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NICTA and ANU
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CPMP 4630/COMP 6463
Assignment "Propositional Logic"

Due date: Monday, August 17, 2015

Marking scheme: Full marks for a formulation that correctly answers the question and clearly shows the steps to obtain the solution.

Solutions to be submitted electronically by email to Peter.Baumgartner@nicta.com.au, or on paper to a lecturer of this course. Neatly hand-written solutions are of course acceptable.

Question 1 (Truth tables, 6 pts). Consider the following formulas:

(a) $P \lor Q \rightarrow \neg P \land Q$

(b) $(P \rightarrow Q) \rightarrow (\neg Q \rightarrow \neg P)$

(1) Evaluate (a) and (b) under the interpretation $I = \{P \mapsto \mathsf{true},\ Q \mapsto \mathsf{false}\}$.
(2) Compute the truth tables for (a) and (b). Are (a) and (b) satisfiable? valid?

Solution.
(1-a) Let $F = P \lor Q \rightarrow \neg P \land Q$. It holds $I(F) = \mathsf{false}$ as per the following table:

P	Q	$\neg P$	$P \lor Q$	$\neg P \land Q$	F
1	0	0	1	0	0

(1-b) Let $F = (P \rightarrow Q) \rightarrow (\neg Q \rightarrow \neg P)$. It holds $I(F) = \mathsf{true}$ as per the following table:

P	Q	$\neg P$	$\neg Q$	$P \rightarrow Q$	$\neg Q \rightarrow \neg P$	F
1	0	0	1	0	0	1

(2-a)

P	Q	$\neg P$	$P \lor Q$	$\neg P \land Q$	F
0	0	1	0	0	1
0	1	1	1	1	1
1	0	0	1	0	0
1	1	0	1	0	0

The formula (a) is satisfiable and not valid.

(2-b)

P	Q	$\neg P$	$\neg Q$	$P \rightarrow Q$	$\neg Q \rightarrow \neg P$	F
0	0	1	1	1	1	1
0	1	1	0	1	1	1
1	0	0	1	0	0	1
1	1	0	0	1	1	1

The formula (b) is satisfiable and valid.

Question 2 (Structural induction, 10 pts). Define recursively a function trans on formulas as follows, where A is a propositional variable and F_1 and F_2 are formulas:

$$
\begin{aligned}
\mathsf{trans}(\top) &= \top & \mathsf{trans}(\bot) &= \bot \\
\mathsf{trans}(A) &= \neg A & \mathsf{trans}(F_1 \lor F_2) &= \mathsf{trans}(F_1) \lor \mathsf{trans}(F_2) \\
\mathsf{trans}(\neg F_1) &= \neg\mathsf{trans}(F_1) & \mathsf{trans}(F_1 \rightarrow F_2) &= \mathsf{trans}(F_1) \rightarrow \mathsf{trans}(F_2) \\
\mathsf{trans}(F_1 \land F_2) &= \mathsf{trans}(F_1) \land \mathsf{trans}(F_2) & \mathsf{trans}(F_1 \leftrightarrow F_2) &= \mathsf{trans}(F_1) \leftrightarrow \mathsf{trans}(F_2)
\end{aligned}
$$

(1) Compute $\mathsf{trans}(P \lor Q \to \neg P \land Q)$.
(2) Let I be an interpretation and I' be the interpretation such that

$$I'[A] = \mathsf{true} \text{ if } I[A] = \mathsf{false}, \text{ and } I'[A] = \mathsf{false} \text{ if } I[A] = \mathsf{true}.$$

Let F be any formula. Prove that $I \models F$ if and only if $I' \models \mathsf{trans}(F)$.

