Notes on Descriptive Set Theory

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Contents

Preface

These are short notes for a 48-hours course on Descriptive Set Theory I taught at the University of Turin, addressed to master students with some (small) background in logic and set theory. Essentially, all the material comes from Kechris' monograph [Kec95], but it is re-organized in a slightly different way to fit the program of the course. Sometimes I give different or more detailed proofs, especially when dealing with facts whose proof is missing (or only sketched) in [Kec95], or when I think that a different approach can be more useful for these specific lectures. When possible, I carefully add precise references to Kechris' textbook: the reader is strongly encouraged to go back and forth between these notes and such reference, as (s)he will surely benefit from following both approaches and presentations. In particular, I remark that the present notes are just a small selection of the material from the book [Kec95], which should then be used to complete the reader's own knowledge of the basic tools and results in descriptive set theory.

Turin, April 23, 2024 *Luca Motto Ros*

1

Polish spaces

1.1 Definition and examples

The notion of a Polish space captures the basic topological properties of $\mathbb{R}^n$, $\mathbb{C}^n$, and their closed subspaces.

Definition 1.1.1. A topological space $X = (X, \tau)$ is called **Polish** if it is second-countable and completely metrizable, where:

- X is **second-countable** if it admits a countable basis;
- X is **(completely) metrizable** if there is a (complete) metric[1] d on X compatible with its topology, i.e. such that τ is generated by the d-open balls $B_d(x, \varepsilon) = \{y \in X \mid d(y, x) < \varepsilon\}$ (for $x \in X$ and $\varepsilon \in \mathbb{R}^+$).

When a specific compatible (complete) metric d on X is singled out, we call $X = (X, d)$ a *Polish metric space.*

Obviously, the class of Polish spaces is closed under homeomorphism. Recall also that any metric space (X, d) admits a (unique, up to isometry) **completion** $(\hat{X}, \hat{d})$. Thus any second-countable metrizable spaces is contained in a Polish space as a dense subspace.

Remark 1.1.2. A topological space X is **separable** if it contains a countable dense set. By the axiom of countable choices, every second-countable space is separable; the converse is true as well whenever X is metrizable (hence, in particular, when X is Polish) — see Proposition 3.6.1 for the details.

It is immediate to check that the class of Polish spaces is closed under the following operations:

Closed subspaces. All closed subspaces of a Polish space are Polish as well (when endowed with the relative topology).

[1] One may always assume d to be bounded: if not, it is enough to replace d with $d' = \frac{d}{1+d}$ and notice that d' is still a compatible (complete) metric on X.

Countable products. If $(X_n)_{n\in\omega}$ is a sequence of Polish spaces, then the product $\prod_{n\in\omega} X_n$ is again Polish when endowed with the product topology. A complete compatible metric d on the product is the one defined by setting for[2] $x, y \in \prod_{n\in\omega} X_n$

$$d(x,y) = \sum_{n=0}^{\infty} 2^{-n} d_n(x(n), y(n)),$$

where each d_n is a compatible complete metric on X_n bounded by 1.

Countable sums. If $(X_n)_{n\in\omega}$ is a sequence of Polish spaces, then their disjoint union $\bigoplus_{n\in\omega} X_n$ is Polish as well, when endowed with the smallest topology refining all the topologies of the X_n's (so that each X_n is clopen in $\bigoplus_{n\in\omega} X_n$). A compatible metric d on the countable sum can be defined by setting

$$d(x,y) = \begin{cases} d_n(x,y) & \text{if } x, y \in X_n \\ 1 & \text{if } x \text{ and } y \text{ belong to different } X_n\text{'s,} \end{cases}$$

where again each d_n is a compatible complete metric on X_n bounded by 1.

Countable intersections. If X is Polish and $(Y_n)_{n\in\omega}$ is a sequence of Polish subspaces of X, then $Y = \bigcap_{n\in\omega} Y_n$ is Polish as well. Indeed, $Z = \prod_{n\in\omega} Y_n$ is Polish and so is its closed subset

$$C = \{z \in Z \mid z(i) = z(j) \text{ for all } i, j \in \mathbb{N}\}.$$

Then the diagonal map $Y \to C$, $y \mapsto (z(n))_{n\in\omega}$ where $z(n) = y$ for all $n \in \omega$ is a homeomorphism, witnessing that Y is a Polish subspace of X.

Obviously, it follows that finitary versions of the above closure properties hold as well.

Example 1.1.3. Here are some natural examples of Polish spaces.

1. $\mathbb{R}$, $\mathbb{C}$, their finite or infinite products $\mathbb{R}^n$, $\mathbb{C}^n$, $\mathbb{R}^\omega$, and $\mathbb{C}^\omega$, the unit interval $[0;1]$, the unit circle $\mathbb{T} = \{x \in \mathbb{C} \mid |x| = 1\}$, the n-dimensional cube $[0;1]^n$, the Hilbert cube $[0;1]^\omega$, the n-dimensional torus $\mathbb{T}^n$, the infinite dimensional torus $\mathbb{T}^\omega$.

2. Any countable set A endowed with the discrete topology, and the product A^ω of countably many copies of such an A. In particular, when $A = 2 = \{0,1\}$ and $A = \omega$ we obtain the **Cantor space** 2^ω and the **Baire space** ω^ω.

[2] When dealing with a product $\prod_{i\in I} X_i$ of topological spaces, we will always denote the i-th coordinate of $x \in \prod_{i\in I} X_i$ by $x(i)$. This is fully justified because, formally, $\prod_{i\in I} X_i$ is the set of functions $x \colon I \to \bigcup_{i\in I} X_i$ such that $x(i) \in X_i$ for all $i \in I$.

3. Equip the set $\mathscr{P}(\mathbb{N}) = \{X \mid X \subseteq \mathbb{N}\}$ with the topology generated by the sets of the form

$$U_n = \{X \subseteq \mathbb{N} \mid n \in X\}$$
$$\hat{U}_n = \{X \subseteq \mathbb{N} \mid n \notin X\}$$

for $n \in \omega$. The resulting space is homeomorphic to 2^ω via the map sending $X \subseteq \mathbb{N}$ into its characteristic function, and thus it is a Polish space.

4. Let $\mathcal{L} = \{R_i \mid i < I\}$ (with $I \leq \omega$) be a(n at most) countable relational first-order language, where each R_i has arity n_i. Every $\mathcal{L}$-structure $\mathcal{A}$ with domain ω can be identified with an element of

$$\mathrm{Mod}_{\mathcal{L}} = \prod_{i \in I} 2^{(\omega^{n_i})}$$

via the characteristic functions of its predicates $R_i^{\mathcal{A}}$. More precisely, $\mathcal{A}$ can be coded as an element $x_{\mathcal{A}} \in \mathrm{Mod}_{\mathcal{L}}$ by setting for $i < I$ and $k_1, \dots, k_{n_i} \in \omega$

$$x_{\mathcal{A}}(i)(k_1, \dots, k_{n_i}) = 1 \iff (k_1, \dots, k_{n_i}) \in R_i^{\mathcal{A}}.$$

The map $\mathcal{A} \mapsto x_{\mathcal{A}}$ is actually a bijection between the $\mathcal{L}$-structures with domain ω and $\mathrm{Mod}_{\mathcal{L}}$, whose inverse map sends an arbitrary $x \in \mathrm{Mod}_{\mathcal{L}}$ to the structure $\mathcal{A}_x$ with domain ω and relations defined by

$$R_i^{\mathcal{A}_x} = \{(k_1, \dots, k_{n_i}) \in \omega^{n_i} \mid x(i)(k_1, \dots, k_{n_i}) = 1\}.$$

Endowing each $2^{(\omega^{n_i})}$ with the (countable) product of the discrete topology on 2, they all become Polish spaces (homeomorphic to the Cantor space). Thus $\mathrm{Mod}_{\mathcal{L}}$ is Polish as well, and can be regarded as the Polish space of all countable $\mathcal{L}$-structures (up to isomorphism).

5. Set $\mathcal{L} = \{R\}$ with R binary, so that $\mathrm{Mod}_{\mathcal{L}} = 2^{\omega \times \omega}$. Consider the set

$$\mathrm{LO} = \{x \in \mathrm{Mod}_{\mathcal{L}} \mid x \text{ codes a linear order}\},$$

that is: $x \in \mathrm{LO}$ if $R^{\mathcal{A}_x}$ is a reflexive, antisymmetric, transitive and total relation. Then

$$\mathrm{LO} = \bigcap_{n \in \omega} R_n \ \cap \bigcap_{\substack{n,m \in \omega \\ n \neq m}} A_{n,m} \ \cap \bigcap_{n,m,k \in \omega} T_{n,m,k} \ \cap \bigcap_{n,m \in \omega} L_{n,m},$$

where

- $R_n = \{x \in 2^{\omega \times \omega} \mid x(n,n) = 1\}$,
- $A_{n,m} = \{x \in 2^{\omega \times \omega} \mid x(n,m) = 0 \vee x(m,n) = 0\}$,
- $T_{n,m,k} = \{x \in 2^{\omega \times \omega} \mid x(n,m) = 0 \vee x(m,k) = 0 \vee x(n,k) = 1\}$,
- $L_{n,m} = \{x \in 2^{\omega \times \omega} \mid x(n,m) = 1 \vee x(m,n) = 1\}$.

Since each of the above sets is clopen (= closed *and* open), it follows that LO is closed in $\mathrm{Mod}_{\mathcal{L}}$, and thus it can be regarded as the Polish space of all countable linear orders (up to isomorphism).

6. Let p be any prime number. Every rational number $q \in \mathbb{Q}$ can be written in a *unique* way as $p^n \frac{a}{b}$ with a and b not divisible by p. Define the **p-adic absolute value** of q as
$$|q|_p = p^{-n}.$$
The space of **p-adic numbers** $\mathbb{Q}_p$ is the completion of $(\mathbb{Q}, d_p)$ where d_p is the metric induced by $|\cdot|_p$, i.e. $d_p(q, q') = |q - q'|_p$. Each p-adic number may be written in a unique way as
$$\sum_{i=k}^{\infty} a_i p^i$$
where $k \in \mathbb{Z}$ is such that $a_k \neq 0$ and each a_i belongs to $\{0, \dots, p-1\}$. The subring of **p-adic integers** $\mathbb{Z}_p$ consists of those p-adic numbers such that $a_i = 0$ for all $i < 0$. Both $\mathbb{Q}_p$ and $\mathbb{Z}_p$ are Polish spaces. Indeed, $\mathbb{Z}_p$ is homeomorphic to the Cantor space 2^ω, while $\mathbb{Q}_p$ is homeomorphic to any subspace of 2^ω obtained by removing a single point.

7. All separable Banach spaces are Polish, when endowed with the topology induced by the norm. These include the ℓ^p spaces ($1 \leq p < \infty$), in particular the **Hilbert space** ℓ^2 (which can be shown to be homeomorphic to $\mathbb{R}^\omega$); c_0, i.e. the space of converging-to-0 sequences with the sup norm; the $L^p(\mu)$ spaces ($1 \leq p < \infty$), where μ is a σ-finite measure on a countably generated σ-algebra; $C(X)$, the space of continuous (real or complex) functions on a compact metrizable space X with the sup norm.

8. Let X, Y be separable Banach spaces, and $L(X, Y)$ be the (generally non-separable) Banach space of bounded linear operators $T\colon X \to Y$ with norm $\|T\| = \sup\{\|Tx\| \mid x \in X, \|x\| \leq 1\}$. Then the unit ball
$$L_1(X, Y) = \{T \in L(X, Y) \mid \|T\| \leq 1\}$$
endowed with the (relative) **strong topology** is Polish. (Recall that the strong topology is the smallest topology on $L(X, Y)$ for which all the maps $f_x\colon L(X, Y) \to Y$ sending T to Tx are continuous, for $x \in X$.) To see this, let $D \subseteq X$ be a countable dense subset of X closed under rational linear combinations. Endow Y^D, the space of sequences indexed by D with values in Y, with the product topology, so that it is Polish. Consider the following closed (hence Polish) subset of Y^D:
$$\begin{aligned} F = \{f \in Y^D \mid \forall x, y \in D\, \forall p, q \in \mathbb{Q}\, (f(px + qy) = pf(x) + qf(y)) \\ \wedge \forall x \in D\, (\|f(x)\| \leq \|x\|)\}. \end{aligned}$$
Then the map $L_1(X, Y) \to Y^D$ sending T to its trace $T \restriction D$ on D is a homeomorphism beween $L_1(X, Y)$ with the strong topology and F.

Closed subspaces are not the unique Polish subspaces of a Polish space. For example, the open interval $(0;1)$, being homeomorphic to $\mathbb{R}$, is a Polish subspace of $\mathbb{R}$, although the usual metric is not complete on it.

Proposition 1.1.4. *If X is Polish and $Y \subseteq X$ is open, then Y is Polish as well (with respect to the topology inherited from X).*

Proof. The open set Y is clearly second-countable, so we only need to show that it is completely metrizable. Let $F = X \setminus Y$, and for any $x \in X$ set $d(x, F) = \inf\{d(x, y) \mid y \in F\}$. Define a new metric d' on Y by letting

$$d'(x, y) = d(x, y) + \left|\frac{1}{d(x, F)} - \frac{1}{d(y, F)}\right|.$$

We claim that d and d' generate the same topology on $Y = X \setminus F$. Since $B_{d'}(x, \varepsilon) \subseteq B_d(x, \varepsilon) \cap Y$ for all $x \in Y$ and $\varepsilon > 0$, it suffices to show that for all $x \in Y$ and $\varepsilon > 0$ there is $\varepsilon' > 0$ such that $B_d(x, \varepsilon') \cap Y \subseteq B_{d'}(x, \varepsilon)$. Choose $0 < \varepsilon' < \frac{\varepsilon}{2}$ such that for all $y \in B_d(x, \varepsilon') \cap Y$

$$\left|\frac{1}{d(x, F)} - \frac{1}{d(y, F)}\right| < \frac{\varepsilon}{2}.$$

Then for all $y \in B_d(x, \varepsilon') \cap Y$ one has

$$d'(x, y) = d(x, y) + \left|\frac{1}{d(x, F)} - \frac{1}{d(y, F)}\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus $B_d(x, \varepsilon') \cap Y \subseteq B_{d'}(x, \varepsilon)$, as desired.

It remains to check that (Y, d') is complete. Any given d'-Cauchy sequence $(y_i)_{i \in \omega}$ in Y is also d-Cauchy, hence $y_i \to y$ for some $y \in X$: we claim that $y \in Y$. Indeed, the sequence $\left(\frac{1}{d(y_i, F)}\right)_{i \in \omega}$ is Cauchy in $\mathbb{R}$ because of the second term in the definition of d', hence it converges in $\mathbb{R}$, and thus the sequence $(d(y_i, F))_{i \in \omega}$ is bounded away from 0. Since $d(y_i, F) \to d(y, F)$ by continuity of the distance map d, this means that $d(y, F) \neq 0$, and hence $y \notin F$, i.e. $y \in Y$. $\square$

It follows that countable intersections of open subsets of a given Polish space are Polish as well.

Example 1.1.5. Consider the symmetric group S_∞ of all permutations of ω. Formally, S_∞ is the subspace of ω^ω consisting of all bijections from ω into itself. Thus S_∞ is the (countable) intersection of the following open sets, where n, m vary over *distinct* natural numbers:

- $\{x \in \omega^\omega \mid x(n) \neq x(m)\}$
- $\bigcup_{k \in \omega} \{x \in \omega^\omega \mid x(k) = n\}$.

Thus S_∞ is a Polish space. Indeed, it is even a **Polish group**, i.e. a topological group[3] whose topology is Polish.

[3] A topological group is a group equipped with a topology turning its operations into continuous functions.

Polish subspaces are characterized by the following notion.

Definition 1.1.6. A subset A of a topological space X is G_δ if it can be written as a countable intersection of open subsets of X, and it is F_σ if it can be written as a countable union of closed sets (equivalently: if its complement is G_δ).

It is immediate to check that the collection of all G_δ subsets of a space X is closed under countable intersections and finite unions, while the collection of all F_σ subsets is closed under countable unions and finite intersections. Moreover, one can show that if X is Polish and $A, B \subseteq X$ are G_δ and dense, then $A \cap B \neq \emptyset$ (this follows e.g. from [Kec95, Theorem 8.4]).

Example 1.1.7. The rationals $\mathbb{Q} = \bigcup_{q \in \mathbb{Q}} \{q\}$ form an F_σ subset of $\mathbb{R}$, hence the irrationals $\mathrm{Irr} = \mathbb{R} \setminus \mathbb{Q}$ form a G_δ set. Since $\mathbb{Q}$ and Irr are both dense, $\mathbb{Q}$ is not G_δ (see Corollary 1.5.6 and Exercise 2.1.25), and hence Irr is not F_σ. The half-open interval $[0;1) = \bigcup_{n \in \omega} [0; 1 - 2^{-n}] = \bigcap_{n \in \omega} (-2^{-n}; 1)$ is an example of a subset of $\mathbb{R}$ which is both F_σ and G_δ, but neither open nor closed.

If X is metrizable, then all its closed subsets F are G_δ (see Proposition 3.6.2 or [Kec95, Proposition 3.7]): indeed, $F = \bigcap_{n \in \omega} U_n$ where $U_n = \bigcup \{B_d(x, 2^{-n}) \mid x \in F\}$ with d any compatible metric on X.

Proposition 1.1.8 ([Kec95, Theorem 3.11])**.** *Let X be a Polish space and $Y \subseteq X$ be endowed with the relative topology. Then the following are equivalent:*

1. *Y is Polish (with respect to the induced topology);*
2. *Y is a G_δ subset of X.*

Proof. The backward direction follows from Proposition 1.1.4 and closure under countable intersections of the class of Polish spaces.

For the other direction of the equivalence we use the following concepts and results. Let Z_1 be a topological space, $Z_2 = (Z_2, d')$ be a metric space, $A \subseteq Z_1$, and $f \colon A \to Z_2$. The **oscillation** of f at $z \in Z_1$ is

$$\mathrm{osc}_f(z) = \inf\{\mathrm{diam}(f(U \cap A)) \mid U \subseteq Z_1 \text{ open}, z \in U\},$$

where for $B \subseteq Z_2$ we set $\mathrm{diam}(B) = \sup\{d'(x,y) \mid x, y \in B\}$ if $B \neq \emptyset$ and $\mathrm{diam}(\emptyset) = 0$. Clearly, if $z \in Z_1 \setminus \mathrm{cl}(A)$ then $\mathrm{osc}_f(z) = 0$, while if $z \in A$ then $\mathrm{osc}_f(z) = 0$ if and only if z is a continuity point of f (i.e. for every open neighborhood $V \subseteq Z_2$ of $f(z)$ there is an open neighborhood $U \subseteq Z_1$ of z such that $f(U \cap A) \subseteq V$). Moreover, for every $\varepsilon > 0$ the set $A_\varepsilon = \{z \in Z_1 \mid \mathrm{osc}_f(z) < \varepsilon\}$ is open, hence $\{z \in Z_1 \mid \mathrm{osc}_f(z) = 0\} = \bigcap_{n \in \omega} A_{2^{-n}}$ is a G_δ set. (In particular, this shows that if Z_1, Z_2 are topological spaces with Z_2 metrizable, then the points of continuity of a function from Z_1 to Z_2 form a G_δ set — see [Kec95, Proposition 3.6].)

Claim 1.1.8.1 ([Kec95, Theorem 3.8])**.** *Let Z_1 be metrizable and Z_2 be completely metrizable, $A \subseteq Z_1$, and $f \colon A \to Z_2$ be continuous. Then there is a G_δ set G with $A \subseteq G \subseteq \mathrm{cl}(A)$ and a continuous function $g \colon G \to Z_2$ with $g \restriction A = f$.*

Proof of the claim. Let d' be a compatible complete metric on Z_2. Let $G = \mathrm{cl}(A) \cap \{z \in Z_1 \mid \mathrm{osc}_f(z) = 0\}$. This is a G_δ set, and since f is continuous on A we have $A \subseteq G \subseteq \mathrm{cl}(A)$. Let $z \in G$, and fix a sequence $(z_n)_{n\in\omega}$ of points of A converging to z (which is possible because $z \in \mathrm{cl}(A)$). Then $(f(z_n))_{n\in\omega}$ is a d'-Cauchy sequence. Indeed, for every $\varepsilon > 0$ there is an open neighborhood U of z with $\mathrm{diam}(f(U \cap A)) < \varepsilon$ because $\mathrm{osc}_f(z) = 0$, and since $z_n \in U$ for all but finitely many n's (because $z_n \to z$) it follows that there is $N \in \omega$ such that $d'(f(z_n), f(z_m)) \leq \mathrm{diam}(f(U \cap A)) < \varepsilon$ for all $n, m \geq N$. Thus $(f(z_n))_{n\in\omega}$ converges in Z_2, and we can set $g(z) = \lim_{n\to\infty} f(z_n)$.

Clearly g is well-defined (i.e. the value of $g(z)$ is independent of the choice of the sequence $z_n \to z$) and extends f. Finally, to see that g is continuous we have to show that $\mathrm{osc}_g(z) = 0$ for all $z \in G$. But given any open $U \subseteq Z_1$, $g(G \cap U) \subseteq \mathrm{cl}(f(A \cap U))$ by definition of g, thus $\mathrm{diam}(g(G \cap U)) \leq \mathrm{diam}(f(A \cap U))$, and hence $\mathrm{osc}_g(z) \leq \mathrm{osc}_f(z) = 0$. □

Assume now that Y is a Polish subspace of X, and let us prove that Y is G_δ. Applying Claim 1.1.8.1 with $Z_1 = X$, $Z_2 = A = Y$, and $f = \mathrm{id}_Y$ the identity function on Y, we get that there is a G_δ subset of X and a continuous function $g\colon G \to Y$ such that $Y \subseteq G \subseteq \mathrm{cl}(Y)$ and $g \restriction Y = \mathrm{id}_Y$. Since Y is dense in G and g is continuous, we have that $g = \mathrm{id}_G$. On the other hand, $\mathrm{rng}(g) \subseteq Y$, hence $G \subseteq Y$. It follows that $Y = G$ is G_δ in X. □

Proposition 1.1.8 yields new examples of Polish spaces: for example, the space $\mathbb{R} \setminus \mathbb{Q}$ of irrationals is Polish (indeed, it is homeomorphic to the Baire space ω^ω). The above characterization can also be used in the other direction: for example, since by Example 1.1.7 the space $\mathbb{Q}$ is not a G_δ subspace of $\mathbb{R}$, it follows that it is not a Polish space. More precisely, since it is obviously separable (it is even countable) this means that not only the usual metric on $\mathbb{Q}$ is not complete: there is no way to endow $\mathbb{Q}$ with a complete metric compatible with its topology.

Remark 1.1.9. Using Claim 1.1.8.1 one can further prove the following fact due to Lavrentiev (see [Kec95, Theorem 3.9]).

> Let Z_1, Z_2 be completely metrizable spaces, $A \subseteq Z_1$, $B \subseteq Z_2$, and $f\colon A \to B$ be a homeomorphism. Then f can be extended to a homeomorphism $h\colon G \to H$ where $G \supseteq A$, $H \supseteq B$, and G, H are both G_δ sets.
>
> In particular, a homeomorphism between dense subsets of X, Y can be extended to a homeomorphism between dense G_δ sets.

1.2 Universal spaces

Theorem 1.2.1 ([Kec95, Theorem 4.14])**.** *Every separable metrizable space is homeomorphic to a subspace of the Hilbert cube $[0;1]^\omega$. In particular, the Polish spaces are, up to homeomorphism, exactly the G_δ subspaces of $[0;1]^\omega$, and the compact metrizable spaces are, up to homeomorphism, the closed (equivalently, compact) subspaces of $[0;1]^\omega$.*

It follows that all compact metrizable spaces are Polish, because they are homeomorphic to a closed subspace of the Polish space $[0;1]^\omega$. Actually one can show that every compatible metric on a compact metrizable space X is complete, and that X is **totally bounded** (i.e. for every $\varepsilon > 0$, X can be covered by finitely many open balls of radius $< \varepsilon$), so that it is secound-countable.

Proof. Let (X, d) be a separable metric space with $d \leq 1$, and let $D = \{z_n \mid n \in \omega\}$ be dense in X. Define

$$f \colon X \to [0;1]^\omega, \qquad x \mapsto (d(x, z_n))_{n\in\omega}.$$

The map f is continuous: if $(a; b)$ is an open subinterval of $[0; 1]$ and $k \in \omega$, the f-preimage of the basic open set $\{y \in [0;1]^\omega \mid a < y(k) < b\}$ is

$$B_d(z_k, b) \setminus B_d^{cl}(z_k, a),$$

where $B_d^{cl}(z_k, a) = \{x \in X \mid d(z_k, x) \leq a\}$ is the "closed" ball of center z_k and radius a. Moreover the map f is injective: if $d(x, y) = \varepsilon > 0$ and $k \in \omega$ is such that $z_k \in B_d(x, \varepsilon/2)$, then $f(x)(k) = d(x, z_k) < \varepsilon/2$ while $f(y)(k) = d(y, z_k) \geq \varepsilon/2$, hence $f(x) \neq f(y)$ because they differ on the k-th coordinate. It remains to show that the inverse map

$$f^{-1} \colon f(X) \to X$$

is continuous as well, i.e. that for any sequence $(x_m)_{m\in\omega}$ of points of X, if $f(x_m) \to f(x)$ then $x_m \to x$. If $f(x_m)$ converges to $f(x)$, then in particular $d(x_m, z_k) \to d(x, z_k)$ for all $k \in \omega$. Fix $\varepsilon > 0$ and let $\bar{k} \in \omega$ be such that $d(x, z_{\bar{k}}) < \varepsilon/2$. Since $d(x_m, z_{\bar{k}}) \to d(x, z_{\bar{k}})$, let M be such that $d(x_m, z_{\bar{k}}) < \varepsilon/2$ for all $m \geq M$. Then for any such m we have $d(x_m, x) \leq d(x_m, z_{\bar{k}}) + d(z_{\bar{k}}, x) < \varepsilon$. So $x_m \to x$.

The additional part concerning Polish spaces follows from Proposition 1.1.8.

As for compact metrizable spaces (X, d), observe that X is totally bounded, hence second-countable and separable. Let $f \colon X \to [0;1]^\omega$ be a topological embedding: then $f(X)$ is compact (a continuous image of a compact set is compact as well), hence closed in $[0;1]^\omega$ because the latter is a Hausdorff space. For the other direction, it is enough to recall that a closed subset of compact space is compact as well: since $[0;1]^\omega$, being a product of the compact space $[0;1]$, is compact, we are done. □

Corollary 1.2.2. *Every separable metrizable space X admits a* ***compactification*** *Y, i.e. a compact metrizable space in which X can be embedded as a dense subset. If X is Polish, then X can be embedded into Y as a dense G_δ (i.e. very large) set.*

Remark 1.2.3. The compactification is not unique. For example, ω^ω can be embedded as a dense subset in both 2^ω and $[0;1]$. (In the former case, consider the map

$$\omega^\omega \to 2^\omega, \qquad x \mapsto \underbrace{0\ldots0}_{x(0)}1\underbrace{0\ldots0}_{x(1)}1\underbrace{0\ldots0}_{x(2)}1\ldots;$$

in the latter, use the fact that ω^ω is homeomorphic to the irrationals $\mathrm{Irr} = \mathbb{R} \setminus \mathbb{Q}$.) However, 2^ω and $[0;1]$ are not homeomorphic: the former is **totally disconnected**, i.e. its only connected components are singletons, while the latter is connected.

If X is Polish, then the set $\mathcal{K}(X) = \{K \subseteq X \mid K \text{ is compact}\}$ can be equipped with the so-called **Vietoris topology**, i.e. with the topology generated by the sets of the form

$$\{K \in \mathcal{K}(X) \mid K \subseteq U\}$$

and

$$\{K \in \mathcal{K}(X) \mid K \cap U \neq \emptyset\}$$

for U open in X. One can show ([Kec95, Section 4.F]) that the space $\mathcal{K}(X)$ is Polish as well. A countable dense subset of $\mathcal{K}(X)$ is given by the collection of all finite subsets of a countable dense subset of X. A compatible complete metric on $\mathcal{K}(X)$ is the **Hausdorff metric** d_H defined as follows:

$$d_H(K,L) = \begin{cases} 0 & \text{if } K = L = \emptyset \\ 1 & \text{if } \textit{exactly} \text{ one of } K, L \text{ is } \emptyset \\ \max\{\delta(K,L), \delta(L,K)\} & \text{if } K, L \neq \emptyset, \end{cases}$$

where

$$\delta(K,L) = \max_{x \in K} d(x, L).$$

Thus Theorem 1.2.1 shows that $\mathcal{K}([0;1]^\omega)$ is a Polish (hyper)space which contains, up to homeomorphism, all compact metrizable spaces.

Theorem 1.2.4 ([Kec95, Theorem 4.17])**.** *Every Polish space is homeomorphic to a closed subspace of* $\mathbb{R}^\omega$*.*

Proof. The argument is reminiscent of the proof of Proposition 1.1.4. By Theorem 1.2.1, we can assume that the given Polish space is a G_δ set $G \subseteq [0;1]^\omega$. Let U_n be open subsets of $[0;1]^\omega$ such that $G = \bigcap_{n\in\omega} U_n$, and let $F_n = [0;1]^\omega \setminus U_n$. Let d be a compatible complete metric on $[0;1]^\omega$. Define $f\colon G \to \mathbb{R}^\omega$ by setting for $x \in G \subseteq [0;1]^\omega$ and $n \in \omega$

$$f(x)(n) = \begin{cases} x(i) & \text{if } n = 2i+1 \\ \frac{1}{d(x,F_i)} & \text{if } n = 2i. \end{cases}$$

The map f is clearly injective, as $x(i) = f(x)(2i+1)$ for all $x \in G$ and $i \in \omega$. Moreover, arguing as in the proof of Theorem 1.2.1 one easily sees that f is continuous. It remains to show that $f(G)$ is closed and that the inverse map $f^{-1}\colon f(G) \to G$ is continuous. For this it is enough to check that if $(x_n)_{n\in\omega}$ is a sequence of points in G such that $f(x_n)$ converges to some $y \in \mathbb{R}^\omega$, then $y \in f(G)$ and x_n converges to some x such that $f(x) = y$. Indeed, because of the definition of the odd coordinates of the values of f we have that $x_n \to x$ where $x \in [0;1]^\omega$ is defined by $x(i) = y(2i+1)$. Moreover,

$f(x_n) \to y$ implies that also that the even coordinates of the x_n's converge, i.e. that $\left(\frac{1}{d(x_n,F_i)}\right)_{n\in\omega}$ converges (in $\mathbb{R}$) for each $i \in \omega$. As in the proof of Proposition 1.1.4, this means that $(d(x_n, F_i))_{n\in\omega}$ is bounded away from 0, and hence $d(x, F_i) = \lim_{n\to\infty} d(x_n, F_i) \neq 0$. Since $i \in \omega$ was arbitrary, this shows that $x \notin F_i$ for all i, so $x \in G = [0;1]^\omega \setminus \bigcup_{i\in\omega} F_i$. The fact that $f(x) = y$ is obvious: for the odd coordinates, this comes from the definition of x, while for the even coordinates one has just to use the fact that $d(x_n, F_i) \to d(x, F_i)$. $\square$

Given a Polish space X, the set $F(X) = \{C \subseteq X \mid C \text{ is closed}\}$ can be endowed with a somewhat "natural" Polish topology, thus $F(\mathbb{R}^\omega)$ can be regarded as the space of all Polish spaces (see Section 2.3 for more details).

Consider now the following problem. Given a Polish space X, consider the collection

$$F_h(X) = \{A \subseteq X \mid A \text{ is homeomorphic to some } C \in F(X)\}$$

of all homeomorphic copies of closed subsets of X. Clearly $F_h(X) \supseteq F(X)$, and by Proposition 1.1.8 all $A \in F_h(X)$ are G_δ. If X is compact, e.g. $X = [0;1]^\omega$ or $X = 2^\omega$, we have that $F_h(X) = F(X) = \mathcal{K}(X)$ because continuous images of compact sets in X are necessarily compact, and thus closed. If X is K_σ, i.e. X can be written as a countable union of compact sets, then each continuous image of a closed (or even just F_σ) subset of X is a countable union of compact sets, and thus F_σ: it follows that each $A \in F_h(X)$ is both F_σ and G_δ, thus $F_h(X)$ does not in general contain all G_δ subsets of X. This applies e.g. to $X = \mathbb{R}^n$ for any $n \in \omega$. Observe that if $X = \mathbb{R}$, then $A = \left\{\frac{1}{n+1} \mid n \in \omega\right\}$ is neither open nor closed (but it is both F_σ and G_δ) and belongs to $F_h(\mathbb{R})$ because it is homeomorphic to $C = \mathbb{N} \subseteq \mathbb{R}$. So $F_h(\mathbb{R}) \supsetneq F(\mathbb{R})$. Remarkably, for spaces X enjoying certain universality properties one gets that $F_h(X)$ is maximal.

Corollary 1.2.5. *Every G_δ subset of $\mathbb{R}^\omega$ is homeomorphic to a closed subset of $\mathbb{R}^\omega$.*

Thus $F_h(\mathbb{R}^\omega)$ coincides with the collection of all G_δ subsets of X.

Proof. If $A \subseteq \mathbb{R}^\omega$ is G_δ then it is Polish by Proposition 1.1.8: the result then follows from Theorem 1.2.4.

We showed $G_\delta \subseteq F_h(\mathbb{R}^\omega)$. The reverse inclusion $F_h(\mathbb{R}^\omega) \subseteq G_\delta$ follows from Proposition 1.1.8 again. $\square$

By Corollary 1.3.14, a similar result holds for the Baire space ω^ω, that is, $F_h(\omega^\omega)$ consists of all G_δ subsets of ω^ω.

1.3 The Cantor and the Baire space

Given a nonempty set A, we endow it with the discrete topology and its countable product (or power) A^ω with the product topology. The elements x of A^ω are countable sequences of elements from A, and for $n \in \omega$ we denote by $x(n)$ or x_n the n-th element of such a sequence, and by $x \restriction n$ the restriction of x

to its first n digits, i.e. the finite sequence $\langle (0), \dots, x(n-1) \rangle$ (when $n = 0$, we stipulate that $x \upharpoonright n = \emptyset$).

Let $A^{<\omega} = \bigcup_{n\in\omega} A^n$ be the set of finite sequences of elements of A. We denote by $\mathrm{lh}(s)$ the **length** of $s \in A^{<\omega}$, i.e. the unique $n \in \omega$ such that $s \in A^n$. We adopt the same notation that we introduced for infinite sequences: $s(i)$ is the i-th element of s (with $i < \mathrm{lh}(s)$), $s \upharpoonright n$ is the restriction of s to its initial segment of length $n \leq \mathrm{lh}(s)$, and so on. When $s \in A^{<\omega}$ and $t \in A^{<\omega} \cup A^\omega$, we denote by $s^\frown t$ the **concatenation** of s and t. When $t = \langle a \rangle$ is of length 1, we write e.g. $s^\frown a$ instead of $s^\frown\langle a \rangle$.

A basis for the topology of A^ω is $\{\boldsymbol{N}_s \mid s \in A^{<\omega}\}$ where

$$\boldsymbol{N}_s = \{x \in A^\omega \mid x \upharpoonright \mathrm{lh}(s) = s\}.$$

This basis has the special property that if two of its elements intersect, then one is contained in the other one. Moreover, $\boldsymbol{N}_s \subseteq \boldsymbol{N}_t$ if and only if $t \subseteq s$. The space A^ω is second-countable (and thus separable) if and only if A is (at most) countable.

The space A^ω is also completely metrizable, as witnessed by the complete metric

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 2^{-(n+1)} & \text{if } x \neq y \text{ and } n \text{ is least such that } x(n) \neq y(n). \end{cases} \tag{1.1}$$

The sets $\boldsymbol{N}_s$ are exactly the nonempty open balls with respect to this d.

The metric d from (1.1) is actually an **ultrametric**, i.e. it satisfies the following strengthening of the triangular inequality: for all $x, y, z \in A^\omega$

$$d(x,y) \leq \max\{d(x,z), d(z,y)\}.$$

This fact implies that A^ω is also **zero-dimensional** (with respect to the small inductive dimension), i.e. that it admits a basis consisting only of clopen sets. This can also be directly checked by observing that each $\boldsymbol{N}_s$ is clopen.

Remark 1.3.1. The fact that a space X is second-countable and zero-dimensional means that there is a basis $\mathcal{B}$ for the topology which is countable and a basis $\mathcal{B}'$ (possibly different from $\mathcal{B}$) consisting of clopen sets. However, by Proposition 3.6.3 this implies that there is a *countable* basis $\mathcal{B}'' \subseteq \mathcal{B}'$, so in all proofs below we will systematically appeal to the existence of such a basis. Actually, in most cases this will be used to obtain that every open covering of X can be refined to a (countable) partition of X consisting of clopen sets. The latter property is usually referred to by saying that X has **Lebesgue covering dimension** 0. (See Proposition 3.6.4 and the definitions preceding it for more details.)

