Nonlinear elastic rod

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1 Introduction

We derive equations of equilibrium for uniaxial deformation of an elastic rod. We approach the problem from three perspectives: strong form (direct equilibrium equation), weak form, and minimization of the total potential energy. We eventually consider the material behavior of the rod as nonlinear, but assume the strains are sufficiently small to neglect geometric nonlinearity. To begin with, however, we start with linear elastic material assumption.

2 Strong formulation

2.1 Fundamental equations

Here we let $u(x)$ represent the axial displacement of a particle whose initial position is x, $\epsilon(x)$ is the axial strain in the rod at location x, and $\sigma(x)$ is the stress acting on the cross section of the rod at location x. $f(x)$ is a body force (i.e. a force per unit volume) assumed to be known in the rod. Here we focus on the special case that the rod has uniform cross sectional area, A, and zero traction along its length. *Equilibrium equation:*

$$\frac{d\sigma}{dx} + f = 0 \tag{1}$$

Constitutive equation:

$$\sigma = \Sigma(\epsilon) = E\epsilon \tag{2}$$

Kinematics equation:

$$\epsilon = \frac{du}{dx} \tag{3}$$

Exercise

Derive equation (1).

2.2 An example $1D$ boundary value problem

We now consider a rod that has length L. It is fixed at $x = 0$ and is acted upon at $x = L$ by a force P. We let $\sigma_o = P/A$. A body force $f(x)$ is known. Therefore, in addition to equations (1-3), we need to add the boundary conditions:

$$u(0) = u_o \tag{4}$$
$$\sigma(L) = P/A = \sigma_o. \tag{5}$$

Strong Form 1

Given u_o, σ_o, E, and $f(x)$, for $0 < x < L$, find $u(x)$, $\epsilon(x)$, $\sigma(x)$ that satisfy:

$$\frac{d\sigma}{dx} + f = 0 \tag{6}$$

$$\sigma = E\epsilon \tag{7}$$

$$\epsilon = \frac{du}{dx} \tag{8}$$

$$u(0) = u_o \tag{9}$$

$$\sigma(L) = \sigma_o. \tag{10}$$

This box summarizes all our fundamental equations and boundary conditions. The unknowns are u, ϵ, σ. To solve for these unknowns, we have two first-order odes, and one algebraic equation (2). If we were to set about to solving this problem, the first thing that we might do is use equations (2) and (3) to eliminate the unknowns σ and ϵ in favor of $u(x)$. This would then give us:

Strong Form 2

Given u_o, σ_o, E, and $f(x)$, for $0 < x < L$, find $u(x)$, that satisfies:

$$\frac{d}{dx}\left(E\frac{du}{dx}\right) + f = 0 \tag{11}$$

$$u(0) = u_o \tag{12}$$

$$E\frac{du}{dx}\bigg|_{x=L} = \sigma_o. \tag{13}$$

Here we have just the one unknown function, $u(x)$ and one differential equation. These problems are equivalent, but they are different. One has three unknown functions; the other one unknown function. One has two first order differential equations; the other had one second order differential equation. One yields all quantities of potential interest (i.e. u, ϵ, σ); the other yields only u. If one uses the Strong Form 2 and wishes to know the stress, then the stress must be computed as a *post-process*. Generally speaking, if one knows any of the functions, u, ϵ, σ, then the problem is considered to be solved. The others can be evaluated *relatively* easily from (2) and/or (3).

3 Minimum potential energy

Here we reformulate the problem using the principle of minimum potential energy.

3.1 Fundamental equations

3.1.1 Stored elastic potential energy

Let W denote the elastic energy density, i.e. the energy per unit volume, contained in a deformed solid. (Normally we think of $W = W(\epsilon)$ to be a pure function of strain (i.e. it depends *only* on strain). One definition of an elastic material is that $W = W(\epsilon)$.) For a linear elastic material in uniaxial tension, $W = \frac{1}{2}E\epsilon^2$. *Elastic potential energy:*

$$\begin{aligned}\Pi_{Elastic}[u] &= \int_V W(\epsilon(u))dV && (14)\\ &= \int_0^L W(\epsilon(u))A\,dx && (15)\end{aligned}$$

Note that in (15), we think of ϵ as determined by $u(x)$ through equation (3).

3.1.2 Work done by loading

The sum of all work done by all externally applied loads is: *External work:*

$$\begin{aligned}\ell[u] &= \int_V f(x)\cdot u(x)dV + P\cdot u(L) && (16)\\ &= \int_0^L f(x)\cdot u(x)A\,dx + P\cdot u(L) && (17)\end{aligned}$$

Again, Note that here $V = (0, L) \times A$, and $dV = A\,dx$. The work done by these loads comes at the loss of some energy. We shall assume that this work represents energy lost by some conservative loading machine, so that

$$\Pi_{Load}[u] = -\ell[u]. \tag{18}$$

That is, work done on the rod is the energy lost by Π_{Load}.

3.1.3 Total potential energy

The total potential energy is, therefore, $\Pi_{Total} = \Pi_{Elastic} + \Pi_{Load}$, or *Total potential energy:*

$$\begin{aligned}\Pi_{Total}[u] &= \Pi_{Elastic}[u] - \ell[u] && (19)\\ &= \int_0^L W(\epsilon(u)) - f(x)\cdot u(x)A\,dx - P\cdot u(L) && (20)\end{aligned}$$

3.2 Principle of minimum potential energy

Principle of minimum potential energy - wordy

Of all admissible functions, $u(x)$, the equilibrium displacement field is given by the $u(x)$ that minimizes $\Pi_{Total}[u]$.

3.2.1 Admissible displacement fields

Admissible displacement fields must satisfy an appropriate degree of continuity and satisfy our *essential* boundary conditions. In our example boundary value problem, the essential boundary condition is (4). The appropriate degree of continuity in $1D$ is that the functions be at least continuous. The set of continuous functions defined for $0 \leq x \leq L$ is denoted $C^0[0, L]$. To say that $u(x)$ is continuous is equivalent to saying that $u(x)$ is in the set C^0, or in math-speak, $u(x) \in C^0[0, L]$. [1] Therefore we introduce the set $\mathcal{S}$ of admissible displacement fields, $u(x)$:

$$\mathcal{S} = \left\{u(x)|u(x) \in C^0[0, L];\; u(0) = u_o.\right\} \tag{21}$$

This reads, "$\mathcal{S}$ is the set of all $u(x)$ such that $u(x)$ is continuous and $u(0) = 0$."

