CS 225

Theorems from Chapter 4

Fall 2020

From 4.1:

Assumptions:

- Basic laws of algebra from appendix A (I assume we do not have to cite these–I listed some on the last page of this document)
- Properties of equality: $x = x$, $x = y \implies y = x$, $x = y$ and $y = z \implies x = z$
- Principle of substitution: if $x = y$, then we can substitute in y wherever x appears.
- The integers are closed under addition, subtraction and multiplication. (The book also mentions that there is no integer between 0 and 1–does that need to be an assumption??)

Definitions:

- An integer n is **even** if and only if it can be written as $n = 2k$ for some integer k.
- An integer n is **odd** if and only if it can be written as $n = 2k + 1$ for some integer k.
- An integer n is **prime** if and only if $n > 1$ and for all positive integers r and s, if $n = rs$, then either r or s equals n.
- An integer n is **composite** if and only if $n > 1$ and $n = rs$ for some integers r and s with $1 < r < n$ and $1 < s < n$.

Theorem 4.1.1

The sum of any two even integers is even.

From 4.2:

Theorem 4.2.1:

The difference of any odd integer and any even integer is odd.

From 4.3:

Definition:

A real number r is **rational** if and only if there are integers a and b with $b \neq 0$ so that $r = \frac{a}{b}$

Property:

The zero product property is: if neither of two real numbers is zero, then their product is also not zero. (this is T11 in appendix A)

Theorem 4.3.1:

Every integer is a rational number.

Theorem 4.3.2:

The sum of any two rational numbers is a rational number.

Corollary 4.2.3:

The double of a rational number is a rational number.

Result of Exercise 12:

The square of any rational number is a rational number.

Result of Exercise 13:

The negative of any rational number is a rational number.

Result of Exercise 14:

The cube of any rational number is a rational number.

Result of Exercise 15:

The product of any two rational numbers is a rational number.

Result of Exercise 17:

The difference of any two rational numbers is a rational number.

From Appendix A:

- Field Axioms for real numbers:

  F1 Commutative laws for addition and multiplication

  F2 Associative laws for addition and multiplication

  F3 Distributive laws

  F4 Existence of an identity for addition and multiplication

  F5 Existence of additive inverses (negative numbers)

  F6 Existence of multiplicative inverses (reciprocals)

- Other theorems:

  T1 Cancellation law for addition: $a + b = a + c \implies b = c$

  T2 Possibility of subtraction: There is a unique solution to $a + x = b$ (namely $x = b - a$)

  T3 $b - a = b + (-a)$

  T4 $-(-a) = a$

  T5 $a(b - c) = ab - ac$

  T6 $0 \cdot a = a \cdot 0 = 0$

  T7 Cancellation law for multiplication: $ab = ac$ and $a \neq 0 \implies b = c$

  T8 Possibility of division: If $a \neq 0$, there is a unique solution to $ax = b$. (namely $x = b/a$)

  T9 If $a \neq 0$, $\dfrac{b}{a} = b \cdot a^{-1}$

  T10 If $a \neq 0$, $(a^{-1})^{-1} = a$

  T11 Zero product property: If $ab = 0$, then $a = 0$ or $b = 0$

  T12 $(-a)b = a(-b) = -(ab)$, $(-a)(-b) = ab$,

  $$-\frac{a}{b} = \frac{-a}{b} = \frac{a}{-b}$$

  T13 Equivalent fractions property: $\dfrac{a}{b} = \dfrac{ac}{bc}$ if $b \neq 0$ and $c \neq 0$

  T14 Fraction addition: $\dfrac{a}{b} + \dfrac{c}{d} = \dfrac{ad + bc}{bd}$ if $b \neq 0$ and $d \neq 0$

  T15 Multiplying fractions: $\dfrac{a}{b} \cdot \dfrac{c}{d} = \dfrac{ac}{bd}$

  T16 Dividing fractions: $\dfrac{a/b}{c/d} = \dfrac{a}{b} \cdot \dfrac{d}{c}$

- Order Axioms (see the appendix on page A-2)

- Order Theorems (see the appendix A-3)

- Least upper bound axiom (see the appendix A-3)