{mgu}(A, B) \qquad \text{[resolution]}$$

$$\frac{C \vee A \vee B}{(C \vee A)\sigma} \quad \text{if } \sigma = \mathrm{mgu}(A, B) \quad \text{[factorization]}$$

In both cases, A and B have to be renamed apart (made variable disjoint).

Example

$$\frac{Q(z) \vee P(z, z) \quad \neg P(x, y)}{Q(x)} \quad \text{where } \sigma = [z \mapsto x, y \mapsto x] \qquad \text{[resolution]}$$

$$\frac{Q(z) \vee P(z, a) \vee P(a, y)}{Q(a) \vee P(a, a)} \quad \text{where } \sigma = [z \mapsto a, y \mapsto a] \quad \text{[factorization]}$$

Completeness of First-Order Resolution

Theorem: Resolution is **refutationally complete**

- That is, if a clause set is unsatisfiable, then Resolution will derive the empty clause $\bot$ eventually
- More precisely: If a clause set is unsatisfiable and closed under the application of the Resolution and Factorization inference rules, then it contains the empty clause $\bot$
- Perhaps easiest proof: Herbrand Theorem + completeness of propositional resolution + Lifting Theorem (see blackboard)

  Lifting Theorem: the conclusion of any propositional inference on ground instances of first-order clauses can be obtained by instantiating the conclusion of a first-order inference on the first-order clauses
- Closure can be achieved by the “Given Clause Loop”

The “Given Clause Loop”

As used in the Otter theorem prover:

Lists of clauses maintained by the algorithm: `usable` and `sos`.

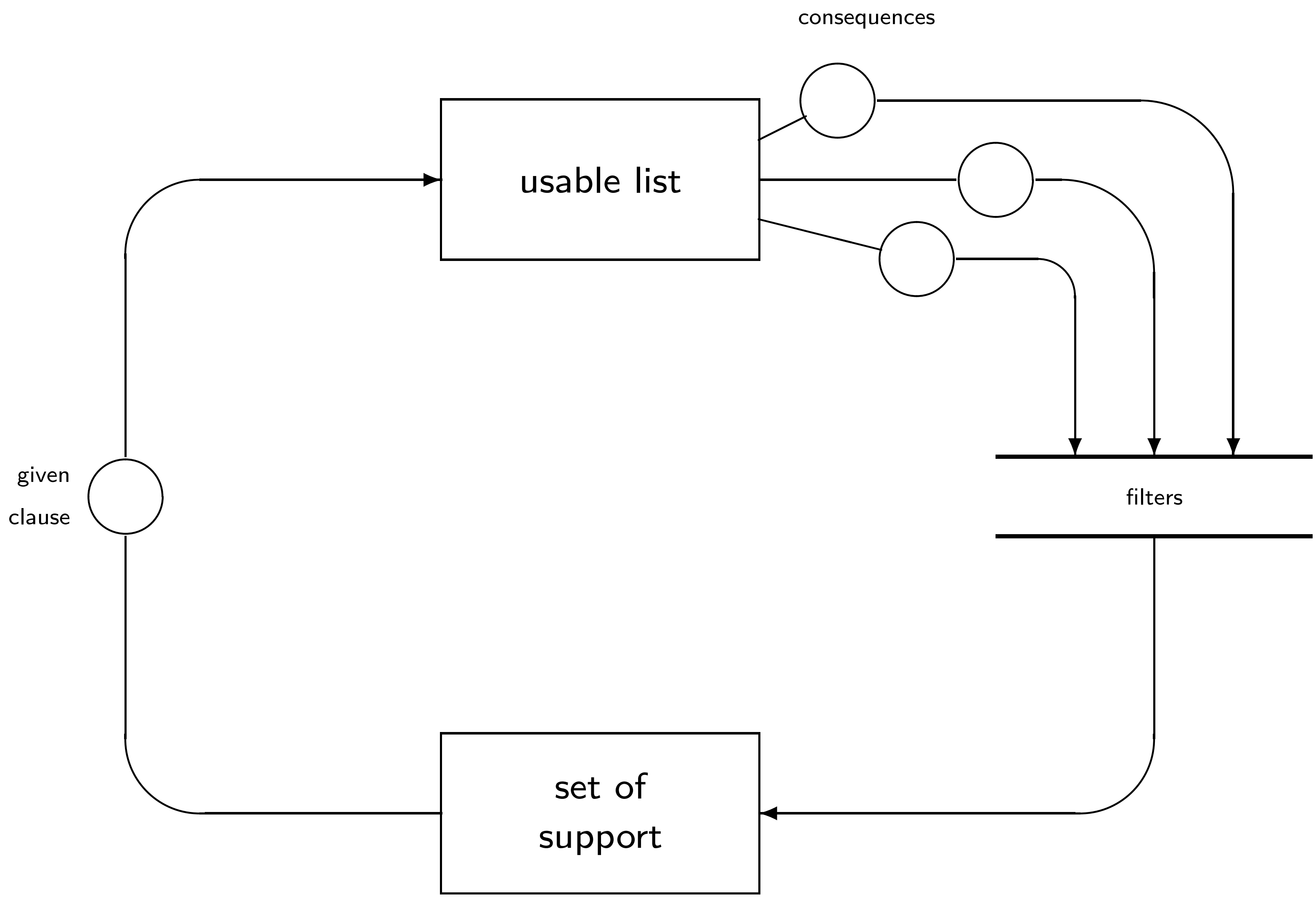
Initialize `sos` with the input clauses, `usable` empty.