Principle of minimum potential energy - mathy

The equilibrium displacement field, u^*, is the minimizer over all $u(x) \in \mathcal{S}$ of:

$$\Pi_{Total}[u] = \int_0^L \frac{1}{2} E \left(\frac{du}{dx}\right)^2 - f(x) \cdot u(x) A\, dx - P \cdot u(L), \tag{22}$$

$$\text{where} \quad \mathcal{S} = \left\{u(x)|u(x) \in C^0[0, L];\; u(0) = u_o.\right\} \tag{23}$$

3.2.2 Admissible variation fields

Pick any $u(x) \in \mathcal{S}$. Then $v(x)$ is an *admissible variation* if (and only if) $u(x) + v(x) \in \mathcal{S}$. We let $\mathcal{V}$ denote the set of all admissible variations.

Exercise:

Use the definition given to show that

$$\mathcal{V} = \left\{v(x)|v(x) \in C^0[0, L];\; v(0) = 0.\right\} \tag{24}$$

3.3 Euler Lagrange Equations: The connection between the strong form and the principle of minimum potential energy

We let $u^* \in \mathcal{S}$ denote the minimizer of $\Pi_{Total}[u]$. Our goal here is to show that u^* satisfies Strong Form 2. We begin by choosing an arbitrary function $v(x) \in \mathcal{V}$. Once we pick that function, it's picked. It doesn't change. To say that it's arbitrary means that it doesn't matter which one we pick, but we have to pick one. In this example, we can imagine it's $v(x) = 8x$. We next define the function $F(\alpha)$, where α is an arbitrary scalar, and

$$F(\alpha) = \Pi_{Total}[u^* + \alpha v]. \tag{25}$$

[1] The superscript 0 indicates how many derivatives of $u(x)$ are required to be continuous. Thus $C^3[0, L]$ is the set of all functions whose third derivative (and therefore second and first and zeroth) is continuous between $0 \leq x \leq L$.

We note that since u^* minimizes $\Pi_{Total}[u]$, then $F(\alpha)$ is minimum at $\alpha = 0$:

$$F(\alpha) = \Pi_{Total}[u^* + \alpha v] \geq \Pi_{Total}[u^*] = F(0). \tag{26}$$

Since $\alpha = 0$ is a minimum of $F(\alpha)$, then the slope of F is zero at that point. Hence $\frac{dF}{d\alpha}\Big|_{\alpha=0} = 0$. Computing gives us:

$$\begin{aligned}
\frac{dF}{d\alpha}\bigg|_{\alpha=0} &= \frac{d}{d\alpha}\bigg|_{\alpha=0} \int_0^L \frac{1}{2}E\left(\frac{du^*}{dx} + \alpha\frac{dv}{dx}\right)^2 - f(x)\cdot(u^* + \alpha v(x))A\,dx - P\cdot(u(L) + \alpha v(L)) \\
&= \left\{\int_0^L E\left(\frac{du^*}{dx} + \alpha\frac{dv}{dx}\right)\frac{dv}{dx} - f(x)\cdot(v(x))A\,dx - P\cdot(v(L))\right\}\bigg|_{\alpha=0} && (27) \\
&= \int_0^L E\frac{du^*}{dx}\frac{dv}{dx} - f(x)\cdot v(x)A\,dx - P\cdot v(L) && (28) \\
&= 0 \qquad \forall v(x) \in \mathcal{V} && (29)
\end{aligned}$$

To see the connection between (28) and (11), we use integration by parts on the first term in (28). Integration by parts is really the product rule of differentiation backwards. Remembering this helps when applying it in multi-dimensional contexts. Therefore, in order to integrate (28) by parts, we first rewrite the product of derivatives using the product rule:

$$\frac{d}{dx}\left(E\frac{du^*}{dx}v(x)\right) = E\frac{du^*}{dx}\frac{dv}{dx} + v(x)\frac{d}{dx}\left(E\frac{du^*}{dx}\right) \tag{30}$$

Therefore,

$$\begin{aligned}
\int_0^L \frac{du^*}{dx}\frac{dv}{dx}A\,dx &= \int_0^L \frac{d}{dx}\left(E\frac{du^*}{dx}v(x)\right) - v(x)\frac{d}{dx}\left(E\frac{du^*}{dx}\right)A\,dx && (31) \\
&= \int_0^L \frac{d}{dx}\left(E\frac{du^*}{dx}v(x)\right)A\,dx - \int_0^L v(x)\frac{d}{dx}\left(E\frac{du^*}{dx}\right)A\,dx && (32) \\
&= \left(E\frac{du^*}{dx}v(x)\right)A\bigg|_{x=0}^{x=L} - \int_0^L v(x)\frac{d}{dx}\left(E\frac{du^*}{dx}\right)A\,dx && (33) \\
\text{since } v \in \mathcal{V}: \quad &= \left(E\frac{du^*}{dx}\right)A\bigg|_{x=L} v(L) - \int_0^L v(x)\frac{d}{dx}\left(E\frac{du^*}{dx}\right)A\,dx && (34)
\end{aligned}$$

Exercise

Get (34) from the line above. When you know how to do it, it's one line.

We now use (34) to rewrite equation (28) in the form:

$$-\int_0^L v(x)\left[\frac{d}{dx}\left(E\frac{du^*}{dx}\right) + f(x)\right]A\,dx + \left[\left(E\frac{du^*}{dx}\right)\bigg|_{x=L}A - P\right]v(L) = 0 \qquad \forall v(x) \in \mathcal{V} \tag{35}$$

Since (35) holds for any choice of $v(x) \in \mathcal{V}$, then we can conclude:

$$\begin{aligned} \frac{d}{dx}\left(E\frac{du^*}{dx}\right) + f(x) &= 0 && (36)\\ \left.\left(E\frac{du^*}{dx}\right)\right|_{x=L} A &= P. && (37) \end{aligned}$$

Equations (36) and (37) are the Euler-Lagrange equations resulting from minimizing the functional $\Pi_{Total}[u]$. We see that $u^*(x)$ satisfies differential equation (36), which is the same differential equation as (11). Furthermore, $u^*(x)$ satisfies boundary condition (37), which is the same boundary condition as (13). Since $u^*(x)$ is required to be in the set $\mathcal{S}$, then u^* also satisfies (12). Therefore u^* satisfies equations (11, 12, 13) of Strong Form 2. Since the solution of Strong Form 2 is unique, then its solution also minimizes $\Pi_{Total}[u]$ over all functions in $\mathcal{S}$. In the process of minimizing $\pi[u]$, we found that boundary condition (37) (or (13)) resulted naturally as a consequence of the minimization. By contrast, boundary condition (12) is an essential part of the formulation of the minimization problem. Without boundary condition (12), the minimization problem wouldn't make sense; there would be no minimum. Therefore, (12) is called an *essential boundary condition*, while (13) is called a *natural boundary condition*.