Solution.
(1) We have

$$\begin{aligned}
\mathsf{trans}(P \lor Q \to \neg P \land Q) &= \mathsf{trans}(P \lor Q) \to \mathsf{trans}(\neg P \land Q) \\
&= \mathsf{trans}(P) \lor \mathsf{trans}(Q) \to \mathsf{trans}(\neg P) \land \mathsf{trans}(Q) \\
&= \neg P \lor \neg Q \to \neg\neg P \land \neg Q
\end{aligned}$$

(2) **Induction start.** If $F = \bot$ or $F = \top$ the result follows trivially. If $F = A$ for some propositonal variable A we have

$$\begin{aligned}
I \models A \text{ iff } & I' \models \neg A && \text{(by definition of } I'\text{)} \\
\text{iff } & I' \models \mathsf{trans}(A) && \text{(by definition of } \mathsf{trans}\text{)}
\end{aligned}$$

Induction step. Let F_1 and F_2 be formulas. In all cases below, assume as the induction hypothesis

$$I \models F_1 \text{ iff } I' \models \mathsf{trans}(F_1) \text{ and } I \models F_2 \text{ iff } I' \models \mathsf{trans}(F_2) \qquad \text{(I.H.)}$$

We carry out a case analysis on the shape of F.
If $F = \neg F_1$ then

$$\begin{aligned}
I \models \neg F_1 \text{ iff } & I \not\models F_1 && \text{(by definition of } \neg\text{)} \\
\text{iff } & I' \not\models \mathsf{trans}(F_1) && \text{(by I.H.)} \\
\text{iff } & I' \models \neg\mathsf{trans}(F_1) && \text{(by definition of } \neg\text{)} \\
\text{iff } & I' \models \mathsf{trans}(\neg F_1) && \text{(by definition of } \mathsf{trans}\text{)} \\
\text{iff } & I' \models F
\end{aligned}$$

If $F = F_1 \land F_2$ then

$$\begin{aligned}
I \models F_1 \land F_2 \text{ iff } & I \models F_1 \text{ and } I \models F_2 && \text{(by definition of } \land\text{)} \\
\text{iff } & I' \models \mathsf{trans}(F_1) \text{ and } I' \models \mathsf{trans}(F_2) && \text{(by I.H.)} \\
\text{iff } & I' \models \mathsf{trans}(F_1) \land \mathsf{trans}(F_2) && \text{(by definition of } \land\text{)} \\
\text{iff } & I' \models \mathsf{trans}(F_1 \land F_2) && \text{(by definition of } \mathsf{trans}\text{)} \\
\text{iff } & I' \models F
\end{aligned}$$

The proof for all other cases is similar and is omitted.

Question 3 (Tableau calculus, 10 pts). Consider again the following formulas:

(a) $P \lor Q \to \neg P \land Q$

(b) $(P \to Q) \to (\neg Q \to \neg P)$

For each of these formulas use the Tableau method to prove their validity or invalidity.

Solution.
(a) Let's assume that $P \lor Q \to \neg P \land Q$ is not valid.

1.	$I \not\models P \lor Q \to \neg P \land Q$	(assumption)
2.	$I \models P \lor Q$	(by 1 and $\to$)
3.	$I \not\models \neg P \land Q$	(by 1 and $\to$)

We have two cases:

4a.	$I \models P$	(by 2 and $\lor$)	4b.	$I \models Q$	(by 2 and $\lor$)

We have two cases below 4a:

5a.	$I \not\models \neg P$	(by 3 and $\land$)	5b.	$I \not\models Q$	(by 3 and $\land$)
6a.	$I \models P$	(by 5a and $\neg$)			

We have (the same) two cases below 4b:

7a.	$I \not\models \neg P$	(by 3 and $\land$)	7b.	$I \not\models Q$	(by 3 and $\land$)
8a.	$I \models P$	(by 5a and $\neg$)	8b.	$\bot$	(by 4b and 7b)

There is a non-contradictory branch. Hence the formula (a) is not valid.

(a) Let's assume that $(P \to Q) \to (\neg Q \to \neg P)$ is not valid.

1.	$I \not\models (P \to Q) \to (\neg Q \to \neg P)$	(assumption)
2.	$I \models P \to Q$	(by 1 and $\to$)
3.	$I \not\models \neg Q \to \neg P$	(by 1 and $\to$)
4.	$I \models \neg Q$	(by 3 and $\to$)
5.	$I \not\models \neg P$	(by 3 and $\to$)
6.	$I \models P$	(by 5 and $\neg$)
7.	$I \not\models Q$	(by 4 and $\neg$)

We have two cases:

8a.	$I \not\models P$	(by 2 and $\to$)	8b.	$I \models Q$	(by 4 and $\to$)
9a.	$I \models \bot$	(by 6 and 8a)	9b.	$\bot$	(by 7 and 8b)

All branches are contradictory. Hence the formula (b) is valid.