Algorithm (straight from the Otter manual):

```
While (sos is not empty and no refutation has been found)
  1. Let given_clause be the ‘lightest’ clause in sos;
  2. Move given_clause from sos to usable;
  3. Infer and process new clauses using the inference rules in
     effect; each new clause must have the given_clause as
     one of its parents and members of usable as its other
     parents;  new clauses that pass the retention tests
     are appended to sos;
End of while loop.
```

Fairness: define clause weight e.g. as “depth + length” of clause.

The “Given Clause Loop” - Graphically



Calculi for First-Order Logic Theorem Proving

Recall:

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- Main problem is the unguided generation of (very many) ground clauses
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    Resolution inferences operate directly on clauses, not on their ground instances

Next: Instance-Based Method "Inst-Gen"

Inst-Gen [Ganzinger&Korovin 2003]

Idea: "semantic" guidance: add only instances that are falsified by a "candidate model"

Eventually, all repairs will be made or there is no more candidate model

Important notation: $\perp$ denotes both a unique constant and a substitution that maps every variable to $\perp$

Example (S is "current clause set"):

$$S: \quad P(x,y) \lor P(y,x) \qquad\qquad S\perp: \quad P(\perp,\perp) \lor P(\perp,\perp)$$

$$\quad \neg P(x,x) \qquad\qquad\qquad\qquad \neg P(\perp,\perp)$$

Analyze $S\perp$:

Case 1: SAT detects unsatisfiability of $S\perp$

Then Conclude S is unsatisfiable

But what if $S\perp$ is satisfied by some model, denoted by $I_\perp$?

Inst-Gen

Main idea: associate to model $I_\perp$ of $S\perp$ a **candidate model** I_S of S.

Calculus goal: add instances to S so that I_S becomes a model of S

Example:

$$S: \quad \underline{P(x)} \lor Q(x) \qquad\qquad S\perp: \quad \underline{P(\perp)} \lor Q(\perp)$$
$$\underline{\neg P(a)} \qquad\qquad\qquad\qquad \underline{\neg P(a)}$$

Analyze $S\perp$:

Case 2: SAT detects model $I_\perp = \{P(\perp), \neg P(a)\}$ of $S\perp$

Case 2.1: candidate model $I_S = \{\neg P(a)\}$ derived from literals <u>selected</u> in S by $I_\perp$ is not a model of S

$$\boxed{\textbf{Add “problematic” instance } P(a) \lor Q(a) \textbf{ to } S \textbf{ to refine } I_S}$$