Summary

The function $u^*(x) \in \mathcal{S}$ minimizes the total potential energy $\Pi_{Total}[u]$, compared to all other functions in $\mathcal{S}$. The minimizing function u^* satisfies equations (11, 12, 13) of Strong Form 2, and likewise, the solution of Strong Form 2 minimizes $\Pi_{Total}[u]$.

4 Weak formulation

4.1 Obtained from minimum potential energy

Above we showed that Strong Form 2 follows from equation (28). We divide (28) through by the constant area A and write that result here for convenience:

$$\int_0^L E\frac{du^*}{dx}\frac{dv}{dx} - f(x)\cdot v(x)\,dx - \sigma_o \cdot v(L) = 0 \qquad \forall v(x) \in \mathcal{V} \tag{38}$$

Equation (38) is known as the weak form the boundary value problem. More precisely, the weak form of the boundary value problem is:

Weak Form

Given u_o, σ_o, E, and $f(x)$, for $0 < x < L$, find $u(x) \in \mathcal{S}$ such that for all $v(x) \in \mathcal{V}$,

$$\int_0^L E\frac{du^*}{dx}\frac{dv}{dx} - f(x)\cdot v(x)\,dx - \sigma_o \cdot v(L) = 0 \tag{39}$$

4.2 Derivation from Strong Form 2

Exercise: Derive the weak form from the strong form

Let $v \in \mathcal{V}$. Compute $\int_0^L (11)\, v(x)dx$. Integrate by parts and simplify to derive (39). In the process of getting (39), you will need to use (12) and (13) various steps. Since you use (11, 12, 13), this implies that the solution of Strong Form 2 also satisfies Weak Form for $\boldsymbol{u} \in \mathcal{S}$. Can you identify where each of these eqns is used? (12) is subtle.

Let's first express $\int_0^L (11) v(x)dx$ using (11):

$$\int_0^L \left[\frac{d}{dx}\left(E\frac{du}{dx} \right) v(x) + f(x)v(x) \right] dx = 0 \tag{40}$$

Evaluate chain rule:

$$\frac{d}{dx}\left(E\frac{du}{dx}v(x) \right) = \frac{d}{dx}\left(E\frac{du}{dx} \right) v(x) + E\frac{du}{dx}\frac{dv}{dx} \tag{41}$$

$$\frac{d}{dx}\left(E\frac{du}{dx} \right) v(x) = \frac{d}{dx}\left(E\frac{du}{dx}v(x) \right) - E\frac{du}{dx}\frac{dv}{dx} \tag{42}$$

$$\int_0^L \frac{d}{dx}\left(E\frac{du}{dx}v(x) \right) - E\frac{du}{dx}\frac{dv}{dx} + fv(x)dx = 0 \tag{43}$$

Evaluate the following integral:

$$\int_0^L \frac{d}{dx}\left(E\frac{du}{dx}v(x) \right) = \left(E\frac{du}{dx}v(x) \right)\Big]_{x=0}^{x=L} = \left(E\frac{du}{dx}\Big]_{x=L} v(L) \right) - \left(E\frac{du}{dx}\Big]_{x=0} v(0) \right) \tag{44}$$

$$\left(E\frac{du}{dx}\Big]_{x=L} v(L) \right) - \left(E\frac{du}{dx}\Big]_{x=0} v(0) \right) - \int_0^L \left[E\frac{du}{dx}\frac{dv}{dx} - fv(x) \right] dx = 0 \tag{45}$$

The first term can be simplified using (13):

$$\left(E\frac{du}{dx}\Big]_{x=L} v(L) \right) = \sigma_o \cdot v(L) \tag{46}$$

Our essential boundary condition is defined in (12). This information helps us simplify the second term in (45) since $v(x)$ must be 0 at all points defined by (12) $\therefore v(0) = 0$ as articulated in (24)

$$v(0) = 0 \therefore \left(E\frac{du}{dx}\Big]_{x=0} v(0) \right) = 0 \tag{47}$$

Finally,

$$\sigma_o \cdot v(L) - \int_0^L \left[E\frac{du}{dx}\frac{dv}{dx} - fv(x) \right] dx = 0 \tag{48}$$

Rewriting slightly, we can see the same form presented in (39):

$$\int_0^L E\frac{du}{dx}\frac{dv}{dx} - f(x)\cdot v(x)dx - \sigma_o \cdot v(L) = 0 \tag{49}$$

5 A mixed weak form

5.1 Derivation from Strong Form 1

5.2 Derivation from a mixed variational principle

6 Non-linear Weak Form

Here we will explore an example in which we consider the material behavior of the rod as nonlinear, as defined below:
Constitutive relationship:

$$\sigma = E\epsilon e^{\beta\epsilon} \tag{50}$$

Nonlinear Strong Form

Given u_0, σ_0, E, and $f(x)$, for $0 < x < L$, find $u(x)$, $\epsilon(x)$,$\sigma(x)$ that satisfy:

$$\frac{d}{dx}\left(E\frac{du}{dx}e^{\beta\frac{du}{dx}}\right) + f = 0 \tag{51}$$

$$u(0) = u_0 \tag{52}$$

$$E\frac{du}{dx}\bigg|_{x=L} = \sigma_o \tag{53}$$

$$\int_0^L \sigma(\epsilon)\frac{dv}{dx} - fvdx - Pv(L) = 0 \tag{54}$$

$$\int_0^L EA\frac{du}{dx}e^{\beta\frac{dv}{dx}}\frac{dv}{dx} - fvdx - Pv(L) = 0 \tag{55}$$