Here are some basic facts (EXERCISE!) concerning the space A^ω and its ultrametric d from (1.1):

- Let $x_n, x \in A^\omega$. Then

$$\begin{aligned} x_n \to x &\iff \forall i \in \omega\, (x_n(i) \to x(i)) \\ &\iff \forall i \in \omega\, (x_n(i) = x(i) \text{ for all but finitely many } n). \end{aligned}$$

- The finite products $(A^\omega)^n$ (for $n \geq 1$) and the countable product $(A^\omega)^\omega$ are all homeomorphic to A^ω.[4]
- If A has at least two points the space A^ω is **perfect**, i.e. it has no isolated point. (Recall that a point x is **isolated** in the topological space X if the singleton $\{x\}$ is open in the topology of X.)
- If $d(x,z) \neq d(y,z)$, then $d(x,y) = \max\{d(x,z), d(y,z)\}$ ("all triangles are isosceles with legs longer than or equal to the basis").
- The "open" balls $B_d(x,\varepsilon) = \{y \in A^\omega \mid d(x,y) < \varepsilon\}$ and the "closed" balls $B_d^{cl}(x,\varepsilon) = \{y \in A^\omega \mid d(x,y) \leq \varepsilon\}$ are both clopen.
- If $y \in B_d(x,\varepsilon)$, then $B_d(y,\varepsilon) = B_d(x,\varepsilon)$ ("all elements of an open ball are centers of it").
- If two open (closed) balls intersect, then one is contained in the other one.
- A sequence $(x_n)_{n\in\omega}$ is d-Cauchy if and only if $d(x_n, x_{n+1}) \to 0$.

One useful feature of the spaces of the form A^ω is that their closed sets and the continuous functions on them can be characterized combinatorially.

Definition 1.3.2. A **tree** on a nonempty set A is a set $T \subseteq A^{<\omega}$ closed under initial segments, i.e. for every $s \in T$ and $n \leq \mathrm{lh}(s)$ it holds $s \restriction n \in T$.

A tree is **pruned** if it has no terminal nodes, where $s \in T$ is called **terminal node** (or a **leaf**) if there is no $t \in T$ which is a proper extension of s (i.e. such that $s \subseteq t$ and $\mathrm{lh}(s) < \mathrm{lh}(t)$).

The **body** of a tree T on A is the following subset of A^ω:

$$[T] = \{x \in A^\omega \mid \forall n \in \omega\, (x \restriction n \in T)\}.$$

A tree T is **well-founded** if $[T] = \emptyset$, otherwise it is **ill-founded**.

Proposition 1.3.3 ([Kec95, Proposition 2.4])**.** *The map $T \mapsto [T]$ is a bijection between pruned trees on A and closed subsets of A^ω. Its inverse is given by*

$$F \mapsto T_F = \{x \restriction n \mid x \in F \wedge n \in \omega\}.$$

*We call T_F the **tree** of F.*

Proof. To check that $[T] \subseteq A^\omega$ is closed, just notice that if $x \notin [T]$ then $x \restriction n \notin T$ for some $n \in \omega$: but then $\boldsymbol{N}_{x\restriction n}$ is an open neighborhood of x disjoint from $[T]$.

The map $T \mapsto [T]$ is injective. Let $S \neq T$ and assume, without loss of generality, that there is some $s \in S \setminus T$. Since S is pruned, we can recursively define a sequence $(s_n)_{n\in\omega}$ such that $s_0 = s$, s_{n+1} properly extends s_n, and $s_n \in S$. Then $x = \bigcup_{n\in\omega} s_n \in [S] \setminus [T]$.

Finally, surjectivity follows from the fact that for all closed sets F, $[T_F] = F$ (i.e. from the fact that $F \mapsto T_F$ is the inverse of $T \mapsto [T]$). The inclusion $F \subseteq [T_F]$ is obvious, so let us prove that $[T_F] \subseteq F$. Consider an arbitrary $x \in [T_F]$. By definition of $[T_F]$, for every $n \in \omega$ there is $y_n \in F$ such that $x \restriction n = y_n \restriction n$. But then $y_n \to x$, hence $x \in F$ because F is closed. □

[4] This remains true even if the discrete space A is replaced by *any* topological space.

Remark 1.3.4. More generally, the proof above shows that if $C \subseteq A^\omega$ is an arbitrary set, then the body of $T_C = \{x \restriction n \mid x \in C \wedge n \in \omega\}$ coincides with the closure $\mathrm{cl}(C)$ of C.

The following notion of A-scheme[5] is used to build continuous functions from the space A^ω to some metrizable space X. In [Kec95], 2-schemes (respectively, ω-schemes) satisfying the hypothesis of Lemma 1.3.6(b) are called **Cantor schemes** (respectively, **Lusin schemes**) with vanishing diameters.

Definition 1.3.5. Let $A \neq \emptyset$ and (X,d) be a metric space. An A**-scheme** on X is a family $\mathcal{S} = \{B_s \mid s \in A^{<\omega}\}$ of subsets of X such that for all $s \in A^{<\omega}$, $a \in A$, and $x \in A^\omega$ the following conditions hold:

(i) $B_{s^\frown a} \subseteq B_s$; *(Monotonicity)*

(ii) $\mathrm{diam}(B_{x \restriction n}) \to 0$. *(Vanishing diameters)*

(The latter will often be ensured by requiring that $\mathrm{diam}(B_s) \leq 2^{-\mathrm{lh}(s)}$.) Every A-scheme canonically induces a function as follows: set

$$D_{\mathcal{S}} = \left\{x \in A^\omega \mid \bigcap\nolimits_{n\in\omega} B_{x\restriction n} \neq \emptyset\right\}$$

and

$$f_{\mathcal{S}} \colon D_{\mathcal{S}} \to X, \qquad x \mapsto f_{\mathcal{S}}(x) \in \bigcap\nolimits_{n\in\omega} B_{x\restriction n}.$$

(When the A-scheme is clear from the context, we drop the subscript from both $f_{\mathcal{S}}$ and $D_{\mathcal{S}}$.)

Condition (ii) in Definition 1.3.5 implies in particular that $\bigcap_{n\in\omega} B_{x\restriction n}$ contains at most one point, thus $f_{\mathcal{S}}$ is well-defined. Moreover, by construction we have $f(\boldsymbol{N}_s \cap D_{\mathcal{S}}) \subseteq B_s \cap f(D_{\mathcal{S}})$ for all $s \in A^{<\omega}$

The next lemma shows that the behavior of $f_{\mathcal{S}}$ can be controlled by imposing further restrictions on the A-scheme.

Lemma 1.3.6. *Let $\mathcal{S} = \{B_s \mid s \in A^{<\omega}\}$ be an A-scheme on a metric space (X,d), and let $f \colon D \to X$ be its induced function.*

(a) *f is continuous.*

(b) *If $B_{s^\frown a} \cap B_{s^\frown a'} = \emptyset$ for all $s \in A^{<\omega}$ and distinct $a, a' \in A$, then f is injective. Moreover, for every $s \in A^{<\omega}$*

$$f(\boldsymbol{N}_s \cap D) = B_s \cap f(D) \tag{1.2}$$

and

$$f(D) = \bigcap_{n\in\omega} \bigcup_{s\in A^n} B_s. \tag{1.3}$$

(c) *If $B_{s^\frown a} \cap B_{s^\frown a'} = \emptyset$ for all $s \in A^{<\omega}$ and distinct $a, a' \in A$ and every B_s is open relatively to $f(D)$ (this happens e.g. if B_s is open in the whole X), then f is a (topological) embedding, i.e. f is a homeomorphism between D and $f(D) = \mathrm{rng}(f)$.*

[5] Our presentation of schemes differs in some details from the Cantor and Lusin schemes introduced in [Kec95].

(d) *If* $B_s = \bigcup_{a \in A} B_{s^\frown a}$ *for all* $s \in A^{<\omega}$*, then* $f(\boldsymbol{N}_s \cap D) = B_s$ *for all* $s \in A^{<\omega}$*.*[6] *In particular,* $f(D) = B_\emptyset$*, thus if additionally we require* $B_\emptyset = X$ *then* f *is surjective.*

(e) *If* (X, d) *is complete and* $\mathrm{cl}(B_{s^\frown a}) \subseteq B_s$ *for all* $s \in A^{<\omega}$ *and* $a \in A$*, then* $D = [T]$ *with* $T = \{s \in A^{<\omega} \mid B_s \neq \emptyset\}$*, thus* D *is a closed subset of* A^ω*. If moreover each* B_s *is nonempty, then* $D = A^\omega$*.*

Observe that if every B_s is closed, the condition $\mathrm{cl}(B_{s^\frown a}) \subseteq B_s$ in part (e) of the lemma is automatically satisfied because in this case $\mathrm{cl}(B_{s^\frown a}) = B_{s^\frown a}$ and $B_{s^\frown a} \subseteq B_s$ by Definition 1.3.5(i).

Proof. (a) We show that f is continuous at every point $x \in D$. Given an open neighborhood $V \subseteq X$ of $f(x)$, let $\varepsilon > 0$ such that $B_d(f(x), \varepsilon) \subseteq V$. By condition (ii) in Definition 1.3.5, there is $n \in \omega$ such that $\mathrm{diam}(B_{x \restriction n}) < \varepsilon$. Since $f(x) \in B_{x \restriction n}$ by definition of f, we have $B_{x \restriction n} \subseteq B_d(f(x), \varepsilon) \subseteq V$, hence the open neighborhood $\boldsymbol{N}_{x \restriction n} \cap D$ of x is such that $f(\boldsymbol{N}_{x \restriction n} \cap D) \subseteq B_{x \restriction n} \cap f(D) \subseteq V$.

(b) Given distinct $x, y \in D \subseteq A^\omega$, let $n \in \omega$ be least such that $x(n) \neq y(n)$. Then setting $s = x \restriction n = y \restriction n$ one has $B_{s^\frown \langle x(n) \rangle} \cap B_{s^\frown \langle y(n) \rangle} = \emptyset$ by our assumption on the scheme: since by definition of f we have $f(x) \in B_{x \restriction (n+1)} = B_{s^\frown \langle x(n) \rangle}$ and $f(y) \in B_{y \restriction (n+1)} = B_{s^\frown \langle y(n) \rangle}$, it follows that $f(x) \neq f(y)$. To prove equation (1.2), the inclusion $f(\boldsymbol{N}_s \cap D) \subseteq B_s \cap f(D)$ is obvious and holds unconditionally. For the reverse inclusion, suppose that $x \in D$ is such that $f(x) \in B_s$, and consider $t = x \restriction \mathrm{lh}(s)$, so that in particular $f(x) \in B_t$. Since they have the same length, if $s \neq t$ then $B_s \cap B_t = \emptyset$ by the previous argument, contradicting $f(x) \in B_s \cap B_t$. Therefore $s = t = x \restriction \mathrm{lh}(s)$, hence $x \in \boldsymbol{N}_s$ and $f(x) \in f(\boldsymbol{N}_s \cap D)$. Finally, let us prove equation (1.3). Observe that the inclusion $f(D) \subseteq \bigcap_{n \in \omega} \bigcup_{s \in A^n} B_s$ holds unconditionally, so it is enough to prove the reverse inclusion. Let $y \in \bigcap_{n \in \omega} \bigcup_{s \in A^n} B_s$. By this and our hypothesis on the scheme, for each $n \in \omega$ there is a *unique* $s_n \in A^n$ such that $y \in B_{s_n}$, so that $s_n \subseteq s_m$ whenever $n \leq m$. Thus $x = \bigcup_{n \in \omega} s_n$ is such that $y \in \bigcap_{n \in \omega} B_{x \restriction n}$ which entails both $x \in D$ and $f(x) = y$, i.e. $y \in f(D)$.

(c) Let $U = \boldsymbol{N}_s \cap D$ (with $s \in A^{<\omega}$) be an arbitrary basic open set of D. Since the hypothesis of part (b) is satisfied, by (1.2) we have $f(U) = B_s \cap f(D)$: by our hypothesis on B_s, the set $f(U)$ is then open in $f(D)$.

(d) It is enough to show that $B_s \subseteq f(\boldsymbol{N}_s \cap D)$. Fix any $y \in B_s$ and set $n = \mathrm{lh}(s)$. For $i < n$, set also $x_i = s(i)$. Since $B_s = \bigcup_{a \in A} B_{s^\frown a}$, by hypothesis there is $x_n \in A$ such that $y \in B_{s^\frown \langle x_n \rangle} = B_{\langle x_0, \dots, x_n \rangle}$. Since $B_{s^\frown \langle x_n \rangle} = \bigcup_{a \in A} B_{s^\frown \langle x_n \rangle^\frown a}$ we then get that $y \in B_{s^\frown \langle x_n, x_{n+1} \rangle} = B_{\langle x_0, \dots x_{n+1} \rangle}$ for some $x_{n+1} \in A$. Continuing this process, we recursively construct a sequence $x = (x_i)_{i \in \omega} \in A^\omega$ such that $s \subseteq x$ and $y \in B_{\langle x_0, \dots, x_{i-1} \rangle} = B_{x \restriction i}$ for all $i \in \omega$. Therefore y witnesses that $x \in D$, and $f(x) = y$ by definition of f. Thus x witnesses $y \in f(\boldsymbol{N}_s \cap D)$.

[6] Compare this with equation (1.2).

(e) It is enough to show that under these hypotheses, for every $x \in A^\omega$

$$x \in D \iff \forall n \in \omega\, (B_{x \restriction n} \neq \emptyset).$$

The forward direction directly follows from the definition of D, so let us assume that for all $n \in \omega$ there is $y_n \in B_{x\restriction n} \neq \emptyset$. We first show that $(y_n)_{n \in \omega}$ is a Cauchy sequence in (X, d). Given $\varepsilon > 0$, let $N \in \omega$ be such that $\operatorname{diam}(B_{x\restriction N}) < \varepsilon$ (such an N exists by Definition 1.3.5(ii)). For all $n, m \geq N$, we have $y_n \in B_{x\restriction n} \subseteq B_{x\restriction N}$ and $y_m \in B_{x\restriction m} \subseteq B_{x\restriction N}$, therefore $d(y_n, y_m) < \varepsilon$. Let now $y = \lim_n y_n$: we claim that $y \in \bigcap_{n\in\omega} B_{x\restriction n}$, so that $x \in D$. Given any $n \in \omega$, the sequence $(y_m)_{m>n}$ is contained in $B_{x\restriction(n+1)}$, and thus

$$y = \lim_n y_n = \lim_{m>n} y_m \in \operatorname{cl}(B_{x\restriction(n+1)}) \subseteq B_{x\restriction n}.$$

Since $n \in \omega$ was arbitrary, this shows $y \in \bigcap_{n\in\omega} B_{x\restriction n}$, as desired. □

Given two families $\mathcal{S} = \{B_s \mid s \in A^{<\omega}\}$ and $\mathcal{S}' = \{B'_s \mid s \in A^{<\omega}\}$ of subsets of a metric space (X, d), we write $\mathcal{S} \sqsubseteq \mathcal{S}'$ if $B_s \subseteq B'_s$ for all $s \in A^{<\omega}$. Moreover, we set $\operatorname{cl}(\mathcal{S}) = \{\operatorname{cl}(B_s) \mid s \in A^{<\omega}\}$. Obviously, $\mathcal{S} \sqsubseteq \operatorname{cl}(\mathcal{S})$.

Lemma 1.3.7. *Let $\mathcal{S} = \{B_s \mid s \in A^{<\omega}\}$ and $\mathcal{S}' = \{B'_s \mid s \in A^{<\omega}\}$ be two A-schemes on the same metric space (X, d).*

(a) *If $\mathcal{S} \sqsubseteq \mathcal{S}'$, then $f_{\mathcal{S}'}$ extends $f_{\mathcal{S}}$, that is, $D_{\mathcal{S}} \subseteq D_{\mathcal{S}'}$ and $f_{\mathcal{S}'}(x) = f_{\mathcal{S}}(x)$ for every $x \in D_{\mathcal{S}}$.*

(b) *The family $\operatorname{cl}(\mathcal{S})$ is an A-scheme, so $f_{\operatorname{cl}(\mathcal{S})}$ extends $f_{\mathcal{S}}$. If moreover (X, d) is complete, then $D_{\operatorname{cl}(\mathcal{S})}$ is closed and hence contains $\operatorname{cl}(D_{\mathcal{S}})$.*

Proof. (a) The inclusion $D_{\mathcal{S}} \subseteq D_{\mathcal{S}'}$ immediately follows that for every $x \in A^\omega$ we have $\bigcap_{n\in\omega} B_{x\restriction n} \subseteq \bigcap_{n\in\omega} B'_{x\restriction n}$. It follows that if $x \in D_{\mathcal{S}}$, then $f_{\mathcal{S}}(x) \in \bigcap_{n\in\omega} B_{x\restriction n} \subseteq \bigcap_{n\in\omega} B'_{x\restriction n} = \{f_{\mathcal{S}'}(x)\}$, thus $f_{\mathcal{S}}(x) = f_{\mathcal{S}'}(x)$.

(b) Since $B_{s^\frown a} \subseteq B_s$ implies $\operatorname{cl}(B_{s^\frown a}) \subseteq \operatorname{cl}(B_s)$ and $\operatorname{diam}(B_s) = \operatorname{diam}(\operatorname{cl}(B_s))$, the family $\operatorname{cl}(\mathcal{S})$ is an A-scheme. Such an A-scheme automatically satisfies the condition in Lemma 1.3.6(e) because its elements are closed, thus $D_{\operatorname{cl}(\mathcal{S})}$ is closed. □

Remark 1.3.8. It follows that not all continuous partial functions from A^ω into a (complete) metric space X are induced by a scheme. For example, set $A = 2$, $X = \omega^\omega$, and consider the inverse f of the embedding from Remark 1.2.3. If it where induced by a 2-scheme $\mathcal{S}$, then f could be extended to a total continuous function $g\colon 2^\omega \to \omega^\omega$ by Lemma 1.3.7(b) and the fact that the domain of f is dense in 2^ω. But this would imply that $g(2^\omega)$ is compact, and since f was already onto ω^ω and the latter is not compact we get a contradiction.

In the opposite direction, given a continuous function $f\colon C \to X$ with $C \subseteq A^\omega$ and X a metric space one can canonically reconstruct a family $\mathcal{S}_f = \{B_s \mid s \in A^{<\omega}\}$ by setting $B_s = f(C \cap \boldsymbol{N}_s)$. It turns out that when C is closed, the family $\mathcal{S}_f$ is an A-scheme inducing exactly the function f, and that the properties of f translate to properties of the scheme.

Lemma 1.3.9. *Let $f\colon C \to X$ be a continuous function from a closed set $C \subseteq A^\omega$ to a metric space (X, d).*

(a) *The family $\mathcal{S}_f$ is an A-scheme inducing f, i.e. it is such that $D_{\mathcal{S}_f} = C$ and $f_{\mathcal{S}_f}(x) = f(x)$ for all $x \in C$.*

(b) *The function f is injective if and only if $B_{s^\frown a} \cap B_{s^\frown a'} = \emptyset$ for all $s \in A^{<\omega}$ and distinct $a, a' \in A$.*

(c) *The function f is a (topological) embedding if and only if $B_{s^\frown a} \cap B_{s^\frown a'} = \emptyset$ for all $s \in A^{<\omega}$ and distinct $a, a' \in A$ and each B_s is open relatively to $f(C)$.*

(d) *For all $s \in A^{<\omega}$ it holds $B_s = \bigcup_{a \in A} B_{s^\frown a}$, and $B_\emptyset = X$ if and only if f is surjective.*

(e) *Every A-scheme $\mathcal{S}'$ such that $\mathcal{S}_f \sqsubseteq \mathcal{S}' \sqsubseteq \mathrm{cl}(\mathcal{S}_f)$ induce the function f, i.e. $f_{\mathcal{S}'} = f$.*

Proof. (a) Since $\boldsymbol{N}_{s^\frown a} \subseteq \boldsymbol{N}_s$ we also have $B_{s^\frown a} = f(C \cap \boldsymbol{N}_{s^\frown a}) \subseteq f(C \cap \boldsymbol{N}_s) = B_s$. Moreover, the fact that f is continuous and with closed domain implies that $\mathrm{osc}_f(x) = 0$ for all $x \in A^{<\omega}$. Since $\mathrm{diam}(B_{x \restriction n})$ decreases when n gets larger and the sets $\boldsymbol{N}_{x \restriction n}$ form a neighborhood basis of x, it follows that $\mathrm{diam}(B_{x \restriction n}) \to 0$. By construction, if $x \in C$ then $f(x) \in \bigcap_{n \in \omega} B_{x \restriction n}$, thus $C \subseteq D_{\mathcal{S}_f}$ and $f_{\mathcal{S}_f}$ extends f. Finally, assume that $x \in D_{\mathcal{S}_f}$, so that, in particular, $B_{x \restriction n} \neq \emptyset$ for all n. Then for each n there is $y_n \in C \cap \boldsymbol{N}_{x \restriction n}$, hence $y_n \to x$ and so $x \in C$ because C is closed.

(b) One direction follows from part (a) and Lemma 1.3.6(b). The other direction easily follows from the fact that $\boldsymbol{N}_{s^\frown a} \cap \boldsymbol{N}_{s^\frown a'} = \emptyset$ if $a \neq a'$ and the fact that by definition $B_s = f(C \cap \boldsymbol{N}_s)$.

(c) The backward direction follows from part (a) and Lemma 1.3.6(c), the forward direction follows from part (b) and the definition of B_s.

(d) If $y \in B_s = f(C \cap \boldsymbol{N}_s)$ then there is $s \subseteq x \in A^\omega$ such that $x \in C$ and $f(x) = y$. But then $y = f(x) \in f(C \cap N_{s^\frown x(\mathrm{lh}(s))})$. So $B_s \subseteq \bigcup_{a \in A} B_{s^\frown a}$. The remaining parts are trivial.

(e) By Lemma 1.3.7 it is clearly enough to consider the special case $\mathcal{S}' = \mathrm{cl}(\mathcal{S}_f)$ and prove that $D_{\mathrm{cl}(\mathcal{S}_f)} = C$. But this easily follows from $C = D_{\mathcal{S}_f} \subseteq D_{\mathrm{cl}(\mathcal{S}_f)}$ and the fact that by $\mathrm{cl}(B_s) \neq \emptyset \iff B_s \neq \emptyset$ we get

$$\begin{aligned} D_{\mathrm{cl}(\mathcal{S}_f)} &\subseteq \{x \in A^\omega \mid \forall n\, (\mathrm{cl}(B_{x \restriction n}) \neq \emptyset)\} \\ &= \{x \in A^\omega \mid \forall n\, (B_{x \restriction n} \neq \emptyset)\} \subseteq C, \end{aligned}$$

where the last inclusion follows from the final part of the proof of part (a). □

Notice that by the (counter)example given in Remark 1.3.8, the condition on the fact that $\mathrm{dom}(f)$ is closed cannot by removed by Lemma 1.3.9.

Using A-schemes above, one can show a number of interesting results. For example, one can use such tool to provide an alternative proof (Exercise!) of the fact that the map $\omega^\omega \to 2^\omega$ introduced in Remark 1.2.3 is a topological embedding. The following result is instead rather technical, but it will turn out to be very useful in the sequel.

Definition 1.3.10. A closed set F in a topological space X is a **retract** of X if there is a continuous function $f\colon X \to F$ (called **retraction**) such that $f(x) = x$ for all $x \in F$ (in particular, f is surjective).

Proposition 1.3.11 ([Kec95, Proposition 2.8])**.** *Let A be a nonempty set. Every nonempty closed subset F of A^ω is a retract of it.*

Proof. Let T_F be the tree of F (see Proposition 1.3.3). We first define by induction a function $\varphi\colon A^{<\omega} \to A^{<\omega}$ such that for all $s \in A^{<\omega}$

(i) $\mathrm{lh}(\varphi(s)) = \mathrm{lh}(s)$;

(ii) if $s \subseteq t$ then $\varphi(s) \subseteq \varphi(t)$;

(iii) $\varphi(s) \in T_F$;

(iv) if $s \in T_F$, then $\varphi(s) = s$.

The definition is by recursion on $\mathrm{lh}(s)$, and the desired properties can inductively be checked along the construction (Exercise!). Set $\varphi(\emptyset) = \emptyset$. Let $s = t^\frown a$ for some $t \in A^{<\omega}$ and $a \in A$, and assume that $\varphi(t)$ has been already defined. Define $\varphi(s)$ as follows: If $s \in T_F$, then set $\varphi(s) = s$. If $s \notin T_F$, then let $\varphi(s)$ be any sequence $\varphi(t)^\frown b \in T_F$, which exists since T_F is pruned and $\varphi(t) \in T_F$ by condition (iii).

Now equip F with the restriction of the complete metric on A^ω defined in (1.1), and notice that such restriction is still a complete metric on F. Consider the A-scheme $\{B_s \mid s \in A^{<\omega}\}$ on F defined by $B_s = \boldsymbol{N}_{\varphi(s)} \cap F$: we claim that the induced map f is the desired retraction of A^ω onto F. Indeed, conditions (i)–(ii) guarantee that the definition of the B_s's yields an A-scheme (hence f is a continuous map), condition (iii) and the fact that B_s is clopen guarantee that f is defined on the whole A^ω (in fact, $\varphi(s) \in T_F$ ensures $B_s \neq \emptyset$), while condition (iv) guarantees that $f(x) = x$ for every $x \in F$. □

Corollary 1.3.12. *Let (X, d) be a complete metric space. Let A be a nonempty set, $C \subseteq A^\omega$ be a closed set, and $f\colon C \to X$. Then f is continuous if and only if there is a total continuous function $g\colon A^\omega \to X$ such that $g \restriction C = f$.*

Remark su approximating functions? Prima o dopo il lemma sulle retrazioni?

We next show that the Cantor and the Baire space are universal for zero-dimensional separable metrizable spaces.

Theorem 1.3.13 ([Kec95, Theorem 7.8]). *Every zero-dimensional separable metrizable space can be embedded into both ω^ω and 2^ω. Every zero-dimensional Polish space is homeomorphic to a closed subspace of ω^ω and to a G_δ subspace of 2^ω.*

Proof. Since by Remark 1.2.3 the space ω^ω is homeomorphic to a (necessarily G_δ) subset of 2^ω, it is enough to consider the case of the Baire space ω^ω.

Let X be a zero-dimensional separable metrizable space, and let $d \leq 1$ be a compatible metric on X. We construct an ω-scheme $\{B_s \mid s \in \omega^{<\omega}\}$ on (X, d) such that for every $s \in \omega^{<\omega}$ and distinct $i, j \in \omega$

(1) $B_{s^\frown i} \cap B_{s^\frown j} = \emptyset$;

(2) $B_\emptyset = X$ and $B_s = \bigcup_{i \in \omega} B_{s^\frown i}$;

(3) B_s is clopen;

(4) $\mathrm{diam}(B_s) \leq 2^{-\mathrm{lh}(s)}$.

By Lemma 1.3.6, this yields an homeomorphism between a subset D of ω^ω and X; moreover, when d can be taken to be complete (i.e. when X is Polish) the set D is closed.

The construction of the scheme is by induction on the length of s. We set $B_\emptyset = X$. Given B_s, we cover it with clopen sets $B'_{s^\frown i}$ in the countable basis of X such that $\mathrm{diam}(B'_{s^\frown i}) \leq 2^{-(\mathrm{lh}(s)+1)}$, and then recursively set $B_{s^\frown 0} = B'_{s^\frown 0} \cap B_s$ and $B_{s^\frown(i+1)} = (B'_{s^\frown(i+1)} \setminus \bigcup_{j \leq i} B'_{s^\frown j}) \cap B_s = (B'_{s^\frown(i+1)} \setminus \bigcup_{j \leq i} B_{s^\frown j}) \cap B_s$. □

Arguing as in the proof of Corollary 1.2.5, from the previous universality result we get

Corollary 1.3.14. *Every G_δ subset of ω^ω is homeomorphic to a closed subset of ω^ω. Thus $F_h(\omega^\omega)$ is constituted by all G_δ subsets of ω^ω.*

Corollary 1.3.15. *Every closed subset of a zero-dimensional Polish space is a retract of it.*

Proof. Let X be a zero-dimensional Polish space and $F \subseteq X$ be closed. By Theorem 1.3.13, we can assume without loss of generality that X is a closed subspace of ω^ω, so that F is closed in such space as well. By Proposition 1.3.11 there is a retraction f of ω^ω onto F. It follows that $f \restriction X$ is a retraction of X onto F. □

The following two results further show that the Cantor and the Baire space are *surjectively* universal for the appropriate class of Polish spaces.

Theorem 1.3.16 ([Kec95, Theorem 4.18]). *Every nonempty compact metrizable space is a continuous image of 2^ω.*

Proof. First we check that the Hilbert cube $[0;1]^\omega$ is a continuous image of 2^ω. The function defined by $f(x) = \sum_{n=0}^\infty x(n)2^{-(n+1)}$ maps 2^ω continuously onto $[0;1]$ (each x is a diadic expansion of a real between 0 and 1). Therefore the map g sending y to $(f(y(i)))_{i\in\omega}$ is a continuous surjection of $(2^\omega)^\omega$ onto $[0;1]^\omega$: since $(2^\omega)^\omega$ and 2^ω are homeomorphic we are done.

Now, since every compact metrizable space X is homeomorphic to a compact subset K of $[0;1]^\omega$ by Theorem 1.2.1, it follows that there is a closed $F \subseteq 2^\omega$ which continuously surjects onto X (indeed, $F = g^{-1}(K)$). Composing this surjection with a retraction of 2^ω onto F (which exists by Proposition 1.3.11) we get the desired result. □

Theorem 1.3.17 ([Kec95, Theorem 7.9])**.** *Let X be a Polish space. Then there are a closed set $F \subseteq \omega^\omega$ and a continuous bijection $f\colon F \to X$. In particular, if X is nonempty, then there is a continuous surjection $g\colon \omega^\omega \to X$ (extending f).*

Proof. The last assertion follows from the first one and Proposition 1.3.11. Moreover, since by Theorem 1.2.4 every Polish space is homeomorphic to a closed subspace of $\mathbb{R}^\omega$, it is enough to show that there is a continuous bijection $f\colon \omega^\omega \to \mathbb{R}^\omega$.

We first show that

Claim 1.3.17.1. *For every $a, b \in \mathbb{R}$ with $a < b$ there is a continuous bijection between ω^ω and $[a;b)$.*

Proof of the claim. Let $\{B_s \mid s \in \omega^{<\omega}\}$ be an ω-scheme on $[a;b]$ (which is a complete metric space) satisfying the following conditions for all $s \in \omega^{<\omega}$ and $n \in \omega$:

(1) $B_s = [a_s; b_s)$ for some $a \leq a_s < b_s \leq b$ (in particular, $\emptyset \neq B_s \subseteq [a;b)$);

(2) $a_\emptyset = a$ and $b_\emptyset = b$ (i.e. $B_\emptyset = [a;b)$);

(3) $a_s \leq a_{s^\frown n}$ and $b_{s^\frown n} < b_s$, so that $\mathrm{cl}(B_{s^\frown n}) = [a_{s^\frown n}; b_{s^\frown n}] \subseteq [a_s; b_s) = B_s$;

(4) $b_{s^\frown n} = a_{s^\frown (n+1)}$;

(5) $\lim_n b_{s^\frown n} = b_s$.

(Such a scheme can easily be constructed by induction on $\mathrm{lh}(s)$ by setting $B_\emptyset = [a;b)$ and $a_{s^\frown n} = b_s - \frac{1}{2^n}(b_s - a_s)$ and $b_{s^\frown n} = b_s - \frac{1}{2^{n+1}}(b_s - a_s)$.) The conditions above ensure that the map h induced by the scheme is a continuous injection with domain ω^ω (i.e. $\bigcap_{n\in\omega} B_{x\restriction n} \neq \emptyset$ for every $x \in \omega^\omega$) and $\mathrm{rng}(h) \subseteq [a;b]$. To check that indeed $\mathrm{rng}(h) = [a;b)$, notice that since $B_s \subseteq [a;b)$ for every $s \in \omega^{<\omega}$, then $\mathrm{rng}(h) \subseteq [a;b)$. Moreover, for the same reason the ω-scheme above can also be construed as an ω-scheme on $[a;b)$, and the above conditions ensure that, when construed in this way, the scheme satisfies also condition (d) of Lemma 1.3.6, hence the induced map h is onto $[a;b)$. □

Fix a bijection $\varphi\colon \omega \to \mathbb{Z}$ and, for each $k \in \omega$, a continuous bijection $f_k\colon \boldsymbol{N}_{\langle k\rangle} \to [\varphi(k); \varphi(k)+1)$ (this can be done because ω^ω is homeomorphic to $\boldsymbol{N}_{\langle k\rangle}$ via the map $x \mapsto k^\frown x$). Then $f = \bigcup_{k\in\omega} f_k\colon \omega^\omega \to \mathbb{R}$ is clearly a bijection, and it is also continuous. Indeed, if $U \subseteq \mathbb{R}$ is open, then $f^{-1}(U) = \bigcup_{k\in\omega} f_k^{-1}(U)$:

since each $f_k^{-1}(U)$ is relatively open with respect to the clopen set $\boldsymbol{N}_{\langle k\rangle}$, the set $f^{-1}(U)$ is open in ω^ω.

Finally, the countable product of $f\colon \omega^\omega \to \mathbb{R}$ is a continuous bijection between $(\omega^\omega)^\omega$ and $\mathbb{R}^\omega$, and since $(\omega^\omega)^\omega$ is homeomorphic to ω^ω we are done. $\square$

Remark 1.3.18. It can be shown that if X has no isolated points (i.e. no $x \in X$ such that $\{x\}$ is open in X) then we can take $F = \omega^\omega$, that is, there is a continuous bijection $f\colon \omega^\omega \to X$. The converse holds as well: if $f\colon \omega^\omega \to X$ is a continuous bijection, then no $x \in X$ can be isolated in it, otherwise $f^{-1}(x)$ would be isolated in ω^ω, which is clearly impossible.

Remark 1.3.19. The continuous bijection f of Theorem 1.3.17 cannot be open if X is not zero-dimensional. However, the specific continuous bijection $f\colon \omega^\omega \to \mathbb{R}$ constructed in the given proof is such that $f(\boldsymbol{N}_s)$ is a half-open interval. Thus the resulting continuous bijection $\omega^\omega \to \mathbb{R}^\omega$ maps open sets to F_σ sets, and therefore we can conclude that the same is true for arbitrary Polish spaces, that is: For any Polish space X there are a closed set $F \subseteq \omega^\omega$ and a continuous bijection $f\colon F \to X$ *such that f maps open sets to F_σ sets.*

Exercise 1.3.20. Observe that the continuous bijection constructed in the proof of Claim 1.3.17.1 is actually an isomorphism between the linear orders $(\omega^\omega, \leq_{\text{lex}})$, where $\leq_{\text{lex}}$ is the lexicographic ordering, and $([0;1), \leq)$, where $\leq$ is the usual ordering on $\mathbb{R}$. Infer that $(\omega^\omega, \tau_{\text{lex}})$, where τ_{lex} is the topology generated by the open $\leq_{\text{lex}}$-intervals, is homeomorphic to $[0;1)$, so it is a Polish space and it is of dimension 1. Argue that τ_{lex} is coarser than the usual product topology on ω^ω, and that each basic open set $\boldsymbol{N}_s$ is both F_σ and G_δ with respect to τ_{lex}, so that every open set in the usual product topology on ω^ω is F_σ with respect to τ_{lex}.

Exercise 1.3.21. Show that for every nonempty Polish space X there is a continuous open surjection $f\colon \omega^\omega \to X$. (Compare this with Remark 1.3.19.)

Hint. First show that if X is a metric space, then for every open set U and every $\varepsilon \in \mathbb{R}^+$ there is a countable covering $(U_n)_{n\in\omega}$ of U such that $\mathrm{cl}(U_n) \subseteq U$ and $\mathrm{diam}(U_n) < \varepsilon$, for all $n \in \omega$. Use this to build an appropriate ω-scheme inducing the function f.

Another important result that can be proved using schemes is the following theorem of Sierpiński, which yields a converse to Exercise 1.3.21.

