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CPMP 4630/COMP 6463
Assignment "First-Order Logic"

Due date: Monday, September 21, 2015

Marking scheme: Full marks for a formulation that correctly answers the question and clearly shows the steps to obtain the solution.

Solutions to be submitted electronically by email to Peter.Baumgartner@nicta.com.au, or on paper to a lecturer of this course. Neatly hand-written solutions are of course acceptable.

Question 1 (Semantics of FOL, 10 pts). Let $F = (P(a) \wedge (\forall x.\ P(x) \rightarrow P(f(x)))$ be a first-order logic formula and I an interpretation. Which of the following statements is true? false? In each case provide a proof.

1. If $I \models F$ then $I \models \exists x.\ P(x)$.
2. If $I \models F$ then $I \models \forall x.\ P(x)$.
3. If $I \models F$ then for every input ground term t wrt. F it holds $I \models P(t)$.

Solution.

(1) The statement is true. Proof: assume $I \models F$ for some interpretation $I = (D_I, \alpha_I)$. It follows $I \models P(a)$, and hence $I \triangleleft \{x \mapsto \alpha_I[a]\} \models P(x)$. By semantics of quantifiers $I \models \exists x.\ P(x)$.

(2) The statement is false. Proof: Take the interpretation $I = (D_I, \alpha_I)$ such that

$$\begin{aligned} D_I &= \mathbb{N} \\ \alpha_I[a] &= 0 \\ \alpha_I[f] &= x \mapsto x + 2 \\ \alpha_I[P] &= x \mapsto x \text{ is even} \end{aligned}$$

Clearly, $I \models F$ but $I \not\models \forall x.P(x)$ as $I \triangleleft \{x \mapsto 1\} \not\models P(x)$.

Alternate proof: let $I = (D_I, \alpha_I)$ be any interpretation such that $I \models F$. Such an interpretation exists. For example, take any domain D_I, set $\alpha_I[P] = x \mapsto \text{true}$ and set $\alpha_I[f]$ arbitrarily. Then construct the interpretation $I' = (D_{I'}, \alpha_{I'})$ as the interpretation obtained from I such that

$$\begin{aligned} D_{I'} &= D_I \cup \{\infty\}, \text{ for some fresh domain element } \infty \text{ not in } D_I \\ \alpha_{I'}[x] &= \alpha_I[x] \text{ for every variable } x \\ \alpha_{I'}[a] &= \alpha_I[a] \\ \alpha_{I'}[f] &= x \mapsto \begin{cases} \infty & \text{if } x = \infty \\ \alpha_I[f](x) & \text{otherwise} \end{cases} \\ \alpha_{I'}[P] &= x \mapsto \begin{cases} \text{false} & \text{if } x = \infty \\ \alpha_I[P](x) & \text{otherwise} \end{cases} \end{aligned}$$

In words, I' is the same as I except for the semantics of f and P on the new domain element ∞. By semantics of first-order logic from $I \models F$ it follows $I' \models F$. In particular it holds $I' \triangleleft \{x \mapsto \infty\} \models P(x) \rightarrow P(f(x))$ as $\alpha_{I'}[P](\infty) = \text{false}$.

On the other hand, the fact $\alpha_{I'}[P](\infty) = \text{false}$ falsifies the formula $\forall x.\ P(x)$, by semantics of quantifiers. In other words, although $I' \models F$ we have $I' \not\models \forall x.\ P(x)$, a direct contradiction to the statement.

(3) The statement is true. Proof: assume $I \models F$ for some interpretation $I = (D_I, \alpha_I)$. Let $t = f^n(a)$ be an input ground term wrt. F, for some $n \geq 0$ ($f^n = \underbrace{f(\cdots f}_{n\times f}(a)\cdots)$).

We prove the statement by induction on n.

Induction start. If $n = 0$ then $P(t) = P(f^0(a)) = P(a)$ and $I \models P(a)$ follows immediately from $I \models F$.

Induction step. If $n > 0$ then $I \models P(f^{n-1}(a))$ by induction. From $I \models F$ it follows $I \models \forall x.P(x) \rightarrow P(f(x))$. By semantics of quantifiers and of $\rightarrow$ conclude $I \models P(f(f^{n-1}(a)))$, equivalently $I \models P(f^n(a))$.

Question 2 (Tableau calculus, 8 pts). Use the tableau calculus to prove that the formula $(\forall x.\ P(x) \rightarrow Q(x)) \rightarrow ((\forall x.\ P(x)) \rightarrow (\forall x.\ Q(x)))$ is valid.

