The Brownian Split method of Sampling Zero-free Intervals of a Brownian Bridge

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November 2024, last update Mar 2025

1 Introduction

The zero-crossings of Brownian motion have long been of interest as a proxy for the time distribution of real-world events. Since the zeros are not isolated, however, direct simulation is usually only performed for a first hitting time (e.g. a Lévy disribution), or the last zero in an interval (an arcsine law).

Here we describe the Brownian "split", which trisects a Brownian bridge into a bridge, excursion and bridge via sampling the last zero before the midpoint, and the first zero after. Such a procedure can be recursively applied until the intervals containing zeros (the bridges) have been trimmed to any desired length.

W.l.o.g. let us consider the standard bridge on the unit interval W_t, $t \in [0,1]$ whose endpoints are constrained to be $W_0 = W_1 = 0$ and define τ_+, τ_- to be the first zero after the midpoint and the last zero before, respectively. The main result here is that a method of jointly sampling τ_+, τ_- is given by

$$\tau_+ = \frac{1}{1+\sin^2\left(\frac{\pi}{2}U_1\right)} \in \left(\frac{1}{2}, 1\right) \tag{1}$$

$$\tau_- = \frac{U_2^2 \tau_+}{2\tau_+ + U_2^2 - 1} \in \left(0, \frac{1}{2}\right) \tag{2}$$

where U_1, U_2 are uniformly distributed random variables over $(0,1)$.

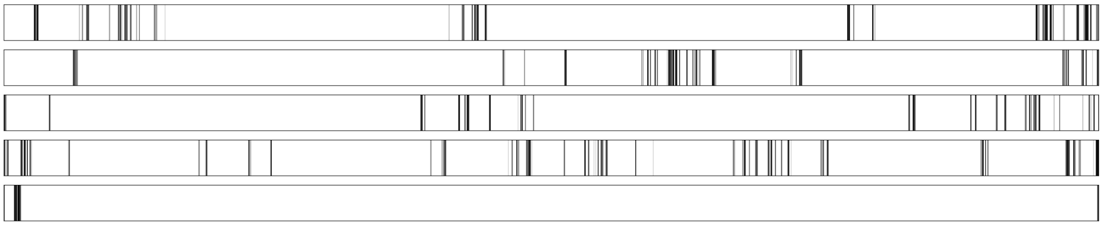

Figure 2: Example distributions of zeros (intervals recursively split until they are $< 0.001\%$ of the original interval) using the sampling method in Eqns (1-2)

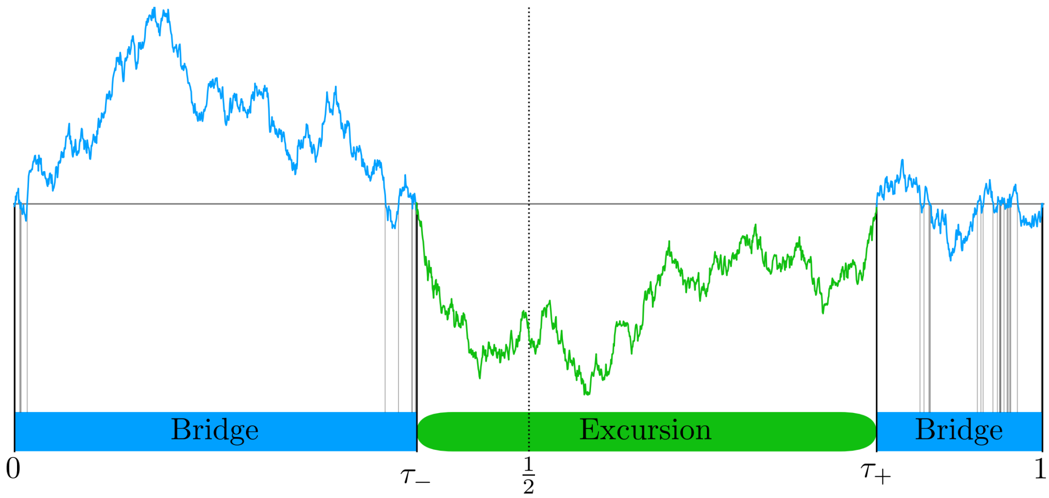

Figure 1: Illustration of finding the last zero crossing (τ_-) before the midpoint, and the first crossing after (τ_+)