Linear shape function:

$$N(x) = \begin{bmatrix} 1-\frac{x}{L} & \frac{x}{L} \end{bmatrix}; B = \frac{dN}{dx} = \begin{bmatrix} \frac{L-1}{L} & \frac{1}{L} \end{bmatrix} \tag{56}$$

6.1 Single Finite Element Approximation

$$u(x) \approx U_1N_1(x) + U_2N_2(x) \tag{57}$$

$$v(x) \approx V_1N_1(x) + V_2N_2(x) \tag{58}$$

$$N_1(x) = \frac{L-x}{L}; N_2(x) = \frac{x}{L}; B_2 = \frac{1}{L} \tag{59}$$

$$\int_0^L \left[EAU_2B_2e^{\beta U_2B_2}V_2B_2 - fV_2N_2\right]dx - PV_2N(L) \tag{60}$$

Solve integral and simplify:

$$AEL\frac{U_2}{L^2}e^{\beta U_2/L} = P \tag{61}$$

Newton's method will be used to solve for U_2

$$F(U_2) = AEL\frac{U_2}{L^2}e^{\beta U_2/L} - P = 0 \tag{62}$$

First guess:

$$U_2 = U_2^{(0)} = 0 \tag{63}$$

Approximate $F(U_2)$ using first guess ($U_2^{(0)}$)

$$F(U_2) = F(U_2^{(0)} + \delta U_2) \tag{64}$$

Using a first-order Taylor Series expansion, (63) can be rewritten as such:

$$F(U_2) \approx F(U_2^{(0)}) + \frac{dF}{dU}\delta U_2 \tag{65}$$

$$F'(U) = \frac{AE}{L}e^{\beta U/L}\left(1 + U\frac{\beta}{L}\right) \tag{66}$$

At $U = 0$ (our first approximation is analogous to linear elastic deformation):

$$F'(0) = \frac{AE}{L} \therefore \delta U = \frac{PL}{AE} \tag{67}$$

For the next Newton iteration (using δU as our guess) :

$$F'(U^{(1)} = P\left(e^{\beta P/AE} - 1\right) \tag{68}$$

$$F'(U^{(1)} + \left.\frac{dF}{dU}\right|_{U^{(1)}}\delta U = 0 \tag{69}$$

$$P\left(e^{\beta P/AE} - 1\right) + \frac{AE^*}{L}\delta U = 0 \tag{70}$$

where $E^* = Ee^{\beta P/AE}\left(1 + \frac{\beta P}{AE}\right)$. This result is analogous to the first (linear elastic) Newton iteration, with an "updated" modulus. This modulus (E^*) is representative of the nonlinear behavior of the material. In this case, the modulus is a function of the applied load and varies with the displacement (strain).

$$\delta U = \frac{-PL}{AE^*}\left(e^{\beta P/AE} - 1\right) \tag{71}$$

6.2 Finite Element Approximation (2 elements)

Next, let's consider the same nonlinear elastic rod solved with two finite elements. The weak form can be restated from (55):

Nonlinear Weak Form

Given u_o, σ_o, E, β, and $f(x)$, for $0 < x < L$, find $u^* \in \mathcal{S}$ such that for all $v(x) \in \mathcal{V}$,

$$\int_0^L \left[E\frac{du^*}{dx}e^{\beta\frac{du^*}{dx}}\frac{dv}{dx} - f(x)\cdot v(x)\right]dx - \sigma_o \cdot v(L) = 0 \tag{72}$$

$$u(x) \approx U_0N_0(x) + U_1N_1(x) + U_2N_2(x) \tag{73}$$

$$v(x) \approx V_0N_0(x) + V_1N_1(x) + V_2N_2(x) \tag{74}$$

Impose essential boundary conditions:

$$u(0) \approx U_0 = 0; v(0) \approx V_0 = 0 \tag{75}$$

$$u(x) \approx U_1N_1(x) + U_2N_2(x); v(x) \approx V_1N_1(x) + V_2N_2(x) \tag{76}$$

$$\sum_{elms}\int_{L_e}\left[EA\frac{du}{dx}e^{\beta\frac{du}{dx}}\frac{dv}{dx} - fv(x)\right]dx - Pv(L) \tag{77}$$

Assume no body forces ($f = 0$)

$$\int_0^{h_1}\left[EA\left(U_1B_1\right)e^{\beta(U_1B_1)}V_1B_1\right]dx$$
$$+\int_0^{h_2}\left[EA\left(U_1B_1+U_2B_2\right)e^{\beta(U_1B_1+U_2B_2)}\left(V_1B_1+V_2B_2\right)\right]dx - P\left[V_1N_1(L)+V_2N_2(L)\right]$$

$$B_1^{(1)} = 1/h_1; B_1^{(2)} = -1/h_2; B_2^{(2)} = 1/h_2 \tag{78}$$

$$\int_0^{h_1}\left[EA\frac{U_1}{h_1}e^{\beta U_1/h_1}\frac{V_1}{h_1}\right]dx + \int_0^{h_2}\left[\frac{EA}{h_2}(-U_1+U_2)e^{\frac{\beta}{h_2}(-U_1+U_2)}\frac{(-V_1+V_2)}{h_2}\right]dx - P(V_1(0)+V_2(1)) \tag{79}$$

$$\left[EA\frac{U_1}{h_1}e^{\beta U_1/h_1}V_1\right] + \left[\frac{EA}{h_2}(-U_1+U_2)e^{\frac{\beta}{h_2}(-U_1+U_2)}(-V_1+V_2)\right] - P(V_2) = 0 \tag{80}$$

$$U' = U_2 - U_1; V' = V_2 - V_1 \tag{81}$$

$$EA\frac{U_1}{h_1}e^{\beta U_1/h_1}V_1 + \frac{EA}{h_2}U'e^{\beta U'/h_2}V' - P(V_2) = 0 \tag{82}$$

Equation 1:

$$V_1 = 1; V_2 = 0 \tag{83}$$

$$F_1(U_1, U_2) = \frac{EAU_1}{h_1}e^{\beta U_1/h_1} - \frac{EAU'}{h_2}e^{\beta U'/h_2} = 0 \tag{84}$$