Solution. Let's assume that the given formula is not valid.

1.	$I \not\models (\forall x.\ P(x) \rightarrow Q(x)) \rightarrow ((\forall x.\ P(x)) \rightarrow (\forall x.\ Q(x)))$	(assumption)
2.	$I \models \forall x.\ P(x) \rightarrow Q(x)$	(by 1 and $\rightarrow$)
3.	$I \not\models (\forall x.\ P(x)) \rightarrow (\forall x.\ Q(x))$	(by 1 and $\rightarrow$)
4.	$I \models \forall x.\ P(x)$	(by 3 and $\rightarrow$)
5.	$I \not\models \forall x.\ Q(x)$	(by 3 and $\rightarrow$)
6.	$I \not\models Q(a)$	(by 5 and $\forall$ ($x \mapsto a$ fresh))
7.	$I \models P(a) \rightarrow Q(a)$	(by 2 and $\forall$ ($x \mapsto a$))

We have two cases:

8a.	$I \not\models P(a)$	(by 7 and $\rightarrow$)
9a.	$I \models P(a)$	(by 4 and $\forall$ ($x \mapsto a$))
10a.	$\perp$	(by 8a and 9a)

8b.	$I \models Q(a)$	(by 7 and $\rightarrow$)
9b.	$\perp$	(by 6 and 8b)

Question 3 (Herbrand interpretation, 8 pts). Let $N = \{P(a),\ \neg P(b),\ \neg P(x) \vee P(f(f(x)))\}$ be a clause set.

1. Describe the Herbrand universe, the Herband Base and the Herbrand expansion of N
2. Find a minimal Herbrand model I of N, i.e., $I \models N$ but $J \not\models N$, for no Herbrand interpretation $J \subsetneq I$. Justify your answer.

Solution. (1)

- Herbrand universe $HU_I = \{a, f(a), f(f(a)), \ldots, b, f(b), f(f(b)), \ldots\}$
- Herbrand base $HB_I = \{P(a), P(f(a)), P(f(f(a))), \ldots, P(b), P(f(b)), P(f(f(b))), \ldots\}$
- Herbrand expansion:

$$\begin{aligned}&\{P(a),\ \neg P(b),\\ &\neg P(a) \vee P(f(f(a))), \neg P(f(a)) \vee P(f(f(f(a)))), \neg P(f(f(a))) \vee P(f(f(f(f(a))))), \ldots\\ &\neg P(b) \vee P(f(f(b))), \neg P(f(b)) \vee P(f(f(f(b)))), \neg P(f(f(b))) \vee P(f(f(f(f(b))))), \ldots\}\end{aligned}$$

(2) As $P(a) \in N$ we must have $P(a) \in I$, otherwise I is not a model of N. As $\neg P(x) \vee P(f(f(x))) \in N$ we must have $P(f^{n+2}(a)) \in I$ whenever $P(f^n(a)) \in I$ for all $n \geq 0$, otherwise I is not a model of N. By induction on n it follows that I must contain $P(f^n(a))$ for all even $n \geq 0$, otherwise I is not a model.

In fact, it is enough to take $I = \{P(a), P(f(f(a))), P(f(f(f(f(a))))), \ldots\}$ (but not any larger). We still need to show that I satisfies (a) all ground instances of $\neg P(f^m(a)) \vee P(f(f(f^m(a))))$ where $m \geq 1$ is odd and (b) all ground instances of $\neg P(f^k(b)) \vee P(f(f(f^k(b))))$ where $k \geq 0$

The proof for both cases (a) and (b) can again be carried out by induction on n.

Question 4 (Clause normal form, 8 pts). Transform the formula $\exists z\ \neg\exists x\ \forall y\ ((R(y,z) \wedge \neg S(x,y)) \vee \forall y\ Q(x,y))$ into clause normal form (CNF). All intermediate formulas (NNF, PNF, SNF, CNF) must be given explicitly.

Solution. Let F be the given formula.

