Mathematics 705

Three.IV.4

Matrix Inverses

Finally for section Three.IV, we consider how to represent the inverse of a linear map. The goal: where a linear transformation h has an inverse, to find the relationship between the matrices $\text{Rep}_{B,D}(h)$ and $\text{Rep}_{D,B}(h^{-1})$.

Let's look at a particular map, $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ and $\iota : \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \stackrel{\pi}{\longmapsto} \begin{pmatrix} x \\ y \end{pmatrix} \qquad \begin{pmatrix} x \\ y \end{pmatrix} \stackrel{\iota}{\longmapsto} \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$$

Then the composition $\pi \circ \iota$ is the identity map $\pi \circ \iota = \text{id}$ on $\mathbb{R}^2$.

$$\begin{pmatrix} x \\ y \end{pmatrix} \stackrel{\iota}{\longmapsto} \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} \stackrel{\pi}{\longmapsto} \begin{pmatrix} x \\ y \end{pmatrix}$$

We say that ι is a *right inverse* of π or, which is the same thing, that π is *left inverse* of ι.

Interestingly, order matters. Look at $\iota \circ \pi$ acting on a particular vector.

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \stackrel{\pi}{\longmapsto} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \stackrel{\iota}{\longmapsto} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Indeed, π has no left inverse at all. We see the problem if we examine what a left inverse of π would have to accomplish.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \stackrel{\pi}{\longmapsto} \begin{pmatrix} x \\ y \end{pmatrix} \stackrel{f}{\longmapsto} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

but now f is not a function, because there fails to be a unique output.

Some functions hava a *two-sided inverse*, another function that is the inverse both from the left and the right. Consider $\vec{v} \mapsto 2 \cdot \vec{v}$. It has a two-sided inverse $\vec{v} \mapsto \frac{1}{2} \cdot \vec{v}$. It can be shown that a function has a two sided inverse iff it is both one-to-one and onto; then this inverse is unique and we call it f^{-1}, the inverse.

Definition. A matrix G is a *left inverse matrix* of the matrix H if GH is the identity matrix. It is a *right inverse* if HG is the identity. A matrix H with a two-sided inverse is an *invertible matrix.* That two-sided inverse is denoted H^{-1}.

Example:

This matrix

$$H = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$$

has a two-sided inverse

$$H^{-1} = \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}$$

To check, just multiply them out.

Example:

To solve the linear system

$$\begin{aligned} 2x + 5y &= -3 \\ x + 3y &= 10 \end{aligned}$$

rewrite as a matrix equation

$$\begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ 10 \end{pmatrix}$$

and multiply both sides from the left by the inverse matrix.

$$\begin{aligned} \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 10 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} -59 \\ 23 \end{pmatrix} \end{aligned}$$

Let's look at this as an arrow diagram.

$$\begin{tikzcd} & \mathcal{W}_{wrt\,C} \arrow[dr, "h^{-1}", "H^{-1}"'] & \\ \mathcal{V}_{wrt\,B} \arrow[ur, "h", "H"'] \arrow[rr, "\mathrm{id}", "I"'] & & \mathcal{V}_{wrt\,B} \end{tikzcd}$$

Lemma. If a matrix has both a left inverse and a right inverse then the two are equal.

Theorem. A matrix is invertible iff it is nonsingular.

Lemma. A product of invertible matrices is invertible: if G and H are invertible and GH is defined, then GH is invertible and $(GH)^{-1} = H^{-1}G^{-1}$.

Take a moment to observe what this lemma is saying about the composition of the maps that G and H represent.

Lemma. A matrix H is invertible iff it can be written as the product of elementary reduction matrices. We can compute the inverse by applying to the identity matrix the same row steps, in the same order, that Gauss-Jordan reduce H.

Now, this gives us a good method for finding inverse matrices. Here's the trick:

Example:

This matrix is nonsingular and thus invertible.

$$A = \begin{pmatrix} 1 & 3 & 1 \\ 2 & 0 & -1 \\ 1 & 2 & 0 \end{pmatrix}$$

The calculation can be rendered easier if we write it in the following fashion and perform Gauss-Jordan elimination:

$$\begin{pmatrix} 1 & 3 & 1 & 1 & 0 & 0 \\ 2 & 0 & -1 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 & 0 & 1 \end{pmatrix} \overset{(many\ steps)}{\longrightarrow} \begin{pmatrix} 1 & 0 & 0 & 2/3 & 2/3 & -1 \\ 0 & 1 & 0 & -1/3 & -1/3 & 1 \\ 0 & 0 & 1 & 4/3 & 1/3 & -2 \end{pmatrix}$$

And the 3×3 matrix on the right side of the augmented matrix is A^{-1}.

Why bother? Because although finding inverse matrices is a lot of work, once you know it we can solve

$$A\vec{x} = \vec{b}$$

easily by noting that

$$\begin{aligned} A^{-1}A\vec{x} &= A^{-1}\vec{b} \\ \vec{x} &= A^{-1}\vec{b} \end{aligned}$$

Use this to solve

$$\begin{aligned} x + 3y + z &= 1 \\ 2x \qquad - z &= 12 \\ x + 2y \qquad &= 4 \end{aligned}$$

(Notice the coefficient matrix is the one for which we just found the inverse.)

Use this trick to find the general form of the inverse of a 2×2 matrix, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}$.

(Should get $\dfrac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.)

Another useful aspect of the inverse matrix approach to solving linear equations is that it can help us get a feel for the effect of small changes in the constants.

Example:

$$
\begin{aligned} 2x_1 - 3x_2 &= 16 \\ x_1 + 2x_2 &= 1 \end{aligned} \quad \Longleftrightarrow \quad \begin{pmatrix} 2 & -3 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 16 \\ 1 \end{pmatrix}
$$

This has inverse

$$
\frac{1}{7} \begin{pmatrix} 2 & 3 \\ -1 & 2 \end{pmatrix}
$$

and solution

$$
\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{1}{7} \begin{pmatrix} 2 & 3 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 16 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ -2 \end{pmatrix}
$$

What happens when we make a small change in one of the constants?

$$
\begin{pmatrix} 2 & -3 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 16.01 \\ 1 \end{pmatrix}
$$

giving us

$$
\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \frac{2 \cdot 16.01 + 3}{7} \\ \frac{-1 \cdot 16.01 + 2}{7} \end{pmatrix}
$$

We see that x_1 changes by a factor of $2/7$ of the small change, while x_2 changes by a factor of $-1/7$ of the small change.

That helps us know how to specify the accuracy of input data to achieve a desired accuracy in the solution.