Some points of note are

- By construction W_t forms a Brownian bridge over $[0, \tau_-]$, a positive/negative Brownian excursion over $[\tau_-, \tau_+]$, and then a Brownian bridge over $[\tau_+, 1]$.
- The crossings τ_- and $1 - \tau_+$ are identically distributed (but not independent). The median τ_+ is $2/3$, and the mean is $\sqrt{1/2}$.
- The Pearson correlation between τ_- and τ_+ is $-4/3 + \sqrt{8/9} \approx -0.39052$. This negative correlation can be intuitively understood by the clustering of zeros: a zero shortly before the midpoint (i.e. a large τ_-) significantly increases the chances of a zero shortly after the midpoint (a small τ_+), and vice-versa.

1.1 Brownian Split derivation

We derive the above results, with a more general splitting point $\alpha \in (0,1)$. The approach here is to introduce the auxiliary variable $w = W_\alpha$, and then integrate over all w.

Let us define τ_+ to be the first crossing of zero after α. The barrier-hitting probability density of a walk starting at $W_\alpha = w$ *without* the constraint at $W_1 = 0$ is given by the standard

$$\rho\left(\tau_+ | W_\alpha = w\right) = \frac{|w|}{\tau_+ - \alpha} \phi\left(w; \tau_+ - \alpha\right) . \tag{3}$$

where $\phi(x;\sigma^2)$ is the density of the normal distribution of zero mean and variance σ^2. The density of arrival at $W_1 = 0$ is given by $\rho\left(W_1 = 0 | W_\alpha = w\right) = \phi(w; 1-\alpha)$, and similarly $\rho\left(W_1 = 0 | \tau_+, W_\alpha = w\right) = \phi(0; 1-\tau_+)$ (using the Markov property), and so by Bayes' theorem for densities we have

$$\begin{aligned} \rho\left(\tau_+ | W_\alpha = w, W_1 = 0\right) &= \frac{\rho(W_1 = 0|\tau_+, W_\alpha = w)\rho(\tau_+|W_\alpha = w)}{\rho(W_1 = 0|W_\alpha = w)} && (4)\\ &= \frac{|w|(1-\alpha)}{(\tau_+ - \alpha)(1-\tau_+)} \phi\left(w; \frac{(1-\alpha)(\tau_+ - \alpha)}{1-\tau_+}\right) . && (5) \end{aligned}$$

Noting that the probability density of w is given by

$$\rho(w|W_1 = 0) = \phi\left(w; \alpha(1-\alpha)\right) \tag{6}$$

(note we assume the condition $W_0 = 0$ for brevity) we can combine with (5) to find the joint probability density of τ_+, w as

$$\rho(\tau_+, w|W_1 = 0) = \frac{|w|}{\tau_+ - \alpha} \frac{1}{\sqrt{2\pi(1-\tau_+)\tau_+}} \phi\left(w; \frac{(\tau_+ - \alpha)\alpha}{\tau_+}\right) \tag{7}$$

and we can integrate over w as

$$\begin{aligned} \rho\left(\tau_+ | W_1 = 0\right) &= \int_{-\infty}^{\infty} \mathrm{d}w \rho\left(\tau_+, w | W_1 = 0\right) && (8)\\ &= \frac{1}{\pi \tau_+} \sqrt{\frac{\alpha}{(1-\tau_+)(\tau_+ - \alpha)}} . && (9) \end{aligned}$$

(As an exercise for the reader, if we remove the Brownian bridge constraint $W_1 = 0$ and repeat this process we recover a Lévy arcsine law [1]). Integrating we can find the CDF

$$\mathrm{CDF}\left(\tau_+ | W_1 = 0\right) = \frac{2}{\pi} \sin^{-1} \sqrt{\frac{\tau_+ - \alpha}{(1-\alpha)\tau_+}} \tag{10}$$

which can be inverted to find a sampling formula

$$\tau_+ = \frac{\alpha}{\alpha + (1-\alpha)\sin^2\left(\frac{\pi}{2} U_1\right)} \in (\alpha, 1) \tag{11}$$

with U_1 a uniformly distributed random variable over $(0,1)$. The mean, variance and median are given by

$$\begin{aligned} \mathbb{E}\left[\tau_+\right] &= \sqrt{\alpha} && (12)\\ \mathrm{Var}\left[\tau_+\right] &= \frac{1}{2}\sqrt{\alpha}\left(1-\sqrt{\alpha}\right)^2 && (13)\\ \mathrm{Med}\left[\tau_+\right] &= \frac{2\alpha}{1+\alpha} . && (14) \end{aligned}$$