Theorem 1.3.22 ([Kec95, Theorem 8.19])**.** *Let X be Polish and Y be a separable metrizable space. If there is a continuous open surjection $f\colon X \to Y$, then Y is Polish.*

Proof. By Exercise 1.3.21 we can assume $X = \omega^\omega$. Let $\hat{Y}$ be the completion of Y. Let $\mathcal{S}_f = \{B_s \mid s \in \omega^{<\omega}\}$ be the ω-scheme on $\hat{Y}$ canonically induced by f (i.e. $B_s = f(\boldsymbol{N}_s)$), and let

$$B'_s = \bigcup\{U \subseteq \hat{Y} \mid U \text{ is open and } U \cap Y \subseteq B_s\}$$

be the largest open subset of $\hat{Y}$ such that $B'_s \cap Y = B_s$. Since Y is dense in $\hat{Y}$, then $B_s \subseteq B'_s \subseteq \mathrm{cl}(B_s)$. The family $\mathcal{S}' = \{B'_s \mid s \in \omega^{<\omega}\}$ is an ω-scheme on $\hat{Y}$: monotonicity follows from the fact that $B'_{s^\frown k} \cap Y = B_{s^\frown k} \subseteq B_s$, hence $B'_{s^\frown k} \subseteq B'_s$ by definition of B'_s; the vanishing-diameters property follows instead from $B'_s \subseteq \mathrm{cl}(B_s)$. Since $\mathcal{S} \sqsubseteq \mathcal{S}' \sqsubseteq \mathrm{cl}(\mathcal{S})$, we have $f_{\mathcal{S}'} = f$ by Lemma 1.3.9(e), and in particular $f_{\mathcal{S}'}(\omega^\omega) = Y$.

We will construct an $\omega \times \omega$-scheme $\hat{\mathcal{S}} = \{\hat{B}_s \mid s \in \omega^{<\omega}\}$ on $\hat{Y}$ such that:

(1) $\hat{B}_{(s_0,s_1)}$ is open and $\hat{B}_{(s_0,s_1)} \cap Y \subseteq B_{s_0}$ for all $(s_0, s_1) \in (\omega \times \omega)^{<\omega}$;

(2) for every $y \in Y$ and $n \in \omega$ there is $(s_0, s_1) \in (\omega\times\omega)^n$ such that $y \in \hat{B}_{(s_0,s_1)}$;

(3) for every $n \in \omega$ and every $z \in \bigcup\{\hat{B}_{(s_0,s_1)} \mid (s_0, s_1) \in (\omega \times \omega)^n\}$ there are only finitely many $(s_0, s_1) \in (\omega \times \omega)^n$ such that $z \in \hat{B}_{(s_0,s_1)}$.

Let us first show how such a scheme allows us to conclude the proof. For every $n \in \omega$, set $W_n = \bigcup\{\hat{B}_{(s_0,s_1)} \mid (s_0, s_1) \in (\omega \times \omega)^n\}$: we show that $Y = \bigcap_{n\in\omega} W_n$, so that Y is a G_δ subset of $\hat{Y}$ and thus a Polish space by Proposition 1.1.8. Condition (2) yields $Y \subseteq \bigcap_{n\in\omega} W_n$. Conversely, condition (3) implies that for every $z \in \bigcap_{n\in\omega} W_n$ the tree $T = \{(s_0, s_1) \in (\omega \times \omega)^{<\omega} \mid z \in \hat{B}_{(s_0,s_1)}\}$ is infinite and finitely branching. By König's Lemma, there is an infinite branch $(x_0, x_1) \in [T]$ through T: clearly, z witnesses $(x_0, x_1) \in D_{\hat{\mathcal{S}}}$, and also $f_{\hat{\mathcal{S}}}(x_0, x_1) = z$. By definition of B'_{s_0}, condition (1) implies that $\hat{B}_{(s_0,s_1)} \subseteq B'_{s_0}$ for all $(s_0, s_1) \in (\omega \times \omega)^{<\omega}$, thus $z = f_{\hat{\mathcal{S}}}(x_0, x_1) = f_{\mathcal{S}'}(x_0) \in Y$.

It remains to recursively construct the $\omega\times\omega$-scheme $\hat{\mathcal{S}}$. Set $\hat{B}_{(\emptyset,\emptyset)} = B'_\emptyset$. Now suppose to have defined $\hat{B}_{(s_0,s_1)}$ for all $(s_0, s_1) \in (\omega\times\omega)^n$ in such a way that (1)–(3) are satisfied. Let $\mathcal{U}$ be the (countable) collection of all nonempty open sets of the form $B'_{s_0{}^\frown k} \cap \hat{B}_{(s_0,s_1)}$, for $(s_0, s_1) \in (\omega \times \omega)^n$ and $k \in \omega$. We claim that $Y \subseteq \bigcup\mathcal{U}$. Fix any $y \in Y$. By inductive hypothesis, there is $(s_0, s_1) \in (\omega \times \omega)^n$ such that $y \in \hat{B}_{(s_0,s_1)}$ because of (2), hence also $y \in B_{s_0}$ because of (1). Since $\bigcup_{k\in\omega} B_{s_0{}^\frown k} = B_{s_0}$ we can find $k \in \omega$ such that $y \in B_{s_0{}^\frown k} = B'_{s_0{}^\frown k} \cap Y$, hence $y \in B'_{s_0{}^\frown k} \cap \hat{B}_{(s_0,s_1)} \in \mathcal{U}$.

Since $\hat{Y}$ is metrizable, we can find a countable family $\mathcal{V}$ consisting of open subsets of $\hat{Y}$ such that

(a) $\bigcup\mathcal{V} = \bigcup\mathcal{U}$,

(b) for every $V \in \mathcal{V}$ there is $U \in \mathcal{U}$ such that $V \subseteq U$, and

(c) for every $z \in \bigcup\mathcal{V}$ there are only finitely many $V \in \mathcal{V}$ such that $z \in V$.

To construct such a $\mathcal{V}$, let $(U_i)_{i<I}$ be an enumeration of $\mathcal{U}$, for the appropriate $I \leq \omega$. Since by metrizability each U_i is F_σ, it can be written as a union $U_i = \bigcup_{j\in\omega} F^j_i$ of an increasing sequence $(F^j_i)_{j\in\omega}$ of closed sets. Let $V_i = U_i \setminus \bigcup_{\ell<i} F^i_\ell$, and let $\mathcal{V} = \{V_i \mid i < I\}$. Since $V_i \subseteq U_i$, part (b) is satisfied and $\bigcup\mathcal{V} \subseteq \bigcup\mathcal{U}$. On the other hand, every $z \in \bigcup\mathcal{U}$ belongs to V_i for $i < I$ smallest such that $z \in U_i$, and since the latter implies that $z \in F^j_i$ for some $j \in \omega$, we also have $z \notin V_\ell$ for all $\ell > i, j$. Thus $\mathcal{V}$ satisfies also (a) and (c).

By removing unnecessary repetitions (and lowering the value of I, if needed), we can assume that $V_i \neq V_{i'}$ if i and i' are distinct. For each $i < I$, pick $(s_0^i, s_1^i) \in (\omega \times \omega)^n$ and $k_i \in \omega$ such that $U_i = B'_{s_0^i {}^\frown k_i} \cap \hat{B}_{(s_0^i, s_1^i)}$, and set $\hat{B}_{(s_0^i {}^\frown k_i, s_1^i {}^\frown i)} = V_i$. All remaining $\hat{B}_{(s_0 {}^\frown k, s_1 {}^\frown k')}$ are set to be the empty set. Since

$$\hat{B}_{(s_0^i {}^\frown k_i, s_1^i {}^\frown i)} \cap V = V_i \cap Y \subseteq U_i \cap Y \subseteq B'_{s_0^i {}^\frown k_i} \cap Y = B_{s_0^i {}^\frown k_i},$$

condition (1) is satisfied. Moreover, $Y \subseteq \bigcup \mathcal{U} = \bigcup \mathcal{V}$, hence also (2) holds at level $n+1$. Finally, $\{\hat{B}_{(s_0,s_1)} \mid (s_0, s_1) \in (\omega \times \omega)^{n+1}\} = \mathcal{V} \cup \{\emptyset\}$, hence (3) is satisfied as well and we are done. □

As a corollary of Exercise 1.3.21 and Theorem 1.3.22, we obtain an interesting characterization of the class of Polish spaces which once again highlights the crucial role played by the Baire space ω^ω in this area.

Corollary 1.3.23. *Let X be a separable metrizable space. Then the following are equivalent:*

(i) *X is Polish;*

(ii) *there is a continuous open surjection from ω^ω onto X.*

The technique of schemes developed so far yields many other interesting results (which will not be proved here), including the following ones. The first two theorems are topological characterizations of the Cantor and the Baire space, respectively.

Theorem 1.3.24 (Brouwer, see [Kec95, Theorem 7.4])**.** *The Cantor space 2^ω is the unique, up to homeomorphism, nonempty, compact metrizable (hence Polish) zero-dimensional space without isolated points.*

Theorem 1.3.25 (Alexandrov-Urysohn, see [Kec95, Theorem 7.7])**.** *The Baire space ω^ω is the unique, up to homeomorphism, nonempty Polish zero-dimensional space for which all compact subsets have empty interior.*

Theorem 1.3.26 (Hurewicz, see [Kec95, Theorem 7.10])**.** *Let X be Polish. Then X contains a closed subspace homeomorphic to ω^ω iff and only if X is not K_σ, i.e. X cannot be written as a countable union of compact sets.*

Example 1.3.27. Since ω^ω is not compact, it is clear that no compact Polish space can contain a homeomorphic copy of ω^ω as a closed subspace. A less trivial example is given by the real line $\mathbb{R} = \bigcup_{n \in \omega} [-n; n]$, which is K_σ: it does contain a subspace homeomorphic to ω^ω, namely the irrationals Irr, but this subspace is not *closed*, and in fact by Theorem 1.3.26 none of its closed subspaces is homeomorphic to the Baire space.

In contrast, in the next section we will see that all uncountable Polish spaces contain a closed subspace homeomorphic to the Cantor space 2^ω.

1.4 Perfect Polish spaces

Definition 1.4.1. A point x of a topological space X is **isolated** if there is an open neighborhood U of it such that $U = \{x\}$. A space is **perfect** if it has no isolated point. A subset $P \subseteq X$ is **perfect in** X if it is closed and perfect with respect to the relative topology.

Remark 1.4.2. If x is not isolated in the metric space (X, d), then every open neighborhood U of x is infinite. Indeed, we can recursively define a sequence $(x_n)_{n\in\omega}$ of distinct points in U as follows. Fix $\varepsilon > 0$ such that $B_d(x, \varepsilon) \subseteq U$ and pick any $x_0 \in B_d(x, \varepsilon) \setminus \{x\}$. Then let x_{n+1} be any point in $B_d(x, d(x, x_n)) \setminus \{x\}$ (such x_n's exist because x is not isolated in X).

Examples of perfect Polish spaces are: $\mathbb{R}^n$, $\mathbb{R}^\omega$, $\mathbb{C}^n$, $\mathbb{C}^\omega$, $[0;1]^n$, $[0;1]^\omega$, 2^ω, ω^ω, $C(X)$ with X compact metrizable, and so on.

Theorem 1.4.3 ([Kec95, Theorem 6.2])**.** *Let X be a nonempty perfect completely metrizable space. Then there is an embedding of 2^ω into X (equivalently: there is a closed subset of X homeomorphic to 2^ω).*

Proof. Fix a complete compatible metric $d \leq 1$ on X. By Lemma 1.3.6, it is enough to build a 2-scheme $\{B_s \mid s \in 2^{<\omega}\}$ such that for every $s \in 2^{<\omega}$ and $i, j \in \{0, 1\}$

(1) $B_{s^\frown i} \cap B_{s^\frown j} = \emptyset$ if $i \neq j$;

(2) B_s is open and nonempty;

(3) $\mathrm{cl}(B_{s^\frown i}) \subseteq B_s$.

The scheme is defined by induction on the length of $s \in 2^{<\omega}$. Let $B_\emptyset$ be an arbitrary nonempty open subset of X of diameter ≤ 1. Given B_s, define $B_{s^\frown 0}$ and $B_{s^\frown 1}$ as follows. Choose two distinct points $x_0, x_1 \in B_s$ (which is possible because $B_s \neq \emptyset$ and X is perfect), let $\varepsilon_i > 0$ be small enough so that $B_d(x_i, \varepsilon_i) \subseteq B_s$, and set $B_{s^\frown i} = B_d(x_i, \varepsilon)$ where $\varepsilon = \frac{1}{2} \min\left\{2^{-(\mathrm{lh}(s)+1)}, \varepsilon_0, \varepsilon_1, d(x_0, x_1)\right\}$. It is easy to check that such $B_{s^\frown i}$ have the required properties. ☐

Corollary 1.4.4 ([Kec95, Corollary 6.3])**.** *Every nonempty perfect Polish space has the cardinality of the continuum $2^{\aleph_0}$. The same is true for nonempty perfect subsets of a Polish space.*

Proof. By Theorem 1.4.3, a nonempty perfect (subset of a) Polish space X contains a copy of 2^ω, and thus has cardinality $\geq 2^{\aleph_0}$. The fact that $|X| \leq 2^{\aleph_0}$ follows from the fact that by Theorem 1.3.17 the space ω^ω surjects onto X, together with the fact that ω^ω has has cardinality $2^{\aleph_0}$. ☐

Remark 1.4.5. One can directly show that ω^ω surjects onto X as follows. Let $D = \{x_n \mid n \in \omega\}$ be a countable dense subset of X. Then the map $f : \omega^\omega \to X$ defined by

$$f(y) = \begin{cases} \lim_{n\to\infty} x_{y(n)} & \text{if } (x_{y(n)})_{n\in\omega} \text{ converges in } X \\ x_0 & \text{otherwise} \end{cases}$$

is clearly surjective. More generally, this argument shows that if a metrizable space X has a dense subset of cardinality κ, then there is a surjection of κ^ω onto X and thus X has cardinality $\leq \kappa^{\aleph_0}$.

Definition 1.4.6. A point x in a topological space X is a **condensation point** if every open neighborhood of x is uncountable.

Theorem 1.4.7 (Cantor-Bendixson, see [Kec95, Theorem 6.4])**.** *Let X be a separable metrizable space. Then X can be written as a disjoint union $X = P \cup C$ with P a perfect subset of X and C a countable open set. If moreover X is Polish, then such a decomposition is unique.*

The uniqueness of such a decomposition may fail if X is not Polish: if e.g. $X = \mathbb{Q}$, then we could equally set $P = \mathbb{Q}$ and $C = \emptyset$, or $P = \emptyset$ and $C = \mathbb{Q}$. When X is Polish, *the* perfect subset P of X given by the above theorem is called **perfect kernel** of X.

Proof. Let
$$X^* = \{x \in X \mid x \text{ is a condensation point of } X\}.$$
Set $P = X^*$ and $C = X \setminus P$. We claim that P and C are as required. First observe that if $\mathcal{B}$ is a countable basis for X then $C = \bigcup\{U \in \mathcal{B} \mid U \text{ is countable}\}$, thus C is open and countable and P is closed. To show that P has no isolated point (with respect to its relative topology!), let $x \in P$ and U be an open neighborhood of x. Then U is uncountable because x is a condensation point, and since C is countable this implies that there is some (in fact, uncountably many) $y \in U \cap P$ witnessing that $U \cap P \neq \{x\}$.

To prove uniqueness, suppose that X is Polish and that $X = P_1 \cup C_1$ is another decomposition as in the statement of the theorem. Notice that if Y is a perfect Polish space then $Y^* = Y$. Indeed, if $y \in Y$ and U is an open neighborhood of y, then $U \cap Y$ is a perfect nonempty Polish space, and thus has cardinality $2^{\aleph_0}$ by Corollary 1.4.4. Thus $P_1^* = P_1$, and hence $P_1 = P_1^* \subseteq X^* = P$ because $Y \subseteq Z$ implies $Y^* \subseteq Z^*$. Moreover, if $x \in C_1$ then $x \in C$, as witnessed by the countable open set C_1 itself: therefore $C_1 \subseteq C$. It follows that $P_1 = P$ and $C_1 = C$. □

Remark 1.4.8. In the proof above we actually showed that if P_1 is a perfect subset of a Polish space X, then $P_1 \subseteq P$, where $P = X^*$ is the perfect kernel of X. Thus the perfect kernel of a Polish space can also be characterized as the largest (with respect to inclusion) perfect subset of it.

Corollary 1.4.9 ([Kec95, Corollary 6.5])**.** *Any uncountable Polish space contains a (necessarily closed) homeomorphic copy of 2^ω and has cardinality $2^{\aleph_0}$.*

Proof. Let P be the perfect kernel of the Polish space X. Since $C = X \setminus P$ is a countable open set, if X is uncountable then $P \neq \emptyset$, and thus P itself is a nonempty perfect Polish space. Thus the result follows from Theorem 1.4.3 and Remark 1.4.5. □

There is an algorithmic way to find the perfect kernel of a Polish space X. The following construction is due to Cantor[7] and is the reason behind the

[7] Actually Cantor was working only with closed subsets of the real line $\mathbb{R}$.

introduction of ordinals.

Definition 1.4.10. For any topological space X, let

$$X' = \{x \in X \mid x \text{ is not isolated in } X\}.$$

We call X' the **Cantor-Bendixson derivative** of X. Clearly, X' is closed and X is perfect if and only if $X = X'$.

Define the **iterated Cantor-Bendixson derivative** by recursion on the ordinals as follows:

$$\begin{aligned} X^{(0)} &= X \\ X^{(\alpha+1)} &= (X^{(\alpha)})' \\ X^{(\lambda)} &= \bigcap_{\alpha<\lambda} X^{(\alpha)}, \qquad \text{if } \lambda \text{ is limit.} \end{aligned}$$

Arguing by induction on α, one can check that the $X^{(\alpha)}$'s form a decreasing sequence of closed subsets of X, and that if X is second-countable and $\alpha < \omega_1$ then $X \setminus X^{(\alpha)}$ is a countable open set.

Lemma 1.4.11 ([Kec95, Theorem 6.9])**.** *Let X be a second countable topological space and $(F_\alpha)_{\alpha<\rho}$ be a strictly decreasing transfinite sequence of closed sets (i.e. $F_\beta \subsetneq F_\alpha$ for all $\alpha < \beta$). Then ρ is a countable ordinal.*

This holds similarly for strictly increasing transfinite sequences of closed sets (and thus for strictly decreasing or incresing transfinite families of opens sets).

Proof. Let $\mathcal{B} = \{U_n \mid n \in \omega\}$ be a countable basis for X. For $F \subseteq X$ closed, let $N(F) = \{n \in \omega \mid U_n \cap F \neq \emptyset\}$. Notice that if $F \subseteq G$ then $N(F) \subseteq N(G)$, and that $F \subsetneq G$ implies $N(G) \setminus N(F) \neq \emptyset$. (Indeed, if $x \in G \setminus F$ then $x \in U_n \subseteq X \setminus F$ for some $n \in \omega$, so that $n \in N(G) \setminus N(F)$.) For each $\alpha < \rho$, pick some $n_\alpha \in N(F_\alpha) \setminus N(F_{\alpha+1})$: since the map $\alpha \mapsto n_\alpha$ is an injection between ρ and ω, we must conclude that ρ is a countable ordinal. □

It follows that if X is a Polish space there is $\alpha_0 < \omega_1$ such that $X^{(\alpha)} = X^{(\alpha_0)}$ for all $\alpha \geq \alpha_0$ (the countable ordinal α_0 is called the **Cantor-Bendixson rank** of X). Then $X^{(\alpha_0)}$, also denoted by X^∞, is the perfect kernel of X, and X is countable if and only if $X^\infty = \emptyset$.

Remark 1.4.12. Notice that the decomposition given by the proof of Theorem 1.4.7 and the one given by the iterated Cantor-Bendixson derivatives can give different decompositions if applied to non-Polish separable metrizable spaces: indeed, if $X = \mathbb{Q}$ then the former gives $P = \emptyset$ and $C = \mathbb{Q}$, while the latter gives $P = \mathbb{Q}$ and $C = \emptyset$.

The results above imply that there is no simple counterexample to the Continuum Hypothesis, but they actually show more.

Definition 1.4.13. A subset A of a topological space X has the **Perfect Set Property** (PSP for short) if either it is countable or[8] there is an embedding of 2^ω into A.

[8] Since 2^ω is compact, this is equivalent to requiring that A contains a closed set homeomorphic to 2^ω, whence the name of the property.

Clearly, if X is separable and A has the PSP then A satisfies the Continuum Hypothesis: either A is countable, or has cardinality $2^{\aleph_0}$. However, the PSP is a stronger property: while it is independent of ZFC that all subsets of $\mathbb{R}$ are either countable or of size $2^{\aleph_0}$ (i.e. that the Continuum Hypothesis CH holds), it can be proved in ZFC that there is a set without the PSP.

Proposition 1.4.14 ([Kec95, Example 8.24])**.** *If X is an uncountable Polish space, then there is an uncountable set $A \subseteq X$ such that $A \cap P \neq \emptyset$ and $P \setminus A \neq \emptyset$ for all nonempty perfect sets $P \subseteq X$. In particular, such an A does not have the* PSP*.*

Sets A as above are called **Bernstein sets**.

Proof. First notice that there are exactly $2^{\aleph_0}$ perfect subsets of X. Indeed, if $\mathcal{B} = \{U_n \mid n \in \omega\}$ is a countable basis for X, then the map sending $x \in \omega^\omega$ to $X \setminus \bigcup_{n \in \omega} U_{x(n)}$ is a surjection of ω^ω onto the closed subsets of X, hence there are at most $2^{\aleph_0}$ perfect subset of X. To show that there are at least $2^{\aleph_0}$ such sets, it is enough to consider the case $X = 2^\omega$ (the general case easily follows from the fact that every uncountable Polish space X contains a closed set homeomorphic to 2^ω by Corollary 1.4.9). For each $x \in 2^\omega$, let $P_x = \{x\} \times 2^\omega \subseteq 2^\omega \times 2^\omega$. It is immediate to check that P_x is perfect in $2^\omega \times 2^\omega$ and that the map $x \mapsto P_x$ is injective. Since $2^\omega \times 2^\omega$ is homeomorphic to 2^ω we are done.

Fix a transfinite enumeration $(P_\xi)_{\xi < 2^{\aleph_0}}$ of the nonempty perfect subsets of X. Find by transfinite recursion on $\xi < 2^{\aleph_0}$ distinct point $a_\xi, b_\xi \in P_\xi$: this is possible because each P_ξ, being a perfect Polish space, has cardinality $2^{\aleph_0}$ by Corollary 1.4.4, while the collection of points $\{a_\nu, b_\nu \mid \nu < \xi\}$ constructed so far has cardinality $|\xi| < 2^{\aleph_0}$. Setting $A = \{a_\xi \mid \xi < 2^{\aleph_0}\}$ we obtain an uncountable set that does not contain any nonempty perfect subset P (and thus 2^ω cannot be embedded into A. Indeed, if P is perfect nonempty then $P = P_\xi$ for some $\xi < 2^{\aleph_0}$, hence $b_\xi \in P \setminus A$ by construction. □

Remark 1.4.15. The proof of Proposition 1.4.14 heavily uses the Axiom of Choice AC. In contrast, it can be shown via a forcing argument that it is consistent with ZF + DC (where DC is the Axiom of Dependent Choice) that all subsets of $\mathbb{R}$ have the PSP.

The existence of a Bernstein set under ZFC naturally leads to the problem of understanding how much complicated such a set must be: the last result of this section shows that simple sets are immune from this "pathological" behaviour.

Theorem 1.4.16. *Every F_σ or G_δ subset A of a Polish space X has the* PSP*, and thus satisfies the Continuum Hypothesis. The same is true for $G_{\delta\sigma}$ subsets (i.e. countable unions of G_δ sets) of X.*

Proof. If A is G_δ, then it is Polish by Proposition 1.1.8. If it is uncountable, then its perfect kernel P is nonempty, and thus 2^ω can be emebedded into $P \subseteq A$ by Theorem 1.4.3.

If now $A = \bigcup_{n \in \omega} A_n$ with each A_n a G_δ set,[9] we distinguish two cases:

[9]Observe that this includes the case of an F_σ set A because in a metrizable space all closed sets are G_δ — see Proposition 3.6.2.

- *Every A_n is countable.* Then A is countable as well and we are done.
- *There is $n \in \omega$ such that A_n is uncountable.* Then 2^ω embeds into $A_n \subseteq A$ by the first part of this proof, and we are done again. □

1.5 Baire category

Let X be a topological space. A set $A \subseteq X$ is **nowhere dense** if there is no open set $U \subseteq X$ such that $A \cap U$ is dense in U. Equivalently, A is nowhere dense if its closure $\mathrm{cl}(A)$ has empty interior. Still equivalently, A is nowhere dense if it is disjoint from an open dense set. Notice that A is nowhere dense if and only if $\mathrm{cl}(A)$ is nowhere dense.

The collection of all nowhere dense subsets of X is closed under subsets and finite unions, that is, it is an ideal of sets. (Closure under finite unions follows from the fact that one can prove by induction that the intersection of finitely many open dense sets is still open dense.) Closing this family under countable unions, we get the so-called meager sets.

Definition 1.5.1. A set $A \subseteq X$ is **meager** if it can be written as a countable union of nowhere dense sets. A set $A \subseteq X$ is **comeager** if its complement is meager; equivalently, A is comeager if and only if it contains the intersection of countably many open dense sets.[10]

It is easy to check that the collection of meager subsets of X is a σ-ideal on X, that is, it is closed under subsets and countable unions. Dually, the collection of comeager subsets of X is a σ-filter on X, i.e. it is closed under supersets and countable intersections.

Example 1.5.2. (1) If $U \subseteq X$ is open, its boundary $\mathrm{cl}(U) \setminus U$ is nowhere dense because every nonempty open set disjoint from U is contained in $X \setminus \mathrm{cl}(U)$. It follows that every second-countable space X can be written as a disjoint union $Z \cup F$ with Z a comeager zero-dimensional G_δ set and F a meager F_σ set. Indeed, it is enough to let $F = \bigcup_{n\in\omega} \mathrm{cl}(U_n) \setminus U_n$ for $\{U_n \mid n \in \omega\}$ a countable basis for X, and $Z = X \setminus F$.

(2) Similarly, if $F \subseteq X$ is closed, then $F \setminus \mathrm{Int}(F)$ is nowhere dense.

(3) The set of rationals $\mathbb{Q}$ is a meager subsets or $\mathbb{R}$, and hence the irrationals are a comeager subset of $\mathbb{R}$. The set of positive reals is neither meager nor comeager in $\mathbb{R}$.

(4) More generally, a countable subset of a *perfect* space is always meager. On the other hand, if x is isolated in X then $\{x\}$ is not meager.

(5) Any dense G_δ set $G \subseteq X$ is comeager.

[10] In general topology, there is an older terminology related to the same concepts, namely: meager sets are called "sets of the first category", non-meager sets are called "sets of the second category", while comeager sets are called "residual sets".

Meager sets are usually regarded as "topologically small sets". However, it might happen that a space is meager in itself, as in the case of the rationals $\mathbb{Q}$: in such spaces, the Baire category notions of meager and comeager are useless — all their subsets are meager. This justifies the following definition.

Definition 1.5.3. A topological space X is **Baire** if it satisfies the following equivalent conditions:

(i) Every nonempty open subset of X is non-meager.

(ii) Every comeager set in X is dense.

(iii) The intersection of countably many dense open subsets of X is dense.

In particular, if X is nonempty and Baire, then the intersection of any two dense G_δ subsets of X must be dense, and hence nonempty.

Lemma 1.5.4 ([Kec95, Proposition 8.3])**.** *If a topological space X is Baire, then so are all its open subspaces.*

Proof. Let $U \subseteq X$ be open, and let $\{U_n \mid n \in \omega\}$ be a family of open dense subsets of U. Then each $V_n = U_n \cup (X \setminus \mathrm{cl}(U))$ is open dense in X. Therefore $\bigcap_{n\in\omega} V_n = (\bigcap_{n\in\omega} U_n) \cup (X \setminus \mathrm{cl}(U))$ is dense in X, hence $\bigcap_{n\in\omega} U_n$ is dense in U. □

The following result is known as Baire Category Theorem.

Theorem 1.5.5 ([Kec95, Theorem 8.4])**.** *Every completely metrizable space X is Baire.*

Proof. Let d be a compatible complete metric on X, and let $U \subseteq X$ be any nonempty open set: we want to show that $U \cap \bigcap_{n\in\omega} U_n \neq \emptyset$. To this aim, we recursively build a sequence of open balls $B_n = B_d(x_n, \varepsilon_n)$ such that $\mathrm{cl}(B_n) \subseteq B_d^{cl}(x_n, \varepsilon_n) \subseteq U \cap U_n \cap B_{n-1}$ (where $B_{-1} = X$) and $\varepsilon_n \leq 2^{-n}$. This implies that $(x_n)_{n\in\omega}$ is a Cauchy sequence in (X, d), and thus $\lim_n x_n \in \bigcap_{n\in\omega} \mathrm{cl}(B_n) \subseteq U \cap \bigcap_{n\in\omega} U_n$.

We define the open balls B_n by recursion on $n \in \omega$. Since U_0 is open dense, the set $U \cap U_0$ is open and nonempty: pick any $x_0 \in U \cap U_0$, and let ε_0 be small enough so that $\varepsilon_0 \leq 2^{-0}$ and $B_d^{cl}(x_0, \varepsilon_0) \subseteq U \cap U_0$. Set $B_0 = B_d(x_0, \varepsilon_0)$. Now suppose that $n > 0$, so that by inductive hypothesis B_{n-1} is a nonempty open subset of U. By density, $B_{n-1} \cap U_n$ is open and nonempty. Pick any $x_n \in B_{n-1} \cap U_n$, let ε_n be small enough so that $\varepsilon_n \leq 2^{-n}$ and $B_d^{cl}(x_n, \varepsilon_n) \subseteq B_{n-1} \cap U_n = U \cap U_n \cap B_{n-1}$, and set $B_n = B_d(x_n, \varepsilon_n)$. This concludes the construction and the proof. □

Using a similar argument, one can also show that every locally compact Hausdorff space is Baire (see [Kec95, Theorem 8.4]).

Corollary 1.5.6. *Let X be a nonempty completely metrizable space. If an F_σ set $A \subseteq X$ is both dense and codense, then it is not G_δ.*

Proof. If both A and $X \setminus A$ were dense G_δ sets, then their intersection should be dense because X is Baire, which is clearly not the case. □

Given $A, B \subseteq X$, we write $A =^* B$ if their symmetric difference $A \triangle B$ is meager in X. The relation $=^*$ is reflexive and symmetric. Moreover, it is also transitive, i.e. an equivalence relation. Indeed, if $A, B, C \subseteq X$ are three sets, then $A \setminus C \subseteq (A \setminus B) \cup (B \setminus C)$ and $C \setminus A \subseteq (C \setminus B) \cup (B \setminus A)$: thus if $A =^* B$ and $B =^* C$, then also $A =^* C$.

Definition 1.5.7. Let X be a topological space. A set $A \subseteq X$ has the **Baire Property** (BP for short) if $A =^* U$ for some open set $U \subseteq X$. We let $\mathsf{BP}(X)$ be the collection of all subsets of X with the Baire property.

By transitivity of $=^*$, if A has the BP and $A =^* B$, then B has the BP too.

Remark 1.5.8. This is an instance of a more general template for regularity properties used in descriptive set theory. Given a σ-ideal $\mathcal{I}$ on X, whose elements are called $\mathcal{I}$-small, we say that a set $A \subseteq X$ is **$\mathcal{I}$-regular** if it equals an open set modulo an $\mathcal{I}$-small set, that is, if $A \triangle U \in \mathcal{I}$ for some open set $U \subseteq X$. An important example of this kind is when we fix some σ-additive measure μ on X, and we consider the σ-ideal $\mathcal{I}$ of all subsets of X with measure 0.

Proposition 1.5.9 ([Kec95, Proposition 8.22])**.** *Let X be a topological space. Then $\mathsf{BP}(X)$ is a σ-algebra on X, that is, it is closed under complements and countable unions (and thus under countable intersections as well). More precisely, $\mathsf{BP}(X)$ is the smallest σ-algebra containing all open sets and all meager sets.*

Proof. Suppose that $A =^* U$ for some open set $U \subseteq X$. Then $(X \setminus A) \setminus (X \setminus U) = U \setminus A$ and $(X \setminus U) \setminus (X \setminus A) = A \setminus U$, and thus $X \setminus A =^* X \setminus U$. But $X \setminus U$ is closed, hence $X \setminus U =^* \operatorname{Int}(X \setminus U)$ by Example 1.5.2(2). By transitivity of $=^*$, it follows that $X \setminus A =^* \operatorname{Int}(X \setminus U)$. This shows that $\mathsf{BP}(X)$ is closed under complements.

Now suppose that $A_n =^* U_n$ with $U_n \subseteq X$ open, for every $n \in \omega$. Since $\bigcup_{n\in\omega} A_n \setminus \bigcup_{n\in\omega} U_n \subseteq \bigcup_{n\in\omega}(A_n \setminus U_n)$ and $\bigcup_{n\in\omega} U_n \setminus \bigcup_{n\in\omega} A_n \subseteq \bigcup_{n\in\omega}(U_n \setminus A_n)$, it follows that $\bigcup_{n\in\omega} A_n =^* V$ with $V = \bigcup_{n\in\omega} U_n$. Thus $\mathsf{BP}(X)$ is also closed under countable unions.

Clearly, $\mathsf{BP}(X)$ contains all open sets, and it contains also all meager sets M because $M =^* \emptyset$. Thus $\mathsf{BP}(X)$ is a σ-algebra containing all open sets and all meager sets. Conversely, every set A with the BP is a Boolean combination of an open set and meager set. Indeed, if $A =^* U$ for $U \subseteq X$ open and $M = A \triangle U$, so that M is meager, then $A = M \triangle U$. □

Remark 1.5.10. The above proof actually shows that the smallest algebra on X containing all open sets and all meager sets is actually a σ-algebra, and it coincides with $\mathsf{BP}(X)$.

Sets with the Baire property can be approximated internally by a G_δ set, and externally by an F_σ set (modulo meager sets).

Proposition 1.5.11 ([Kec95, Proposition 8.23]). *Let X be a topological space and $A \subseteq X$. Then the following are equivalent:*

(i) $A \in \mathsf{BP}(X)$;

(ii) $A = G \cup M$ *for some* G_δ *set* G *and some meager set* M;

(iii) $A = F \setminus M$ *for some* F_σ *set* F *and some meager set* M.

Proof. The implications (ii) $\Rightarrow$ (i) and (iii) $\Rightarrow$ (i) follow from Proposition 1.5.9.

For (i) $\Rightarrow$ (ii), pick any open set $U \subseteq X$ such that $A =^* U$. Write $A \,\Delta\, U$ as $\bigcup_{n\in\omega} D_n$ with each D_n nowhere dense, and set $F = \bigcup_{n\in\omega} \mathrm{cl}(D_n)$. Then F is a meager F_σ set with $A \,\Delta\, U \subseteq F$. Set $G = U \setminus F$ and $M = A \setminus G$. Since $G \subseteq A$, we have $A = G \cup M$. Moreover, the set G is G_δ by definition, and M is meager because $M \subseteq F$, hence we are done.

We finally show that (i) $\Rightarrow$ (iii). The proof in the previous paragraph works for every set with the BP: in particular, it can be applied to $X \setminus A$ because $X \setminus A \in \mathsf{BP}(X)$ by (i) and Proposition 1.5.9. Therefore $X \setminus A = G \cup M$ with M meager and G a G_δ set. Let $F = X \setminus G$, so that F is an F_σ set. Then $A = F \cap (X \setminus M) = F \setminus M$. $\square$

Corollary 1.5.12. *Let X be a nonempty perfect Polish space, and let $A \in \mathsf{BP}(X)$ be a non-meager set. Then there is an embedding of 2^ω into A.*

Proof. By Proposition 1.5.11, A contains a non-meager G_δ set G. In particular, G is Polish, and it is uncountable by Exercise 1.5.2(4). Therefore 2^ω embeds into $G \subseteq A$ by Corollary 1.4.9. $\square$

Working in ZFC, it is possible to show that there are sets without the Baire property.

Proposition 1.5.13 (Cfr. [Kec95, Example 8.24]). *If X is a nonempty perfect Polish space, then $\mathsf{BP}(X) \neq \mathscr{P}(X)$.*

Proof. Let $A \subseteq X$ be a Bernstein set, whose existence is granted by Proposition 1.4.14. We claim that $A \notin \mathsf{BP}(X)$. If not, since either A or $X \setminus A$ is non-meager (because X is Baire by Theorem 1.5.5), the Cantor space would embed in one of A or $X \setminus A$ by Corollary 1.5.12, contradicting the fact that A is a Bernstein set. $\square$

The notions of meagerness and comeagerness can be “localized” to an open set $U \subseteq X$. Notice that a set $D \subseteq U$ is nowhere dense in U (with the relative topology) if and only if it is nowhere dense in X. Indeed, if $V \subseteq U$ is open dense in U and such that $V \cap D = \emptyset$, then $W = V \cup (X \setminus \mathrm{cl}(U))$ is open dense in X and such that $W \cap D = \emptyset$. Conversely, if $W \subseteq X$ is open dense in X and such that $W \cap D = \emptyset$, then $V = W \cap U$ is open dense in U and such that $V \cap D = \emptyset$. It follows that a set $M \subseteq U$ is meager in U if and only if it is meager in X.

Definition 1.5.14. Let X be a topological space and $U \subseteq X$ be an open set. We say that $A \subseteq X$ is **meager in** U if its trace $A \cap U$ is meager in U or, equivalently, in X. The set A is **comeager in** U if its complement is meager in U, that is, if $U \setminus A$ is meager.