Question 4 (Normal form, 4pts). Compute the NNF and the CNF of $\neg(P \land \neg(Q \lor \neg R))$.

Solution. NNF:

$$\begin{aligned} &\neg(P \land \neg(Q \lor \neg R)) && \text{(given)} \\ &\neg P \lor \neg\neg(Q \lor \neg R) && \text{(De Morgan law)} \\ &\neg P \lor (Q \lor \neg R) && \text{(remove double negation)} \end{aligned}$$

The CNF is $\neg P \lor Q \lor \neg R$.

Question 5 (Semantic trees and DPLL, 12pts). Consider the clause set

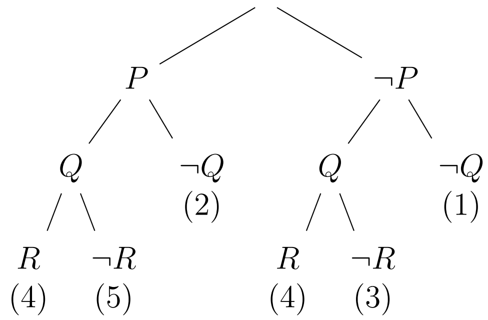
$$N = \{P \lor Q,\ \neg P \lor Q,\ \neg Q \lor R \lor P,\ \neg R,\ \neg P \lor R\}\ .$$

(1) Compute a closed semantic tree for N.
(2) Compute $\mathsf{simplify}(N, \neg R)$.
(3) Is there a clause $C \in N$ such that $N \setminus \{C\}$ is unsatisfiable? If so, which one?

Solution. (1) Let the clauses in N be numbered as follows:

$$P \vee Q \tag{1}$$
$$\neg P \vee Q \tag{2}$$
$$\neg Q \vee R \vee P \tag{3}$$
$$\neg R \tag{4}$$
$$\neg P \vee R \tag{5}$$

Closed semantic tree for N:



(2)

1. Remove all clauses from N that contain $\neg R$:

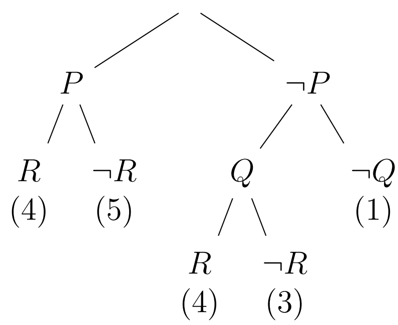
$$N' = \{P \vee Q,\ \neg P \vee Q,\ \neg Q \vee R \vee P,\ \neg P \vee R\}$$

2. Delete R from all remaining clauses N':

$$N'' = \{P \vee Q,\ \neg P \vee Q,\ \neg Q \vee P,\ \neg P\}$$

That is, $\mathsf{simplify}(N, \neg R) = N''$

(3) Yes, $N \setminus \{\neg P \vee Q\}$ is unsatisfiable. This is demonstrated by the following closed semantic tree for $N \setminus \{\neg P \vee Q\}$:



Question 6 (Resolution calculus, 8 pts). Consider the clause set

$$N = \{P \vee Q \vee Q,\ \neg P \vee Q,\ \neg Q \vee R \vee P,\ \neg R \vee \neg R,\ \neg P \vee R\}$$

Find a resolution refutation of N.

Solution.

1.	$P \vee Q \vee Q$	(given)
2.	$\neg P \vee Q$	(given)
3.	$\neg Q \vee R \vee P$	(given)
4.	$\neg R \vee \neg R$	(given)
5.	$\neg P \vee R$	(given)
6.	$P \vee Q$	(Fact. 1)
7.	$Q \vee Q$	(Res. 6 into 2)
8.	Q	(Fact. 7)
9.	$R \vee P$	(Res. 8 into 3)
10.	$R \vee R$	(Res. 9 into 5)
11.	R	(Fact. 10)
12.	$\neg R$	(Res. 11 into 4)
13.	$\square$	(Res. 11 into 12)