Now by symmetry w.r.t. time we find for the distribution of τ_- of the last zero before α,

$$\begin{aligned} \mathbb{E}\left[\tau_-\right] &= 1-\sqrt{1-\alpha} && (15)\\ \mathrm{Var}\left[\tau_-\right] &= \frac{1}{2}\sqrt{1-\alpha}\left(1-\sqrt{1-\alpha}\right)^2 && (16)\\ \mathrm{Med}\left[\tau_-\right] &= \frac{\alpha}{2-\alpha}\,. && (17) \end{aligned}$$

and with (5) we can find

$$\rho\left(\tau_-|W_\alpha=w\right)=\frac{|w|\alpha}{\tau_-(\alpha-\tau_-)}\phi\left(w;\frac{\alpha(\alpha-\tau_-)}{\tau_-}\right)\,. \tag{18}$$

and combine with (7) to give the joint density

$$\rho(\tau_-,w,\tau_+)=\frac{w^2}{2\pi}\frac{\phi\left(w;\frac{(\alpha-\tau_-)(\tau_+-\alpha)}{\tau_+-\tau_-}\right)}{(\alpha-\tau_-)(\tau_+-\alpha)\sqrt{\tau_-(1-\tau_+)(\tau_+-\tau_-)}} \tag{19}$$

(we omit the condition $W_1=0$ for brevity) and integrating over w

$$\rho(\tau_-,\tau_+)=\frac{1}{2\pi\sqrt{\tau_-(1-\tau_+)(\tau_+-\tau_-)^3}} \tag{20}$$

then by Bayes (and the Markov property)

$$\rho(\tau_-|\tau_+)=\frac{\rho(\tau_-,\tau_+)}{\rho(\tau_+|W_1=0)}=\frac{\tau_+}{2}\sqrt{\frac{\tau_+-\alpha}{\alpha(\tau_+-\tau_-)^3\tau_-}} \tag{21}$$

and integrate to the c.d.f.

$$\mathrm{CDF}(\tau_-|\tau_+)=\sqrt{\frac{(\tau_+-\alpha)\tau_-}{\alpha(\tau_+-\tau_-)}} \tag{22}$$

to give a (conditional) sampling formula

$$\tau_-=\frac{\alpha U_2^2\tau_+}{\alpha U_2^2+\tau_+-\alpha}\in(0,\alpha)\ . \tag{23}$$

For the correlation we must perform a (fairly unpleasant) integral to find the moment

$$\mathbb{E}\left[\tau_-\tau_+\right]=\frac{1}{3}\left(2+\alpha^{3/2}-(\alpha+2)\sqrt{1-\alpha}\right) \tag{24}$$

which gives Pearson correlation coefficient

$$\mathrm{Corr}\left[\tau_-,\tau_+\right]=\frac{2}{3}\frac{\left(1-\sqrt{\alpha}-\sqrt{1-\alpha}\right)}{\alpha^{1/4}\left(1-\alpha\right)^{1/4}} \tag{25}$$

By splitting at the midpoint $\alpha=1/2$ into Eqs. (11),(23) we recover the formulae in Eqs. (1-2).

2 Longest interval between zeros

An interesting application of (20) is in terms of the longest interval between zeros of the Brownian bridge, a problem studied by [2, 3]. Denoting the longest the longest interval l, then if $l > 1/2$ then it is straightforward from the above analysis that this can only occur iff $\tau_+ - \tau_- > 1/2$, and hence the cumulative probability is given by

$$\mathbb{P}(l < r) = 2 - \sqrt{\frac{1}{r}},\ r \in \left[\frac{1}{2}, 1\right] \tag{26}$$

a result known to Rosén (via [2]).