Clearly, if A is (co)meager in some open set $U \subseteq X$, then A is also (co)meager in every open set V such that $V \subseteq U$. An important fact concerning localization is the following.

Proposition 1.5.15 ([Kec95, Proposition 8.26])**.** *Let X be a topological space and $A \in \mathsf{BP}(X)$. Then either A is meager, or else it is comeager in some nonempty open set $U \subseteq X$. If moreover X is Baire, then exactly one of these alternatives holds.*

Proof. Let $U \subseteq X$ be an open set such that $A =^* U$. If $U = \emptyset$, then A is meager. Otherwise, A is comeager in U by definition of $=^*$. □

Finally, we also mention the Kuratowski-Ulam Theorem, a quite powerful result which is the analogue in the context of Baire category theory of the well-known Fubini's theorem about double integrals.

Theorem 1.5.16 ([Kec95, Theorem 8.4])**.** *Let X, Y be second countable topological spaces, and suppose that $A \subseteq X \times Y$ has the* BP *(with respect to the product topology).*

(i) *The set of $x \in X$ for which $A_x = \{y \in Y \mid (x, y) \in A\} \in \mathsf{BP}(Y)$ is comeager in X. Similarly, the set of $y \in Y$ for which $A^y = \{x \in X \mid (x, y) \in A\} \in \mathsf{BP}(X)$ is comeager.*

(ii) *The set A is meager if and only if for comeagerly-many $x \in X$ the set A_x is meager in Y, if and only if for comeagerly-many $y \in Y$ the set A^y is meager in X.*

(iii) *The set A is comeager if and only if for comeagerly-many $x \in X$ the set A_x is comeager in Y, if and only if for comeagerly-many $y \in Y$ the set A^y is comeager in X.*

It follows that if the spaces X and Y are second-countable and Baire, then their product $X \times Y$ is Baire as well. The Kuratowski-Ulam Theorem can e.g. be used to show that any well-ordering of a non-empty perfect Polish space X does not have the BP as a subset of X^2.

2

Borel sets and functions, and their stratification

2.1 Borel sets and the Borel hierarchy

Definition 2.1.1. A subset A of a topological space $X = (X, \tau)$ is called **Borel** if it belongs to the σ-algebra on X generated by τ, i.e. if it is in the smallest collection of subsets of X containing all open sets and closed under complements and countable unions. The collection of all Borel subsets of X is denoted by $\mathbf{Bor}(X)$, possibly suppressing X from the notation if the space is clear from the context. We instead write $\mathbf{Bor}(X, \tau)$ or simply $\mathbf{Bor}(\tau)$ when we want to make explicit the topology τ we started with.

Fact 2.1.2 (see [Kec95, Theorem 10.1])**.** $\mathbf{Bor}(X)$ *is the smallest collection of subsets of X containing all open and closed sets, and closed under countable unions and countable intersections.*

Proof. Let $\mathcal{E}$ be the smallest collection of subsets of X containing all open and closed sets, and closed under countable unions and countable intersections. Notice that $\mathbf{Bor}(X)$ is also closed under countable intersections, as $\bigcap_{n \in \omega} A_n = X \setminus (\bigcup_{n \in \omega} (X \setminus A_n))$. Since $\mathbf{Bor}(X)$ contains all open and closed sets and is closed under countable unions by definition, we obtain $\mathcal{E} \subseteq \mathbf{Bor}(X)$.

Conversely, let $\mathcal{E}' = \{A \in \mathcal{E} \mid X \setminus A \in \mathcal{E}\}$: we claim that $\mathbf{Bor}(X) \subseteq \mathcal{E}' \subseteq \mathcal{E}$, so that $\mathbf{Bor}(X) = \mathcal{E}$. Indeed, $\mathcal{E}'$ contains all open sets, so it is enough to show that it is a σ-algebra because by definition $\mathbf{Bor}(X)$ is the *smallest* σ-algebra containing the open sets. Closure of $\mathcal{E}'$ under complements directly follows from its definition. To see that it is also closed under countable unions, let $A_n \in \mathcal{E}'$. On the one hand $\bigcup_{n \in \omega} A_n \in \mathcal{E}$ because $\mathcal{E}' \subseteq \mathcal{E}$ and the latter is closed under countable unions. On the other hand $X \setminus \bigcup_{n \in \omega} A_n = \bigcap_{n \in \omega} (X \setminus A_n) \in \mathcal{E}$ because $X \setminus A_n \in \mathcal{E}$ (by definition of $\mathcal{E}'$) and the latter is closed under countable intersections. $\square$

Remark 2.1.3. If X is a metrizable space, then $\mathbf{Bor}(X)$ can also be characterized as the smallest collection of subsets of X containing all open sets and closed under countable unions and countable intersections; indeed, under our additional assumption on X the latter class already contains all closed sets because they are actually G_δ. The same observation applies when X is zero-dimensional (but not necessarily metrizable).

The complexity of a Borel set can be measured by counting how many times we have to apply the operations of countable unions and countable intersections to generate it.

Definition 2.1.4. Let $X = (X, \tau)$ be a topological space. The (boldface) pointclasses $\boldsymbol{\Sigma}^0_\alpha(X)$ and $\boldsymbol{\Pi}^0_\alpha(X)$ are defined by (simultaneous) induction on the ordinals $\alpha \geq 1$ as follows:

$$\boldsymbol{\Sigma}^0_1(X) = \tau = \Big\{U \subseteq X \mid U \text{ is open}\Big\} \qquad \boldsymbol{\Pi}^0_1(X) = \Big\{C \subseteq X \mid C \text{ is closed}\Big\}$$

$$\boldsymbol{\Sigma}^0_\alpha(X) = \left\{\bigcup_{n\in\omega} A_n \mid A_n \in \bigcup_{1\leq\beta<\alpha} \boldsymbol{\Pi}^0_\beta\right\} \qquad \boldsymbol{\Pi}^0_\alpha(X) = \left\{\bigcap_{n\in\omega} A_n \mid A_n \in \bigcup_{1\leq\beta<\alpha} \boldsymbol{\Sigma}^0_\beta\right\}$$

We also set $\boldsymbol{\Delta}^0_\alpha(X) = \boldsymbol{\Sigma}^0_\alpha(X) \cap \boldsymbol{\Pi}^0_\alpha(X)$ for every $\alpha \geq 1$. As in Definition 2.1.1, we remove X from the notation if clear from the context, while we add a reference to τ between the parentheses when we need to specify the topology under consideration.

In particular, $\boldsymbol{\Sigma}^0_1(X)$ consists of all open sets, $\boldsymbol{\Pi}^0_1(X)$ consists of all closed sets, $\boldsymbol{\Delta}^0_1(X)$ consists of all clopen sets, $\boldsymbol{\Sigma}^0_2(X)$ is the collection of all F_σ sets, $\boldsymbol{\Pi}^0_2(X)$ is the collection of all G_δ sets, $\boldsymbol{\Sigma}^0_3(X)$ is the collection of all $G_{\delta\sigma}$ sets, and so on. Pointclasses of the form $\boldsymbol{\Sigma}^0_\alpha(X)$ (respectively: $\boldsymbol{\Pi}^0_\alpha(X)$, $\boldsymbol{\Delta}^0_\alpha(X)$) are called **additive** (respectively: **multiplicative**, **ambiguous**) **classes**.

Lemma 2.1.5. *Let X be a metrizable space and $\alpha \geq 1$.*

(i) $\boldsymbol{\Pi}^0_\alpha(X) = \{X \setminus A \mid A \in \boldsymbol{\Sigma}^0_\alpha(X)\}$.

(ii) $\boldsymbol{\Sigma}^0_\alpha(X), \boldsymbol{\Pi}^0_\alpha(X) \subseteq \boldsymbol{\Delta}^0_{\alpha+1}(X) \subseteq \boldsymbol{\Sigma}^0_{\alpha+1}(X), \boldsymbol{\Pi}^0_{\alpha+1}(X)$.

(iii) $\mathbf{Bor}(X) = \bigcup_{1\leq\alpha<\omega_1} \boldsymbol{\Sigma}^0_\alpha(X) = \bigcup_{1\leq\alpha<\omega_1} \boldsymbol{\Pi}^0_\alpha(X) = \bigcup_{1\leq\alpha<\omega_1} \boldsymbol{\Delta}^0_\alpha(X)$.

(iv) $\boldsymbol{\Sigma}^0_\alpha$ *is closed under continuous preimages (i.e. it is a* ***boldface*** *pointclass), namely: if Y is a metrizable space, $A \in \boldsymbol{\Sigma}^0_\alpha(Y)$, and $f \colon X \to Y$ is continuous, then $f^{-1}(A) \in \boldsymbol{\Sigma}^0_\alpha(X)$. The same is true for $\boldsymbol{\Pi}^0_\alpha$, $\boldsymbol{\Delta}^0_\alpha$, and* $\mathbf{Bor}$ *(hence they are all boldface pointclasses).*

(v) $\boldsymbol{\Sigma}^0_\alpha(X)$ *is closed under countable unions and finite intersections, $\boldsymbol{\Pi}^0_\alpha(X)$ is closed under countable intersections and finite unions, and $\boldsymbol{\Delta}^0_\alpha(X)$ is a Boolean algebra, i.e. it is closed under complements, finite unions, and finite intersections.*

(vi) *If $Y \subseteq X$, then*

$$\boldsymbol{\Sigma}^0_\alpha(Y) = \boldsymbol{\Sigma}^0_\alpha(X) \restriction Y,$$

where $\boldsymbol{\Sigma}^0_\alpha(X) \restriction Y = \{A \cap Y \mid A \in \boldsymbol{\Sigma}^0_\alpha(X)\}$. Similarly for $\boldsymbol{\Pi}^0_\alpha$ and $\mathbf{Bor}$.

Proof. All proofs are by induction on $\alpha \geq 1$ (Exercise!). For the first equality of part (iii) observe that $\mathbf{\Sigma}^0_\alpha(X) \subseteq \mathbf{Bor}(X)$ (this can be proved again by induction on $\alpha \geq 1$) and that $\mathbf{Bor}(X) \subseteq \bigcup_{1 \leq \alpha < \omega_1} \mathbf{\Sigma}^0_\alpha(X)$ because the latter is a σ-algebra containing all open sets of X. □

Remark 2.1.6. Part (vi) of Lemma 2.1.5 fails for $\mathbf{\Delta}^0_\alpha$. For example, let $E_{\frac{1}{3}} \subseteq \mathbb{R}$ be the Cantor ternary set. Since it is homeomorphic to the Cantor space 2^ω, it is zero-dimensional, and thus there is $A \in \mathbf{\Delta}^0_1(E_{\frac{1}{3}})$ such that $A \neq \emptyset, E_{\frac{1}{3}}$. However $A \notin \mathbf{\Delta}^0_1(\mathbb{R}) \restriction E_{\frac{1}{3}}$ because $\mathbb{R}$ is connected.

For another example, consider any $A \subseteq \mathbb{Q}$ such that both A and $\mathbb{Q} \setminus A$ are dense in $\mathbb{R}$. Then $A \in \mathbf{\Delta}^0_2(\mathbb{Q})$ but there is no $B \subseteq \mathbb{R}$ in $\mathbf{\Delta}^0_2(\mathbb{R})$ such that $B \cap \mathbb{Q} = A$. (This uses a category argument, see Section 1.5).

However, if both X and Y are Polish and $\alpha \geq 2$, then $\mathbf{\Delta}^0_\alpha(Y) = \mathbf{\Delta}^0_\alpha(X) \restriction Y$. This is easy if $\alpha \geq 3$, as $Y \in \mathbf{\Pi}^0_2(X)$ (Exercise!). For $\alpha = 2$, this follows from [Kec95, Theorem 22.27]. Notice that the first counterexample considered in this remark shows that it is not possible to include the case $\alpha = 1$ even when X and Y are assumed to be Polish.

Thus the boldface pointclasses $\mathbf{\Sigma}^0_\alpha(X)$, $\mathbf{\Pi}^0_\alpha(X)$, $\mathbf{\Delta}^0_\alpha(X)$ constitute a hierarchy of complexity for the Borel subsets of X of length at most ω_1 — this is the so-called **Borel hierarchy**. Given $A \in \mathbf{Bor}(X)$ we call **Borel rank** of A the smallest $1 \leq \alpha < \omega_1$ such that $A \in \mathbf{\Sigma}^0_\alpha(X) \cup \mathbf{\Pi}^0_\alpha(X)$. The Borel rank of a Borel set measures its complexity. The smallest (with respect to inclusion) of the classes $\mathbf{\Sigma}^0_\alpha(X)$, $\mathbf{\Pi}^0_\alpha(X)$, and $\mathbf{\Delta}^0_\alpha(X)$ which contains $A \subseteq X$ is called the **Borel class** of A.

Example 2.1.7. (1) Every countable subset A of a Polish space X is $\mathbf{\Sigma}^0_2(X)$, and its complement is $\mathbf{\Pi}^0_2(X)$. Indeed, $A = \bigcup_{x \in A}\{x\}$. In particular, $\mathbb{Q} \in \mathbf{\Sigma}^0_2(\mathbb{R})$ and $\mathrm{Irr} \in \mathbf{\Pi}^0_2(\mathbb{R})$.

(2) Semi-open intervals $[a; b)$ and $(a; b]$ in $\mathbb{R}$ are $\mathbf{\Delta}^0_2(\mathbb{R})$, but they are neither $\mathbf{\Sigma}^0_1(\mathbb{R})$ nor $\mathbf{\Pi}^0_1(\mathbb{R})$ (Exercise!).

(3) Let $C_0 = c_0 \cap [0; 1]^\omega = \{(x_n)_{n\in\omega} \in [0; 1]^\omega \mid x_n \to 0\}$. Then $C_0 \in \mathbf{\Pi}^0_3([0; 1]^\omega)$. In fact,

$$\begin{aligned}(x_n)_{n\in\omega} \in C_0 &\iff \forall \varepsilon \in \mathbb{R}^+ \, \exists n \in \omega \, \forall m \geq n \, (x_m \leq \varepsilon) \\ &\iff \forall \varepsilon \in \mathbb{Q}^+ \, \exists n \in \omega \, \forall m \geq n \, (x_m \leq \varepsilon),\end{aligned}$$

therefore

$$C_0 = \bigcap_{\varepsilon \in \mathbb{Q}^+} \bigcup_{n \in \omega} \bigcap_{m \geq n} A_{\varepsilon,m},$$

where $A_{\varepsilon,m} = \{(x_n)_{n\in\omega} \mid x_m \leq \varepsilon\}$. Since $A_{\varepsilon,m}$ is the preimage of the closed interval $[0; \varepsilon]$ under the projection on the m-th coordinate, which is a continuous map, we get that $A_{\varepsilon,m} \in \mathbf{\Pi}^0_1([0; 1]^\omega)$, hence $C_0 \in \mathbf{\Pi}^0_3([0; 1]^\omega)$ by Lemma 2.1.5.

(4) Given $f \in C([0;1])$, let $D_f = \{x \in [0;1] \mid f' \text{ exists }\}$ (at endpoints we take one-side derivatives). Then $D_f \in \mathbf{\Pi}^0_3([0;1])$ (EXERCISE!).

Hint. Observe that $x \in D_f$ if and only if for all $\varepsilon \in \mathbb{Q}^+$ there is $\delta \in \mathbb{Q}^+$ such that for all $p, q \in [0;1] \cap \mathbb{Q}$

$$|p-x|, |q-x| < \delta \Rightarrow \left|\frac{f(p)-f(x)}{p-x} - \frac{f(q)-f(x)}{q-x}\right| \leq \varepsilon.$$

Notice that we could replace $\leq$ with $<$ at the end of the previous formula: however, we would then obtain only $D_f \in \mathbf{\Pi}^0_4([0;1])$ rather than $D_f \in \mathbf{\Pi}^0_3([0;1])$.

Remark 2.1.8 (The Tarski-Kuratowski algorithm)**.** In the previous examples we implicitly used the so-called **Tarski-Kuratowski algorithm**, which allows us to compute the Borel class of a set by looking at the logical form of one of its (optimal) definitions. In particular, we take advantage of the following well-known correspondence between logical symbols and set-theoretical operations:

- $\neg$ (negation) $\rightsquigarrow$ complementation $X \setminus \cdot$
- $\wedge$ (conjunction) $\rightsquigarrow$ intersection $\cap$
- $\vee$ (disjunction) $\rightsquigarrow$ union $\cup$
- $\forall n$ (universal quantification over a *countable* set) $\rightsquigarrow$ countable intersection $\bigcap_n$
- $\exists n$ (existential quantification over a *countable* set) $\rightsquigarrow$ countable union $\bigcup_n$

The implication $\Rightarrow$ and bi-implication $\iff$ are treated exploiting the fact that $\varphi \Rightarrow \psi$ is equivalent to $\neg\varphi \vee \psi$ and $\varphi \iff \psi$ is equivalent to $(\varphi \Rightarrow \psi) \wedge (\psi \Rightarrow \varphi)$.

For example, consider the set $A \subseteq 2^\omega$ defined by

$$x \in A \iff \varphi(x)$$

where $\varphi(x)$ is the formula $\exists n \, \forall m \geq n \, \neg(x(m) = 1)$ (i.e. A is the set of binary sequences which are eventually equal to 0). Applying the Tarski-Kuratowski algorithm to $\varphi(x)$ yields the fact that

$$A = \bigcup_{n \in \omega} \bigcap_{m \geq n} (2^\omega \setminus B_m)$$

where $B_m = \{x \in 2^\omega \mid x(m) = 1\}$. Since the latter is a clopen set (for each $m \in \omega$), we get that A is a $\mathbf{\Sigma}^0_2$ set. Similarly, one can check that

$$B = \{x \in [0;1]^\omega \mid x \text{ is eventually constant}\}$$

is a $\mathbf{\Sigma}^0_2$ subset of $[0;1]^\omega$.

When X is countable, its Borel hierarchy collapses to level 2 because all of its subsets are $\mathbf{\Sigma}^0_2$ by Example 2.1.7(1).

Fact 2.1.9. *Let X be a metrizable space. If X is countable, then* $\mathbf{Bor}(X) = \mathbf{\Delta}^0_2(X) = \mathscr{P}(X)$.

In contrast, if X is an uncountable Polish space, then its Borel hierarchy never collapses (before ω_1). To prove this, we use a diagonalization argument and the existence of (X-)universal sets for the classes $\mathbf{\Sigma}^0_\alpha(X)$ and $\mathbf{\Pi}^0_\alpha(X)$.

To state the following concepts and results in their full generality, it is convenient to introduce the following notation and terminology. A **boldface pointclass** $\mathbf{\Gamma}$ is an operator assigning to each topological space X a collection $\mathbf{\Gamma}(X) \subseteq \mathscr{P}(X)$ which is closed under continuous preimages in the following strong sense: if $A \in \mathbf{\Gamma}(Y)$ and $f \colon X \to Y$ is continuous, then $f^{-1}(A) \in \mathbf{\Gamma}(X)$. We denote by $\check{\mathbf{\Gamma}}$ the **dual** of $\mathbf{\Gamma}$, where $\check{\mathbf{\Gamma}}(X) = \{X \setminus A \mid A \in \mathbf{\Gamma}(X)\}$, and by $\mathbf{\Delta_\Gamma}$ the **ambiguous class** associated to $\mathbf{\Gamma}$, where $\mathbf{\Delta_\Gamma}(X) = \mathbf{\Gamma}(X) \cap \check{\mathbf{\Gamma}}(X)$. (Notice that the dual of $\check{\mathbf{\Gamma}}$ is $\mathbf{\Gamma}$ itself, and that $\mathbf{\Delta_\Gamma} = \mathbf{\Delta}_{\check{\mathbf{\Gamma}}}$.) When $\mathbf{\Gamma}(\omega^\omega) = \check{\mathbf{\Gamma}}(\omega^\omega)$ (so that $\mathbf{\Gamma}(\omega^\omega) = \mathbf{\Delta_\Gamma}(\omega^\omega)$) we say that $\mathbf{\Gamma}$ is **selfdual**, otherwise we say that $\mathbf{\Gamma}$ is **nonselfdual**.

Example 2.1.10. By Lemma 2.1.5, all of $\mathbf{\Sigma}^0_\alpha$, $\mathbf{\Pi}^0_\alpha$, $\mathbf{\Delta}^0_\alpha$, and $\mathbf{Bor}$ are boldface pointclasses. The dual of $\mathbf{\Sigma}^0_\alpha$ is $\mathbf{\Pi}^0_\alpha$, and viceversa. The pointclasses $\mathbf{\Delta}^0_\alpha$ and $\mathbf{Bor}$ are selfdual.

Definition 2.1.11. A set $\mathcal{U} \subseteq Y \times X$ is **Y-universal for $\mathbf{\Gamma}(X)$** if $\mathcal{U} \in \mathbf{\Gamma}(Y \times X)$ and $\mathbf{\Gamma}(X) = \{\mathcal{U}_y \mid y \in Y\}$, where $\mathcal{U}_y = \{x \in X \mid (y,x) \in \mathcal{U}\}$ is the **vertical section** of $\mathcal{U}$ at y.

Such a universal set $\mathcal{U}$ provides a **parametrization** (or **coding**) of the sets in $\mathbf{\Gamma}(X)$, where we view y as a parameter (or code) of $\mathcal{U}_y$. (Notice however that the code of a set in $\mathbf{\Gamma}(X)$ is in general not unique.)

Remark 2.1.12. If $\mathcal{U}$ is Y-universal for $\mathbf{\Gamma}(X)$, then its complement $\mathcal{U}^c = (Y \times X) \setminus \mathcal{U}$ is Y universal for $\check{\mathbf{\Gamma}}(X)$.

Remark 2.1.13. When $\mathbf{\Gamma}$ is a boldface pointclass, to show that $\mathcal{U} \subseteq Y \times X$ is Y-universal for $\mathbf{\Gamma}(X)$ it is enough to show that $\mathcal{U} \in \mathbf{\Gamma}(Y \times X)$ and that for every $A \in \mathbf{\Gamma}(X)$, $A = \mathcal{U}_y$ for some $y \in Y$. Indeed, the remaining condition "$\mathcal{U}_y \in \mathbf{\Gamma}(X)$ for all $y \in Y$" already follows from the fact that each $\mathcal{U}_y$ is the continuous preimage of $\mathcal{U}$ via the map $x \mapsto (y,x)$.

Theorem 2.1.14 ([Kec95, Theorem 22.3])**.** *Let X be a separable metrizable space. Then for each $1 \leq \alpha < \omega_1$ there is a 2^ω-universal set for $\mathbf{\Sigma}^0_\alpha(X)$, and similarly for $\mathbf{\Pi}^0_\alpha(X)$. Moreover, we can replace 2^ω with any uncountable Polish space.*

Proof. By (simultaneous) induction on $\alpha \geq 1$.

Let $\{V_n \mid n \in \omega\}$ be a countable basis for the topology of X. Let $\mathcal{U} \subseteq 2^\omega \times X$ be defined by

$$(y,x) \in \mathcal{U} \iff x \in \bigcup\{V_n \mid y(n) = 1\}.$$

Clearly, $\mathcal{U} \in \mathbf{\Sigma}^0_1(2^\omega \times X)$: if $(y,x) \in \mathcal{U}$ and $n \in \omega$ is such that $x \in V_n$ and $y(n) = 1$, then $(y,x) \in \boldsymbol{N}_{y \restriction (n+1)} \times V_n \subseteq \mathcal{U}$. Given $U \subseteq X$ open, let[1] $y \in 2^\omega$ be

[1] Actually, one can let y be the characteristic funtction of *any* set $A \subseteq \omega$ such that $U = \bigcup_{n \in A} V_n$. Our definition consider the largest among such A's.

such that $y(n) = 1 \iff V_n \subseteq U$, so that $U = \bigcup\{V_n \mid y(n) = 1\}$; then y is a code for U, i.e. $\mathcal{U}_y = U$. By Remark 2.1.13, this shows that $\mathcal{U}$ is 2^ω-universal for $\mathbf{\Sigma}^0_1(X)$, and hence $\mathcal{U}^c$ is 2^ω-universal for $\mathbf{\Pi}^0_1(X)$ by Remark 2.1.12.

Let now $\alpha > 1$, and fix a sequence $(\eta_n)_{n\in\omega}$ such that $1 \leq \eta_n < \alpha$, $\eta_n \leq \eta_{n+1}$, and $\sup\{\eta_n + 1 \mid n \in \omega\} = \alpha$. (In particular, when $\alpha = \beta + 1$ we can set $\eta_n = \beta$ for every $n \in \omega$.) Let $\mathcal{U}_n$ be 2^ω-universal for $\mathbf{\Pi}^0_{\eta_n}(X)$. Finally, let $\rho\colon 2^\omega \to (2^\omega)^\omega$ be a homeomorphism, and for each $k \in \omega$ let $\pi_k\colon (2^\omega)^\omega \to 2^\omega$ be the projection on the k-th coordinate, so that $\pi_k((y_n)_{n\in\omega}) = y_k$ for each $(y_n)_{n\in\omega}$. Notice that each $f_k = \pi_k \circ \rho$ is continuous, and that if $y = \rho^{-1}((y_k)_{k\in\omega})$ then $f_k(y) = y_k$. Define $\mathcal{U} \subseteq 2^\omega \times X$ by setting

$$(y, x) \in \mathcal{U} \iff \exists k \in \omega\, [(f_k(y), x) \in \mathcal{U}_k].$$

We claim that $\mathcal{U}$ is 2^ω-universal for $\mathbf{\Sigma}^0_\alpha(X)$ (so that $\mathcal{U}^c$ is 2^ω-universal for $\mathbf{\Pi}^0_\alpha(X)$ by Remark 2.1.12). First observe that each $A_k = \{(y, x) \in 2^\omega \times X \mid (f_k(y), x) \in \mathcal{U}_k\}$ is in $\mathbf{\Pi}^0_{\eta_k}(2^\omega \times X)$ because it is the preimage of $\mathcal{U}_k$ under the continuous function $(y, x) \mapsto (f_k(y), x)$. It follows that $\mathcal{U}$ is in $\mathbf{\Sigma}^0_\alpha(2^\omega \times X)$. Let $A = \bigcup_{n\in\omega} A_n$ with each $A_n \in \bigcup_{1\leq\beta<\alpha} \mathbf{\Pi}^0_\beta(X)$. Without loss of generality, we may assume that $A_n \in \mathbf{\Pi}^0_{\eta_n}$. (Indeed, if this is not the case fix an injection $\iota\colon \omega \to \omega$ such that $A_n \in \mathbf{\Pi}^0_{\eta_{\iota(n)}}(X)$ for every $n \in \omega$, and set $A'_{\iota(n)} = A_n$ for every $n \in \omega$ and $A'_k = \emptyset$ for all $k \in \omega$ which are not in the range of ι: clearly $A = \bigcup_{k\in\omega} A'_k$ and $A'_k \in \mathbf{\Pi}^0_{\eta_k}$ for all $k \in \omega$.) For every $k \in \omega$, let $y_k \in 2^\omega$ be such that $(\mathcal{U}_k)_{y_k} = A_k$. Finally, let $y = \rho^{-1}((y_k)_{k\in\omega})$. Then $\mathcal{U}_y = A$. Indeed,

$$\begin{aligned} x \in A &\iff \exists k \in \omega\, [x \in A_k] \\ &\iff \exists k \in \omega\, [(y_k, x) \in \mathcal{U}_k] \\ &\iff \exists k \in \omega\, [(f_k(y), x) \in \mathcal{U}_k] \\ &\iff (y, x) \in \mathcal{U}. \end{aligned}$$

Finally, let Y be an uncountable Polish space, and let $f\colon 2^\omega \to Y$ be an embedding. By Remark 2.1.12, it is enough to show that for every $\alpha \geq 1$ there is a Y-universal set $\mathcal{U}$ for $\mathbf{\Pi}^0_\alpha(X)$. Let $\mathcal{U}'$ be 2^ω-universal for $\mathbf{\Pi}^0_\alpha(X)$, and let $\mathcal{U} \subseteq Y \times X$ be defined by

$$(y, x) \in \mathcal{U} \iff y \in \operatorname{rng}(f) \wedge (f^{-1}(y), x) \in \mathcal{U}'.$$

Then

$$\begin{aligned} \mathbf{\Pi}^0_\alpha(X) = \{\mathcal{U}'_z \mid z \in 2^\omega\} &= \{\mathcal{U}_{f(z)} \mid z \in 2^\omega\} \\ &= \{\mathcal{U}_y \mid y \in \operatorname{rng}(f)\} = \{\mathcal{U}_y \mid y \in Y\}, \end{aligned}$$

where the last equality holds because $\mathcal{U}_y = \emptyset$ when $y \notin \operatorname{rng}(f)$, and $\emptyset$ belongs to any $\mathbf{\Pi}^0_\alpha(X)$. Moreover, $\mathcal{U}$ is the image of $\mathcal{U}'$ under the homeomorphism between $2^\omega \times X$ and $\operatorname{rng}(f) \times X$ given by $(y, x) \mapsto (f(y), x)$, hence it is in $\mathbf{\Pi}^0_\alpha(\operatorname{rng}(f) \times X)$. Since $\operatorname{rng}(f) \times X$ is closed in $Y \times X$, it follows that $\mathcal{U}$ is in $\mathbf{\Pi}^0_\alpha(Y \times X)$ as well, hence it is as required. $\square$

Lemma 2.1.15 ([Kec95, Exercise 22.7])**.** *Let X be any metrizable space, and let $\boldsymbol{\Gamma}$ be a boldface pointclass such that $\boldsymbol{\Gamma}(X) = \check{\boldsymbol{\Gamma}}(X)$ (equivalently: $\boldsymbol{\Gamma}(X)$ is closed under complements). Then there is no X-universal set for $\boldsymbol{\Gamma}(X)$.*

Proof. Suppose towards a contradiction that $\mathcal{U} \subseteq X \times X$ is X-universal for $\boldsymbol{\Gamma}(X)$. Consider the diagonal map $f \colon X \to X \times X$ defined by $x \mapsto (x, x)$, and set

$$D = \{x \in X \mid f(x) \notin \mathcal{U}\} = f^{-1}((X \times X) \setminus \mathcal{U}) = X \setminus f^{-1}(\mathcal{U}).$$

By the hypotheses on $\boldsymbol{\Gamma}$ and the fact that $\mathcal{U} \in \boldsymbol{\Gamma}(X \times X)$, we have $D \in \boldsymbol{\Gamma}(X)$. Let y_0 be such that $D = \mathcal{U}_{y_0}$. Then we reach a contradiction by observing that

$$(y_0, y_0) \in \mathcal{U} \iff y_0 \in D \iff (y_0, y_0) \notin \mathcal{U},$$

where the first equivalence follows from the fact that y_0 is a code for D, while the second one follows from the definition of D. $\square$

Corollary 2.1.16. *Let X be a separable metrizable space. Then there is no X-universal set for $\boldsymbol{\Delta}^0_\alpha(X)$, $1 \leq \alpha < \omega_1$, and for $\mathbf{Bor}(X)$. If moreover X is Polish and uncountable, then there is no 2^ω-universal set for $\boldsymbol{\Delta}^0_\alpha(X)$, $\alpha \geq 2$, and for $\mathbf{Bor}(X)$.*

Proof. By Theorem 2.1.14 and Lemma 2.1.15. The additional part concerning uncountable Polish spaces follows from the argument used at the end of the proof of Theorem 2.1.14. $\square$

Theorem 2.1.17 ([Kec95, Theorem 22.4])**.** *Let X be an uncountable Polish space. Then $\boldsymbol{\Sigma}^0_\alpha(X) \neq \boldsymbol{\Pi}^0_\alpha(X)$ for each $1 \leq \alpha < \omega_1$.*

Proof. Assume towards a contradiction that for some α as in the statement it holds $\boldsymbol{\Sigma}^0_\alpha(X) = \boldsymbol{\Pi}^0_\alpha(X)$. Then $\boldsymbol{\Sigma}^0_\alpha(X)$ would be closed under complements: since there is an X-universal set for $\boldsymbol{\Sigma}^0_\alpha(X)$ by Theorem 2.1.14, this contradicts Lemma 2.1.15. $\square$

Corollary 2.1.18 ([Kec95, Theorem 22.4, Exercise 22.5, and Exercise 22.8])**.** *Let X be an uncountable Polish space, and $1 \leq \alpha < \omega_1$.*

(i) $\boldsymbol{\Delta}^0_\alpha(X) \subsetneq \boldsymbol{\Sigma}^0_\alpha(X) \subsetneq \boldsymbol{\Delta}^0_{\alpha+1}(X)$*, and similarly for* $\boldsymbol{\Pi}^0_\alpha(X)$.

(ii) *If α is limit, then*

$$\bigcup_{1 \leq \beta < \alpha} \boldsymbol{\Sigma}^0_\beta(X) \left(= \bigcup_{1 \leq \beta < \alpha} \boldsymbol{\Pi}^0_\beta(X) = \bigcup_{1 \leq \beta < \alpha} \boldsymbol{\Delta}^0_\beta(X) \right) \subsetneq \boldsymbol{\Delta}^0_\alpha(X).$$

(iii) $\boldsymbol{\Sigma}^0_\alpha(X)$ *is not closed under either complements or countable intersections. Also $\boldsymbol{\Pi}^0_\alpha(X)$ is not closed under either complements or countable unions and, for $\alpha \geq 2$ or $\alpha = 1$ and X zero-dimensional, $\boldsymbol{\Delta}^0_\alpha(X)$ is not closed under either countable unions or intersections.*

Proof. (i) If one of the inclusion were not strict, then $\boldsymbol{\Sigma}^0_\alpha(X)$ would be closed under complements, hence $\boldsymbol{\Sigma}^0_\alpha(X) = \boldsymbol{\Pi}^0_\alpha(X)$, contradicting Theorem 2.1.17.

(ii) Let $(C_n)_{n\in\omega}$ be a sequence of pairwise disjoint uncountable closed subsets of X. (For example, let $C_n = f(\boldsymbol{N}_{0^{(n)}\smallfrown 1})$, where f is an embedding of 2^ω into X and $0^{(n)}$ is the binary sequence of length n constantly equal to 0.) Fix a sequence $(\alpha_n)_{n\in\omega}$ of nonzero ordinals cofinal in α, and for each $n \in \omega$ pick $A_n \in \boldsymbol{\Pi}^0_{\alpha_n}(C_n) \setminus \boldsymbol{\Sigma}^0_{\alpha_n}(C_n)$. Finally, let $A = \bigcup_{n\in\omega} A_n$. Then A is in $\boldsymbol{\Sigma}^0_\alpha(X)$, and the same is true for

$$X \setminus A = \bigcup_{n\in\omega}(C_n \setminus A_n) \cup \left(X \setminus \bigcup_{n\in\omega} C_n\right)$$

because $\alpha > 2$, therefore $A \in \boldsymbol{\Delta}^0_\alpha(X)$. If A where in $\boldsymbol{\Pi}^0_\beta(X)$ for some $1 \le \beta < \alpha$, then $A_n = A \cap C_n \in \boldsymbol{\Pi}^0_\beta(X) \restriction C_n = \boldsymbol{\Pi}^0_\beta(C_n)$ for every $n \in \omega$. But when $n \in \omega$ is such that $\beta < \alpha_n$, this contradicts the choice of A_n.

(iii) If $\boldsymbol{\Sigma}^0_\alpha(X)$ were closed under complements, then $\boldsymbol{\Sigma}^0_\alpha(X) = \boldsymbol{\Pi}^0_\alpha(X)$, contradicting Theorem 2.1.17. If $\boldsymbol{\Sigma}^0_\alpha(X)$ were closed under countable intersections, then $\boldsymbol{\Pi}^0_{\alpha+1}(X) = \boldsymbol{\Sigma}^0_\alpha(X)$. But then $\boldsymbol{\Pi}^0_\alpha(X) \subseteq \boldsymbol{\Pi}^0_{\alpha+1}(X) = \boldsymbol{\Sigma}^0_\alpha(X)$, and hence $\boldsymbol{\Sigma}^0_{\alpha+1}(X) = \boldsymbol{\Sigma}^0_\alpha(X) = \boldsymbol{\Pi}^0_{\alpha+1}(X)$, contradicting Theorem 2.1.17 again.

The case of $\boldsymbol{\Pi}^0_\alpha(X)$ follows from the case of $\boldsymbol{\Sigma}^0_\alpha(X)$ and the fact that the corresponding pointclasses are one the dual of the other.