1. $F_1 = NNF(F) = \exists z\ \forall x\ \exists y\ ((\neg R(y,z) \vee S(x,y)) \wedge \exists y\ \neg Q(x,y))$
2. $F_2 = PNF(F_1) = \exists z\ \forall x\ \exists y\ \exists w\ ((\neg R(y,z) \vee S(x,y)) \wedge \neg Q(x,w))$
3. $F_3 = SNF(F_2) = \forall x\ ((\neg R(f(x),a) \vee S(x,f(x))) \wedge \neg Q(x,g(x)))$
   where $\{z \mapsto a\}$, $\{y \mapsto f(x)\}$ and $\{w \mapsto g(x)\}$ are the substitutions used for Skolemization.
4. $F_4 = CNF(F_3) = (\neg R(f(x),a) \vee S(x,f(x))) \wedge \neg Q(x,g(x))$

Question 5 (Unification, 8 pts). Apply the rule based unification algorithm to the unification problems E below and read off the result, i.e., either $\bot$ or the mgu (a is a constant, x, y and z are variables):

1. $E = \{y \doteq x,\ g(f(a,y)) \doteq g(x)\}$
2. $E = \{g(f(x,y)) \doteq g(z),\ g(f(y,a)) \doteq g(z)\}$

Solution. (1)

$$\begin{array}{rlll}
E: & y \doteq x,\ \underline{g(f(a,y)) \doteq g(x)} & & \text{(given)} \\
E_1: & y \doteq x,\ \underline{f(a,y) \doteq x} & & \text{(by Decompose)} \\
E_2: & y \doteq x,\ \underline{x \doteq f(a,y)} & & \text{(by Orient)} \\
E_3: & \underline{y \doteq f(a,y)},\ x \doteq f(a,y) & & \text{(by Apply } x \mapsto f(a,y)\text{)} \\
E_4: & \bot & & \text{(by Occur Check)}
\end{array}$$

There is no unifier of E.

(2)

$$\begin{array}{rlll}
E: & \underline{g(f(x,y)) \doteq g(z)},\ g(f(y,a)) \doteq g(z) & & \text{(given)} \\
E_1: & \underline{f(x,y) \doteq z},\ g(f(y,a)) \doteq g(z) & & \text{(by Decompose)} \\
E_2: & \underline{z \doteq f(x,y)},\ g(f(y,a)) \doteq g(z) & & \text{(by Orient)} \\
E_3: & z \doteq f(x,y),\ \underline{g(f(y,a)) \doteq g(f(x,y))} & & \text{(by Apply } z \mapsto f(x,y)\text{)} \\
E_4: & z \doteq f(x,y),\ \underline{f(y,a) \doteq f(x,y)} & & \text{(by Decompose)} \\
E_5: & z \doteq f(x,y),\ \underline{y \doteq x},\ a \doteq y & & \text{(by Decompose)} \\
E_6: & z \doteq f(x,x),\ y \doteq x,\ \underline{a \doteq x} & & \text{(by Apply } y \mapsto x\text{)} \\
E_7: & z \doteq f(x,x),\ y \doteq x,\ \underline{x \doteq a} & & \text{(by Orient)} \\
E_8: & z \doteq f(a,a),\ y \doteq a,\ x \doteq a & & \text{(by Apply } x \doteq a\text{)}
\end{array}$$

Result is mgu $\sigma = \{z \mapsto f(a,a),\ y \mapsto a,\ x \mapsto a\}$.

Question 6 (First-order resolution, 8 pts). Find a Resolution refutation of the following clause set. As for the mgus used, it suffices to only state them, you do not need to show the details of the runs of the unification algorithm. As in question 5, a is a constant, x and y are variables.

(1)	$P(x) \vee Q(f(x,y)) \vee Q(f(y,x))$	(3)	$\neg P(a) \vee \neg P(x)$
(2)	$\neg Q(x) \vee R(x)$	(4)	$\neg R(f(a,y))$

Solution.

1.	$P(x) \vee Q(f(x,y)) \vee Q(f(y,x))$	(given)
2.	$\neg Q(x) \vee R(x)$	(given)
3.	$\neg P(a) \vee \neg P(x)$	(given)
4.	$\neg R(f(a,y))$	(given)
5.	$P(y) \vee Q(f(y,y))$	(Fact. 1, $\{x \mapsto y\}$)
6.	$Q(f(a,a)) \vee \neg P(x)$	(Res. 5 into 3, $\{y \mapsto a\}$)
7.	$Q(f(a,a)) \vee Q(f(y,y))$	(Res. 6 into 5, $\{x \mapsto y\}$)
8.	$Q(f(a,a))$	(Fact. 7, $\{y \mapsto a\}$)
9.	$R(f(a,a))$	(Res. 8 into 2, $\{x \mapsto f(a,a)\}$)
10.	$\square$	(Res. 9 into 4, $\{y \mapsto a\}$)