Inst-Gen

Clause set after adding $P(a) \vee Q(a)$

$$S: \quad \underline{P(x)} \vee Q(x) \qquad\qquad S\bot: \quad \underline{P(\bot)} \vee Q(\bot)$$

$$P(a) \vee \underline{Q(a)} \qquad\qquad P(a) \vee \underline{Q(a)}$$

$$\underline{\neg P(a)} \qquad\qquad \underline{\neg P(a)}$$

Analyze $S\bot$:

Case 2: SAT detects model $I_\bot = \{P(\bot), Q(a), \neg P(a)\}$ of $S\bot$

Case 2.2: candidate model $I_S = \{Q(a), \neg P(a)\}$ derived from literals selected in S by $I_\bot$ **is** a model of S

Then conclude S is satisfiable

How to derive candidate model I_S?

Inst-Gen - Model Construction

It provides (partial) interpretation for S_{ground} for given clause set S

$$
S: \quad \underline{P(x)} \lor Q(x) \qquad\qquad \Sigma = \{a, b\},\ S_{\text{ground}}: \quad \underline{P(b)} \lor Q(b)
$$

$$
P(a) \lor \underline{Q(a)} \qquad\qquad\qquad\qquad P(a) \lor \underline{Q(a)}
$$

$$
\underline{\neg P(a)} \qquad\qquad\qquad\qquad \underline{\neg P(a)}
$$

- For each $C_{\text{ground}} \in S_{\text{ground}}$ find most specific $C \in S$ that can be instantiated to C_{ground}
- <u>Select literal</u> in C_{ground} corresponding to <u>selected literal</u> in that C
- Add <u>selected literal</u> of that C_{ground} to I_S if not in conflict with I_S

Thus, $I_S = \{P(b), Q(a), \neg P(a)\}$

Model Generation

Scenario: no "theorem" to prove, or disprove a "theorem"

A model provides further information then

Why compute models?

Planning: Can be formalised as propositional satisfiability problem.

[Kautz& Selman, AAAI96; Dimopolous et al, ECP97]

Diagnosis: Minimal models of *abnormal* literals (circumscription). [Reiter, AI87]

Databases: View materialisation, View Updates, Integrity Constraints.

Nonmonotonic reasoning: Various semantics (GCWA, Well-founded, Perfect, Stable,...), all based on minimal models. [Inoue et al, CADE 92]

Software Verification: Counterexamples to conjectured theorems.

Theorem proving: Counterexamples to conjectured theorems.

Finite models of quasigroups, (MGTP/G). [Fujita et al, IJCAI 93]

Model Generation

Why compute models (cont'd)?

Natural Language Processing:

- Maintain models $\mathcal{I}_1, \ldots, \mathcal{I}_n$ as different readings of discourses:

$$\mathcal{I}_i \models \mathit{BG\text{-}Knowledge} \cup \mathit{Discourse\_so\_far}$$

- Consistency checks ("Mia's husband loves Sally. She is not married.")

$$\begin{array}{ll} & \mathit{BG\text{-}Knowledge} \cup \mathit{Discourse\_so\_far} \not\models \neg \mathit{New\_utterance} \\ \mathit{iff} & \mathit{BG\text{-}Knowledge} \cup \mathit{Discourse\_so\_far} \cup \mathit{New\_utterance} \text{ is } \textbf{satisfiable} \end{array}$$

- Informativity checks ("Mia's husband loves Sally. She is married.")

$$\begin{array}{ll} & \mathit{BG\text{-}Knowledge} \cup \mathit{Discourse\_so\_far} \not\models \mathit{New\_utterance} \\ \mathit{iff} & \mathit{BG\text{-}Knowledge} \cup \mathit{Discourse\_so\_far} \cup \neg \mathit{New\_utterance} \text{ is } \textbf{satisfiable} \end{array}$$

Example - Group Theory

The following axioms specify a group

$$
\begin{array}{llllll}
\forall x, y, z & : & (x * y) * z & = & x * (y * z) & (\mathrm{associativity}) \\
\forall x & : & e * x & = & x & (\mathrm{left-identity}) \\
\forall x & : & i(x) * x & = & e & (\mathrm{left-inverse})
\end{array}
$$

Does

$$
\forall x, y \quad : \quad x * y \quad = \quad y * x \qquad (\mathrm{commutat.})
$$

follow?