Finally, if $\boldsymbol{\Delta}^0_\alpha(X)$ were closed under countable unions, then $\boldsymbol{\Sigma}^0_\alpha(X) = \boldsymbol{\Delta}^0_\alpha(X)$, hence also $\boldsymbol{\Pi}^0_\alpha(X) = \boldsymbol{\Delta}^0_\alpha(X)$ because the latter is closed under complements; this contradicts Theorem 2.1.17 again. The case of countable intersections is similar but with the role of $\boldsymbol{\Sigma}^0_\alpha(X)$ and $\boldsymbol{\Pi}^0_\alpha(X)$ interchanged. □

Theorem 2.1.14 also allows us to compute, for X any infinite Polish space, the cardinality of the boldface pointclasses $\boldsymbol{\Sigma}^0_\alpha(X)$, $\boldsymbol{\Pi}^0_\alpha(X)$, $\boldsymbol{\Delta}^0_\alpha(X)$, and $\mathbf{Bor}(X)$, and to show that if X is uncountable then there is a non-Borel set $A \subseteq \mathbf{Bor}(X)$ (compare this with Fact 2.1.9).

Proposition 2.1.19. *Let X be an infinite Polish space. For every $1 \le \alpha < \omega_1$*

$$|\boldsymbol{\Sigma}^0_\alpha(X)| = |\boldsymbol{\Pi}^0_\alpha(X)| = |\mathbf{Bor}(X)| = 2^{\aleph_0}.$$

(In particular, if X is uncountable then there is a non-Borel subset of X.)

Moreover, $2 \le |\boldsymbol{\Delta}^0_\alpha(X)| \le 2^{\aleph_0}$ for all $1 \le \alpha < \omega_1$, and $|\boldsymbol{\Delta}^0_\alpha(X)| = 2^{\aleph_0}$ if $\alpha \ge 2$.

Proof. The equality $|\boldsymbol{\Sigma}^0_\alpha(X)| = |\boldsymbol{\Pi}^0_\alpha(X)|$ is cleary witnessed by the bijective map $A \mapsto X \setminus A$, so we will only consider the classes $\boldsymbol{\Sigma}^0_\alpha(X)$ and $\mathbf{Bor}(X)$.

For a lower bound, by Lemma 2.1.5 it is enough to show that $|\boldsymbol{\Sigma}^0_1(X)| \ge 2^{\aleph_0}$. We distinguish two cases. If X is uncountable, then $|X| = 2^{\aleph_0}$ by Corollary 1.4.9, hence the sets of the form $X \setminus \{x\}$ for any $x \in X$ form a family of size $2^{\aleph_0}$ of pairwise distinct open sets. If instead X is countable, then $X \setminus X'$ is an infinite open set carrying the discrete topology. (Indeed, if $X \setminus X'$ is finite then X' would be already perfect and nonempty, so that X would be uncountable.) It follows that $\mathscr{P}(X \setminus X')$ is again a family of size $2^{\aleph_0}$ of pairwise distinct open sets.

For the upper bound, we first consider the additive classes $\mathbf{\Sigma}^0_\alpha(X)$. Let $\mathcal{U}$ be 2^ω-universal for $\mathbf{\Sigma}^0_\alpha(X)$: then the map $y \mapsto \mathcal{U}_y$ is a surjection of 2^ω onto $\mathbf{\Sigma}^0_\alpha(X)$, hence $|\mathbf{\Sigma}^0_\alpha(X)| \leq 2^{\aleph_0}$. As for $\mathbf{Bor}(X)$, by Lemma 2.1.5 again we have

$$|\mathbf{Bor}(X)| = \left| \bigcup_{1 \leq \alpha < \omega_1} \mathbf{\Sigma}^0_\alpha(X) \right| \leq 2^{\aleph_0} \cdot \aleph_1 = 2^{\aleph_0}.$$

The existence of a non-Borel set when X is uncountable follows from a cardinality argument, as

$$|\mathbf{Bor}(X)| = 2^{\aleph_0} < 2^{(2^{\aleph_0})} = |\mathscr{P}(X)|.$$

Finally, the cardinality (in)equalities concerning the classes $\mathbf{\Delta}^0_\alpha(X)$ follow from the fact that $\{\emptyset, X\} \subseteq \mathbf{\Delta}^0_1(X) \subseteq \mathbf{Bor}(X)$, and that $\mathbf{\Sigma}^0_1(X) \subseteq \mathbf{\Delta}^0_\alpha(X)$ when $\alpha \geq 2$. □

Corollary 2.1.20. *If τ is a Polish topology on a set X, then either $|\tau| = 2^n$ for some $n \in \omega$, or else $|\tau| = 2^{\aleph_0}$. In particular, there is no countably infinite Polish topology.*

The same applies to the classes $\mathbf{\Sigma}^0_\alpha(X,\tau)$, $\mathbf{\Pi}^0_\alpha(X,\tau)$, $\mathbf{\Delta}^0_\alpha(X,\tau)$, and $\mathbf{Bor}(X,\tau)$.

Proof. If X is finite of cardinality $n \in \omega$, then τ is discrete because so is any Hausdorff topology on a finite space. It follows that $\tau = \mathscr{P}(X)$, therefore $|\tau| = 2^n$. If instead X is infinite, then $|\tau| = 2^{\aleph_0}$ by Proposition 2.1.19. □

Remark 2.1.21. More generally, using the same arguments one can show that there is no countably infinite completely metrizable topology.

Definition 2.1.22 ([Kec95, Definition 22.9])**.** Let $\mathbf{\Gamma}$ be a boldface pointclass, and X be a Polish space. A set $A \subseteq X$ is called **$\mathbf{\Gamma}$-hard** if for all $B \in \mathbf{\Gamma}(\omega^\omega)$ there is a continuous $f \colon \omega^\omega \to X$ such that $f^{-1}(A) = B$. The set A is **$\mathbf{\Gamma}$-complete** if it is $\mathbf{\Gamma}$-hard and $A \in \mathbf{\Gamma}(X)$.

The next lemma summarize the basic properties about $\mathbf{\Gamma}$-hard and $\mathbf{\Gamma}$-complete sets. The proof is immediate. (Exercise!)

Lemma 2.1.23. *Let $\mathbf{\Gamma}$ be a boldface pointclass, and X be any Polish space.*

(i) *If $\mathbf{\Gamma}$ is nonselfdual, no $\mathbf{\Gamma}$-hard set $A \subseteq X$ is in $\check{\mathbf{\Gamma}}(X)$.*

(ii) *A set $A \subseteq X$ is $\mathbf{\Gamma}$-hard (respectively, $\mathbf{\Gamma}$-complete) if and only if $X \setminus A$ is $\check{\mathbf{\Gamma}}$-hard (respectively, $\check{\mathbf{\Gamma}}$-complete).*

(iii) *If $\mathcal{U}$ is Y-universal for $\mathbf{\Gamma}(\omega^\omega)$, then $\mathcal{U}$ is $\mathbf{\Gamma}$-complete. In particular, there are $\mathbf{\Sigma}^0_\alpha$-complete and $\mathbf{\Pi}^0_\alpha$-complete sets.*

(iv) *If A is $\mathbf{\Gamma}$-hard and $A = f^{-1}(A')$ for some continuous $f \colon X \to X'$ and $A' \subseteq X'$ with X' Polish, then A' is $\mathbf{\Gamma}$-hard as well.*

Remark 2.1.24. Since universality implies completeness, one may wonder whether there are 2^ω-universal sets for $\mathbf{\Gamma}(X)$ whenever there is a $\mathbf{\Gamma}$-complete subset of X. This is not true in general: every nontrivial clopen subset of ω^ω is $\mathbf{\Delta}^0_1$-complete, but there is no 2^ω-universal set for $\mathbf{\Delta}^0_1(\omega^\omega)$ by Corollary 2.1.16. However, it can be shown that for every $\mathbf{\Gamma} \neq \mathbf{\Delta}^0_1$ consisting of Borel sets, there is a 2^ω-universal set for $\mathbf{\Gamma}(\omega^\omega)$ if and only if there is a $\mathbf{\Gamma}$-complete subset of ω^ω. In particular, this is the case for all *nonselfdual* pointclasses $\mathbf{\Delta}^0_1 \subseteq \mathbf{\Gamma} \subseteq \mathbf{Bor}$.

The previous lemma gives us some methods to show that a set $A \in \mathbf{\Gamma}(X)$ is not in $\check{\mathbf{\Gamma}}(X)$ (so that, if successful, we have determined the Borel class and the Borel rank of A).

- Directly show that A is $\mathbf{\Gamma}$-hard by constructing, for an arbitrary $B \in \mathbf{\Gamma}(\omega^\omega)$, a continuous function $f \colon \omega^\omega \to X$ such that $B = f^{-1}(A)$.
- If we already know that some $C \subseteq Y$ is $\mathbf{\Gamma}$-hard, then it is enough to show that there is a continuous $f \colon Y \to X$ such that $C = f^{-1}(A)$.

Exercise 2.1.25. Let X be a perfect[2] Polish space. Show that every countable (infinite) dense subset D of X is $\mathbf{\Sigma}^0_2$-complete, hence in particular it is an F_σ set which is not G_δ. Conclude that the same is true if D is **somewhere dense**, i.e. if there is some open set $U \subseteq X$ such that D is dense in U.

Solution. Let d be a compatible complete metric on X. Fix an enumeration $(x_n)_{n\in\omega}$ of D without repetitions, and let $A = \bigcup_{n\in\omega} F_n \subseteq \omega^\omega$ be any set in $\mathbf{\Sigma}^0_2(\omega^\omega)$. Let $T_n = T_{F_n}$ be the tree of F_n (see Proposition 1.3.3), and for each $s \in \omega^{<\omega}$ let $n(s)$ be the least n such that $s \in T_n$ if $s \in \bigcup_{n\in\omega} T_n$, and $n(s) = \infty$ otherwise (notice that $s \subseteq t$ implies $n(s) \leq n(t)$). We will define an ω-scheme $\{B_s \mid s \in \omega^{<\omega}\}$ on X and an auxiliary function $\sigma \colon \omega^{<\omega} \to \omega$ such that for every $s \subsetneq t \in \omega^{<\omega}$:

- $\sigma(s) \leq \sigma(t)$, and $\sigma(s) = \sigma(t) \iff n(s) = n(t) \wedge n(s) < \infty$;
- $B_s = B_d(x_{\sigma(s)}, \varepsilon_s)$ for some $\varepsilon_s > 0$, and $x_i \notin B_s$ for all $i < \sigma(s)$;
- $\mathrm{cl}(B_t) \subseteq B_s$.

Then the function induced by such a scheme is a total continuous map $f \colon \omega^\omega \to X$ such that $f^{-1}(D) = A$. Indeed, if $y \in A$, then there is a least $\bar{n} \in \omega$ such that $y \in F_{\bar{n}}$, which implies that eventually $n(y \restriction k) = \bar{n} < \infty$. By the first condition on the scheme, this means that there is $\bar{k} \in \omega$ such that $\sigma(y \restriction k) = \bar{k}$ for all large enough k's: since $B_{y\restriction k} = B_d(x_{\sigma(y\restriction k)}, \varepsilon_{y\restriction k})$, this implies that $f(y) = x_{\bar{k}} \in D$. If instead $y \notin A$, then $n(y \restriction k) \to \infty$ (including the degenerate case where $n(y \restriction k) = \infty$ for some $k \in \omega$): then $\sigma(y \restriction k) \to \infty$ and the second condition on the scheme ensures that in this case $f(y) \neq x_i$ for every $i \in \omega$, so that $f(y) \notin D$.

[2]The hypothesis that X be perfect is necessary. It is possible to construct uncountable Polish spaces whose set of isolated points D is dense in the space: thus in that case D would be just an open set rather than a $\mathbf{\Sigma}^0_2$-complete set.

The construction of the desired scheme and the definition of the auxiliary function σ is by recursion on $\mathrm{lh}(s)$. Set $\sigma(\emptyset) = 0$ and $B_\emptyset = B_d(x_0, 1)$. Now suppose that $\sigma(s)$ and $B_s = B_d(x_{\sigma(s)}, \varepsilon_s)$ have been defined, and consider any $t = s^\frown m$. If $n(s) = n(t) < \infty$, then set $\sigma(t) = \sigma(s)$ and $B_t = B_d(x_{\sigma(s)}, \varepsilon_t)$ with $\varepsilon_t = \frac{\varepsilon_s}{2}$. In the remaining cases, let $\sigma(t)$ be the smallest $n > \sigma(s)$ such that $x_n \in B_s$ (which exists by density of D and the fact that X is perfect), and let ε_t be small enough so that $\varepsilon_t \leq \frac{\varepsilon_s}{2}$, $\mathrm{cl}(B_d(x_{\sigma(t)}, \varepsilon_t)) \subseteq B_s$ and $x_i \notin B_d(x_{\sigma(t)}, \varepsilon_t)$ for all $i < \sigma(t)$ (this can be achieved because there are just finitely many such i's).

In particular, by the previous exercise the set

$$Q_2 = \{x \in 2^\omega \mid \exists n \in \omega \, \forall k \geq n \, (x(k) = 0)\}$$

is $\mathbf{\Sigma}^0_2$-complete. It also follows that $\mathbb{Q}$ is a $\mathbf{\Sigma}^0_2$-complete subset of $\mathbb{R}$ (so that in particular $\mathbb{Q}$ is not G_δ and Irr is not F_σ).

Exercise 2.1.26. Let P_3 be the set of infinite binary matrices[3] (i.e. $\omega \times \omega$-matrices) with all rows eventually 0, i.e.

$$\begin{aligned} P_3 &= \{x \in 2^{\omega\times\omega} \mid \forall m \, [(x(m,n))_{n\in\omega} \in Q_2]\} \\ &= \{x \in 2^{\omega\times\omega} \mid \forall m \, \exists n \, \forall k \geq n \, (x(m,k) = 0)\}. \end{aligned}$$

Show that P_3 is $\mathbf{\Pi}^0_3$-complete.

Solution. Clearly, $P_3 \in \mathbf{\Pi}^0_3(2^{\omega\times\omega})$. To check that it is $\mathbf{\Pi}^0_3$-hard, let $A = \bigcap_{n\in\omega} A_n \subseteq \omega^\omega$ with $A_n \in \mathbf{\Sigma}^0_2(\omega^\omega)$. For each $n \in \omega$, fix a continuous function $f_n \colon \omega^\omega \to 2^\omega$ such that $f^{-1}(Q_2) = A_n$, which exists because Q_2 is $\mathbf{\Sigma}^0_2$-complete. Then the map g sending $x \in \omega^\omega$ to the unique matrix in $2^{\omega\times\omega}$ whose n-th rows is $f_n(x)$ is continuous and such that $x \in A = \bigcap_{n\in\omega} A_n \iff f(x) \in P_3$.

Exercise 2.1.27. The set

$$C_3 = \{x \in \omega^\omega \mid \lim_{n\to\infty} x(n) = \infty\}$$

is $\mathbf{\Pi}^0_3$-complete.

Solution. We have

$$x \in C_3 \iff \forall k \, \exists n \, \forall m \geq n \, (x(m) \geq k),$$

so $C_3 \in \mathbf{\Pi}^0_3(\omega^\omega)$. Now define a function $f \colon 2^{\omega\times\omega} \to \omega^\omega$ as follows. Given $x \in 2^{\omega\times\omega}$ and $n \in \omega$, let $f(x)(n)$ be the smallest $i \leq n$ such that $x(i,n) = 1$ if there is any such i, and $f(x)(n) = n$ otherwise. It is easy to check that f is continuous and that $x \in P_3 \iff f(x) \in C_3$, therefore C_3 is $\mathbf{\Pi}^0_3$-complete because P_3 is.

[3] The space $2^{\omega\times\omega}$ of infinite binary matrices is equipped with the product over $\omega \times \omega$ of the discrete topology on 2. It is homeomorphic to $(2^\omega)^\omega$ (hence also to 2^ω) via the map sending an infinite matrix to the sequence of its rows.

Exercise 2.1.28 (Ki-Linton)**.** A subset $A \subseteq \omega$ has *density* 0 if

$$\lim_{n\to\infty} \frac{|A \cap \{0, \ldots, n-1\}|}{n} = 0.$$

Show that the set Z of all characteristic functions $y \in 2^\omega$ of sets with density 0 is $\mathbf{\Pi}^0_3$-complete.

Solution. For $t \in 2^{<\omega}$ let

$$\delta(t) = \frac{|\{\ell < \mathrm{lh}(t) \mid t(\ell) = 1|}{\mathrm{lh}(t)}$$

be the "partial density" of the (finite) set coded by s, so that $y \in Z$ if and only if $\delta(y \restriction \ell) \to 0$. It follows that $y \in Z$ if and only if

$$\forall \varepsilon \in \mathbb{Q}^+ \, \exists n \in \omega \, \forall \ell \geq n \, (\delta(y \restriction \ell) \leq \varepsilon),$$

so that $Z \in \mathbf{\Pi}^0_3(2^\omega)$.

For the hardness part, define a monotone function $\varphi \colon \omega^{<\omega} \to 2^{<\omega}$ such that:

1. $\mathrm{lh}(\varphi(s)) \geq \mathrm{lh}(s)$;
2. $\delta(\varphi(s^\frown n)) = \frac{1}{n+2}$;
3. $\delta(t) \leq \max\{\delta(\varphi(s)), \delta(\varphi(s^\frown n))\}$ for all $\varphi(s) \subseteq t \subseteq \varphi(s^\frown n)$.

This can be done recursively as follows. Set $\varphi(\emptyset) = \emptyset$ and for $s = \langle n \rangle$ set $\varphi(s) = \langle \underbrace{0, \ldots, 0}_{n+1 \text{ times}}, 1 \rangle$. Consider now $s \in \omega^{<\omega}$ with $\mathrm{lh}(s) > 0$ and an arbitrary $n \in \omega$. The idea is to define the desired $\varphi(s^\frown n)$ by adding to $\varphi(s)$ a certain number of consecutive 0's followed by a certain number of consecutive 1's ensuring that $\delta(\varphi(s^\frown n)) = \frac{1}{n+2}$. This entails that the last condition is automatically satisfied, as $\delta(t) > \delta(t^\frown 0)$ while $\delta(t) \leq \delta(t^\frown 1)$ (because $\frac{i}{j} \leq \frac{i+1}{j+1}$ if $j \geq i$). So let $\varphi(s^\frown n)$ be the unique sequence extending $\varphi(s)$ with length $\mathrm{lh}(\varphi(s))(n+2)$ (so that we are adding to $\varphi(s)$ at least $\mathrm{lh}(\varphi(s))$-many digits) so that the final block of consecutive 1's has length $\mathrm{lh}(\varphi(s)) - n_s$ where $n_s = |\{\ell < \mathrm{lh}(\varphi(s)) \mid \varphi(s)(\ell) = 1\}|$ (this is possible because $n_s \leq \mathrm{lh}(\varphi(s))$). In this way $\delta(\varphi(s^\frown n)) = \frac{\mathrm{lh}(\varphi(s))}{\mathrm{lh}(\varphi(s))(n+2)} = \frac{1}{n+2}$, as required.

Consider now the map f sending $x \in \omega^\omega$ to $\cup_{\ell \in \omega} \varphi(x \restriction \ell)$. Clearly f is continuous. Moreover

$$\begin{aligned} x \in C_3 &\iff x(n) \to \infty \\ &\iff \frac{1}{x(n)+2} \to 0 \\ &\iff \delta(\varphi(x \restriction \ell)) \to 0 \\ &\iff \delta(f(x) \restriction \ell) \to 0, \end{aligned}$$

where the third and fourth equivalence follow from the second and third condition on φ, respectively.

Exercise 2.1.29. For $0 < p < \infty$, let

$$\ell_p \cap [0;1]^\omega = \left\{ (x_n)_{n\in\omega} \in [0;1]^\omega \;\middle|\; \|x\|_p = \left(\sum_{n=0}^{\infty} |x_n|^p \right)^{\frac{1}{p}} < \infty \right\}.$$

Also let

$$C_0 = c_0 \cap [0;1]^\omega = \{(x_n)_{n\in\omega} \in [0;1]^\omega \mid x_n \to 0\}$$
$$C = \{(x_n)_{n\in\omega} \in [0;1]^\omega \mid (x_n)_{n\in\omega} \text{ converges}\}.$$

Then $\ell_p \cap [0;1]^\omega$ is $\mathbf{\Sigma}^0_2$-complete and C_0, C are $\mathbf{\Pi}^0_3$-complete.

Most of the "natural" examples of Borel sets have finite Borel rank: this is because an explicit definition of a set usually involves a finite number of quantifiers, which in turn yields a finite alternation of countable intersections and unions. However, the next example shows that indeed there are reasonable examples of sets which are complete for arbitrarily high classes in the Borel hierarchy.

Example 2.1.30 (Stern)**.** Let LO be the Polish space of (codes for) countable linear orders. For $\omega < \alpha < \omega_1$, let

$$\mathrm{WO}^{<\alpha} = \{x \in 2^{\omega\times\omega} \mid L_x \text{ is a well-order of order type } < \alpha\}.$$

Then $\mathrm{WO}^{<\omega^\alpha}$ is $\mathbf{\Sigma}^0_{2\cdot\alpha}$-complete (for every $\alpha \geq 2$). If instead $\omega^\alpha < \beta < \omega^{\alpha+1}$, then $\mathrm{WO}^{<\beta} \in \mathbf{\Delta}^0_{2\cdot\alpha+2} \setminus \mathbf{\Sigma}^0_{2\cdot\alpha+1}$.

Finally, the following proposition gives a way to inductively contruct $\mathbf{\Sigma}^0_\alpha$-complete sets: these sets can in turn be used as canonical objects to show that other sets are complete in the relevant class (see Lemma 2.1.23, and Exercise 2.1.27 for an example of how this technique can be applied).

Proposition 2.1.31 ([Kec95, Exercise 23.3])**.** *Let $\rho\colon 2^\omega \to (2^\omega)^\omega$ be an homeomorphism, and let $\rho_n = \pi_n \circ \rho$, where π_n is the projection on the n-th coordinate. Inductively define*

$$C_1 = \{x \in 2^\omega \mid \exists n\,(x(n) = 0)\}$$
$$C_{\alpha+1} = \{x \in 2^\omega \mid \exists n\,(\rho_n(x) \notin C_\alpha)\}$$
$$C_\lambda = \{x \in 2^\omega \mid \exists n\,(\rho_n(x) \notin C_{\alpha_n}\} \qquad \textit{if } \lambda \textit{ is limit,}$$

where in the last case $(\alpha_n)_{n\in\omega}$ is any increasing sequence of nonzero ordinals cofinal in λ. Then C_α is $\mathbf{\Sigma}^0_\alpha$-complete for $\alpha \geq 1$.

Proof. The result can be proved by an easy induction on $1 \leq \alpha < \omega_1$ using the homeomorphism ρ, the fact that $A \subseteq X$ is $\mathbf{\Sigma}^0_\alpha$-complete if and only if $X \setminus A$ is $\mathbf{\Pi}^0_\alpha$-complete, and the following claim.

Claim 2.1.31.1. *Let $1 \leq \alpha < \omega_1$, $(X_n)_{n\in\omega}$ be a sequence of Polish spaces, and $(\alpha_n)_{n\in\omega}$ be a sequence of nonzero countable ordinals such that $\limsup\{\alpha_n + 1 \mid n \in \omega\} = \alpha$. If $A_n \subseteq X_n$ is $\mathbf{\Sigma}^0_{\alpha_n}$-complete for every $n \in \omega$, then $\prod_{n\in\omega} A_n \subseteq \prod_{n\in\omega} X_n$ is $\mathbf{\Pi}^0_\alpha$-complete.*

To prove the claim, first notice that

$$x \in \prod_{n\in\omega} A_n \iff \forall n\, (\pi_n(x) \in A_n),$$

where π_n is the projection on the n-th coordinate. Since each π_n is continuous, $\prod_{n\in\omega} A_n \in \mathbf{\Pi}^0_\alpha(\prod_{n\in\omega} X_n)$. Let now $B = \bigcap_{n\in\omega} B_n \subseteq \omega^\omega$ with $B_n \in \bigcup_{1\leq\beta<\alpha} \mathbf{\Sigma}^0_\beta(\omega^\omega)$. Pick an injective function $\iota\colon \omega \to \omega$ such that $B_n \in \mathbf{\Sigma}^0_{\alpha_{\iota(n)}}(\omega^\omega)$ — such a ι can be found because $\limsup\{\alpha_n + 1 \mid n \in \omega\} = \alpha$. For each $n \in \omega$, let $f_n\colon \omega^\omega \to X_{\iota(n)}$ be a continuous function such that $f_n^{-1}(A_{\iota(n)}) = B_n$, and fix any $y_n \in A_n$. Finally, let g be the function sending each $x \in \omega^\omega$ to the unique $(x_n)_{n\in\omega} \in \prod_{n\in\omega} X_n$ such that $x_{\iota(n)} = f_n(x)$ and $x_n = y_n$ if $n \notin \mathrm{rng}(\iota)$. Then g is continuous and such that $x \in B = \bigcap_{n\in\omega} B_n \iff g(x) \in \prod_{n\in\omega} A_n$. □

2.2 Changes of topologies

Lemma 2.2.1 ([Kec95, Lemma 13.3])**.** *Let $X = (X,\tau)$ be Polish, and let $(\tau_n)_{n\in\omega}$ be a sequence of Polish topologies on X with $\tau \subseteq \tau_n$ for every $n \in \omega$. Then the topology τ_∞ generated by $\bigcup_{n\in\omega} \tau_n$ is Polish. Moreover, if $\tau_n \subseteq \mathbf{\Sigma}^0_\alpha(X,\tau)$ for every $n \in \omega$, then $\tau_\infty \subseteq \mathbf{\Sigma}^0_\alpha(X,\tau)$ as well.*

Notice that if all τ_n are zero-dimensional, then so is τ_∞.

Proof. Let $X_n = (X,\tau_n)$, and consider the map $f\colon X \to \prod_{n\in\omega} X_n$ defined by $f(x) = (x,x,x,\dots)$. Notice that $f(X)$, i.e. the diagonal of $\prod_{n\in\omega} X_n$, is closed in $\prod_{n\in\omega} X_n$. Indeed, if $(x_n)_{n\in\omega} \notin f(X)$, then there are distinct $i,j \in \omega$ such that $x_i \neq x_j$. Let $U,V \in \tau$ be disjoint open sets such that $x_i \in U$ and $x_j \in V$, and notice that $U \in \tau_i$ and $V \in \tau_j$ because $\tau \subseteq \tau_i, \tau_j$. Then the set

$$W = X_0 \times \dots \times X_{i-1} \times U \times X_{i+1} \times \dots \times X_{j-1} \times V \times X_{j+1} \times \dots$$

is an open neighborhood of $(x_n)_{n\in\omega}$ which is disjoint from $f(X)$ because for any $(y_n)_{n\in\omega} \in W$ one has $y_i \in U$ and $y_j \in V$, so that $y_i \neq y_j$ by $U \cap V = \emptyset$. It follows that $f(X)$ is a Polish space, and it is immediate to check that f is a homeomorphism between (X,τ_∞) and $f(X)$.

For the additional part, fix for each $n \in \omega$ a countable basis $\mathcal{B}_n$: then $\bigcup_{n\in\omega} \mathcal{B}_n$ is a countable subbasis for τ_∞ contained in $\mathbf{\Sigma}^0_\alpha(X,\tau)$, hence $\tau_\infty \subseteq \mathbf{\Sigma}^0_\alpha(X,\tau)$ because the latter pointclass is closed under finite intersections and countable unions. □

Theorem 2.2.2 (Kuratowski, see [Kec95, Theorem 22.18 and Exercises 22.19 and 22.20])**.** *Let $X = (X,\tau)$ be a Polish space, let $1 \leq \alpha < \omega_1$, and let $A_n \subseteq X$ be in $\mathbf{\Delta}^0_\alpha(X,\tau)$ for every $n \in \omega$. Then there is a Polish topology $\tau' \supseteq \tau$ on X such that $\tau' \subseteq \mathbf{\Sigma}^0_\alpha(X,\tau)$ and $A_n \in \mathbf{\Delta}^0_1(X,\tau')$ for all $n \in \omega$.*

Moreover, when $\alpha > 1$ we can require τ' to be zero-dimensional, and if $\alpha > 1$ is a successor ordinal and all the A_n coincide with the same set A, then we may require $\tau' \subseteq \mathbf{\Delta}^0_\alpha(X,\tau)$ (dropping zero-dimensionality, unless τ was already zero-dimensional).

Proof. The proof is by induction on $\alpha \geq 1$. If $\alpha = 1$, then it is enough to set $\tau' = \tau$. If $\alpha = 2$, then each A_n and its complement $X \setminus A_n$ are G_δ, hence Polish. Let τ_n be the direct sum of the relative topologies on A_n and $X \setminus A_n$: then $\tau_n \supseteq \tau$ is still Polish, $A_n \in \mathbf{\Delta}^0_1(X, \tau_n)$, and $\tau_n \subseteq \mathbf{\Sigma}^0_2(X, \tau)$ because it consists of the sets of the form $(U \cap A_n) \cup (V \setminus A_n)$ for $U, V \in \tau$. Letting τ' be the topology generated by $\bigcup_{n \in \omega} \tau_n$, by Lemma 2.2.1 we get the desired result. Clearly, if all the A_n coincide with the same set A, then τ' is just the topology consisting of all sets of the form $(U \cap A) \cup (V \setminus A)$ for some $U, V \in \tau$, hence $\tau' \subseteq \mathbf{\Delta}^0_2(X, \tau)$. To see instead that τ' can in general be required to be zero-dimensional, it is enough to observe that without loss of generality we may assume that $\{A_n \mid n \in \omega\}$ contains $X \setminus U_k$ for every $k \in \omega$, where $\{U_k \mid k \in \omega\}$ is any countable basis for τ: in fact, it then follows that the subbasis $\{U_k \cap A_n, U_k \setminus A_n \mid n, k \in \omega\}$ of τ' consists of τ'-clopen sets.

Let now $\alpha > 1$ be a limit ordinal. Then $A_n = \bigcup_{i \in \omega} A_{n,i} = \bigcap_{i \in \omega} B_{n,i}$, with $A_{n,i}, B_{n,i} \in \mathbf{\Delta}^0_{\alpha_{n,i}}(X, \tau)$ for the appropriate $1 < \alpha_{n,i} < \alpha$. By inductive hypothesis, let $\tau'_{n,i}$ and $\tau''_{n,i}$ be (zero-dimensional) Polish topologies that work as in the statement of the theorem for the sets $A_{n,i}$ and $B_{n,i}$, respectively, so that in particular

$$A_{n,i} \in \mathbf{\Delta}^0_1(X, \tau'_{n,i}) \quad \text{and} \quad \tau \subseteq \tau'_{n,i} \subseteq \mathbf{\Sigma}^0_{\alpha_{n,i}}(X, \tau) \subseteq \mathbf{\Sigma}^0_\alpha(X, \tau),$$

and similarly for $B_{n,i}$ and $\tau''_{n,i}$. Then letting τ' be the topology generated by $\bigcup_{n,i \in \omega} (\tau'_{n,i} \cup \tau''_{n,i})$, we get from Lemma 2.2.1 that $\tau' \supseteq \tau$ is a Polish topology such that $\tau' \subseteq \mathbf{\Sigma}^0_\alpha(X, \tau)$. Moreover, $A_n = \bigcup_{i \in \omega} A_{n,i} = \bigcap_{i \in \omega} B_{n,i} \in \mathbf{\Delta}^0_1(X, \tau')$ because all the $A_{n,i}$ and $B_{n,i}$ are τ'-clopen.

Finally, let $\alpha = \beta + 1 \geq 3$ be a successor ordinal. Then $A_n = \bigcup_{i \in \omega} \bigcap_{j \in \omega} A_{n,i,j} = \bigcap_{i \in \omega} \bigcup_{j \in \omega} B_{n,i,j}$ with $A_{n,i,j}, B_{n,i,j} \in \mathbf{\Delta}^0_\beta(X, \tau)$. By inductive hypothesis, for each $n, i, j \in \omega$ there are Polish topologies $\tau'_{n,i,j}$ and $\tau''_{n,i,j}$ refining τ such that $A_{n,i,j}$ is $\tau'_{n,i,j}$-clopen, $B_{n,i,j}$ is $\tau''_{n,i,j}$-clopen, and $\tau'_{n,i,j}, \tau''_{n,i,j} \subseteq \mathbf{\Sigma}^0_\beta(X, \tau)$. Let τ_∞ be the topology generated by $\bigcup_{n,i,j \in \omega} (\tau'_{n,i,j} \cup \tau''_{n,i,j})$, so that all $A_{n,i,j}$ and $B_{n,i,j}$ are τ_∞-clopen, $\tau \subseteq \tau_\infty \subseteq \mathbf{\Sigma}^0_\beta(X, \tau)$, and τ_∞ is Polish by Lemma 2.2.1. It follows that $A_n \in \mathbf{\Delta}^0_2(X, \tau_\infty)$. Applying now the case $\alpha = 2$ to the A_n's viewed as subsets of (X, τ_∞), we get that there is a (zero-dimensional) Polish topology $\tau' \supseteq \tau_\infty \supseteq \tau$ such that each A_n is τ'-clopen and $\tau' \subseteq \mathbf{\Sigma}^0_2(X, \tau_\infty) \subseteq \mathbf{\Sigma}^0_{\beta+1}(X, \tau)$, so that τ' is as desired because $\beta + 1 = \alpha$. Finally, notice that if all the A_n's coincide with the same set A, then in the last step of the proof we may require $\tau' \subseteq \mathbf{\Delta}^0_2(X, \tau_\infty) \subseteq \mathbf{\Delta}^0_\alpha(X, \tau)$. ☐

Corollary 2.2.3 (essentially [Kec95, Theorem 13.1 and Exercise 13.5])**.** *Let $X = (X, \tau)$ be a Polish space and $A_n \in \mathbf{Bor}(X, \tau)$ for every $n \in \omega$. Then there is a Polish topology $\tau' \supseteq \tau$ such that $\mathbf{Bor}(X, \tau') = \mathbf{Bor}(X, \tau)$ and each A_n is clopen with respect to τ'. Moreover, τ' can be taken to be zero-dimensional.*

Proof. Let $\alpha > 1$ be such that $A_n \in \mathbf{\Sigma}^0_\alpha(X, \tau)$ for every $n \in \omega$, and let τ' be the (zero-dimensional) topology given by Theorem 2.2.2. By induction on $1 \leq \beta < \omega_1$, one can easily show that

$$\mathbf{\Sigma}^0_\beta(X, \tau) \subseteq \mathbf{\Sigma}^0_\beta(X, \tau') \subseteq \mathbf{\Sigma}^0_{\alpha+\beta}(X, \tau),$$

hence $\mathbf{Bor}(X,\tau') = \bigcup_{1 \leq \beta < \omega_1} \mathbf{\Sigma}^0_\beta(X,\tau') = \bigcup_{1 \leq \beta < \omega_1} \mathbf{\Sigma}^0_\beta(X,\tau) = \mathbf{Bor}(X,\tau)$. $\square$

There is no analogue of Corollary 2.2.3 working for $\aleph_1$-many Borel sets $\mathcal{A} = \{A_\alpha \mid \alpha < \omega_1\}$ at once, even if we further require that all these sets have Borel rank bounded by the same $1 < \beta < \omega_1$ (a condition trivially satisfied when considering only countably many Borel sets). Indeed, let X be an uncountable Polish space and $A \subseteq X$ be any set of size $\aleph_1$. Let $\mathcal{A} = \{\{x\} \mid x \in A\}$, so that this is a family of $\aleph_1$-many closed (hence $\mathbf{\Delta}^0_\beta(X)$ for any $1 < \beta < \omega_1$) sets. Then any topology τ' such that $\mathcal{A} \subseteq \tau'$ cannot be separable, since all points in A would be τ'-isolated.

Also the possibility of having a "good" change of topology turning a non-Borel set into an open (or even just Borel) one is hopeless. We will show in Theorem 3.2.5 that if f is an *injective* continuous (in fact, even just Borel: see Section 2.4 and Corollary 3.2.6) function between two Polish spaces X and Y, then $f(A) \in \mathbf{Bor}(Y)$ for every $A \in \mathbf{Bor}(X)$. Now suppose towards a contradiction that there is a set $A \subseteq X$ which is not Borel (with respect to the Polish topology τ on X), but it is such that there is a Polish topology $\tau' \supseteq \tau$ such that $A \in \mathbf{Bor}(X,\tau')$. Then the identity function $\mathrm{id}_X \colon (X,\tau') \to (X,\tau)$ would be continuous and injective, and since $A \in \mathbf{Bor}(X,\tau')$ then $\mathrm{id}_X(A) = A$ would be in $\mathbf{Bor}(X,\tau)$, a contradiction.

We now use the previous result to show that all Borel subsets of a Polish space have the PSP (thus there is no Borel Bernstein set).