Equation 2:

$$V_1 = 0; V_2 = 1 \tag{85}$$

$$F_2(U_1, U_2) = \frac{EAU'}{h_2} e^{\beta U'/h_2} - P = 0 \tag{86}$$

Newton's method iterations: Given $F_1(U_1, U_2)$ and $F_2(U_1, U_2)$, find U_1^* and U_2^* so that $F_1(U_1^*, U_2^*) = 0$ and $F_2(U_1^*, U_2^*) = 0$

For the first Newton's method iteration, we will be solving solving the following linear problem to find δU_1 and δU_2. Our first guess for U^* is 0, and we will repeat this process for the next iteration with an improved guess using δU_1 and δU_2.

$$\begin{bmatrix} \frac{\partial F_1}{\partial U_1} & \frac{\partial F_1}{\partial U_2} \\ \frac{\partial F_2}{\partial U_1} & \frac{\partial F_2}{\partial U_2} \end{bmatrix} \begin{bmatrix} \delta U_1 \\ \delta U_2 \end{bmatrix} + \begin{bmatrix} F_1(0,0) \\ F_2(0,0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{87}$$

The four components for $\frac{\partial F}{\partial U}$*:*

$$\frac{\partial F_1}{\partial U_1} = \frac{EA}{h_1} e^{\frac{\beta U_1}{h_1}} \left(\frac{U_1 \beta}{h_1} + 1 \right) + \frac{EA}{h_2} e^{\frac{\beta U'}{h_2}} \left(\frac{U' \beta}{h_2} + 1 \right) \tag{88}$$

$$\frac{\partial F_1}{\partial U_2} = -\frac{EA}{h_2} e^{\frac{\beta U'}{h_2}} \left(\frac{U' \beta}{h_2} + 1 \right) \tag{89}$$

$$\frac{\partial F_2}{\partial U_1} = -\frac{EA}{h_2} e^{\frac{\beta U'}{h_2}} \left(\frac{U' \beta}{h_2} + 1 \right) \tag{90}$$

$$\frac{\partial F_2}{\partial U_2} = \frac{EA}{h_2} e^{\frac{\beta U'}{h_2}} \left(\frac{U' \beta}{h_2} + 1 \right) \tag{91}$$

Evaluating $F(U_1, U_2)$ *(*$U^{(0)} = 0$*):*

$$F_1(U_1, U_2) \approx F_1(U_1^{(0)}, U_2^{(0)}) + \frac{dF_1}{dU} \delta U = \frac{\partial F_1}{\partial U_1} \delta U_1 + \frac{\partial F_1}{\partial U_2} \delta U_2 \tag{92}$$

$$F_2(U_1, U_2) \approx F_2(U_1^{(0)}, U_2^{(0)}) + \frac{dF_2}{dU} \delta U = -mg + \frac{\partial F_2}{\partial U_1} \delta U_1 + \frac{\partial F_2}{\partial U_2} \delta U_2 \tag{93}$$

Plugging these into the above system:

$$\begin{bmatrix} \frac{EA}{h_1} + \frac{EA}{h_2} & -\frac{EA}{h_2} \\ -\frac{EA}{h_2} & \frac{EA}{h_2} \end{bmatrix} \begin{bmatrix} \delta U_1^{(0)} \\ \delta U_2^{(0)} \end{bmatrix} + \begin{bmatrix} 0 \\ -mg \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{94}$$

Solving this system yields the following definitions for δU_1 *and* δU_2*:*

$$\delta U_1^{(0)} = \frac{P h_1}{EA} \tag{95}$$

$$\delta U_2^{(0)} = \frac{P}{EA}(h_1 + h_2) \tag{96}$$

Let's now improve our initial guess:

$$U^{(1)} = U^{(0)} + \delta U^{(0)} \tag{97}$$

$$U' = U_2 - U_1 = \frac{Ph_2}{EA} \tag{98}$$

$$F_1(U_1^{(1)}, U'^{(1)}) = F_1\left(\frac{Ph_1}{EA}, \frac{Ph_2}{EA}\right) = 0 \tag{99}$$

$$F_2(U_1^{(1)}, U'^{(1)}) = F_2\left(\frac{Ph_1}{EA}, \frac{Ph_2}{EA}\right) = P\left(e^{\frac{\beta P}{EA}} - 1\right) \tag{100}$$

$$\begin{bmatrix} \left[\frac{1}{h_1} + \frac{1}{h_2}\right] EAe^{\frac{\beta P}{EA}}\left(\frac{\beta P}{EA} + 1\right) & -\frac{EA}{h_2}e^{\frac{\beta P}{EA}}\left(\frac{\beta P}{EA} + 1\right) \\ -\frac{EA}{h_2}e^{\frac{\beta P}{EA}}\left(\frac{\beta P}{EA} + 1\right) & \frac{EA}{h_2}e^{\frac{\beta P}{EA}}\left(\frac{\beta P}{EA} + 1\right) \end{bmatrix} \begin{bmatrix} \delta U_1^{(1)} \\ \delta U_2^{(1)} \end{bmatrix} + \begin{bmatrix} 0 \\ P\left(e^{\frac{\beta P}{EA}} - 1\right) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{101}$$

Our unknowns $\delta U^{(1)}$ can be solved using the above linearly independent system of equations

$$\delta U_2^{(1)} = \frac{h_1 + h_2}{h_1}\delta U_1^{(1)} \tag{102}$$

$$\delta U_1^{(1)} = \frac{\sigma_o h_1}{E}\left(1 - e^{\frac{\beta\sigma_o}{E}}\right)\left[e^{\frac{\beta\sigma_o}{E}}\left(\frac{\beta\sigma_o}{E} + 1\right)\right]^{-1} \tag{103}$$

$$\delta U_2^{(1)} = \frac{\sigma_o(h_1 + h_2)}{E}\left(1 - e^{\frac{\beta\sigma_o}{E}}\right)\left[e^{\frac{\beta\sigma_o}{E}}\left(\frac{\beta\sigma_o}{E} + 1\right)\right]^{-1} \tag{104}$$