For smaller values of r the cumulative probability is only piecewise analytic [3], on the intervals $r^{-1} \in (n, n+1]$. Defining

$$F_n(r) = \mathbb{P}(l < r),\ r^{-1} \in (n, n+1] \tag{27}$$

then the F_n can be written in terms of Wendel's factorial moments $M_n^\star$,

$$F_n(r) \quad = \quad \sum_{k=0}^{n} (-1)^k M_k^\star(r)\,. \tag{28}$$

where $M_n^\star(r)$ is the expectation of $\binom{N(r)}{n}$, where $N(r)$ is the number of zero free intervals of length $> r$. The first of these is given by $M_0^\star(r) = 1$ and recurse to larger n via Laplace transform,

$$M_{n+1}^\star(r) = \begin{cases} \frac{1}{\pi}\int_{nr}^{1-r} \frac{\mathrm{d}x}{1-x} M_n^\star\left(\frac{r}{x}\right)\sqrt{\frac{1-r-x}{rx}}, & 0 < r < \frac{1}{n+1}, \\ 0, & \text{otherwise} \end{cases} \tag{29}$$

(note the lower limit in [2] is given as zero rather than nr, however $M_n^\star(r)$ zero for $nr < 1$, so this is equivalent).

Application of this recurrence gives us straightforward expressions for

$$M_1^\star(r) \quad = \quad \frac{1}{\sqrt{r}} - 1 \tag{30}$$

$$M_2^\star(r) \quad = \quad \frac{2}{\pi}\left[\frac{\sqrt{1-2r}}{r} + \cos^{-1}\frac{r}{1-r} - \frac{2}{\sqrt{r}}\cos^{-1}\sqrt{\frac{r}{1-r}}\right] \tag{31}$$

after which the integrals are problematic.

An alternative approach to finding the cumulative probability of the longest interval is to recursively subdivide Brownian bridges via Eqns. (20). Taking l_- to be the longest interval in $[0, \tau_-]$, and l_+ to be the longest interval in $[\tau_+, 1]$, we may write

$$F_n(r) \quad = \quad \int_{\frac{1}{2}-r}^{\frac{1}{2}} \mathrm{d}\tau_- \int_{\frac{1}{2}}^{\tau_-+r} \mathrm{d}\tau_+ \mathbb{P}\left(l_- < \frac{r}{\tau_-}\right)\mathbb{P}\left(l_+ < \frac{r}{1-\tau_+}\right)\rho(\tau_-, \tau_+) \tag{32}$$

where the integration domain is restricted, since $\tau_+ - \tau_- > r$ excludes the longest interval being shorter than r.

Eqn. (32) may be decomposed as in the diagrams in Fig. 3 for even $2n$,

$$\begin{aligned} F_{2n}(r) \quad = \quad & 2\int_{nr}^{\frac{1}{2}} \mathrm{d}\tau_- \int_{1-nr}^{\tau_-+r} \mathrm{d}\tau_+ F_n\left(\frac{r}{\tau_-}\right) F_{n-1}\left(\frac{r}{1-\tau_+}\right)\rho(\tau_-, \tau_+) + \\ & \int_{1-(n+1)r}^{nr} \mathrm{d}\tau_- \int_{1-nr}^{\tau_-+r} \mathrm{d}\tau_+ F_{n-1}\left(\frac{r}{\tau_-}\right) F_{n-1}\left(\frac{r}{1-\tau_+}\right)\rho(\tau_-, \tau_+) + \\ & \int_{nr}^{\frac{1}{2}} \mathrm{d}\tau_- \int_{\frac{1}{2}}^{1-nr} \mathrm{d}\tau_+ F_n\left(\frac{r}{\tau_-}\right) F_n\left(\frac{r}{1-\tau_+}\right)\rho(\tau_-, \tau_+) \end{aligned} \tag{33}$$