Theorem 2.2.4 (Alexandrov, Hausdorff, see [Kec95, Theorem 13.6])**.** *Let $X = (X,\tau)$ be Polish and $A \in \mathbf{Bor}(X)$. Then A has the* PSP*. In particular, every uncountable Borel subset of a Polish space has cardinality $2^{\aleph_0}$.*

Proof. By Corollary 2.2.3, let $\tau' \supseteq \tau$ be a Polish topology on X such that $A \in \mathbf{\Delta}^0_1(X,\tau')$, so that $(A, \tau' \restriction A)$ is Polish (where $\tau' \restriction A$ is the relative topology of τ' on the set A). If A is uncountable, then by Corollary 1.4.9 there is a continuous injection $f \colon 2^\omega \to (A, \tau' \restriction A) \subseteq (X,\tau')$. But since $\tau \subseteq \tau'$, the function f is continuous also as a function from 2^ω to (X,τ), and thus it is an embedding of 2^ω into A (with respect to the original topology τ of X). $\square$

Theorem 2.2.4 is just an instance of a general phenomenon, namely, that Borel subsets of Polish spaces enjoy all regularity properties of interest in descriptive set theory. Another example is given by Proposition 1.5.9, which implies that every Borel set has the Baire property.

As another application of Corollary 2.2.3, we get a nice representation of Borel sets.

Proposition 2.2.5 (Lusin-Souslin, see [Kec95, Theorem 13.7])**.** *Let X be Polish and $A \subseteq X$ be Borel. There is a closed set $F \subseteq \omega^\omega$ and a continuous bijection $f \colon F \to A$. In particular, if $A \neq \emptyset$, there is also a continuous surjection $g \colon \omega^\omega \to A$ extending f.*

Proof. Apply Corollary 2.2.3 to get a Polish topology τ' refining the topology τ of X such that A is τ'-clopen (hence Polish with respect to the relative topology of τ'). Then by Theorem 1.3.17 there are F and f (or even g if $A \neq \emptyset$) as in the statement, except that f (respectively, g) is continuous as a function between F (respectively, ω^ω) and (X, τ'). But since $\tau \subseteq \tau'$, the function remains continuous when equipping X with its original topology τ, hence we are done. □

A converse to the first part of the above proposition will be given in Corollary 3.2.7.

2.3 Standard Borel spaces

Definition 2.3.1. A **Borel space** is a pair $X = (X, \mathbf{B})$ where $\mathbf{B}$ is a σ-algebra on X such that $\mathbf{B} = \mathbf{Bor}(X, \tau)$ for some separable metrizable topology τ on X.

A function $f \colon X \to X'$ between two Borel spaces $X = (X, \mathbf{B})$ and $X' = (X', \mathbf{B}')$ is called **Borel** if $f^{-1}(B) \in \mathbf{B}$ for every $B \in \mathbf{B}'$. The function f is a **Borel isomorphism** if it is bijective and both f and f^{-1} are Borel, i.e. for every $A \subseteq X$

$$A \in \mathbf{B} \iff f(A) \in \mathbf{B}';$$

in this case we say that X and X' are **Borel isomorphic**.

Borel spaces can be characterized as follows. Recall that a σ-algebra $\mathbf{B}$ on X is **countably generated** if there is a countable family $\mathcal{A} = \{A_n \mid n \in \omega\} \subseteq \mathscr{P}(X)$ such that $\mathbf{B}$ is the smallest σ-algebra containing $\mathcal{A}$, and it **separates points** if for all pairs of distinct points in X there is $A \in \mathbf{B}$ which contains exactly one of those points (i.e. separates one of the points from the other one). Notice that if $\mathcal{A}$ is a set of generators of $\mathbf{B}$, then $\mathbf{B}$ separates points if and only if for all distinct x and y there is a generator $A \in \mathcal{A}$ separating one from the other. This can be proved observing that the collection of all $A \in \mathbf{B}$ that *do not* separate a given pair of points is a σ-algebra: therefore, if all of $\mathcal{A}$ were contained in such σ-algebra, then $\mathbf{B}$ itself could not separate those two points.

Proposition 2.3.2 ([Kec95, Proposition 12.2]). *Let $\mathbf{B}$ be a σ-algebra on a set X. Then the following are equivalent:*

(i) *$(X, \mathbf{B})$ is a Borel space;*

(ii) *$(X, \mathbf{B})$ is Borel isomorphic to $(Y, \mathbf{Bor}(Y))$ for some $Y \subseteq 2^\omega$ (and thus to some $Y \subseteq Z$ for any uncountable Polish space Z);*

(iii) *$\mathbf{B}$ is countably generated and separates points.*

Proof. The implications (ii) ⇒ (i) and (i) ⇒ (iii) are obvious, so let us show that (iii) ⇒ (ii). Let $\mathcal{A} = \{A_n \mid n \in \omega\}$ be a set of generators for $\mathbf{B}$. Define $f \colon X \to 2^\omega$ by setting for each $x \in X$ and $n \in \omega$

$$f(x)(n) = 1 \iff x \in A_n,$$

i.e. $f(x) = (\chi_{A_n})_{n \in \omega}$ where χ_{A_n} is the characteristic function of A_n. Then f is injective because $\mathbf{B}$ separates points, and it is Borel by definition (here we use the fact that $\mathbf{Bor}(2^\omega)$ is generated by the elements of the canonical subbase for the product topology on 2^ω). Let $Y = f(X) \subseteq 2^\omega$, so that f is a bijection between X and Y. Then $(f^{-1})^{-1}(A_n) = f(A_n) = \{y \in 2^\omega \mid y(n) = 1\} \cap Y$, hence also f^{-1} is Borel and f is a Borel isomorphism. □

Definition 2.3.3 ([Kec95, Definition 12.5])**.** A Borel space $X = (X, \mathbf{B})$ is called **standard** if there is a Polish topology τ on X such that $\mathbf{B} = \mathbf{Bor}(X, \tau)$. Equivalently, X is standard if and only if it is Borel isomorphic to $(Y, \mathbf{Bor}(Y))$ for some Polish space Y.

Proposition 2.3.4. *A Borel space $X = (X, \mathbf{B})$ is standard if and only if there is a Polish space Y and $A \in \mathbf{Bor}(Y)$ such that X is Borel isomorphic to $(A, \mathbf{Bor}(Y) \restriction A)$.*

In particular, if $X = (X, \mathbf{B})$ is a standard Borel space and $A \in \mathbf{B}$, then $(A, \mathbf{B} \restriction A)$ is a standard Borel space as well, where $\mathbf{B} \restriction A = \{C \cap A \mid C \in \mathbf{B}\}$.

Proof. For the nontrivial direction of the equivalence, let $A \subseteq Y$ be as in the statement, and let $f \colon X \to A$ be a Borel isomorphism. By Corollary 2.2.3 there is a Polish topology τ' on Y such that the Borel sets remain the same but A becomes τ'-clopen, hence Polish: since f is a Borel isomorphism also with respect to this new topology, we are done. □

Standard Borel spaces retain all the properties of Polish spaces (for what concerns their Borel structure), and are quite useful. For example, given a Polish space X equip the set

$$F(X) = \{F \subseteq X \mid F \text{ is closed}\}$$

with the σ-algebra $\mathbf{B}_{F(X)}$ (called the **Effros Borel structure** on $F(X)$) generated by the sets

$$\{F \in F(X) \mid F \cap U \neq \emptyset\}$$

where U varies over the open subsets of X. The resulting space $F(X) = (F(X), \mathbf{B}_{F(X)})$ is called **Effros Borel space**.

Theorem 2.3.5 ([Kec95, Theorem 12.6])**.** *Let X be a Polish space. Then the Effros Borel space $F(X)$ is a standard Borel space.*

Aggiungere nuova dimostrazione!

In particular, let us consider the space $\mathbb{R}^\omega$. Since its closed subspaces coincide, up to homeomorphism, with the collection of all Polish spaces, we can regard $F(\mathbb{R}^\omega)$ as the standard Borel space of all Polish spaces, and the same is true for $F(\ell_2)$ (recall that $\mathbb{R}^\omega$ and ℓ_2 are homeomorphic). Similarly, a basic result of Banach space theory shows that every separable Banach space is isometrically isomorphic to a closed subspace of $C(2^\omega)$, i.e. there is a linear isometry between the given space and a closed subspace of $C(2^\omega)$. Since one can easily show (using Theorem 2.3.6 below) that the collection $\mathrm{Subs}(C(2^\omega))$ of all closed linear subspaces of $C(2^\omega)$ is a Borel set in $F(C(2^\omega))$, we can regard $\mathrm{Subs}(C(2^\omega))$ as the standard Borel space of all separable Banach spaces. Similarly, one can form the standard Borel space of all Polish groups, the standard

Borel space of all von Neumann algebras, and so on. All these spaces can be dealt with the techniques and methods of descriptive set theory.

A basic yet fundamental result concerning the Effros Borel spaces is the following selection theorem for $F(X)$.

Theorem 2.3.6 (Kuratowski-Ryll-Nardzewski, see [Kec95, Theorem 12.13])**.** *Let X be Polish. Then there is a sequence of Borel functions $d_n \colon F(X) \to X$ (where $F(X)$ is equipped with its Effros Borel structure and X with the σ-algebra of its Borel sets), $n \in \omega$, such that for every nonempty $F \in F(X)$, the set $\{d_n(F) \mid n \in \omega\}$ is a dense subset of F.*

2.4 Borel functions and the Baire stratification

Definition 2.4.1 ([Kec95, Definition 24.2])**.** Let X, Y be metrizable spaces, and $\mathbf{\Gamma}$ be a boldface pointcass. A function $f \colon X \to Y$ if **$\mathbf{\Gamma}$-measurable** if $f^{-1}(U) \in \mathbf{\Gamma}(X)$ for every open $U \subseteq Y$.

Notice that if $\mathbf{\Gamma}(X)$ is closed under countable unions and finite intersections and Y is second-countable, then in the previous definition it is enough to restrict U to any countable subbasis for Y. Clearly, the notion of $\mathbf{\Sigma}^0_1$-measurability coincides with continuity.

Definition 2.4.2. Let X, Y be metrizable spaces. A function $f \colon X \to Y$ is called **Borel** if it is **Bor**-measurable. Equivalently, f is Borel if and only if $f^{-1}(B) \in \mathbf{Bor}(X)$ for every $B \in \mathbf{Bor}(Y)$, i.e. if it is Borel as a function between the Borel spaces $(X, \mathbf{Bor}(X))$ and $(Y, \mathbf{Bor}(Y))$.

A function $f \colon X \to Y$ is a **Borel isomorphism** if it is a bijection and both f and f^{-1} are Borel; when such a function exists, we say that X and Y are **Borel isomorphic**.

Remark 2.4.3. If Y is second-countable, then $f \colon X \to Y$ is Borel if and only if there is $1 \leq \alpha < \omega_1$ such that f is $\mathbf{\Sigma}^0_\alpha$-measurable. (Indeed, it is enough to set $\alpha = \sup\{\alpha_n \mid n \in \omega\}$, where the α_n are such that $f^{-1}(U_n) \in \mathbf{\Sigma}^0_{\alpha_n}(X)$ for $\{U_n \mid n \in \omega\}$ a countable basis for Y.) Since $\mathbf{\Sigma}^0_\alpha(X) \subseteq \mathbf{\Sigma}^0_\beta(X)$ when $\alpha \leq \beta$, $\mathbf{\Sigma}^0_\alpha$-measurability gives a stratification of the Borel functions in at most ω_1-many levels. If X is an uncountable Polish space and $|Y| \geq 2$, this hierarchy does not collapse before ω_1. To see this, fix distinct $y_0, y_1 \in Y$, and given $1 \leq \alpha < \omega_1$ pick $A \in \mathbf{\Delta}^0_\alpha(X) \setminus \bigcup_{1 \leq \beta < \alpha}(\mathbf{\Sigma}^0_\beta(X) \cup \mathbf{\Pi}^0_\beta(X))$: then the function defined by $f(x) = y_0$ if $x \in A$ and $f(x) = y_1$ if $x \notin A$ is $\mathbf{\Sigma}^0_\alpha$-measurable but not $\mathbf{\Sigma}^0_\beta$-measurable for any $1 \leq \beta < \alpha$.

Here are some basic facts (Exercise!) concerning $\mathbf{\Sigma}^0_\alpha$-measurable functions.

- If $f \colon X \to Y$ is $\mathbf{\Sigma}^0_\alpha$-measurable, then for every $\beta < \omega_1$ we have that if $A \in \mathbf{\Sigma}^0_{1+\beta}(Y)$ (respectively, $A \in \mathbf{\Pi}^0_{1+\beta}(Y)$ or $A \in \mathbf{\Delta}^0_{1+\beta}(Y)$) then $f^{-1}(A) \in \mathbf{\Sigma}^0_{\alpha+\beta}(X)$ (respectively, $f^{-1}(A) \in \mathbf{\Pi}^0_{\alpha+\beta}(X)$ or $f^{-1}(A) \in \mathbf{\Delta}^0_{\alpha+\beta}(X)$). It follows that if $f \colon X \to Y$ is $\mathbf{\Sigma}^0_\alpha$-measurable and $g \colon Y \to Z$ is $\mathbf{\Sigma}^0_{1+\beta}$-measurable, then $g \circ f \colon X \to Z$ is $\mathbf{\Sigma}^0_{\alpha+\beta}$-measurable.

- If X_i and Y_i are metrizable spaces with Y_i second-countable, $i < I \leq \omega$, and each $f_i \colon X_i \to Y_i$ is $\mathbf{\Sigma}^0_\alpha$-measurable, then the product function
$$\prod_{i<I} f_i \colon \prod_{i<I} X_i \to \prod_{i<I} Y_i$$
is $\mathbf{\Sigma}^0_\alpha$-measurable as well.
- Similarly, if X, Y_i are metrizable with Y_i second-countable, $i < I \leq \omega$, and each $f_i \colon X \to Y_i$ is $\mathbf{\Sigma}^0_\alpha$-measurable, then the function $g \colon X \to \prod_{i<I} Y_i$ sending $x \in X$ to $(f_i(x))_{i<I}$ is $\mathbf{\Sigma}^0_\alpha$-measurable as well.

The next result generalizes to all countable α's the following well-know topological fact (which corresponds to the case $\alpha = 1$): if X, Y are topological spaces with Y Hausdorff and $f \colon X \to Y$ is continuous, then the graph of f is a closed set.

Proposition 2.4.4. *Let X, Y be metrizable spaces with Y separable. If $f \colon X \to Y$ is $\mathbf{\Sigma}^0_\alpha$-measurable, then its graph*
$$\operatorname{graph}(f) = \{(x,y) \in X \times Y \mid f(x) = y\}$$
is in $\mathbf{\Pi}^0_\alpha(X \times Y)$. In particular, the graph of a Borel function is Borel.

Proof. The set $\operatorname{graph}(f)$ is the preimage of the diagonal of Y, which is a closed set, via the $\mathbf{\Sigma}^0_\alpha$-measurable function $f \times \mathrm{id}_Y \colon X \times Y \to Y \times Y$. □

We will later show that when X and Y are both Polish, then a partial converse is true: if $f \colon X \to Y$ has a Borel graph, then f is Borel (see 3.2.4).

Theorem 2.4.5 (Lebesgue, Hausdorff, Banach, see [Kec95, Theorem 24.3])**.** *Let X, Y be metrizable spaces, with Y separable. Let $1 < \alpha < \omega_1$.*

(i) *If α is a successor ordinal, then f is $\mathbf{\Sigma}^0_{\alpha+1}$-measurable if and only if $f = \lim_{n\to\infty} f_n$, where each $f_n \colon X \to Y$ is $\mathbf{\Sigma}^0_\alpha$-measurable.*

(ii) *If α is a limit ordinal, then f is $\mathbf{\Sigma}^0_{\alpha+1}$-measurable if and only if $f = \lim_{n\to\infty} f_n$, where each $f_n \colon X \to Y$ is $\mathbf{\Sigma}^0_{\beta_n}$-measurable for some $1 \leq \beta_n < \alpha$.*

Remark 2.4.6. When λ is a (countable) limit ordinal, there is no natural way to obtain the collection of $\mathbf{\Sigma}^0_\lambda$-measurable functions as pointwise limits of simpler functions. Indeed, by part (ii) of the previous theorem the closure under pointwise limits of all the preceding classes (that is, of the collection of all functions which are $\mathbf{\Sigma}^0_\beta$-measurable for some $1 \leq \beta < \lambda$) already coincide with the collection of all $\mathbf{\Sigma}^0_{\lambda+1}$-measurable, a class which is in general strictly larger than the class of $\mathbf{\Sigma}^0_\lambda$-measurable functions by Remark 2.4.3.

The situation when $\alpha = 1$ is more delicate. It is still true that a limit of $\mathbf{\Sigma}^0_1$-measurable (i.e. continuous) functions is $\mathbf{\Sigma}^0_2$-measurable, but the converse may fail. Indeed, if X is connected and Y is totally disconnected, then any continuous functions $f\colon X \to Y$ is constant, and therefore also a limit of continuous functions must be constant; however, if both X and Y contain at least two points, then there are non-constant $\mathbf{\Sigma}^0_2$-measurable functions (for example, we can let $f = \chi_{\{0\}}\colon \mathbb{R} \to \{0,1\}$ be the characteristic function of the singleton $\{0\}$.) This problem can be overcome by requiring that either Y is a well-behaved space, or X is far from being connected.

Theorem 2.4.7 (Lebesgue, Hausdorff, Banach, see [Kec95, Theorem 24.10])**.** *Let X, Y be separable metrizable and $f\colon X \to Y$ be $\mathbf{\Sigma}^0_2$-measurable. If either $Y = \mathbb{R}$ (or $Y = \mathbb{R}^n$, $Y = \mathbb{C}^n$, Y is an interval in $\mathbb{R}$, and so on), or else X is zero-dimensional, then f is the (pointwise) limit of a sequence of continuous functions.*

Definition 2.4.8. Suppose that X and Y satisfy the hypothesis of Theorem 2.4.7. Let $\mathcal{B}_0(X,Y)$ be the collection of all continuous functions $f\colon X \to Y$, and for $\alpha < \omega_1$ inductively define $\mathcal{B}_\alpha(X,Y) = \{\lim_{n\to\infty} f_n \mid f_n \in \bigcup_{\nu<\alpha} \mathcal{B}_\nu(X,Y)\}$. Functions in $\mathcal{B}_\alpha(X,Y)$ are called **Baire class** α functions.

Remark 2.4.9. By definition, $\mathcal{B}_\alpha(X,Y) \subseteq \mathcal{B}_\beta(X,Y)$ whenever $\alpha \leq \beta < \omega_1$, and $\bigcup_{\alpha<\omega_1} \mathcal{B}_\alpha(X,Y)$ is the smallest collection of functions containing the continuous ones and closed under (pointwise) limits.

Baire class 1 functions are ubiquitous in analysis and in mathematics.

Example 2.4.10.
- Upper semicontinuous and lower semicontinuous functions $f\colon X \to \mathbb{R}$ (where X is an arbitrary Polish space) are Baire class 1 functions.
- If X is Polish and $f\colon X \to \mathbb{R}$ has only countably many discontinuities, then f is of Baire class 1. In particular, all $f\colon [0;1] \to \mathbb{R}$ which are monotone or of bounded variation are of Baire class 1.
- Let $F\colon [0;1] \to \mathbb{R}$ be differentiable (at endpoints we take one-sided derivatives). Then its derivative F' is a Baire class 1 function.

The following result is a corollary of Theorems 2.4.5 and 2.4.7.

Theorem 2.4.11 (Lebesgue, Hausdorff, see [Kec95, Theorem 11.6])**.** *Let X, Y be spaces satisfying the hypotheses of Theorem 2.4.7. Then $f\colon X \to Y$ is of Baire class α (for $\alpha < \omega_1$) if and only if it is $\mathbf{\Sigma}^0_{\alpha+1}$-measurable. Moreover, $\mathcal{B}_\alpha(X,Y) \subsetneq \mathcal{B}_\beta(X,Y)$ for any $\alpha < \beta < \omega_1$, and the class of all Borel functions between X and Y is the smallest collection of functions containing the continuous ones and closed under (pointwise) limits.*

The technique of changes of topologies can be applied also to turn Borel functions into continuous ones.

Proposition 2.4.12 ([Kec95, Exercise 24.5])**.** *Let (X,τ) be a Polish space, Y be a separable metrizable space, and $1 \leq \alpha < \omega_1$. Then $f\colon (X,\tau) \to Y$ is $\mathbf{\Sigma}^0_\alpha$-measurable if and only if there is a Polish topology $\tau' \supseteq \tau$ on X such that $\tau' \subseteq \mathbf{\Sigma}^0_\alpha(X,\tau)$ and $f\colon (X,\tau') \to Y$ is continuous.*

In particular, $f\colon (X,\tau) \to Y$ is Borel if and only if there is a Polish topology $\tau' \supseteq \tau$ on X such that $\mathbf{Bor}(X,\tau') = \mathbf{Bor}(X,\tau)$ and $f\colon (X,\tau') \to Y$ is continuous.

Proof. One direction is obvious, so let us assume that f is $\boldsymbol{\Sigma}^0_\alpha$-measurable for some $\alpha > 1$ (the case $\alpha = 1$ is trivial). Let $\{U_n \mid n \in \omega\}$ be a countable basis for Y, and let $B_{n,i} \in \boldsymbol{\Delta}^0_\alpha(X,\tau)$ be such that $f^{-1}(U_n) = \bigcup_{i\in\omega} B_{n,i}$. Apply Theorem 2.2.2 to these $B_{n,i}$: then the resulting topology τ' is as required. $\square$

All Baire-measurable functions, hence all Borel functions, are continuous on a comeager set.

The following result connects the Baire and the Borel hierarchy; indeed, originally the stratification of Borel sets was actually *defined* through the Baire hierarchy as in the statement of the next theorem.

Theorem 2.4.13. *Let X be a Polish space and $\alpha < \omega_1$. Then $A \subseteq X$ is $\boldsymbol{\Pi}^0_{\alpha+1}$ if and only if it is the zero-set of a real-valued Baire class α function.*

Proof. One direction is obvious: if $f\colon X \to \mathbb{R}$ is of Baire class α, then f is $\boldsymbol{\Sigma}^0_{\alpha+1}$-measurable by Theorem 2.4.11, hence $f^{-1}(0) \in \boldsymbol{\Pi}^0_{\alpha+1}(X)$ because $\{0\}$ is closed.

Conversely, consider first the case $\alpha = 0$. Let d be a compatible metric on X. Given $A \subseteq X$, define $f\colon X \to \mathbb{R}$ by setting $f(x) = d(x,A) = \inf\{d(x,y) \mid y \in A\}$; it is easy to check that f is continuous, and moreover $A \subseteq f^{-1}(0)$. On the other hand, $f(x) = 0$ implies that x is a limit point of A, hence if A is closed $x \in A$: therefore in this case $A = f^{-1}(0)$.

Let now $\alpha \geq 1$, and let $A \in \boldsymbol{\Pi}^0_{\alpha+1}(X)$, so that $A = \bigcap_{n\in\omega} A_n$ with $A_n \in \boldsymbol{\Delta}^0_{\alpha+1}(X)$. By Theorem 2.2.2, there is a Polish topology τ' refining the topology τ of X such that each A_n is τ'-clopen and $\tau' \subseteq \boldsymbol{\Sigma}^0_{\alpha+1}(X,\tau)$. Then A is τ'-closed, so there if a continuous $f\colon (X,\tau') \to \mathbb{R}$ such that $f^{-1}(0) = A$. But then $f\colon (X,\tau) \to \mathbb{R}$ is $\boldsymbol{\Sigma}^0_{\alpha+1}$-measurable, therefore f is of Baire class α by Theorem 2.4.11. $\square$

3

Analytic and projective sets

3.1 Analytic sets

Definition 3.1.1 ([Kec95, Definition 14.1])**.** Let X be a separable metrizable space. A set $A \subseteq X$ is called **analytic** if there is a Polish space Y and a continuous surjection from Y onto A, i.e. a function $f\colon Y \to X$ such that $f(Y) = A$. (The empty set is analytic, by taking $Y = \emptyset$.) A set $A \subseteq X$ is called **coanalytic** if $X \setminus A$ is analytic, and it is called bi-analytic if both A and its complement are analytic. The collection of analytic (respectively, coanalytic, bi-analytic) subsets of X is denoted by $\boldsymbol{\Sigma}^1_1(X)$ (respectively, $\boldsymbol{\Pi}^1_1(X)$, $\boldsymbol{\Delta}^1_1(X)$).

Proposition 3.1.2. *Let X be Polish and $\emptyset \neq A \subseteq X$. The following are equivalent:*

(i) *A is analytic;*

(ii) *A is a continuous image of ω^ω;*

(iii) *$A = \pi_X(C)$ for some nonempty $C \in \boldsymbol{\Pi}^0_1(X \times \omega^\omega)$, where π_X is the projection on X;*

(iv) *$A = \pi_X(C)$ for some Polish Y and nonempty $C \in \mathbf{Bor}(X \times Y)$;*

(v) *$A = f(C)$ for some nonempty $C \in \mathbf{Bor}(Y)$ and $f\colon Y \to X$ Borel, where Y is any Polish space (or even just $Y = \omega^\omega$).*

In particular, when $X = \omega^\omega$, then $A \subseteq X$ is analytic if and only it is the projection of the body of a (pruned) tree on $\omega \times \omega$.

Proof. (i) $\Rightarrow$ (ii). Let $f\colon Y \to A$ be a continuous surjection with Y Polish (necessarily nonempty). By Theorem 1.3.17, there is a continuous surjection $g\colon \omega^\omega \to Y$: thus A is the image of ω^ω under the continuous function $f \circ g$.

(ii) $\Rightarrow$ (iii). Let $f\colon \omega^\omega \to A$ be a continuous surjection. Then $C = (\mathrm{graph}(f))^{-1} \subseteq X \times \omega^\omega$ is closed, and $A = \pi_X(C)$.

(iii) $\Rightarrow$ (iv) and (iv) $\Rightarrow$ (v) are obvious.

(v) $\Rightarrow$ (i). First observe that we can assume $C = Y = \omega^\omega$ because every nonempty Borel subset of a Polish space is a continuous image of ω^ω by Proposition 2.2.5, so let $f\colon \omega^\omega \to X$ be a Borel function such that $A = f(\omega^\omega)$. By

Proposition 2.4.12, there is a Polish topology τ' on ω^ω such that $f\colon (\omega^\omega, \tau') \to X$ is continuous, hence A is analytic. $\square$

Remark 3.1.3. By Corollary 3.6.11, in (ii)–(iii) of Proposition 3.1.2 we cannot replace ω^ω with 2^ω or with any σ-compact space (like the real line $\mathbb{R}$). However, since ω^ω is homeomorphic to a G_δ subset of 2^ω part (iii) could be replaced by

(iii′) $A = \pi_X(G)$ for some nonempty $G \in \mathbf{\Pi}^0_2(X \times 2^\omega)$.

By part (v) the notion of analytic set depends only on the Borel structure of Polish spaces rather than on their topology, and it is thus immune from changes of topologies as those considered in Sections 2.2 and 2.4. Indeed, such a formulation of analyticity can be used to define $\mathbf{\Sigma}^1_1$ subsets of standard Borel spaces. Moreover, since any Borel set is the image of itself through the identity function, we easily get

Corollary 3.1.4. *For any Polish space X,*

$$\mathbf{Bor}(X) \subseteq \mathbf{\Sigma}^1_1(X).$$

Since $\mathbf{Bor}(X)$ is closed under complements, it also follows $\mathbf{Bor}(X) \subseteq \mathbf{\Delta}^1_1(X)$.

Notice that Corollary 3.1.4 also follows from Proposition 2.2.5.

Proposition 3.1.5. *Let X be a Polish space. Then $\mathbf{\Sigma}^1_1(X)$ is closed under countable unions and countable intersections. Moreover, analytic sets are closed under Borel images and preimages, that is: if X, Y are Polish and $f\colon X \to Y$ is Borel, then for $A \subseteq X$ and $B \subseteq Y$ analytic, both $f(A)$ and $f^{-1}(B)$ are analytic. In particular, $\mathbf{\Sigma}^1_1$ is a boldface pointclass.*

Proof. Let $A_n \subseteq X$, $n \in \omega$, be analytic sets, and let $f_n\colon Y_n \to X$ be continuous functions witnessing this, i.e. such that $f_n(Y_n) = A_n$. Without loss of generality, we may assume the spaces Y_n to be pairwise disjoint. Then $\bigcup_{n\in\omega} f_n$ is a continuous surjection from the countable sum $\bigoplus_{n\in\omega} Y_n$ onto $\bigcup_{n\in\omega} A_n$, hence the latter set is analytic. For the countable intersection, let $Z = \{(y_n)_{n\in\omega} \in \prod_{n\in\omega} Y_n \mid f_n(y_n) = f_m(y_m)\}$, Then Z is the preimage of the diagonal of X^ω under the (continuous) product of the f_n's, and thus it is closed (hence Polish). Then the map $g\colon Z \to X$ sending $(y_n)_{n\in\omega}$ to $f_0(y_0)$ is continuous and such that $g(Z) = \bigcap_{n\in\omega} A_n$, thus the intersection of the A_n's is analytic.

Closure under Borel images follows from Proposition 3.1.2(v), so let us show that if $B \in \mathbf{\Sigma}^1_1(Y)$ and $f\colon X \to Y$ is Borel then $f^{-1}(B) \in \mathbf{\Sigma}^1_1(X)$. Consider $\operatorname{graph}(f) \subseteq X \times Y$, which is Borel by Proposition 2.4.4 and thus analytic by Proposition 2.2.5. Notice that $x \in f^{-1}(B)$ if and only if there is $y \in Y$ such that $y \in B$ and $(x, y) \in \operatorname{graph}(f)$, i.e.

$$f^{-1}(B) = \pi_X(F)$$

where $F = \{(x, y) \in X \times Y \mid y \in B\} \cap \operatorname{graph}(f)$. Let $g\colon Z \to Y$ be a continuous function on a Polish space Z such that $g(Z) = B$. Then the map $X \times Z \to X \times Y$ defined by $(x, z) \mapsto (x, g(z))$ is continuous, and its range is

$\{(x, y) \in X \times Y \mid y \in B\}$, which is thus an analytic set. It follows that F is analytic because $\mathbf{\Sigma}^1_1(X \times Y)$ is closed under (countable) intersections, and therefore $f^{-1}(B)$ is analytic because it is the continuous image of an analytic set. □

Remark 3.1.6. Closure of $\mathbf{\Sigma}^1_1(X)$ under countable unions and intersections yields an alternative proof the Borel sets are (bi-)analytic. Indeed, all open and closed subsets of a Polish space X are analytic because they are Polish, and thus can be written as a continuous image of themselves via the identity function. Since $\mathbf{Bor}(X)$ is the smallest class of subsets of X containing open and closed sets and closed under countable unions and intersection, the result follows.

Remark 3.1.7. It is clear that according to Definition 3.1.1, being analytic does not depend too much on the ambient space: if $Y \subseteq X$ are separable metrizable spaces and $A \subseteq Y$, then $A \in \mathbf{\Sigma}^1_1(Y)$ if and only if $A \in \mathbf{\Sigma}^1_1(X)$ (any continuous function witnessing one of the two statements, witnesses the other one as well). Using this and closure under finite intersections we easily get that if $Y \subseteq X$ are Polish, then $\mathbf{\Sigma}^1_1(Y) = \mathbf{\Sigma}^1_1(X) \restriction Y$. This equality holds even if we just have $Y \in \mathbf{\Sigma}^1_1(X)$, but it fails if Y is *not* analytic: in such case, we still have $Y \in \mathbf{\Sigma}^1_1(X) \restriction Y$ because Y can be written as $X \cap Y$ and $X \in \mathbf{\Sigma}^1_1(X)$, but $Y \notin \mathbf{\Sigma}^1_1(Y)$ because otherwise it would be the image of a Polish space, and so $Y \in \mathbf{\Sigma}^1_1(X)$.

Example 3.1.8. Recall from Example 2.1.30 the Polish space $\mathrm{LO} \subseteq 2^{\omega \times \omega}$ of (codes for) countable linear orders. Set

$$\mathrm{WO} = \{x \in \mathrm{LO} \mid x \text{ is a well-order}\}$$

and

$$\mathrm{NWO} = \mathrm{LO} \setminus \mathrm{WO}.$$

Then $x \in \mathrm{NWO} \iff \exists z \in \omega^\omega \, \forall n < m \, (z(n) \neq z(m) \wedge x(z(m), z(n)) = 1)$. Therefore $\mathrm{NWO} = \pi_{\mathrm{LO}} \left(\bigcap_{n<m} C_{n,m} \right)$, where $C_{n,m} = \{(x, z) \in \mathrm{LO} \times \omega^\omega \mid z(n) \neq z(m) \wedge x(z(m), z(n)) = 1\}$. Since each $C_{n,m}$ is clopen, NWO is analytic, hence WO is coanalytic.

Example 3.1.9. Let $2^{(\omega^{<\omega})}$ be endowed with the product over the countable index set $\omega^{<\omega}$ of the discrete topology on $2 = \{0, 1\}$. Let $\mathrm{Tr} \subseteq 2^{(\omega^{<\omega})}$ be the set consisting of all characteristic functions of trees on ω. Then Tr is closed in $2^{(\omega^{<\omega})}$ (Exercise!), hence a Polish space. Let $\mathrm{IF} \subseteq \mathrm{Tr}$ be the set of (characteristic functions of) ill-founded trees, i.e. of those tree T on ω such that $[T] \neq \emptyset$. Fix a bijection $h \colon \omega \to \omega^{<\omega}$. Then

$$x \in \mathrm{IF} \iff \exists z \in \omega^\omega \, \forall n < m \, (x(h(z(n))) = 1 \wedge h(z(n)) \subsetneq h(z(m))).$$

Therefore

$$\mathrm{IF} = \pi_{\mathrm{Tr}} \left(\bigcap_{n \leq m} (A_n \cap B_{n,m}) \right),$$

where $A_n = \{(x, z) \in \mathrm{Tr} \times \omega^\omega \mid x(h(z(n))) = 1\}$ and $B_{n,m} = \{(x, z) \in \mathrm{Tr} \times \omega^\omega \mid h(z(n)) \subseteq h(z(m))\}$. Since all the A_n and $B_{n,m}$ are clopen, if follows that IF is the projection of a closed subset of $\mathrm{Tr} \times \omega^\omega$, hence $\mathrm{IF} \in \mathbf{\Sigma}^1_1(\mathrm{Tr})$.

Remark 3.1.10. In the previous examples we again implicitly used the (expanded version of the) Tarski-Kuratowski algorithm presented in Remark 2.1.8, where we further exploit the fact that the existential quantification $\exists x$ with x varying over an uncountable Polish (or even just standard Borel) space corresponds to a projection. Thus, if a subset of a Polish space X is defined by a formula $\psi(x)$ of the form

$$\exists y_0 \dots \exists y_n \, \varphi(x, y_0, \dots, y_n)$$

with each y_i varying over an uncountable Polish space Y_i and $\varphi(x, y_0, \dots, y_n)$ a formula involving only quantifications over countable sets, connectives, and "atomic formulas" defining (at most) Borel subsets of $X \times Y_0 \times \dots \times Y_n$, then $\varphi(x, y_0, \dots, y_n)$ defines a Borel set $B \subseteq X \times Y_0 \times \dots \times Y_n$ by Remark 2.1.8, hence $\psi(x)$ defines the $\mathbf{\Sigma}^1_1(X)$ set $A = \pi_X(B)$.

The universal quantification $\forall x$, with x varying over an uncountable Polish or standard Borel space, is equivalent to $\neg\exists x \neg$. Therefore, if $\varphi(x, y_0, \dots, y_n)$ defines a Borel subset of $X \times Y_0 \times \dots \times Y_n$, then

$$\forall y_0 \dots \forall y_n \, \varphi(x, y_0, \dots, y_n)$$

defines a coanalytic subset of X (namely: the complement of the projection on X of the set defined by $\neg\varphi(x, y_0, \dots, y_n)$).

Finally, observe that by the closure properties of $\mathbf{\Sigma}^1_1$ and $\mathbf{\Pi}^1_1$ it follows that:

- if φ defines a set in $\mathbf{\Sigma}^1_1(X)$ (respectively, $\mathbf{\Pi}^1_1(X)$), then both $\exists n \, \varphi$ and $\forall n \, \varphi$, with n varying over a countable set, define an analytic (respectively, coanalytic) subset of X;
- if φ defines a set in $\mathbf{\Sigma}^1_1(X)$ (respectively, $\mathbf{\Pi}^1_1(X)$), then $\neg\varphi$ defines a coanalytic (respectively, analytic) subset of X.

Exercise 3.1.11. Let X be a Polish space. Show that

$$A = \{(x_n)_{n\in\omega} \in X^\omega \mid (x_n)_{n\in\omega} \text{ has a convergent subsequence}\}$$

is analytic.