7 Newton's method applied to functionals and functions

7.1 Newton's method for a scalar unknown

Here we suppose we wish to find x^*, the solution of the nonlinear equation

$$F(x^*) = 0. \tag{105}$$

We know that $x^* \approx x_0$, but this approximation is (for whatever reason) not satisfactory for us. We seek to improve it. We can improve our approximation for x^* by approximating the equation itself. To that end, we use a truncated Taylor expansion about the point x_0:

$$F(x) \approx F^L(x; x_0) = F(x_0) + \left.\frac{dF}{dx}\right|_{x=x_0}(x - x_0). \tag{106}$$

Here, $F^L(x; x_0)$ stands for the "linear approximation" to $F(x)$ around the point $x = x_0$. Since F^L is a linear function, it is easy to find its root. Therefore, we choose x_1, our improved approximation to x^*, to be the solution of

$$F^L(x_1; x_0) = 0. \tag{107}$$

In detail, that means:

$$F(x_0) + \left.\frac{dF}{dx}\right|_{x=x_0}(x_{-}x_0) \;\; = \;\; 0 \tag{108}$$

$$\left[\left.\frac{dF}{dx}\right|_{x=x_0}\right](x_1 - x_0) \;\; = \;\; -F(x_0). \tag{109}$$

We note that here, $\frac{dF}{dx}\Big|_{x=x_0}$ is just a scalar, and so solving the linear equation (109) for $x_1 - x_0$ is straightforward. We also notice that we're solving for $(x_1 - x_0)$, really, which is the amount we need to change x_0 to get a new-and-improved x_1. We call this quantity "the update". When done, we have an improved approximation $x^* \approx x_1$.[2] Having figured out how to improve our initial guess once, we can repeat the process to get a sequence of (we hope) more accurate approximations to x^*, by solving:

$$\begin{aligned} F^L(x_{n+1}; x_n) &= 0, && (110)\\ \text{or:} \qquad \left[\frac{dF}{dx}\Big|_{x=x_n}\right](\delta x_{n+1}) &= -F(x_n). && (111)\end{aligned}$$

Here, we introduced the update, δx_{n+1}, defined as:

$$\delta x_{n+1} = x_{n+1} - x_n. \tag{112}$$

7.2 Newton's method for multiple scalar unknowns

Now we suppose we wish to find $\boldsymbol{x}^* = [x^*{}_1, x^*{}_2, x^*{}_3, \ldots, x^*{}_{N_{unk}}]^T$, the column vector that represents the N_{unk} solutions of the N_{eqns} simultaneous nonlinear equations:

$$F_j(\boldsymbol{x}^*) = 0. \qquad j = 1, 2, \ldots, N_{eqns}. \tag{113}$$

Typically, we need to have the same number of equations as unknowns, and so we will require $N_{unk} = N_{eqns}$. As before, we will suppose that we know $\boldsymbol{x}^* \approx \boldsymbol{x}_n$, and seek $\boldsymbol{x}_{n+1}$ which is an improvement to this approximation To that end, we again use a truncated Taylor expansion about the point $\boldsymbol{x}_n$,

$$F_j(\boldsymbol{x}) \approx F_j^L(\boldsymbol{x}; \boldsymbol{x}_n) = F_j(\boldsymbol{x}_n) + \sum_{i=1}^{N_{unk}} \frac{\partial F_j}{\partial x_i}\Big|_{\boldsymbol{x}=\boldsymbol{x}_n} (x_i - x_{i_n}). \tag{114}$$

Here, $F_j^L(\boldsymbol{x}; \boldsymbol{x}_n)$ stands for the "linear approximation" to $F_j(\boldsymbol{x})$ around the point $\boldsymbol{x} = \boldsymbol{x}_n$. The numbers $\frac{\partial F_j}{\partial x_i}\Big|_{\boldsymbol{x}=\boldsymbol{x}_n}$ represent entries in a square matrix:

$$\begin{aligned} \boldsymbol{J_n} &= \nabla \boldsymbol{F} && (115)\\ \boldsymbol{J_n}_{ji} &= \frac{\partial F_j}{\partial x_i}\Big|_{\boldsymbol{x}=\boldsymbol{x}_n}. && (116)\end{aligned}$$

This matrix is often referred to as the "Jacobian matrix" for the system of equations. (It's not related to the Jacobian of the deformation from continuum mechanics, except that both involve the derivative of a bunch of functions with respect to a bunch of variables.) As

[2]We *hope* the approximation is improved, but it's not always the case. For us to be sure that it is an improvement, $F(x)$ has to have nice properties in the neighborhood of x^*, *and* our initial guess needs to be near enough to x^* that those nice properties still apply to x_0. It's not always practical to check those properties, but we can to things to make sure our intial guess is very close to x^*.

with the linear case, we choose our next approximation by requiring it to solve the linear equations, linearized about the current approximation:

$$F_j^L(\boldsymbol{x}_{n+1};\boldsymbol{x}_n) = 0, \qquad j = 1, 2, \ldots, N_{eqns}. \tag{117}$$

We can write (117) out in more detail but still succinctly by introducing the column vector $\boldsymbol{F}_n = [F_1(\boldsymbol{x}_n), F_2(\boldsymbol{x}_n), \ldots, F_{N_{eqns}}(\boldsymbol{x}_n)]^T$. Similarly, we can introduce a vector of unknowns, $\delta\boldsymbol{x}_{n+1}$, whose entries are given by:

$$\delta x_{i_{n+1}} = (x_{i_{n+1}} - x_{i_n}). \tag{118}$$

Then, equation (117) can be written as the matrix equation:

$$\boldsymbol{J_n}\delta\boldsymbol{x}_{n+1} = -\boldsymbol{F}_n. \tag{119}$$

7.3 Newton's method with function unknowns

First we recall our nonlinear weak form:

Nonlinear Weak Form

Given u_o, σ_o, E, β, and $f(x)$, for $0 < x < L$, find $u^*(x) \in \mathcal{S}$ such that for all $v(x) \in \mathcal{V}$,

$$\int_0^L E\frac{du^*}{dx}e^{\beta\frac{du^*}{dx}}\frac{dv}{dx} - f(x)\,v(x)\,dx - \sigma_o v(L) = 0 \tag{120}$$

