Exercise sheet 4

Descriptive Set Theory

Academic year 2023-24

1. Prove that for any Polish space X and $x \in X$, the singleton $\{x\}$ is $\mathbf{\Pi}^0_1$-complete if and only if x is not isolated in X. Conclude that the set
$$C_1 = \{x \in 2^\omega \mid \exists n\,(x(n) = 0)\}$$
from Proposition 2.1.31 of the notes is $\mathbf{\Sigma}^0_1$-complete.

2. Prove that for any Polish space and $A \subseteq X$, if A is not open then it is $\mathbf{\Pi}^0_1$-hard. Conclude that a set A is truly closed (i.e. closed but not open) if and only if it is $\mathbf{\Pi}^0_1$-complete, and similarly for $\mathbf{\Sigma}^0_1$.

3. Prove that the sets
$$C_0 = c_0 \cap [0;1]^\omega = \{(x_n)_{n\in\omega} \in [0;1]^\omega \mid x_n \to 0\}$$
$$C = \{(x_n)_{n\in\omega} \in [0;1]^\omega \mid (x_n)_{n\in\omega} \text{ converges}\}$$
are both $\mathbf{\Pi}^0_3$-complete.

   [*Hint.* For the hardness part, compare these sets with the $\mathbf{\Pi}^0_3$-complete set C_3 from Exercise 2.1.27 in the notes.]

4. Prove that for any $0 < p < \infty$ the set
$$\ell_p \cap [0;1]^\omega = \left\{ (x_n)_{n\in\omega} \in [0;1]^\omega \;\middle|\; \|x\|_p = \left(\sum_{n=0}^{\infty} |x_n|^p \right)^{\frac{1}{p}} < \infty \right\}$$
is $\mathbf{\Sigma}^0_2$-complete.

   [*Hint.* Recall that a positive terms series converges if and only if the sequence of partial sums is bounded from above. For the hardness part, compare this sets with the $\mathbf{\Sigma}^0_2$-complete set Q_2 from the notes.]

5. Show that the collection of all sequences $(x_n)_{n\in\omega} \in [0;1]^\omega$ having an irrational accumulation point is analytic.