Solution. Let d be a compatible metric on X. Notice that

$$\begin{aligned}(x_n)_{n\in\omega} \in A \iff & \exists y \in \omega^\omega \, (\forall n, m \in \omega \, (y(n) \neq y(m)) \wedge \\ & \forall \varepsilon \in \mathbb{Q}^+ \, \exists N \in \omega \, \forall n, m \geq N \, (d(x_{y(n)}, x_{y(m)}) < \varepsilon)),\end{aligned}$$

and then use the Tarski-Kuratowski algorithm from Remark 3.1.10. Alternatively, observe that

$$(x_n)_{n\in\omega} \in A \iff \exists y \in X \, \forall \varepsilon \in \mathbb{Q}^+ \, \exists n \in \omega \, (d(x_n, y) < \varepsilon).$$

Exercise 3.1.12. Show that both

$$\mathrm{CN} = \{(f_n)_{n\in\omega} \in C([0;1])^\omega \mid (f_n)_{n\in\omega} \text{ converges pointwise}\}$$

and

$$\mathrm{CN}_0 = \{(f_n)_{n\in\omega} \in C([0;1])^\omega \mid f_n \to 0 \text{ pointwise}\}$$

are coanalytic subsets of $C([0;1])^\omega$.

Solution. We have

$$(f_n)_{n\in\omega} \in \mathrm{CN} \iff \forall x \in [0;1]\, \forall \varepsilon \in \mathbb{Q}^+\, \exists N \in \omega \\ \forall n, m \geq N\, (d(f_n(x), f_m(x)) < \varepsilon)$$

and

$$(f_n)_{n\in\omega} \in \mathrm{CN}_0 \iff \forall x \in [0;1]\, \forall \varepsilon \in \mathbb{Q}^+\, \exists N \in \omega \\ \forall n \geq N\, (d(f_n(x), 0) < \varepsilon).$$

Exercise 3.1.13. Let A be the set of sequences $(x_n)_{n\in\omega} \in \mathbb{R}^\omega$ such that all the accumulation points of $(x_n)_{n\in\omega}$ are multiple of the same integer $k \in \mathbb{Z}$ (where such k may vary depending on the sequence $(x_n)_{n\in\omega}$ under consideration). Show that $A \in \mathbf{\Pi}^1_1(\mathbb{R}^\omega)$.

Solution. We have that $(x_n)_{n\in\omega} \in A$ if and only if

$$\exists k \in \mathbb{Z}\, \forall y \in \mathbb{R}\, (\forall \varepsilon \in \mathbb{Q}^+\, \exists n \in \omega\, (d(y, x_n) < \varepsilon) \to \exists j \in \mathbb{Z}\, (y = k \cdot j)).$$

Exercise 3.1.14. Show that the collection of sequences $(x_n)_{n\in\omega} \in [0;1]^\omega$ having an irrational among their accumulation points is analytic.

Theorem 3.1.15 (Souslin, see [Kec95, Theorem 14.2])**.** *Let X be a Polish space. Then for every uncountable Polish space Y there is a Y-universal set for $\mathbf{\Sigma}^1_1(X)$. Therefore, if X is infinite then $|\mathbf{\Sigma}^1_1(X)| = 2^{\aleph_0}$. Moreover, if X is uncountable then $\mathbf{\Sigma}^1_1(X) \neq \mathbf{\Pi}^1_1(X)$, $\mathbf{\Sigma}^1_1(X)$ is not closed under complements, and $\mathbf{Bor}(X) \subseteq \mathbf{\Delta}^1_1(X) \subsetneq \mathbf{\Sigma}^1_1(X)$.*

Proof. The additional parts follow from the first one by Lemma 2.1.15 and using standard arguments (see Section 2.1), so let us show that there is a Y-universal set for $\mathbf{\Sigma}^1_1(X)$ for any uncountable Polish space Y. Let $\mathcal{V}$ be Y-universal for $\mathbf{\Pi}^0_1(X \times \omega^\omega)$, and let $\mathcal{U} = \pi_{Y\times X}(\mathcal{V})$. Then $\mathcal{U}$ and all its vertical sections are clearly analytic. Conversely, given $A \in \mathbf{\Sigma}^1_1(X)$ let $C \in \mathbf{\Pi}^0_1(X \times \omega^\omega)$ be such that $A = \pi_X(C)$, and let $y \in Y$ be such that $C = \mathcal{V}_y$. Then $A = \mathcal{U}_y$. Indeed, for every $x \in X$

$$\begin{aligned} x \in A &\iff \exists z\, ((x, z) \in C) \\ &\iff \exists z\, ((y, x, z) \in \mathcal{V}) \\ &\iff (y, x) \in \mathcal{U} \\ &\iff x \in \mathcal{U}_y. \end{aligned}$$

□

Corollary 3.1.16. *The collection $\mathbf{\Pi}^1_1$ and $\mathbf{\Delta}^1_1$ are boldface pointclasses, and* $\mathbf{Bor}(X) \subseteq \mathbf{\Delta}^1_1(X) \subseteq \mathbf{\Pi}^1_1(X)$ *for any Polish space X; if moreover X is uncountable, then $\mathbf{\Delta}^1_1(X) \subsetneq \mathbf{\Pi}^1_1(X)$. Finally, $\mathbf{\Pi}^1_1(X)$ is closed under countable unions and countable intersections but not under complements, while $\mathbf{\Delta}^1_1(X)$ is a σ-algebra; moreover both $\mathbf{\Pi}^1_1$ and $\mathbf{\Delta}^1_1$ are closed under Borel preimages but, if X is uncountable, they are not closed under continuous (hence neither Borel) images.*

Proof. All the statements easily follow from the analogous closure properties of $\mathbf{\Sigma}^1_1(X)$ given in Proposition 3.1.5 and from Theorem 3.1.15. To see that when X is uncountable $\mathbf{\Pi}^1_1$ is not closed under continuous images, observe that if this were true then $\mathbf{\Sigma}^1_1(X) \subseteq \mathbf{\Pi}^1_1(X)$ by $\mathbf{Bor}(X) \subseteq \mathbf{\Pi}^1_1(X)$ and Proposition 3.1.2(v); thus it would follow that $\mathbf{\Sigma}^1_1(X) = \mathbf{\Delta}^1_1(X)$, contradicting Theorem 3.1.15. The case of $\mathbf{\Delta}^1_1(X)$ is similar. □

Since universal sets for $\mathbf{\Gamma}(X)$ are in fact $\mathbf{\Gamma}$-complete (Lemma 2.1.23), it follows that there are $\mathbf{\Sigma}^1_1$-complete and $\mathbf{\Pi}^1_1$-complete sets. We now consider some particularly important examples of this phenomenon.

Example 3.1.17 (see [Kec95, Theorem 27.1])**.** The set IF of (characteristic functions of) ill-founded trees on ω is a $\mathbf{\Sigma}^1_1$-complete subset of Tr (see Example 3.1.9). Indeed, let $A \subseteq \omega^\omega$ be an arbitrary analytic set, and let T be a (pruned) tree on $\omega \times \omega$ such that A is the projection of $[T]$. Consider the continuous map $f \colon \omega^\omega \to \mathrm{Tr}$ sending $x \in \omega^\omega$ to the (characteristic function of the) tree $T(x) = \{t \in \omega^{<\omega} \mid (x \restriction \mathrm{lh}(t), t) \in T\}$. Then $x \in A \iff T(x) \in \mathrm{IF}$, so that $A = f^{-1}(\mathrm{IF})$.

It follows that the set

$$\mathrm{WF} = \{x \in \mathrm{Tr} \mid x \text{ is (the characteristic function of) a well-founded tree}\}$$

is $\mathbf{\Pi}^1_1$-complete.

Example 3.1.18 (Lusin-Sierpiński, see [Kec95, Theorem 27.12])**.** Recall the setup from Example 3.1.8. The set NWO of (codes for) non-well-founded linear orders of ω is a $\mathbf{\Sigma}^1_1$-complete subset of LO (hence the set WO of countable well-orders is $\mathbf{\Pi}^1_1$-complete). To see this, it suffices to show that there is a continuous $f \colon \mathrm{Tr} \to \mathrm{LO}$ such that $\mathrm{IF} = f^{-1}(\mathrm{NWO})$. Order $\omega^{<\omega}$ with the so-called **Kleene-Brouwer ordering** $<_{KB}$, where for $s, t \in \omega^{<\omega}$

$$s <_{KB} t \iff s \supsetneq t \vee \exists i < \mathrm{lh}(s), \mathrm{lh}(t)\, (s \restriction i = t \restriction i \wedge s(i) < t(i)).$$

Notice that a tree T on ω is well-founded if and only if the restriction of $<_{KB}$ to T is a well-order ([Kec95, Proposition 2.12]). Let $h \colon \omega \to \omega^{<\omega}$ be any bijection. Given $x \in \mathrm{Tr}$, let T_x be the tree on ω whose characteristic function is x and define the (code for a) linear order $f(T_x)$ by setting

$$\begin{aligned} f(T_x)(n,m) = 1 \iff & (h(n), h(m) \in T_x \wedge h(n) <_{KB} h(m)) \vee \\ & (h(n) \in T_x \wedge h(m) \notin T_x) \vee \\ & (h(n), h(m) \notin T_x \wedge n < m). \end{aligned}$$

Thus $f(T_x)$ is a linear ordering on ω isomorphic (via h) to the ordering of $\omega^{<\omega}$ in which all elements of T_x precede those in $\omega^{<\omega} \setminus T_x$, the elements of T_x are ordered by $<_{KB}$, and the elements of $\omega^{<\omega} \setminus T_x$ are ordered using the bijection h (thus this last part is always well-ordered with order type $\leq \omega$). Then clearly $x \in \mathrm{IF} \iff f(x) \in \mathrm{NWO}$, hence NWO is $\mathbf{\Sigma}^1_1$-hard (and thus also $\mathbf{\Sigma}^1_1$-complete) by Example 3.1.18.

Here are some further examples of $\mathbf{\Sigma}^1_1$- and $\mathbf{\Pi}^1_1$-complete sets (we omit the relevant proofs here, referring the reader to [Kec95] for more details).

(1) The set of compact $K \subseteq [0;1]$ such that K contains an irrational number is a $\mathbf{\Sigma}^1_1$-complete subset of $K([0;1])$ (Hurewicz, see [Kec95, Exercise 27.4]).

(2) If X is an uncountable Polish space, then the set of all countable compact subsets of X is a $\mathbf{\Pi}^1_1$-complete subset of $K(X)$ (Hurewicz, see [Kec95, Theorem 27.5]).

(3) Let X be a Polish space which is not σ-compact. Then the set of all sequences $(x_n)_{n\in\omega} \in X^\omega$ having a convergent subsequence is $\mathbf{\Sigma}^1_1$-complete (see [Kec95, Exercise 27.15]).

(4) The set of all differentiable $f \in C([0;1])$ is $\mathbf{\Pi}^1_1$-complete (Mazurkiewicz, see [Kec95, Theorem 33.9]).

(5) The set of all $f \in C([0;1])$ satisfying Rolle's theorem (i.e. those f for which for all $a < b$ in $[0;1]$, if $f(a) = f(b)$, there is $c \in (a;b)$ with $f'(c) = 0$) is $\mathbf{\Sigma}^1_1$-complete (Woodin).

(6) Both

$$\mathrm{CN} = \{(f_n)_{n\in\omega} \in C([0;1])^\omega \mid (f_n)_{n\in\omega} \text{ converges pointwise}\}$$

and

$$\mathrm{CN}_0 = \{(f_n)_{n\in\omega} \in C([0;1])^\omega \mid f_n \to 0 \text{ pointwise}\}$$

are $\mathbf{\Pi}^1_1$-complete subsets of $C([0;1])^\omega$ (see [Kec95, Theorem 33.11]).

The descriptive set theoretic complexity can be used e.g. to show that two classes of objects are different (because they have different complexity). As an example of such applications, we consider the following old problem in Banach space theory. Given a class $\mathcal{F}$ of separable Banach spaces, a separable Banach space Y is called **universal** for $\mathcal{F}$ if all its closed subspace are in $\mathcal{F}$ and, conversely, every $X \in \mathcal{F}$ is isomorphic to some closed subspace of Y. For example, $C(2^\omega)$ is universal for the class of all separable Banach spaces. Problem 49 in the Scottish Book, due to Banach and Mazur, asks whether there is a separable Banach space with separable dual, which is universal for the class of separable Banach spaces with separable dual. Wojtaszczyk answered this negatively using methods of Szlenk. Here we sketch an easier proof due to Burgain involving the computation of the complexity of the relevant classes of Banach spaces.

Theorem 3.1.19 ([Kec95, Section 33.K])**.** *There is no universal space for the class of separable Banach spaces having separable dual.*

Sketch of the proof. For the sake of definiteness, we consider the case of real Banach spaces (the case of complex Banach spaces is similar). Given $K \in K(2^\omega) \setminus \{\emptyset\}$, consider the separable Banach space $C(K)$. The dual $C^*(K)$ of $C(K)$ is the space of signed Borel measures on K (see [Kec95, Exercise 17.32]). If K is countable, then $C^*(K)$ is isomorphic to l^1 if K is infinite, and to $\mathbb{R}^n$ if K has cardinality $n \in \omega$. Therefore $C^*(K)$ is separable when K is countable. On the other hand, if K is uncountable then $C^*(K)$ is non-separable. (Consider, for example, the Dirac measures δ_x for $x \in K$: if $x \neq y$, then $\|\delta_x - \delta_y\| = 2$.)

As shown in the proof of [Kec95, Theorem 33.24], the map $K \mapsto C(K)$ can be realized as a Borel map between the Polish space $K(2^\omega) \setminus \{\emptyset\}$ and the standard Borel space of all separable Banach spaces $\mathrm{Subs}(C(2^\omega))$ (see page 52). Thus the class

$$\mathcal{F} = \{X \in \mathrm{Subs}(C(2^\omega)) \mid X^* \text{ is separable}\}$$

is not analytic because, as we noticed, the collection of countable compacts $K \subseteq 2^\omega$ is a $\mathbf{\Pi}^1_1$-complete subset of $K(2^\omega)$ (and $\mathbf{\Pi}^1_1(K(2^\omega)) \neq \mathbf{\Sigma}^1_1(K(2^\omega))$ because $K(2^\omega)$ is an uncountable Polish space).

Assume now towards a contradiction that there is $Y \in \mathrm{Subs}(C(2^\omega))$ which is universal for $\mathcal{F}$. Using Theorem 2.3.6, one can see that the relation $\sqsubseteq$ of embedding (i.e. being isomorphic to a closed subspace) between separable Banach spaces is analytic, i.e. it is a $\mathbf{\Sigma}^1_1$ subset of the square $\mathrm{Subs}(C(2^\omega)) \times \mathrm{Subs}(C(2^\omega))$ (see [Kec95, Exercise 33.26]). It would then follow that

$$\mathcal{F} = \{X \in \mathrm{Subs}(C(2^\omega)) \mid X \sqsubseteq Y\}$$

is an analytic subset of $\mathrm{Subs}(C(2^\omega))$, a contradiction. □

3.2 Souslin's theorem

We noticed in Corollary 3.1.16 that $\mathbf{Bor}(X) \subseteq \mathbf{\Delta}^1_1(X)$ for every Polish space X: the next results show that in fact the two classes coincide.

Theorem 3.2.1 (The Lusin separation theorem, see [Kec95, Theorem 14.7]). *Let X be a Polish space, and let $A, B \subseteq X$ be two disjoint analytic sets. Then there is a Borel set $C \subseteq X$* ***separating*** *A from B, i.e. such that $A \subseteq C$ and $C \cap B = \emptyset$.*

Proof. Call $P, Q \subseteq X$ **Borel separable** if there is a Borel set R separating P from Q. Notice that if $P = \bigcup_{m\in\omega} P_m$, $Q = \bigcup_{n\in\omega} Q_n$, and P_m, Q_n are Borel separable for every $m, n \in \omega$, then P and Q are Borel separable as well. Indeed, If $R_{m,n}$ is a Borel set separating P_m from Q_n, then $\bigcup_{m\in\omega}\bigcap_{n\in\omega} R_{m,n}$ is a Borel set separating P from Q.

Assuming now, without loss of generality, that both A and B are nonempty, let $f\colon \omega^\omega \to A$ and $g\colon \omega^\omega \to B$ be continuous surjections. For $s \in \omega^{<\omega}$, set $A_s = f(\boldsymbol{N}_s)$ and $B_s = g(\boldsymbol{N}_s)$, so that $A = A_\emptyset$, $B = B_\emptyset$, $A_s = \bigcup_{m\in\omega} A_{s^\frown m}$, and $B_s = \bigcup_{n\in\omega} B_{s^\frown n}$. Let

$$T_{A,B} = \{(s,t) \in \omega^{<\omega} \times \omega^{<\omega} \mid \mathrm{lh}(s) = \mathrm{lh}(t) \wedge A_s, B_t \text{ are not Borel separable}\}.$$

By the first paragraph, $T_{A,B}$ is a pruned tree. Suppose towards a contradiction that A and B are not Borel separable. Then $T_{A,B}$ is nonempty, hence $[T_{A,B}] \neq \emptyset$. Pick any $(x, y) \in [T_{A,B}]$: since $A \cap B = \emptyset$, $f(x) \in A$, and $g(y) \in B$, we get $f(x) \neq g(y)$. Let U, V be disjoint open neighborhoods of $f(x)$ and $g(y)$, respectively, and use the continuity of f and g to find $n \in \omega$ large enough so that $\boldsymbol{N}_{x \restriction n} \subseteq f^{-1}(U)$ and $\boldsymbol{N}_{y \restriction n} \subseteq g^{-1}(V)$, i.e. $A_{x \restriction n} \subseteq U$ and $B_{y \restriction n} \subseteq V$. Then U would be a Borel set separating $A_{x \restriction n}$ from $B_{y \restriction n}$, contradicting $(x \restriction n, y \restriction n) \in T_{A,B}$. □

Corollary 3.2.2 ([Kec95, Corollary 14.9])**.** *Let X be a Polish space, and let $(A_n)_{n \in \omega}$ be a sequence of pairwise disjoint analytic sets. Then there are pairwise disjoint Borel sets B_n with $B_n \supseteq A_n$.*

Proof. For distinct $n, m \in \omega$, let $C_{n,m}$ be a Borel set separating A_n from A_m. Inductively set $B_0 = \bigcap_{m>0} C_{0,m}$ and $B_{n+1} = \bigcap_{m \neq n+1} C_{n+1,m} \setminus \bigcup_{i \leq n} B_i$. Since by definition $B_n \cap A_m = \emptyset$ for $m \neq n$, it follows that $A_n \subseteq B_n$, hence we are done. □

Theorem 3.2.3 (Souslin's theorem, see [Kec95, Theorem 14.11])**.** *Let X be a Polish space. Then* $\mathbf{Bor}(X) = \boldsymbol{\Delta}^1_1(X)$.

Proof. Given $A \in \boldsymbol{\Delta}^1_1(X)$, take $B = X \setminus A$ in Theorem 3.2.1. □

A consequence of this is the following result, which provides a partial converse to Proposition 2.4.4.

Theorem 3.2.4 ([Kec95, Theorem 14.12])**.** *Let $f \colon X \to Y$ be a function between the Polish spaces X and Y. Then the following are equivalent:*

(i) *f is a Borel function;*

(ii) $\mathrm{graph}(f)$ *is Borel;*

(iii) $\mathrm{graph}(f)$ *is analytic.*

In particular, if f is a Borel bijection, then f is a Borel isomorphism (i.e. f^{-1} is also Borel).

Proof. It is enough to show that if $\mathrm{graph}(f)$ is analytic then f is a Borel function. Given any Borel $B \subseteq Y$, for every $x \in X$

$$x \in f^{-1}(B) \iff \exists y\, ((x, y) \in \mathrm{graph}(f) \wedge y \in B),$$

hence $f^{-1}(B)$ is analytic. Since $\mathrm{Bor}(Y)$ is closed under complements, it follows that $f^{-1}(B) \in \boldsymbol{\Delta}^1_1(X) = \mathrm{Bor}(X)$ for every $B \in \mathbf{Bor}(Y)$. □

Although the continuous image of a Borel set need not be Borel, we have the following basic fact.

Theorem 3.2.5 (Lusin-Souslin, see [Kec95, Theorem 15.1])**.** *Let X, Y be Polish spaces and $f \colon X \to Y$ be continuous. If $A \subseteq X$ is Borel and $f \restriction A$ is injective, then $f(A)$ is Borel.*

Proof. By Proposition 2.2.5, we can assume that $X = \omega^\omega$ and A is closed. By Lemma 1.3.9(a)–(b) the ω-scheme $\mathcal{S}_f = \{B_s \mid s \in \omega^{<\omega}\}$, where $B_s = f(A \cap \boldsymbol{N}_s)$, induces f, and moreover $B_{s^\frown n} \cap B_{s^\frown m} = \emptyset$ for all $s \in \omega^{<\omega}$ and distinct $n, m \in \omega$ because f is injective. Since by definition each B_s is analytic, we can apply level-by-level Corollary 3.2.2 to each sequence $(B_{s^\frown n})_{n\in\omega}$ in order to find a family $\{B'_s \mid s \in \omega^{<\omega}\}$ with B'_s Borel and such that $B_s \subseteq B'_s$ and $B'_{s^\frown n} \cap B'_{s^\frown m} = \emptyset$ for all $s \in \omega^{<\omega}$ and distinct $n, m \in \omega$. Finally, we turn the family $\{B'_s \mid s \in \omega^{<\omega}\}$ into an ω-scheme $\mathcal{S}^*$ such that $\mathcal{S}_f \sqsubseteq \mathcal{S}^* \sqsubseteq \mathrm{cl}(\mathcal{S}_f)$. This is obtained by recursively defining the sets B^*_s as follows:

$$
\begin{aligned}
B^*_\emptyset &= B'_\emptyset \cap \mathrm{cl}(B_\emptyset) \\
B^*_{s^\frown n} &= B'_{s^\frown n} \cap B^*_s \cap \mathrm{cl}(B_{s^\frown n}).
\end{aligned}
$$

(The fact that $B_s \subseteq B^*_s$ is proved by induction on $\mathrm{lh}(s)$, where in the inductive step we use in particular that $B_{s^\frown n} \subseteq B_s \subseteq B^*_s$.) By Lemma 1.3.9(e), the ω-scheme $\mathcal{S}^* = \{B^*_s \mid s \in \omega^{<\omega}\}$ still induces the function f. Moreover, we still have $B^*_{s^\frown n} \cap B^*_{s^\frown m} = \emptyset$ for all distinct $n, m \in \omega$ because $B^*_s \subseteq B'_s$ for all $s \in \omega^{<\omega}$. Thus by Lemma 1.3.6(b) we have $f(A) = \bigcap_{n\in\omega} \bigcup_{s\in\omega^n} B^*_s$, and since the B^*_s are all Borel sets by construction, so is $f(A)$. □

Corollary 3.2.6 ([Kec95, Corollary 15.2]). *Let $f \colon X \to Y$ be a Borel function between the Polish spaces X and Y. If $A \subseteq X$ is Borel and $f \restriction A$ is injective, then $f(A)$ is Borel and f is a Borel isomorphism between A and $f(A)$.*

Proof. Let τ be the topology of X, and apply Proposition 2.4.12 to get a Polish topology $\tau' \supseteq \tau$ on X such that $\mathbf{Bor}(X, \tau') = \mathbf{Bor}(X, \tau)$ and $f \colon (X, \tau') \to Y$ is continuous. Then $A \in \mathbf{Bor}(X, \tau')$, hence $f(A)$ is Borel by Theorem 3.2.5. Moreover, for every τ-open $U \subseteq X$, we have $U \cap A \in \mathbf{Bor}(X, \tau) = \mathbf{Bor}(X, \tau')$, hence $f(U \cap A)$ is Borel. Therefore $(f \restriction A)^{-1}$ is Borel and we are done. □

Corollary 3.2.7 ([Kec95, Exercise 15.3]). *Let X be a Polish space. Then the following are equivalent for a set $A \subseteq X$:*

(i) *A is Borel;*

(ii) *A is a continuous injective image of a closed subset of ω^ω;*

(iii) *A is a Borel injective image of a Borel subset of a Polish space.*

Proof. By Proposition 2.2.5 and Corollary 3.2.6. □

The next results allow us to classify Polish spaces up to Borel isomorphism.

Theorem 3.2.8 (The Borel Schröder-Bernstein theorem, see [Kec95, Theorem 15.7]). *Let X, Y be Polish spaces and $f \colon X \to Y$ and $g \colon Y \to X$ be Borel injections. Then X and Y are Borel isomorphic.*

Proof. It is enough to show that there are Borel sets $A \subseteq X$ and $B \subseteq Y$ such that $f(A) = Y \setminus B$ and $g(B) = X \setminus A$, because then $(f \restriction A) \cup (g^{-1} \restriction (X \setminus A))$ is a Borel isomorphism between X and Y.

Recursively define X_n, Y_n, $n \in \omega$, by setting $X_0 = X$, $Y_0 = Y$, $X_{n+1} = (g \circ f)(X_n)$, and $Y_{n+1} = (f \circ g)(Y_n)$. Let $X_\infty = \bigcap_{n\in\omega} X_n$ and $Y_\infty = \bigcap_{n\in\omega} Y_n$. Then $f(X_\infty) = Y_\infty$, $f(X_n \setminus g(Y_n)) = f(X_n) \setminus Y_{n+1}$, and $g(Y_n \setminus f(X_n)) = g(Y_n) \setminus X_{n+1}$. Finally, let $A = X_\infty \cup \bigcup_{n\in\omega}(X_n \setminus g(Y_n))$ and $B = \bigcup_{n\in\omega}(Y_n \setminus f(X_n))$. All these sets are Borel by Corollary 3.2.6. □

Theorem 3.2.9 (The Borel isomorphism theorem, see [Kec95, Theorem 15.6])**.** *Two Polish spaces X and Y are Borel isomorphic if and only if they have the same cardinality. In particular, two uncountable Polish spaces are Borel isomorphic.*

Proof. If X and Y are countable, then any bijection $f \colon X \to Y$ is a Borel isomorphism, so it is enough to show that every uncountable Polish space X is Borel isomorphic to 2^ω. By Theorem 1.3.17 and Corollary 3.2.6, X is Borel isomorphic to a closed subset of ω^ω, and hence to a $\mathbf{\Pi}^0_2$ subset of 2^ω. On the other hand, 2^ω is homeomorphic to a closed subset of X by Corollary 1.4.9. Therefore the result follows from Theorem 3.2.8. □

In all the results of this sections which depend only on the Borel structure of the Polish spaces under consideration, we could as well consider standard Borel spaces instead of Polish spaces. In particular, we get that any two uncountable standard Borel spaces are always Borel isomorphic (and the same for countable standard Borel spaces). In particular

Corollary 3.2.10. *Let $X = (X, \tau_X)$ and $Y = (Y, \tau_Y)$ be Polish spaces, and let $A \in \mathbf{Bor}(X)$ and $B \in \mathbf{Bor}(Y)$. Then there is a Borel isomorphism $f \colon A \to B$ if and only if A and B have the same cardinality. In particular, if A and B are both uncountable then they are Borel isomorphic.*

Proof. By Corollary 2.2.3 there are Polish topologies τ_A on X and τ_B on Y such that $\mathbf{Bor}(X, \tau_X) = \mathbf{Bor}(X, \tau_A)$, A is τ_A-clopen, $\mathbf{Bor}(Y, \tau_Y) = \mathbf{Bor}(Y, \tau_B)$, and B is τ_B-clopen. In particular, A and B are Polish spaces when endowed with the relative topologies of τ_A and τ_B, respectively. Therefore the result follows from Theorem 3.2.9 and the fact that our new topologies do not change the notion of Borelness in the corresponding spaces. □

3.3 Projective sets

Definition 3.3.1. Let X be a separable metrizable space. For $n \geq 1$, let $\mathbf{\Sigma}^1_{n+1}(X)$ be the collection of all $A \subseteq X$ for which there is a Polish space Y and a continuous $f \colon Y \to X$ such that $A = f(B)$ for some $B \in \mathbf{\Pi}^1_n(Y)$. Moreover we set

$$\mathbf{\Pi}^1_{n+1}(X) = \{X \setminus A \mid A \in \mathbf{\Sigma}^1_{n+1}(X)\} = \check{\mathbf{\Sigma}}^1_{n+1}(X)$$
$$\mathbf{\Delta}^1_{n+1}(X) = \mathbf{\Sigma}^1_{n+1}(X) \cap \mathbf{\Pi}^1_{n+1}(X) = \mathbf{\Delta}_{\mathbf{\Sigma}^1_{n+1}}(X).$$

A set $A \subseteq X$ is called **projective** if it belongs to $\mathbf{\Sigma}^1_n(X)$ for some $n \geq 1$; the collection of all projective subsets of X is denoted by $\mathbf{Proj}(X)$.

Proposition 3.3.2. *Let X be a Polish space. Then for every $n \geq 1$*

$$\mathbf{\Sigma}^1_n(X), \mathbf{\Pi}^1_n(X) \subseteq \mathbf{\Delta}^1_{n+1}(X) \subseteq \mathbf{\Sigma}^1_{n+1}(X), \mathbf{\Pi}^1_{n+1}(X).$$

In particular, $\mathbf{Bor}(X)$ *is contained in all these classes.*

Proof. By induction on $n \geq 1$. The inclusion $\mathbf{\Sigma}^1_n(X) \subseteq \mathbf{\Sigma}^1_{n+1}(X)$ follows from $\mathbf{Bor}(X) \subseteq \mathbf{\Pi}^1_1(X)$ in the case $n = 1$, and by the inductive hypothesis $\mathbf{\Pi}^1_{n-1}(X) \subseteq \mathbf{\Pi}^1_n(X)$ when $n > 1$. The inclusion $\mathbf{\Pi}^1_n(X) \subseteq \mathbf{\Sigma}^1_{n+1}(X)$ follows from the fact that every set is the image of itself under the identity function. □

Proposition 3.3.3. *Let X be a Polish space. Then for every $n \geq 1$, the classes $\mathbf{\Sigma}^1_n(X)$ and $\mathbf{\Pi}^1_n(X)$ are closed under countable unions and countable intersections. Moreover, the pointclasses $\mathbf{\Sigma}^1_n$ and $\mathbf{\Pi}^1_n$ are closed under Borel preimages (hence they are boldface pointclasses), and $\mathbf{\Sigma}^1_n$ is also closed under Borel images. Therefore, $\mathbf{\Delta}^1_n(X)$ is a σ-algebra and $\mathbf{\Delta}^1_n$ is closed under Borel preimages (hence a boldface pointclass).*

It can be shown ([Kec95, Exercise 37.8]) that if X is uncountable, then $\mathbf{\Delta}^1_{n+1}(X)$ is *not* the smallest σ-algebra containing $\mathbf{\Sigma}^1_n(X)$.

Proof. By induction on $n \geq 1$. The basic case $n = 1$ (corresponding to analytic, co-analytic, and bi-analytic sets) has already been treated, so let us consider the case $n > 1$. It is clearly enough to consider the case of $\mathbf{\Sigma}^1_n(X)$ — the stated properties for its associated dual and ambiguous pointclasses follows immediately. Let $A_k \in \mathbf{\Sigma}^1_n(X)$, and for each $k \in \omega$ let Y_k be a Polish space and $f_k \colon Y_k \to X$ be a continuous function such that $A_k = f(B_k)$ for some $B_k \in \mathbf{\Pi}^1_{n-1}(Y_k)$. Let $Y = \bigoplus_{k\in\omega} Y_k$ and $B = \bigcup_{k\in\omega} B_k$. Since $\mathbf{\Pi}^1_{n-1}$ is a boldface pointclass and $\mathbf{\Pi}^1_{n-1}(Y)$ is closed under countable unions, it follows that $B \in \mathbf{\Pi}^1_{n-1}(Y)$. Then $\bigcup_{k\in\omega} A_k = f(B)$ where $f \colon Y \to X$ is the continuous function $f = \bigcup_{k\in\omega} f_k$. Now consider the closed (hence Polish) subspace $F \subseteq \prod_{k\in\omega} Y_k$ defined by

$$(y_k)_{k\in\omega} \in F \iff \forall k, k' \, (f_k(y_k) = f_{k'}(y_{k'})).$$

Let $B = F \cap \prod_{k\in\omega} B_k = \bigcap_{k\in\omega} \pi_k^{-1}(B_k)$, where $\pi_k \colon F \to Y_k$ is the restriction to F of the projection function on the k-th coordinate. By the closure properties of $\mathbf{\Pi}^1_{n-1}$, we have $B \in \mathbf{\Pi}^1_{n-1}(F)$. Finally, the map $f = f_0 \circ \pi_0$ is continuous and such that $\bigcap_{k\in\omega} A_k = f(B)$.

Finally, let us show that if $f \colon X \to Y$ is a Borel function between the Polish spaces X and Y, and $A \in \mathbf{\Sigma}^1_n(X)$ and $B \in \mathbf{\Sigma}^1_n(Y)$, then both $f^{-1}(B) \in \mathbf{\Sigma}^1_n(X)$ and $f(A) \in \mathbf{\Sigma}^1_n(Y)$, starting with the former (the other one is similar). We have

$$\begin{aligned} f^{-1}(B) &= \{x \in X \mid \exists y \, ((x,y) \in \operatorname{graph}(f) \wedge y \in B\} \\ &= \{x \in X \mid \exists y \, ((x,y) \in \operatorname{graph}(f) \wedge (x,y) \in X \times B\}. \end{aligned}$$

Since $\operatorname{graph}(f)$ is Borel (and thus in $\mathbf{\Sigma}^1_n$), the pointclass $\mathbf{\Sigma}^1_n$ is closed under continuous functions (by definition), and we already proved its closure under (countable) intersections, we only need to check that $X \times B \in \mathbf{\Sigma}^1_n(X \times Y)$.

But if $C \in \mathbf{\Pi}^1_{n-1}(Z)$ with Z Polish and $f \colon Z \to Y$ is continuous and such that $f(C) = B$, then $\mathrm{id}_X \times f \colon X \times Z \to X \times Y$ is such that $f(X \times C) = X \times B$. But $X \times C = \pi_Z^{-1}(C)$ is in $\mathbf{\Pi}^0_{n-1}$ because by inductive hypothesis $\mathbf{\Pi}^0_{n-1}$ is closed under continuous preimages, hence we are done. The case of $f(A)$ is similar and is left to the reader as an exercise. $\square$

Remark 3.3.4. As for analytic sets (see Remark 3.1.7), if $Y \subseteq X$ are Polish, then $\mathbf{\Sigma}^1_n(Y) = \mathbf{\Sigma}^1_n(X) \restriction Y$ (for any $n \geq 1$), and similarly for $\mathbf{\Pi}^1_n$ and $\mathbf{\Delta}^1_n$. Indeed, the same if true when $Y \in \mathbf{\Sigma}^1_n(X)$ with X Polish. It follows that if Y is a Polish subspace (or even just a projective subset) of the Polish space X, then $\mathbf{Proj}(Y) = \mathbf{Proj}(X) \restriction Y$.

We now provide equivalent reformulations of the notion of a $\mathbf{\Sigma}^1_n$ set analoguous to the one considered in Proposition 3.1.2.

Proposition 3.3.5. *Let X be a Polish space. For every nonzero $n \in \omega$ and $A \subseteq X$ the following are equivalent:*

(i) $A \in \mathbf{\Sigma}^1_{n+1}(X)$*;*

(ii) $A = f(B)$*, where* $f \colon Y \to X$ *is a Borel function with* Y *Polish and* $B \in \mathbf{\Pi}^1_n(Y)$*;*

(iii) $A = f(B)$*, where* $f \colon \omega^\omega \to X$ *is a Borel function and* $B \in \mathbf{\Pi}^1_n(\omega^\omega)$*;*

(iv) $A = \pi_X(C)$ *for some* $C \in \mathbf{\Pi}^1_n(X \times \omega^\omega)$*;*

(v) $A = \pi_X(C)$ *for some Polish space* Y *and* $C \in \mathbf{\Pi}^1_n(X \times Y)$*.*

Moreover, in parts (iii)–(iv) we can replace ω^ω by any *uncountable Polish space Z.*

Proof. (i) $\Rightarrow$ (ii), (iv) $\Rightarrow$ (v), and (v) $\Rightarrow$ (i) are obvious. Moreover, (ii) $\Rightarrow$ (iii) because every nonempty Polish space Y is a continuous image of ω^ω by Theorem 1.3.17 and $\mathbf{\Pi}^1_n$ is a boldface pointclass by Proposition 3.3.3. Finally, let us prove that (iii) $\Rightarrow$ (iv). Let A, f, and B be as in (iii). Then it is enough to set $C = \{(x,y) \in X \times \omega^\omega \mid (y,x) \in \mathrm{graph}(f) \wedge y \in B\}$ and notice that C is as required by the closure properties of $\mathbf{\Pi}^1_n$ (Propositions 3.3.2 and 3.3.3).

The additional part follows from the fact that Z and ω^ω are Borel isomorphic by Theorem 3.2.9 and that $\mathbf{\Pi}^1_n$ is closed under Borel preimages by Theorem 3.3.3 (and thus under Borel isomorphisms in both directions). $\square$

Definition 3.3.6. Let $\mathbf{\Gamma}$ be a boldface pointclass. Then $\mathrm{proj}(\mathbf{\Gamma})$ is the pointclass defined for every space X by

$$\mathrm{proj}(\mathbf{\Gamma})(X) = \{\pi_X(C) \mid C \in \mathbf{\Gamma}(X \times \omega^\omega)\},$$

where π_X is the projection function on X.