We can rewrite (120) using the abstract notation:

$$\begin{aligned} a(u^*, v) &= l(v) &(121)\\ a(u, v) &= \int_0^L E\frac{du}{dx}e^{\beta\frac{du}{dx}}\frac{dv}{dx}\,dx &(122)\\ l(v) &= f(x)\,v(x)\,dx + \sigma_o v(L) &(123)\end{aligned}$$

We suppose we have an initial approximation, $u_n(x)$, for the solution $u^*(x)$. We linearize equation (122) around $u = u_n$ as follows:

$$\begin{aligned} a(u, v) = a(u_n + \delta u, v) &\approx a_L(\delta u, v; u_n) = a(u_n, v) + b(\delta u, v; u_n) &(124)\\ b(\delta u, v; u_n) &= \lim_{\alpha\to 0}\frac{d}{d\alpha}a(u_n + \alpha\delta u, v) &(125)\end{aligned}$$

In perfect analogy with equation (117), we find the update by solving *the linear problem*:

$$a_L(\delta u_{n+1}, v; u_n) = l(v), \qquad \forall v \in \mathcal{V}. \tag{126}$$

Alternatively, (124) may be used to rewrite (126) as:

Newton's Method in Weak Form

Given $u_0(x)$. Let $n = 0$. Solve:

$$b(\delta u_{n+1}, v; u_n) = l(v) - a(u_n, v), \qquad \forall v \in \mathcal{V}. \tag{127}$$

Update:

$$u_{n+1} = u_n + \delta u_{n+1} \tag{128}$$

Let $n \leftarrow n + 1$. Loop to (127).

7.3.1 Specific form of $b(\delta u, v; u_n)$

To figure out what $b(\delta u, v; u_n)$ is for our problem, we need to use its definition (125) and the explicit expression for $a(u, v)$ in (122).

$$\begin{aligned}
b(\delta u, v; u_n) &= \lim_{\alpha\to 0} \frac{d}{d\alpha} a(u_n + \alpha\delta u, v) && (129)\\
&= \lim_{\alpha\to 0} \frac{d}{d\alpha} \int_0^L E\frac{d}{dx}(u_n + \alpha\delta u) e^{\beta\frac{d}{dx}(u_n+\alpha\delta u)} \frac{dv}{dx}\, dx && (130)\\
&= \lim_{\alpha\to 0} \int_0^L E\frac{d}{d\alpha}\left[\frac{d}{dx}(u_n + \alpha\delta u)\right] e^{\beta\frac{d}{dx}(u_n+\alpha\delta u)} \frac{dv}{dx}\, dx \\
&\quad + \lim_{\alpha\to 0} \int_0^L E\frac{d}{dx}(u_n + \alpha\delta u) \frac{d}{d\alpha}\left[e^{\beta\frac{d}{dx}(u_n+\alpha\delta u)}\right] \frac{dv}{dx}\, dx && (131)\\
&= \lim_{\alpha\to 0} \int_0^L E\frac{d}{dx}\left[\frac{d}{d\alpha}(u_n + \alpha\delta u)\right] e^{\beta\frac{d}{dx}(u_n+\alpha\delta u)} \frac{dv}{dx}\, dx \\
&\quad + \lim_{\alpha\to 0} \int_0^L E\frac{d}{dx}(u_n + \alpha\delta u) e^{\beta\frac{d}{dx}(u_n+\alpha\delta u)} \frac{d}{d\alpha}\left(\beta\frac{d}{dx}(u_n + \alpha\delta u)\right) \frac{dv}{dx}\, dx && (132)\\
&= \int_0^L E\frac{d\delta u}{dx} e^{\beta\frac{du_n}{dx}} \frac{dv}{dx}\, dx \\
&\quad + \int_0^L E\frac{du_n}{dx} e^{\beta\frac{du_n}{dx}} \beta\frac{d\delta u}{dx} \frac{dv}{dx}\, dx && (133)\\
&= \int_0^L E e^{\beta\frac{du_n}{dx}} \left[1 + \beta\frac{du_n}{dx}\right] \frac{d\delta u}{dx} \frac{dv}{dx}\, dx && (134)
\end{aligned}$$

Exercise

Show that when $u_n = 0$, then (127) simplifies to (39).

8 Scaling of stress and strain our $1D$ model problem

Recall equation (50) that gives the stress in terms of the strain as:

$$\sigma = E\epsilon e^{\beta\epsilon}. \tag{135}$$

Exercise

Show that the stress (135) corresponds to the strain energy density (in $1D$):

$$W = \frac{E}{\beta^2}\left[1 + \exp(\beta\epsilon)(\beta\epsilon - 1)\right] \tag{136}$$

We note that equation (135) can alternatively be written as

$$\frac{\sigma\beta}{E} = (\beta\epsilon)e^{(\beta\epsilon)}. \tag{137}$$

In equation (137), we see that ϵ appears ONLY when multiplied by β. Similarly, σ appears ONLY in the form on the left. Also, this equation is non-dimensional, in that each group of parameters is non-dimensional. These observations motivate the introduction of two new variables, $\overline{\sigma}$ and $\overline{\epsilon}$ such that:

$$\overline{\sigma} = \overline{\epsilon}e^{\overline{\epsilon}} \tag{138}$$

where:

$$\overline{\sigma} = \frac{\sigma\beta}{E} \tag{139}$$

$$\overline{\epsilon} = \beta\epsilon. \tag{140}$$

Exercise

Create some examples and/or an argument that can be used to teach an undergraduate why equation (138) can be used to think about stress and strain for *any* choice of E and β.
Example questions:

1. For example, how big does the ϵ need to be to see significant curvature in the stress-strain curve?

2. How big is the stress at that strain level?

1. Choose several values of E and β. Think of each pair (E, β) as a different material. Label them $1, 2, 3, \ldots, N_{mats}$.

2. For each material, do a simulated experiment: Compute a family of points (ϵ_i, σ_i) as if you were measuring the stress-strain curve for that material. (Space the ϵ_i rather broadly so that you can see the lines between the markers.)

3. Graph all of your stress strain curves on a single graph. Use a different color for each material.

4. For each material, compute the values of $(\overline{\epsilon}_i, \overline{\sigma}_i)$.

5. Plot all the $(\overline{\epsilon}_i, \overline{\sigma}_i)$ points on a single graph (as a scatter plot - no lines). Use a different color for each material.