No, it does not

Example - Group Theory

Counterexample: a group with finite domain of size 6, where the elements 2 and 3 are not commutative: Domain: $\{1, 2, 3, 4, 5, 6\}$

$e : 1$

$i :$

1	2	3	4	5	6
1	2	3	5	4	6

$* :$

	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	1	4	3	6	5
3	3	5	1	6	2	4
4	4	6	2	5	1	3
5	5	3	6	1	4	2
6	6	4	5	2	3	1

Finite Model Finders - Idea

- Assume a fixed domain size n.
- Use a tool to decide if there exists a model with domain size n for a given problem.
- Do this starting with $n = 1$ with increasing n until a model is found.
- Note: domain of size n will consist of $\{1, \ldots, n\}$.

1. Approach: SEM-style

- Tools: SEM, Finder, Mace4
- Specialized constraint solvers.
- For a given domain generate all ground instances of the clause.
- Example: For domain size 2 and clause $p(a, g(x))$ the instances are $p(a, g(1))$ and $p(a, g(2))$.

1. Approach: SEM-style

- Set up multiplication tables for all symbols with the whole domain as cell values.
- Example: For domain size 2 and function symbol g with arity 1 the cells are $g(1) = \{1, 2\}$ and $g(2) = \{1, 2\}$.
- Try to restrict each cell to exactly 1 value.
- The clauses are the constraints guiding the search and propagation.
- Example: if the cell of a contains $\{1\}$, the clause $a = b$ forces the cell of b to be $\{1\}$ as well.

2. Approach: Mace-style

- Tools: Mace2, Paradox
- For given domain size n transform first-order clause set into equisatisfiable propositional clause set.
- Original problem has a model of domain size n iff the transformed problem is satisfiable.
- Run SAT solver on transformed problem and translate model back.

Paradox - Example

Domain:	$\{1, 2\}$
Clauses:	$\{p(a) \vee f(x) = a\}$
Flattened:	$p(y) \vee f(x) = y \vee a \neq y$
Instances:	$p(1) \vee f(1) = 1 \vee a \neq 1$
	$p(2) \vee f(1) = 1 \vee a \neq 2$
	$p(1) \vee f(2) = 1 \vee a \neq 1$
	$p(2) \vee f(2) = 1 \vee a \neq 2$
Totality:	$a = 1 \vee a = 2$
	$f(1) = 1 \vee f(1) = 2$
	$f(2) = 1 \vee f(2) = 2$
Functionality:	$a \neq 1 \vee a \neq 2$
	$f(1) \neq 1 \vee f(1) \neq 2$
	$f(2) \neq 1 \vee f(2) \neq 2$

A model is obtained by setting the blue literals true

Theory Reasoning

Let T be a first-order theory of signature Σ

Let L be a class of Σ-formulas

The T-validity Problem

- Given ϕ in L, is it the case that $T \models \phi$? More accurately:
- Given ϕ in L, is it the case that $T \models \forall\ \phi$?

Examples

- "$0/0,\ s/1,\ +/2,\ =/2,\ \leq/2$" $\models \exists y . y > x$
- The theory of equality E $\models \phi$ (ϕ arbitrary formula)
- "An equational theory" $\models \exists\ s_1 = t_1 \wedge \cdots \wedge s_n = t_n$ (E-Unification problem)
- "Some group theory" $\models s = t$ (Word problem)

The T-validity problem is decidably only for restricted L and T

Approaches to Theory Reasoning

Theory-Reasoning in Automated First-Order Theorem Proving

- Semi-decide the T-validity problem, $T \models \phi$?
- ϕ arbitrary first-order formula, T universal theory
- Generality is strength and weakness at the same time
- Really successful only for specific instance:
  $T =$ equality, inference rules like paramodulation

Satisfiability Modulo Theories (SMT)

- Decide the T-validity problem, $T \models \phi$?
- Usual restriction: ϕ is quantifier-free, i.e. all variables implicitly universally quantified
- Applications in particular to formal verification