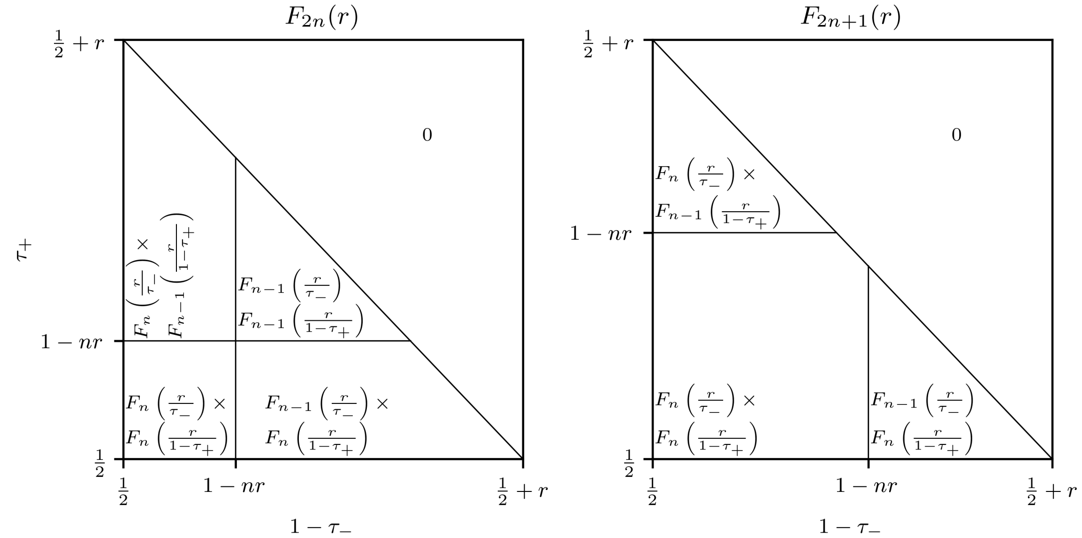

Figure 3: Diagrams of the decomposition of the integrand in Eqn. (32) as a function of $1-\tau_-$ and τ_+. *Left* is the even recurrence in Eqn. (33), *right* the odd recurrences in Eqn. (34).

where we take the "natural" convention $F_0(r) = 1$. Similarly for odd $2n+1$ we have

$$\begin{aligned} F_{2n+1}(r) &= \int_{nr}^{\frac{1}{2}} \mathrm{d}\tau_- \int_{\frac{1}{2}}^{\min(\tau_-+r,1-nr)} \mathrm{d}\tau_+ F_n\left(\frac{r}{\tau_-}\right) F_n\left(\frac{r}{1-\tau_+}\right)\rho(\tau_-,\tau_+) + \\ & \quad 2\int_{\frac{1}{2}-r}^{nr} \mathrm{d}\tau_- \int_{\frac{1}{2}}^{\tau_-+r} \mathrm{d}\tau_+ F_{n-1}\left(\frac{r}{\tau_-}\right) F_n\left(\frac{r}{1-\tau_+}\right)\rho(\tau_-,\tau_+) \end{aligned} \tag{34}$$

These integrals allow us to go slightly further than the Laplace transform approach, giving

$$F_1(r) = 2 - \sqrt{\frac{1}{r}} \tag{35}$$

$$F_2(r) = 2 - \frac{1}{\sqrt{r}} + \frac{2}{\pi}\left[\frac{1}{r}\sqrt{1-2r} - \frac{2}{\sqrt{r}}\cos^{-1}\sqrt{\frac{r}{1-r}} + \cos^{-1}\frac{r}{1-r}\right] \tag{36}$$

$$F_3(r) = \frac{2}{\sqrt{r}} + \frac{1}{\pi}\left[\frac{8}{r}\sqrt{1-2r} - \frac{3r+1}{r\sqrt{r}} + 8\cos^{-1}\frac{r}{1-r} - \frac{16}{\sqrt{r}}\cos^{-1}\sqrt{\frac{r}{1-r}}\right] \tag{37}$$

before the we can integrate no further. Interestingly the correspondence implies the 3rd factorial moment will be (via Eqn. 28)

$$M_3^\star(r) = F_2(r) - F_3(r) = \frac{3}{\sqrt{r}} - 1 + \frac{3r+1}{\pi r^{\frac{3}{2}}} - \frac{6}{\pi}\left[\frac{\sqrt{1-2r}}{r} - \sin^{-1}\frac{r}{1-r} + \frac{2}{\sqrt{r}}\sin^{-1}\sqrt{\frac{r}{1-r}}\right]. \tag{38}$$

References

[1] Lévy, P 1940 *Compositio Mathematica* **7**

[2] Wendel J G 1964 *Math. Scand.* **14** 21

[3] Godréche, C 2017 *Journal of Physics A: Mathematical and Theoretical* **50** 19