9 Exposing the structure of our formulation to make it easier to go to $3D$

9.1 Weak form

The nonlinear weak form (72) can be written:

$$\int_0^L \left[\sigma\left(\epsilon\right)\frac{dv}{dx} - f(x)\cdot v(x)\right]dx - \sigma_o\cdot v(L) = 0 \tag{141}$$

or

$$\int_0^L \left[\sigma\left(\frac{du}{dx}\right)\frac{dv}{dx} - f(x)\cdot v(x)\right]dx - \sigma_o\cdot v(L) = 0 \tag{142}$$

9.2 Newton iteration:

Letting $u \leftarrow u + \delta u$:

$$\begin{aligned}
\int_0^L \left[\sigma\left(\frac{du}{dx}+\frac{d\delta u}{dx}\right)\frac{dv}{dx}\right]dx &= \int_0^L \left[\sigma\left(\frac{du}{dx}\right)\frac{dv}{dx}\right]dx \\
&\quad + \int_0^L \left[\frac{d\sigma}{d\epsilon}\left(\frac{du}{dx}\right)\times\frac{d\delta u}{dx}\frac{dv}{dx}\right]dx && (143)\\
&= \int_0^L \left[\sigma\left(\frac{du}{dx}\right)\frac{dv}{dx}\right]dx \\
&\quad + \int_0^L \left[E^*\times\frac{d\delta u}{dx}\frac{dv}{dx}\right]dx && (144)\\
\text{where} \qquad E^* &= \frac{d\sigma}{d\epsilon}\left(\frac{du}{dx}\right) && (145)\\
&= \text{tangent modulus} && (146)
\end{aligned}$$

9.3 Extending to multi-dimensions

- $\int_0^L dx \leftarrow \int_V dV$.
- $\sigma \leftarrow \sigma_{ij}$.
- $\epsilon \leftarrow \epsilon_{ij}$.
- $\frac{dv}{dx} \leftarrow \frac{\partial v_i}{\partial x_j}$.
- $2\boldsymbol{\epsilon} = \nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T$.
- $2\epsilon_{ij} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}$.
- $\frac{d\sigma}{d\epsilon} \leftarrow \frac{\partial\sigma_{ij}}{\partial\epsilon_{kl}}$.
- $u \approx u^h = \sum_A d_A N_A(x)$ becomes $u_i \approx u_i^h = \sum_A d_{iA} N_A(x)$.

9.4 Weak form in multiple dimensions

Nonlinear Weak Form

Given u_{oi}, T_i, the function $\sigma(\epsilon)$, and $f_i(x)$, find $u^* \in \mathcal{S}$ such that for all $v(x) \in \mathcal{V}$,

$$\int_V \left[\sigma_{ij}\left(\boldsymbol{\epsilon}\right)\frac{\partial v_i}{\partial x_j} - f_i(x)v_i(x)\right]dV - \int_{\Gamma_T} T_i \cdot v_i \, dS = 0 \tag{147}$$

where

$$\mathcal{S} = \left\{\boldsymbol{u}(x)|\boldsymbol{u}(x) \in C^0[V];\ u_i|_{\Gamma_u} = u_{oi}.\right\} \tag{148}$$

$$\mathcal{V} = \left\{\boldsymbol{v}(x)|\boldsymbol{v}(x) \in C^0[V];\ v_i|_{\Gamma_u} = 0.\right\} \tag{149}$$

9.5 A sample $3D$ constitutive equation

The classical form of the linear elastic isotropic constititive equation in $3D$ is:

$$\epsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\delta_{ij}\sigma_{kk}. \tag{150}$$

When expressing stress in terms of strain, the Lamé parameters, λ, μ, are often used as material properties. In terms of the Lamé parameters, we may write the stress-strain relation as:

$$\sigma_{ij} = \lambda\delta_{ij}\epsilon_{kk} + 2\mu\epsilon_{ij}. \tag{151}$$

Exercise

1. Look up relation between $E, \nu, \lambda, \mu \equiv G, K, M$ at label
2. Use (151) and (150) to derive the expression for μ in terms of E and ν, and also for E in terms of λ and μ.
3. Derive (152).

In general, strain can be decomposed into two parts: volume change and shape change. For large deformation, that decomposition is a multiplicative decomposition of the deformation gradient. For small (linearized) deformations, the decomposition simplifies to an additive decomposition. First recall that for small strains, the volume change per unit volume is:

$$\frac{\Delta V}{V} \equiv \Delta = \epsilon_{kk} \tag{152}$$

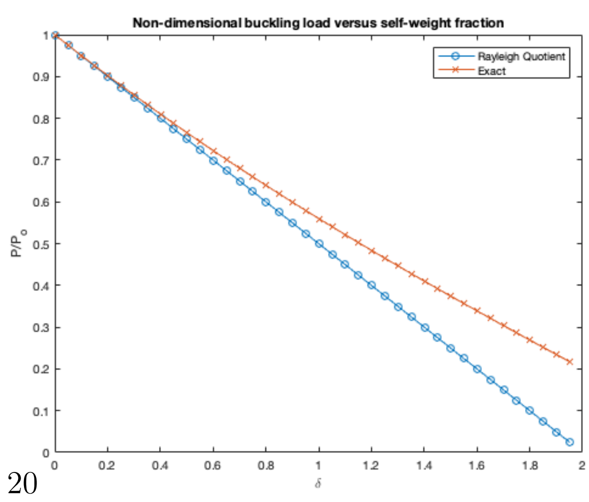

Figure 1: This is a random figure placed for Andre.

Now we introduce the deviatoric strain, e_{ij}, so that:

$$\epsilon_{ij} = \frac{\Delta}{3}\delta_{ij} + e_{ij}. \tag{153}$$

In a similar way, we can introduce the deviatoric stress s_{ij} and pressure p, so that:

$$\begin{aligned} \sigma_{ij} &= -p\delta_{ij} + s_{ij} && (154) \\ p &= -\frac{1}{3}\sigma_{kk} && (155) \end{aligned}$$

One way to think about why an isotropic material has two different material properties is that it has one “spring" constant associated with volume change, and a different one associated with shape change. Therefore:

$$\begin{aligned} p &= -K\Delta && (156) \\ \boldsymbol{s} &= 2\mu\boldsymbol{e} = 2G\boldsymbol{e} && (157) \end{aligned}$$