Checking Satisfiability Modulo Theories

Given: A quantifier-free formula ϕ (implicitly existentially quantified)
Task: Decide whether ϕ is T-satisfiable
(T-validity via "$T \models \forall\ \phi$" iff "$\exists\ \neg\phi$ is not T-satisfiable")

Approach: eager translation into SAT

- Encode problem into a T-equisatisfiable propositional formula
- Feed formula to a SAT-solver
- Example: $T =$ equality (Ackermann encoding)

Approach: lazy translation into SAT

- Couple a SAT solver with a given decision procedure for T-satisfiability of ground literals
- For instance if T is "equality" then the Nelson-Oppen congruence closure method can be used

Lazy Translation into SAT

$$g(a) = c \quad \wedge \quad f(g(a)) \neq f(c) \ \vee \ g(a) = d \quad \wedge \quad c \neq d$$

Theory: Equality

Lazy Translation into SAT

$$\underbrace{g(a) = c}_{1} \quad \wedge \quad \underbrace{f(g(a)) \neq f(c)}_{\overline{2}} \vee \underbrace{g(a) = d}_{3} \quad \wedge \quad \underbrace{c \neq d}_{\overline{4}}$$

Lazy Translation into SAT

$$\underbrace{g(a) = c}_{1} \quad \wedge \quad \underbrace{f(g(a)) \neq f(c)}_{\overline{2}} \vee \underbrace{g(a) = d}_{3} \quad \wedge \quad \underbrace{c \neq d}_{\overline{4}}$$

- Send $\{1,\ \overline{2} \vee 3,\ \overline{4}\}$ to SAT solver.

Lazy Translation into SAT

$$\underbrace{g(a) = c}_{1} \quad \wedge \quad \underbrace{f(g(a)) \neq f(c)}_{\overline{2}} \vee \underbrace{g(a) = d}_{3} \quad \wedge \quad \underbrace{c \neq d}_{\overline{4}}$$

- Send $\{1,\ \overline{2} \vee 3,\ \overline{4}\}$ to SAT solver.
- SAT solver returns model $\{1,\ \overline{2},\ \overline{4}\}$.
  Theory solver finds $\{1,\ \overline{2}\}$ E-unsatisfiable.

Lazy Translation into SAT

$$\underbrace{g(a) = c}_{1} \quad \wedge \quad \underbrace{f(g(a)) \neq f(c)}_{\overline{2}} \vee \underbrace{g(a) = d}_{3} \quad \wedge \quad \underbrace{c \neq d}_{\overline{4}}$$

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- Send $\{1, \overline{2} \vee 3, \overline{4}, \overline{1} \vee 2\}$ to SAT solver.

Lazy Translation into SAT

$$\underbrace{g(a) = c}_{1} \quad \wedge \quad \underbrace{f(g(a)) \neq f(c)}_{\overline{2}} \vee \underbrace{g(a) = d}_{3} \quad \wedge \quad \underbrace{c \neq d}_{\overline{4}}$$

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- Send $\{1,\ \overline{2} \vee 3,\ \overline{4},\ \overline{1} \vee 2\}$ to SAT solver.
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Lazy Translation into SAT

$$\underbrace{g(a) = c}_{1} \quad \wedge \quad \underbrace{f(g(a)) \neq f(c)}_{\overline{2}} \vee \underbrace{g(a) = d}_{3} \quad \wedge \quad \underbrace{c \neq d}_{\overline{4}}$$

- Send $\{1, \overline{2} \vee 3, \overline{4}\}$ to SAT solver.
- SAT solver returns model $\{1, \overline{2}, \overline{4}\}$.
  Theory solver finds $\{1, \overline{2}\}$ E-unsatisfiable.
- Send $\{1, \overline{2} \vee 3, \overline{4}, \overline{1} \vee 2\}$ to SAT solver.
- SAT solver returns model $\{1, 2, 3, \overline{4}\}$.
  Theory solver finds $\{1, 3, \overline{4}\}$ E-unsatisfiable.
- Send $\{1, \overline{2} \vee 3, \overline{4}, \overline{1} \vee 2, \overline{1} \vee \overline{3} \vee 4\}$ to SAT solver.
  SAT solver finds $\{1, \overline{2} \vee 3, \overline{4}, \overline{1} \vee 2, \overline{1} \vee \overline{3} \vee 4\}$ unsatisfiable.