Notice that since every Polish space is a continuous image of ω^ω by Theorem 1.3.17, then $A \in \mathrm{proj}(\boldsymbol{\Gamma})(X)$ if and only if there is *some* Polish space Z and some $C \in \boldsymbol{\Gamma}(X \times Z)$ such that $A = \pi_X(C)$. Moreover, it is not difficult to show that $\mathrm{proj}(\boldsymbol{\Gamma})$ is a boldface pointclass as well. Indeed, if $A = \pi_Y(C)$ for some $C \in \boldsymbol{\Gamma}(Y \times \omega^\omega)$ and $f \colon X \to Y$ is continuous, then $f^{-1}(A) = \pi_X(C')$ where $C' = (f \times \mathrm{id}_{\omega^\omega})^{-1}(C) \in \boldsymbol{\Gamma}(X \times \omega^\omega)$.

The following lemma generalizes the proof of Theorem 3.1.15.

Lemma 3.3.7. *Let $\boldsymbol{\Gamma} \supseteq \boldsymbol{\Pi}^0_1$ be a boldface pointclass closed under intersections. Let X be a Polish space, and assume that there is a Y-universal set $\mathcal{V}$ for $\boldsymbol{\Gamma}(X \times \omega^\omega)$ (where Y is any Polish space). Then there is a Y-universal set $\mathcal{U}$ for* $\mathrm{proj}(\boldsymbol{\Gamma})$.

Proof. Let $\mathcal{U} = \pi_{Y \times X}(\mathcal{V})$. Then $\mathcal{U} \in \mathrm{proj}(\boldsymbol{\Gamma})(Y \times X)$, and for all $y \in Y$ also $\mathcal{U}_y \in \mathrm{proj}(\boldsymbol{\Gamma})(X)$ because under our assumptions $\mathrm{proj}(X)$ is a boldface pointclass and $\mathcal{U}_y$ is the preimage of $\mathcal{U}$ under the continuous function $x \mapsto (y, x)$. Conversely, given $A \in \mathrm{proj}(\boldsymbol{\Gamma})(X)$ let $C \in \boldsymbol{\Gamma}(X \times \omega^\omega)$ be such that $A = \pi_X(C)$, and let $y \in Y$ be such that $C = \mathcal{V}_y$. Then $A = \mathcal{U}_y$. Indeed, for every $x \in X$

$$
\begin{aligned}
x \in A &\iff \exists z\, ((x, z) \in C) \\
&\iff \exists z\, ((y, x, z) \in \mathcal{V}) \\
&\iff (y, x) \in \mathcal{U} \\
&\iff x \in \mathcal{U}_y.
\end{aligned}
$$

□

The projective hierarchy does not collapse on uncountable Polish spaces. This is proved through the existence of suitable universal sets.

Theorem 3.3.8. *Let X be a Polish space. Then for every $n \geq 1$ and every uncountable Polish space Y there is a Y-universal set for $\boldsymbol{\Sigma}^1_n(X)$ (and hence also for $\boldsymbol{\Pi}^1_n(X)$). Theorefore, if X is infinite then*

$$|\boldsymbol{\Sigma}^1_n(X)| = |\boldsymbol{\Pi}^1_n(X)| = |\boldsymbol{\Delta}^1_n(X)| = |\mathbf{Proj}(X)| = 2^{\aleph_0}.$$

Moreover, if X is uncountable then $\boldsymbol{\Sigma}^1_n(X) \neq \boldsymbol{\Pi}^1_n(X)$, neither $\boldsymbol{\Sigma}^1_n(X)$ nor $\boldsymbol{\Pi}^1_n(X)$ are closed under complements, $\boldsymbol{\Pi}^1_n(X)$ is not closed under projections (equivalently, under continuous or Borel images),

$$\boldsymbol{\Sigma}^1_n(X) \subsetneq \boldsymbol{\Delta}^1_{n+1}(X) \subsetneq \boldsymbol{\Sigma}^1_{n+1}(X),$$

and the same for $\boldsymbol{\Pi}^1_{n(+1)}(X)$ in place of $\boldsymbol{\Sigma}^1_{n(+1)}(X)$.

Proof. By induction on $n \geq 1$, using Theorem 3.1.15, Lemma 3.3.7, and the fact that $\boldsymbol{\Sigma}^1_{n+1} = \mathrm{proj}(\boldsymbol{\Pi}^1_n)$ by Proposition 3.3.5. □

Using the fact that existential quantifications correspond to projections and that $\forall y\, \psi$ is logically equivalent to $\neg \exists y \neg \psi$, we can now see that the Tarski-Kuratowski algorithm introduced in Remarks 2.1.8 and 3.1.10 can be extended to include definitions with arbitrary quantifications. More precisely, we can add to the algorithm the following "rules":

- if $\psi(x,y)$ defines a Borel subset of $X \times Y$, then $\exists y\, \psi(x,y)$ and $\forall y\, \psi(x,y)$ define, respectively, a $\mathbf{\Sigma}^1_1$ and a $\mathbf{\Pi}^1_1$ subset of X;

- if $\psi(x,y)$ defines a $\mathbf{\Sigma}^1_n$ subset of $X \times Y$ for some $n \geq 1$, then $\exists y\, \psi(x,y)$ and $\forall y\, \psi(x,y)$ define, respectively, a $\mathbf{\Sigma}^1_n$ and a $\mathbf{\Pi}^1_{n+1}$ subset of X;

- if $\psi(x,y)$ defines a $\mathbf{\Pi}^1_n$ subset of $X \times Y$ for some $n \geq 1$, then $\exists y\, \psi(x,y)$ and $\forall y\, \psi(x,y)$ define, respectively, a $\mathbf{\Sigma}^1_{n+1}$ and a $\mathbf{\Pi}^1_n$ subset of X;

- if $\psi(x,y)$ defines a $\mathbf{\Delta}^1_n$ subset of $X \times Y$ for some $n \geq 1$, then $\exists y\, \psi(x,y)$ and $\forall y\, \psi(x,y)$ define, respectively, a $\mathbf{\Sigma}^1_{n+1}$ and a $\mathbf{\Pi}^1_{n+1}$ subset of X.

It follows that a subset A of a Polish space is projective if and only if it can be defined by a formula φ whose bounded variables range over Polish (or standard Borel) spaces and whose atomic formulas define Borel (or even just) projective sets. In fact, the Tarski-Kuratowski algorithm shows that there is a level-by-level correspondence between the topological complexity of A and the complexity of the formula φ which defines it — see Appendix B for more on this issue.

Here are some natural examples of projective sets which are neither analytic nor coanalytic.

Example 3.3.9 (Woodin)**.** Consider the set

$$\mathrm{MV} = \{f \in C([0;1]) \mid f \text{ satisfies the Mean Value Theorem}\},$$

where $f \in C([0;1])$ satisfies the Mean Value Theorem if for all $a < b$ in $[0;1]$ there is c with $a < c < b$ such that $f'(c)$ exists and $f'(c) = \frac{f(b)-f(a)}{b-a}$. Then MV is a $\mathbf{\Pi}^1_2$ subset of $C([0;1])$ (Exercise!), and it can be shown that indeed it is $\mathbf{\Pi}^1_2$-complete (hence it is not a $\mathbf{\Sigma}^1_2$ set).

Example 3.3.10 (Becker)**.** Let $\mathcal{U} \subseteq C([0;1])^\omega \times C([0;1])$ be given by

$$((f_n)_{n\in\omega}, f) \in \mathcal{U} \iff \text{there is a subsequence } (f_{n_i})_{i\in\omega} \text{ converging pointwise to } f.$$

Then $\mathcal{U}$ is $C([0;1])^\omega$-universal for $\mathbf{\Sigma}^1_2(C([0;1]))$, and therefore it is a $\mathbf{\Sigma}^1_2$-complete set.

Example 3.3.11. Say that $(f_n)_{n\in\omega} \in C([0;1])^\omega$ is **quasidense** in $C([0;1])$ if every $h \in C([0;1])$ is the pointwise limit of a subsequence of $(f_n)_{n\in\omega}$. Then the set of quasidense $(f_n)_{n\in\omega} \in C([0;1])^\omega$ is $\mathbf{\Pi}^1_3$-complete.

3.4 The PSP in the projective hierarchy

The next theorem shows in particular that there is no analytic Bernstein set.

Theorem 3.4.1. *Let X be a Polish space. Then every analytic $A \subseteq X$ has the* PSP*. In particular,* CH *holds for analytic sets.*

Proof. Let $A \subseteq X$ be an analytic set, and let $C \subseteq X \times \omega^\omega$ be a closed set such that $A = \pi_X(C)$. Let $C' = C \setminus \bigcup\{U \in \mathcal{B} \mid \pi_X(C \cap U) \text{ is countable}\}$, where $\mathcal{B}$ is any countable base for $X \times \omega^\omega$. Notice that $A' = \pi_X(C')$ is an analytic subset of A, and that $A \setminus A'$ is countable. If A' is empty, this means that A is countable and we are done; otherwise, $C' \neq \emptyset$ and by construction we have that for every $U \subseteq X \times \omega^\omega$ such that $U \cap C' \neq \emptyset$, the set $\pi_X(U \cap C')$ is uncountable.

Fix a complete compatible metric d on C' (this exists since C' is Polish). We will build a 2-scheme $\{B_s \mid s \in 2^{<\omega}\}$ on C' such that for all $s \in 2^{<\omega}$ and $i \in \{0,1\}$:

(1) B_s is open and nonempty;

(2) $\mathrm{cl}(B_{s^\frown i}) \subseteq B_s$;

(3) $\pi_X(B_{s^\frown 0}) \cap \pi_X(B_{s^\frown 1}) = \emptyset$;

(4) $\mathrm{diam}(B_s) \leq 2^{-\mathrm{lh}(s)}$.

Let $f \colon 2^\omega \to C' \subseteq X \times \omega^\omega$ be the embedding of 2^ω into C' induced by such a scheme. Then $\pi_X \circ f$ is an embedding of 2^ω into A' (and hence also into A): indeed, it is continuous (being the composition of two continuous functions), injective (by (3)), and such that $\mathrm{rng}(\pi_X \circ f) \subseteq A' \subseteq A$ (because $\pi_X(C') = A'$).

The above scheme is constructed by induction on $\mathrm{lh}(s)$. Choose any nonempty $B_\emptyset \in \mathbf{\Sigma}^0_1(C')$ with diameter ≤ 1. Next suppose that B_s has been defined. Pick $z_0 = (x_0, y_0)$ and $z_1 = (x_1, y_1)$ in B_s such that $x_0 \neq x_1$ — this is possible because $\pi_X(B_s)$ is uncountable by definition of C'. Let $U_0, U_1 \subseteq X$ be disjoint open sets such that $x_i \in U_i$, for $i = 0, 1$. Choose $\varepsilon > 0$ small enough so that $B^{cl}_d(z_i, \varepsilon) \subseteq B_s \cap \pi_X^{-1}(U_i)$ and $\varepsilon < 2^{-(\mathrm{lh}(2)+2)}$: then setting $B_{s^\frown i} = B_d(z_i, \varepsilon)$ we have that (1)–(4) are satisfied. □

Iit can be shown that if all coanalytic subsets of, say, ω^ω have the PSP, then $\aleph_1$ is inaccessible in Gödel's constructible universe L, and hence, in particular, $\mathrm{V} \neq \mathrm{L}$. Thus it is consistent with ZFC that there is a coanalytic subset of ω^ω without the PSP (this happens e.g. in any model of $\mathsf{ZFC} + \mathsf{V} = \mathsf{L}$), and the consistency strength of the statement "all $\mathbf{\Pi}^1_1$ sets have the PSP" is at least (in fact, equal to) that of the existence of an inaccessible cardinal.

Conversely, if we assume the existence of an inaccessible cardinal λ, then by collapsing λ to ω_1 with the Lévy collapsing forcing we get a model of ZFC in which all projective sets have the PSP. Furthermore, it can be shown using game-theoretic techniques that large cardinal assumptions (e.g. the existence of infinitely many Woodin cardinals with a measurable above) directly imply that all projective sets have the PSP, and thus that there is no projective counterexample to the continuous hypothesis. The same is true if one assumes strong forcing axioms, like the Proper Forcing Axiom PFA.

The above discussion shows in particular that it is independent of ZFC whether all coanalytic (i.e. $\mathbf{\Pi}^1_1$) sets have the PSP, thus Theorem 3.4.1 is optimal when working in ZFC alone.

3.5 Projective functions

Definition 3.5.1. A function $f\colon X \to Y$ between Polish spaces is called **projective** if it is $\mathbf{\Sigma}^1_n$-measurable for some $n \geq 1$, i.e. if $f^{-1}(U) \in \mathbf{\Sigma}^1_n(X)$ for every open set $U \subseteq Y$.[1]

As for Borel functions, to check that $f\colon X \to Y$ is $\mathbf{\Sigma}^1_n$-measurable it is enough to consider any countable basis $\mathcal{B} = \{U_k \mid k \in \omega\}$ of Y and check that $f^{-1}(U_k) \in \mathbf{\Sigma}^1_n(X)$ for all $k \in \omega$ (this follows from the fact that $\mathbf{\Sigma}^1_n(X)$ is closed under countable unions).

Definition 3.5.1 natural induces a stratification of projective functions according to their level of measurability. Indeed $\mathbf{\Sigma}^1_n$-measurable functions are obviously $\mathbf{\Sigma}^1_m$-measurable for all $m \geq n$. Moreover, if X is uncountable and $|Y| \geq 2$, then the hierarchy does not collapse before ω, that is, for each $n \geq 1$ there are $\mathbf{\Sigma}^1_{n+1}$-measurable functions which are not $\mathbf{\Sigma}^1_n$-measurable. To see this, pick any $A \in \mathbf{\Delta}^1_{n+1} \setminus (\mathbf{\Sigma}^1_n(X) \cup \mathbf{\Pi}^1_n(X))$ and fix distinct points $y_0, y_1 \in Y$: then the function $f\colon X \to Y$ defined by $f(x) = y_0$ if $x \in A$ and $f(x) = y_1$ otherwise is as required. Finally, we notice that this hierarchy "continuously" extend the Baire stratification of all Borel functions, in the sense that its first level coincides with the collection of Borel functions. (This is a sort of counterpart in term of functions of Souslin's Theorem 3.2.3.)

Lemma 3.5.2. *Let X, Y be Polish space. A function $f\colon X \to Y$ is $\mathbf{\Sigma}^1_1$-measurable if and only if it is Borel.*

Proof. Since Borel sets are analytic, the backward direction is obvious. For the forward direction, since f is $\mathbf{\Sigma}^1_1$-measurable, then for every G_δ set $A = \bigcap_{n\in\omega} U_n$ we have $f^{-1}(A) = \bigcap_{n\in\omega} f^{-1}(U_n) \in \mathbf{\Sigma}^1_1(X)$ because analytic sets are closed under countable intersections. In particular, this applies to closed sets, hence $f^{-1}(U) \in \mathbf{\Delta}^1_1(X)$ for all open sets $U \subseteq Y$. The result thus follows from Theorem 3.2.3. $\square$

The following proposition provides equivalent reformulations of Definition 3.5.1 and generalizes Theorem 3.2.4 (which corresponds to the case $n = 1$ once we have Lemma 3.5.2).[2]

Proposition 3.5.3. *Let X and Y be Polish spaces, and fix a nonzero $n \in \omega$. Then for every $f\colon X \to Y$ the following are equivalent:*

(i) *f is $\mathbf{\Sigma}^1_n$-measurable;*

(ii) *$\mathrm{graph}(f) \in \mathbf{\Delta}^1_n(X \times Y)$;*

(iii) *$\mathrm{graph}(f) \in \mathbf{\Sigma}^1_n(X \times Y)$;*

(iv) *$f^{-1}(A) \in \mathbf{\Sigma}^1_n(X)$ for every $A \in \mathbf{\Sigma}^1_n(Y)$;*

[1] By closure of $\mathbf{\Sigma}^1_n(X)$ under countable unions, it is actually enough to require that the preimages of sets in any countable basis for Y belong to $\mathbf{\Sigma}^1_n(X)$.

[2] In the opposite direction, Proposition 3.5.3 for $n = 1$ together with Theorem 3.2.4 provide an alternative proof of Lemma 3.5.2.

(v) $f^{-1}(A) \in \mathbf{\Sigma}^1_n(X)$ *for every* $A \in \mathbf{\Delta}^1_n(Y)$*;*

(vi) $f^{-1}(A) \in \mathbf{\Sigma}^1_n(X)$ *for every* $A \in \mathbf{Bor}(Y)$*;*

In particular, f is a projective function if and only if has a projective graph, i.e. $\mathrm{graph}(f) \in \mathbf{Proj}(X \times Y)$*.*

Proof. (i) $\Rightarrow$ (ii). Fix a countable basis $\{U_n \mid n \in \omega\}$ for Y. Then for every $x \in X$ and $y \in Y$

$$\begin{aligned}(x,y) \in \mathrm{graph}(f) &\iff \forall n\,(y \in U_n \Rightarrow f(x) \in U_n)\\ &\iff \forall n\,(f(x) \in U_n \Rightarrow y \in U_n).\end{aligned}$$

Using the Tarski-Kuratowski algorithm, one sees that the first formulation shows that $\mathrm{graph}(f) \in \mathbf{\Sigma}^1_n(X \times Y)$, while the second one witnesses $\mathrm{graph}(f) \in \mathbf{\Pi}^1_n(X \times Y)$.

(ii) $\Rightarrow$ (iii). Obvious.

(iii) $\Rightarrow$ (iv). For every $x \in X$

$$x \in f^{-1}(A) \iff \exists y\,((x,y) \in \mathrm{graph}(f) \wedge y \in A).$$

Using again the Tarksi-Kuratowski algorithm, we see that this witnesses $f^{-1}(A) \in \mathbf{\Sigma}^1_n(X)$.

(iv) $\Rightarrow$ (v), (v) $\Rightarrow$ (vi), and (vi) $\Rightarrow$ (i). By $\mathbf{\Sigma}^0_1(Y) \subseteq \mathbf{Bor}(Y) \subseteq \mathbf{\Delta}^1_n(Y) \subseteq \mathbf{\Sigma}^1_n(Y)$. $\square$

Corollary 3.5.4. *All the pointclasses $\mathbf{\Sigma}^1_n$, $\mathbf{\Pi}^1_n$, and $\mathbf{\Delta}^1_n$ (for $n \geq 1$) are closed under preimages via $\mathbf{\Sigma}^1_n$-measurable functions. Moreover, $\mathbf{\Sigma}^1_n$ is also closed under images by $\mathbf{\Sigma}^1_n$-measurable functions.*

Proof. The first part follows from Proposition 3.5.3(iv). For the second one, argue as in the proof of Proposition 3.3.3 using Proposition 3.5.3(iii). $\square$

3.6 Beyond projective sets

The projective sets constitute the traditional field of study in descriptive set theory, but they only form a part, albeit one that is very important, of the domain of "definable" sets in Polish spaces. For example, one can notice that the boldface pointclass **Proj** of projective sets is an algebra (i.e. it is closed under complements and finite unions and intersections), but it is not closed under countable unions or countable intersections. To see this, for each $n \in \omega$ pick $A_n \in \mathbf{\Sigma}^1_{n+1}(Y_n) \setminus \mathbf{\Pi}^1_{n+1}(Y_n)$, where Y_n is an uncountable Polish space, and consider the Polish space $Y = \bigoplus_{n \in \omega} Y_n$. Then each A_n belongs to $\mathbf{Proj}(Y)$, but $A = \bigcup_{n \in \omega} A_n$ is not projective. (Otherwise there would be $n \geq 1$ such that $A \in \mathbf{\Sigma}^1_n(Y)$, hence $A_n = \mathrm{id}^{-1}_{Y_n}(A) = A_n \in \mathbf{\Sigma}^1_n(Y_n)$, a contradiction.) Thus it is natural (and interesting) to further consider e.g. the collection of **σ-projective** subsets of a Polish space X, i.e. sets in the smallest σ-algebra containing $\mathbf{Proj}(X)$.

In the last 45 years or so the range of classical descriptive set theory has been greatly expanded, under suitable set-theoretic assumptions, to encompass vastly more extensive hierarchies of “definable sets”, such as, for example, those belonging to $L(\mathbb{R})$, that is, the smallest model of ZF set theory containing all the ordinals and reals. (The projective, the σ-projective, as well as the more complex hyperprojective sets belong to this model.)

Appendix A: Some basic topological facts

This chapter contains a miscellanea of topological results (sometimes with proofs) that are needed for this course.

Proposition 3.6.1. *Every second-countable topological space is separable. If X is a metrizable separable space, then it is also second-countable.*

Proof. If $\mathcal{B}$ is a countable basis for X, then choosing an element from each nonempty $U \in \mathcal{B}$ we obtain a countable set which is dense in X. Conversely, if d is a compatible metric on X and $D \subseteq X$ is countable dense, then the collection of all open balls of the form $B_d(x, q)$ with $x \in D$ and $q \in \mathbb{Q}^+ = \{q \in \mathbb{Q} \mid q > 0\}$ is a countable basis for X. □

Proposition 3.6.2. *If (X, d) is a metric space, then all its closed subsets F are G_δ.*

Proof. It is enough to check that if $F \subseteq X$ is closed then $F = \bigcap_{n\in\omega} U_n$ where $U_n = \bigcup\{B_d(x, 2^{-n}) \mid x \in F\}$. The inclusion $F \subseteq \bigcap_{n\in\omega} U_n$is obvious. For the other direction, let $y \in \bigcap_{n\in\omega} U_n$ and for each $n \in \omega$ choose $x_n \in F$ such that $y \in B_d(x_n, s^{-n})$. Then $x_n \to y$, and hence $y \in F$ because F is closed. □

Proposition 3.6.3. *If $\mathcal{B} = \{U_\alpha \mid \alpha < \lambda\}$ is a basis of size λ of the topological space X, then every basis $\mathcal{B}'$ of X contains a basis $\mathcal{B}''$ of size $\leq \lambda$.*

Proof. For every $\alpha, \beta < \lambda$ choose an element $V_{\alpha,\beta}$ in $A_{\alpha,\beta} = \{V \in \mathcal{B}' \mid U_\alpha \subseteq V \subseteq U_\beta\}$ whenver this set is nonempty, and let $\mathcal{B}''$ be the collection of these $V_{\alpha,\beta}$'s. The size of $\mathcal{B}''$ is $\leq \lambda \times \lambda = \lambda$, thus it is enough to check that $\mathcal{B}''$ is a base. Fix any $\beta < \lambda$ and a point $x \in U_\beta$: we want to show that there is $V_{\alpha,\beta} \in \mathcal{B}''$ such that $x \in V_{\alpha,\beta} \subseteq U_\beta$. Since $\mathcal{B}'$ is a base, there is $V \in \mathcal{B}'$ with $x \in V \subseteq U_\beta$, and since $\mathcal{B}$ is also a base there is in turn $\alpha < \lambda$ such that $x \in U_\alpha \subseteq V$. Thus V witnesses that $A_{\alpha,\beta} \neq \emptyset$, so that there is $V_{\alpha,\beta} \in \mathcal{B}''$ with $x \in U_\alpha \subseteq V_{\alpha,\beta} \subseteq U_\beta$, as desired. □

A topological space X has **small inductive dimension** 0 (or, briefly, X is **zero-dimensional**) if it admits a basis consisting of clopen set. The space X has **Lebesgue covering dimension** 0 if every open covering of X can be refined to a clopen partition, that is: If $\mathcal{V}$ is a collection of open sets of X such that $X = \bigcup \mathcal{V}$, then there is a family $\mathcal{U}$ of open sets such that $\mathcal{U}$ refines $\mathcal{V}$ (i.e.

for every $U \in \mathcal{U}$ there is $V \in \mathcal{V}$ such that $U \subseteq V$), $X = \bigcup \mathcal{U}$, and for every $x \in X$ there is at most one $U \in \mathcal{U}$ such that $x \in U$. (This means that $\mathcal{U}$ is a partition of X, hence its elements are necessarily clopen.)

Proposition 3.6.4. *If X is second-countable and zero-dimensional, then X has Lebesgue covering dimension 0. More precisely, every open covering of X can be refined to a* countable *clopen partition.*

Conversely, if X is metrizable with density character λ and has Lebesgue covering dimension 0, then it admits a clopen basis of size λ and is thus zero-dimensional.

Proof. Fix a clopen basis $\mathcal{B}$ of X, which might be assumed to be countable by Proposition 3.6.3. Let $\mathcal{V}$ be an open covering of X, and let $\mathcal{V}' = \{U \in \mathcal{B} \mid \exists V \in \mathcal{V}\,(U \subseteq V)\}$. Then $\mathcal{V}'$ is a countable covering refining $\mathcal{V}$ consisting of clopen sets: write it as $\mathcal{V}' = \{V'_n \mid n \in \omega\}$. Finally, recursively set $U_n = V'_n \setminus \bigcup_{m<n} V'_m$ and notice that $\mathcal{U} = \{U_n \mid n \in \omega\}$ is then a countable partition of X into clopen sets refining $\mathcal{V}'$ (and hence $\mathcal{V}$).

Let $D = \{x_\alpha \mid \alpha < \lambda\}$ be a dense subset of X and, for each $n \in \omega$, let $\mathcal{V}_n$ be the collection of all open balls centered in D with radius 2^{-n}. Each $\mathcal{V}_n$ is an open cover of X, hence it can be refined to a clopen partition $\mathcal{U}_n$ because X has Lebesgue covering dimension 0. It follows that $\mathcal{B} = \bigcup_{n\in\omega} \mathcal{U}_n$ is a clopen basis for X, which can then be reduced to a clopen basis of size λ by Proposition 3.6.3 again. □

Recall that the elements of a product $\prod_{i\in I} X_i$ are functions $z\colon I \to \bigcup_{i\in I} X_i$ such that $z(i) \in X_i$ for all $i \in I$).

Proposition 3.6.5. *Let $y^n, y \in \prod_{i\in I} X_i$, where I is an arbitrary set and the X_i are arbitrary nonempty topological spaces. Then $y^n \to y$ if and only if $y^n(i) \to y(i)$ for all $i \in I$.*

Proof. Suppose first that $y^n \to y$, and fix any $i \in I$. Given an open neighborhood $U \subseteq X_i$ of $y(i)$, let $V = \pi_i^{-1}(U)$, where π_i denotes the projection on the i-th cooridnate. Then $y^n \in V$ for all but finitely many n, i.e. $y^n(i)$ for all large enoguh n. Therefore $y^n(i) \to y(i)$. Conversely, assume that $y^n(i) \to y(i)$ for all $i \in I$. Let $U \subseteq \prod_{i\in I} X_i$ be an open neighborhood of y, and let I_0 be a finite subset of I and U_i, $i \in I_0$, be open neighborhoods of $y(i)$ such that the basic open set $V = \{z \in \prod_{i\in I} X_i \mid \forall i \in I_0\,(z(i) \in U_i)\}$ is contained in U. Then by our assumption for every $i \in I_0$ there is $N_i \in \omega$ with $y^n(i) \in U_i$ for $n \geq N_i$. Therefore $y^n \in V \subseteq U$ for all $n \geq \max\{N_i \mid i \in I_0\}$. Thus $y^n \to y$. □

Corollary 3.6.6. *Consider a family of functions $f_i\colon X_i \to Y_i$, $i \in I$, between topological spaces, and its product $\prod_{i\in I} f_i\colon \prod_{i\in I} X_i \to \prod_{i\in I} Y_i$ defined by $\prod_{i\in I} f_i(z)(i) = f_i(z(i))$. Then $\prod_{i\in I} f_i$ is continuous if and only if all the functions f_i, $i \in I$, are continuous.*

Similarly, if $g_i\colon X \to Y_i$, $i \in I$, are functions with the same domain X, then the map $X \mapsto \prod_{i\in I} Y_i$ sending $x \in X$ to the element $y \in \prod_{i\in I} Y_i$ defined by $y(i) = f_i(x)$ (for all $i \in I$) is continuous if and only if all the f_i, $i \in I$, are continuous.

Proposition 3.6.7. *Let X be a Hausdorff space. Then the diagonal of $X \times X$ is a closed set. More generally, if X^I is the product of X over the index set I, then the diagonal of X^I*

$$\Delta(X^I) = \{x \in X^I \mid \forall i, j \in I\, (x(i) = x(j))\}$$

is closed with respect to the product topology on X^I.

Proof. Given $y \notin \Delta(X^I)$, let $i, j \in I$ be such that $y(i) \neq y(j)$. Since X is Hausdorff, there are disjoint open neighborhoods U_i and U_j of $y(i)$ and $y(j)$, respectively. Then the product $\prod_{k \in I} U_k$ where $U_k = X$ if $k \neq i, j$ is an open neighborhood of y disjoint from $\Delta(X^I)$. □

Proposition 3.6.8. *If X is a Hausdorff topological space, then all its compact subsets are closed. Moreover, if d is a compatible topology on X then all the compact subsets of X have bounded diameter.*[3]

Conversely, all closed subsets of a compact (non necessarily Hausdorff) space are compact.

Proof. Let $K \subseteq X$ be compact. Suppose that $x \in X \setminus K$. For every $y \in K$, let U_y and V_y be disjoint open sets such that $x \in U_y$ and $y \in V_y$. Since $\{V_y \mid y \in K\}$ is an open covering of K, there is some finite $K' \subseteq K$ such that $K \subseteq \bigcup_{y \in K'} V_y$. It follows that $\bigcap_{y \in K'} U_y$ is an open neighborhood of x disjoint from K. Let now d be a compatible metric on X, and assume towards a contradiction that there is a compact $K \subseteq X$ with unbounded diameter: then for any $x \in K$, $\{B_d(x, n) \mid n \in \omega\}$ would be an open covering of K without a finite subcover, a contradiction.

Let K be a compact set and $C \subseteq K$ be closed. Given an arbitrary open covering $\mathcal{U}$ of C, we get that $\mathcal{U}' = \mathcal{U} \cup \{K \setminus C\}$ is an open covering of K. Let $\mathcal{V}' \subseteq \mathcal{U}'$ be a finite subcover of K: then $\mathcal{V} = \mathcal{V}' \setminus \{K \setminus C\} \subseteq \mathcal{U}$ is a finite subcover of C. □

Proposition 3.6.9. *A continuous image of a compact set is compact. In particular, if K is compact and Y is Hausdorff, then every continuous injection $f \colon K \to Y$ is automatically a topological embedding.*

Proof. Let K be compact and $f \colon K \to Y$ be continuous. Given an open cover $\mathcal{U}$ of $F(K)$, the collection $\{f^{-1}(U) \mid U \in \mathcal{U}\}$ is an open cover of K. Therefore there is a finite $\mathcal{U}_0 \subseteq \mathcal{U}$ such that $\{f^{-1}(U) \mid U \in \mathcal{U}_0\}$ is a subcover of K. But then $\mathcal{U}_0$ is a finite subcover of $f(K)$, as required. Assume now that Y is Hausdorff. Then for every closed $C \subseteq K$, C is compact by Proposition 3.6.8, so that $f(C)$ is compact in Y by the first part of this proposition, and hence $f(C)$ is closed in Y by Proposition 3.6.8 again. It follows that if $U \subseteq K$ is open then $f(U)$ is relatively open in $f(K)$, i.e. f is a topological embedding. □

[3]The converse is not true in general, i.e. there are metric spaces (X, d) for which not all of their closed and bounded subsets are compact. Metric spaces for which a subset is compact if and only if it is closed and bounded are called **Heine-Borel**. Typical examples of Heine-Borel spaces are the finite-dimensional Euclidean spaces $\mathbb{R}^n$.

Proposition 3.6.10. *If X and Y are topological spaces and Y is compact, then the projection map $\pi_X : X \times Y \to X$ is a closed map (i.e. the image of a closed set is closed).*

Proof. Suppose $C \subseteq X \times Y$ is closed, and take any $x_0 \in X \setminus \pi_X(C)$. For any $y \in Y$ we have $(x_0, y) \notin C$, and since the latter is a closed set there are open sets $U_y \subseteq X$ and $V_y \subseteq Y$ such that $(x_0, y) \in U_y \times V_y$ and $U_y \times V_y \cap C = \emptyset$. But then $\{V_y \mid y \in Y\}$ is an open covering of Y, hence by compactness there is a finite $Y_0 \subseteq Y$ such that $\{V_y \mid y \in Y_0\}$ is a subcovering of Y. Then $U = \bigcap_{y \in Y_0} U_y$ is an open neighborhood of x_0 disjoint from $\pi_X(C)$. □

Corollary 3.6.11. *A continuous image of a σ-compact space is σ-compact (hence also F_σ, if the codomain of the function is Hausdorff).*

If X and Y are topological spaces and Y is σ-compact, then $\pi_X(F)$ is F_σ for every F_σ set $F \subseteq X \times Y$.

Proof. The first part follows from Proposition 3.6.9, while the second one follows from Proposition 3.6.10. □

Appendix B: Projective sets and definability

The following exercises give mathematically precise formulations of the observation on page 71 on the existing link between projective sets and definability. (The second one requires a certain familiarity with the basic definitions and ideas of second order arithmetic, which will be omitted here.)

Exercise 3.6.12 ([Kec95, Exercise 37.6])**.** Consider the structure $\mathcal{R} = (\mathbb{R}, +, \cdot, \mathbb{Z})$ in the language $\mathcal{L} = (f, g, P)$, where f and g are binary function symbols and P is a unary predicate symbol. Show that a set $A \subseteq \mathbb{R}^n$ is projective if and only if it is first-order definable with parameters in $\mathcal{R}$, i.e. there are a first-order $\mathcal{L}$-formula $\varphi(u_1, \dots, u_n, w_1, \dots, w_m)$ and $r_1, \dots, r_m \in \mathbb{R}$ such that for all $(x_1, \dots, x_n) \in \mathbb{R}^n$

$$(x_1, \dots, x_n) \in A \iff \mathcal{R} \models \varphi[x_1, \dots, x_n, r_1, \dots, r_m].$$

Exercise 3.6.13. Identify the subsets of ω with their characteristic functions, so that $\mathscr{P}(\omega)$ is identified with the Cantor space 2^ω. Then a set $A \subseteq 2^\omega$ is projective if and only if it is definable (with parameters) in second-order arithmetic. More precisely, for every nonzero $n \in \omega$ we have the following.

- $A \in \mathbf{\Sigma}^0_n(2^\omega)$ if and only if A is definable by a Σ^0_n**-formula** (with parameters), i.e. a formula of the form

  $$\exists k_1 \forall k_2 \exists k_3 \dots Q_n k_n \, \psi(X, k_1, \dots, k_n, P_1, \dots, P_m),$$

  where ψ is a quantifier-free formula, $P_1, \dots, P_m$ are parameters (i.e. subsets of ω), and $\exists k_1 \forall k_2 \exists k_3 \dots Q_n k_n$ is a block of alternated quantifiers on variables ranging over ω (clearly, whether the last quantifier Q_n is $\exists$ or $\forall$ depends on the parity of n).

- $A \in \mathbf{\Sigma}^1_n(2^\omega)$ if and only if A is definable by a Σ^1_n**-formula** (with parameters), i.e. a formula of the form

  $$\exists Y_1 \forall Y_2 \exists Y_3 \dots Q_n Y_n \, \psi(X, Y_1, \dots, Y_n, P_1, \dots, P_m),$$

  where ψ is a formula with only first-order quantifiers (i.e. quantifications over natural numbers), $P_1, \dots, P_m$ are parameters, and $\exists Y_1 \forall Y_2 \exists Y_3 \dots Q_n Y_n$ is a block of alternated quantifiers on variables ranging over $\mathscr{P}(\omega)$ (as before, whether the last quantifier Q_n is $\exists$ or $\forall$ depends on the parity of n).

Exercise 3.6.14 (See [Jec03, Lemma 25.25 and the ensuing comment]). Let H_{ω_1} be the collection of hereditarily countable sets, and let $k \geq 1$. A set $A \subseteq (\omega^\omega)^n$, $n \geq 1$, is in the pointclass $\mathbf{\Sigma}^1_{k+1}$ if and only if it is definable in the structure $\langle H_{\omega_1}, \in \rangle$ via a Σ_k-formula φ with parameters in ω^ω, that is: there are a formula $\varphi(u_1, \ldots, u_n, w_1, \ldots, w_m)$ of level Σ_k according to the Lévy hierarchy (see [Jec03, p. 183]) and parameters $r_1, \ldots, r_m \in \omega^\omega$ such that for all $(x_1, \ldots, x_n) \in (\omega^\omega)^n$

$$(x_1, \ldots, x_n) \in A \iff \langle H_{\omega_1}, \in \rangle \models \varphi[x_1, \ldots, x_n, r_1, \ldots, r_m].$$

Bibliography

[Jec03] T. Jech. *Set theory*, the third millennium edition, revised and expanded, Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003.

[Kec95] Alexander S. Kechris. *Classical descriptive set theory*, volume 156 of Graduate Texts in Mathematics. Springer-Verlag, New York, 1995.