Lazy Translation into SAT: Summary

- Abstract T-atoms as propositional variables
- SAT solver computes a model, i.e. satisfying boolean assignment for propositional abstraction (or fails)
- Solution from SAT solver may not be a T-model. If so,
  - Refine (strengthen) propositional formula by incorporating reason for false solution
  - Start again with computing a model

Optimizations

Theory Consequences

- The theory solver may return consequences (typically literals) to guide the SAT solver

Online SAT solving

- The SAT solver continues its search after accepting additional clauses (rather than restarting from scratch)

Preprocessing atoms

- Atoms are rewritten into normal form, using theory-specific atoms (e.g. associativity, commutativity)

Several layers of decision procedures

- "Cheaper" ones are applied first

Combining Theories

Theories:

- $\mathcal{R}$: theory of rationals
  $\Sigma_{\mathcal{R}} = \{\leq, +, -, 0, 1\}$
- $\mathcal{L}$: theory of lists
  $\Sigma_{\mathcal{L}} = \{=, \mathrm{hd}, \mathrm{tl}, \mathrm{nil}, \mathrm{cons}\}$
- $\mathcal{E}$: theory of equality
  Σ: free function and predicate symbols

Problem: Is

$$x \leq y \land y \leq x + \mathrm{hd}(\mathrm{cons}(0, \mathrm{nil})) \land P(h(x) - h(y)) \land \neg P(0)$$

satisfiable in $\mathcal{R} \cup \mathcal{L} \cup \mathcal{E}$?

Nelson-Oppen Combination Method

G. Nelson and D.C. Oppen: *Simplification by cooperating decision procedures*, ACM Trans. on Programming Languages and Systems, 1(2):245-257, 1979.

Given:

- $\mathcal{T}_1$, $\mathcal{T}_2$ first-order theories with signatures Σ_1, Σ_2
- $\Sigma_1 \cap \Sigma_2 = \emptyset$
- ϕ quantifier-free formula over $\Sigma_1 \cup \Sigma_2$

Obtain a decision procedure for satisfiability in $\mathcal{T}_1 \cup \mathcal{T}_2$ from decision procedures for satisfiability in $\mathcal{T}_1$ and $\mathcal{T}_2$.

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Variable abstraction + *equality propagation*:

$$x \leq y \wedge y \leq x + \mathrm{hd}(\mathrm{cons}(0, \mathrm{nil})) \wedge P(h(x) - h(y)) \wedge \neg P(0)$$

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$\mathcal{R}$	$\mathcal{L}$	$\mathcal{E}$
$x \leq y$		$P(v_2)$
$y \leq x + v_1$		$\neg P(v_5)$

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$x \leq y$		$P(v_2)$
$y \leq x + v_1$		$\neg P(v_5)$
$v_2 = v_3 - v_4$	$v_1 = \mathrm{hd}(\mathrm{cons}(v_5, \mathrm{nil}))$	$v_3 = h(x)$
$v_5 = 0$		$v_4 = h(y)$

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$x = y$	$v_1 = v_5$	$v_3 = v_4$
$v_2 = v_5$		$\perp$

Conclusions

- Talked about the role of first-order theorem proving
- Talked about some standard techniques (Normal forms of formulas, Resolution calculus, unification, Instance-based method, Model computation)
- Talked about DPLL and Satisfiability Modulo Theories (SMT)

Further Topics

- Redundancy elimination, efficient equality reasoning, adding arithmetics to first-order theorem provers
- FOTP methods as decision procedures in special cases
  E.g. reducing planning problems and temporal logic model checking problems to function-free clause logic and using an instance-based method as a decision procedure
- Implementation techniques
- Competition CASC and TPTP problem library
- Instance-based methods (a lot to do here, cf. my home page)
  Attractive because of complementary features to more established methods