Polish spaces

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Recall. . .

A topological space is

- **separable** if it contains a countable dense set,
- **first countable** if the filter of neighborhoods of any point has a countable base,
- **second countable** if the topology has a countable base,
- **metrizable** if there is a metric that induces the topology.

If d is a metric then

- d is **complete** if every Cauchy sequence converges,
- $\frac{d}{1+d}$ is a metric compatible with d, and it is bounded by 1.
  (Two metrics are compatible if they induce the same topology.)

Polish spaces

Definition 1.1.1

$X = (X, \tau)$ is **Polish** if it is second-countable and completely metrizable, that is there is a complete metric d on X compatible with its topology, i.e. such that τ is generated by the d-open balls
$\mathrm{B}_d(x, r) = \{y \in X \mid d(y, x) < r\}$.
When a specific compatible (complete) metric d on X is singled out, we call $X = (X, d)$ a **Polish metric space**.

The class of Polish spaces is closed under homeomorphism. Any metric space (X, d) admits a (unique, up to isometry) **completion** $(\hat{X}, \hat{d})$ so any second-countable metrizable spaces is contained in a Polish space as a dense subspace.

Closure properties of Polish spaces

- Closed subspaces
- Countable products: if the X_n are Polish, then $\prod_{n \in \mathbb{N}} X_n$ with the product topology is Polish. A complete compatible metric d is

$$d(x, y) = \sum_{n=0}^{\infty} 2^{-n} d_n(x(n), y(n))$$

  where each d_n is a compatible complete metric on X_n bounded by 1.
- Countable sums: if the X_n are Polish, then their disjoint union $\bigoplus_{n \in \mathbb{N}} X_n$ is Polish with with the smallest topology refining all the topologies of the X_n's (so that each X_n is clopen in $\bigoplus_{n \in \mathbb{N}} X_n$).

$$d(x, y) = \begin{cases} d_n(x, y) & \text{if } x, y \in X_n \\ 1 & \text{if } x \text{ and } y \text{ belong to different } X_n\text{'s.} \end{cases}$$

Closure properties of Polish spaces

- Countable intersections: if $Y_n \subseteq X$ are Polish, then $Y = \bigcap_{n\in\mathbb{N}} Y_n$ is Polish. Indeed, $Z = \prod_{n\in\mathbb{N}} Y_n$ is Polish and so is its closed subset

$$C = \{z \in Z \mid z(i) = z(j) \text{ for all } i, j \in \mathbb{N}\}$$

and the diagonal map $Y \to C$, $y \mapsto (z(n))_{n\in\mathbb{N}}$ where $z(n) = y$ for all $n \in \mathbb{N}$ is a homeomorphism.

Examples of Polish spaces

1. $\mathbb{R}^n$, $\mathbb{C}^n$, $\mathbb{R}^{\mathbb{N}}$, and $\mathbb{C}^{\mathbb{N}}$, $[0;1]$, $\mathbb{T} = \{x \in \mathbb{C} \mid |x| = 1\}$, the Hilbert cube $[0;1]^{\mathbb{N}}$, ...
2. $A^{\mathbb{N}}$ with the product topology, where A is a countable set with the discrete topology. In particular, when $A = 2 = \{0, 1\}$ and $A = \mathbb{N}$ we obtain the **Cantor space** $2^{\mathbb{N}}$ and the **Baire space** $\mathbb{N}^{\mathbb{N}}$.
3. $\mathscr{P}(\mathbb{N}) = \{X \mid X \subseteq \mathbb{N}\}$ with the topology generated by

$$U_n = \{X \subseteq \mathbb{N} \mid n \in X\} \qquad \hat{U}_n = \{X \subseteq \mathbb{N} \mid n \notin X\}$$

The resulting space is homeomorphic to $2^{\mathbb{N}}$.

4. Let $\mathcal{L} = \{R_i \mid i \in I\}$ (with I an initial segment of $\mathbb{N}$) be countable relational language, where each R_i has arity n_i. Every $\mathcal{L}$-structure $\mathcal{A}$ with domain $\mathbb{N}$ can be identified with an element of

$$\mathrm{Mod}_{\mathcal{L}} = \prod_{i \in I} 2^{(\mathbb{N}^{n_i})}$$

via the characteristic functions of its predicates $R_i^{\mathcal{A}}$.
Endowing each $2^{(\mathbb{N}^{n_i})}$ with the (countable) product of the discrete topology on 2, they all become Polish spaces (homeomorphic to the Cantor space). Thus $\mathrm{Mod}_{\mathcal{L}}$ is Polish, and can be regarded as the Polish space of all countable $\mathcal{L}$-structures (up to isomorphism).

5. Set $\mathcal{L} = \{R\}$ with R binary, so that $\mathrm{Mod}_{\mathcal{L}} = 2^{\mathbb{N}\times\mathbb{N}}$. Consider the set

$$\mathrm{LO} = \{x \in \mathrm{Mod}_{\mathcal{L}} \mid x \text{ codes a linear order}\},$$

that is: $x \in \mathrm{LO}$ if $R^{\mathcal{A}_x}$ is a reflexive, antisymmetric, transitive and total relation. Then

$$\mathrm{LO} = \bigcap_{n\in\mathbb{N}} R_n \;\cap\; \bigcap_{\substack{n,m\in\mathbb{N}\\ n\neq m}} A_{n,m} \;\cap\; \bigcap_{n,m,k\in\mathbb{N}} T_{n,m,k} \;\cap\; \bigcap_{n,m\in\mathbb{N}} L_{n,m},$$

where

- $R_n = \{x \in 2^{\mathbb{N}\times\mathbb{N}} \mid x(n,n) = 1\}$,
- $A_{n,m} = \{x \in 2^{\mathbb{N}\times\mathbb{N}} \mid x(n,m) = 0 \vee x(m,n) = 0\}$,
- $T_{n,m,k} = \{x \in 2^{\mathbb{N}\times\mathbb{N}} \mid x(n,m) = 0 \vee x(m,k) = 0 \vee x(n,k) = 1\}$,
- $L_{n,m} = \{x \in 2^{\mathbb{N}\times\mathbb{N}} \mid x(n,m) = 1 \vee x(m,n) = 1\}$.

Since each of the above sets is clopen (= closed *and* open), it follows that LO is closed in $\mathrm{Mod}_{\mathcal{L}}$, and thus it can be regarded as the Polish space of all countable linear orders (up to isomorphism).

6. Let p be any prime number. Every $q \in \mathbb{Q}$ can be written in a unique way as $p^n \frac{a}{b}$ with a and b not divisible by p. Define the p-**adic absolute value** of q as

$$|q|_p = p^{-n}.$$

The space of p-**adic numbers** $\mathbb{Q}_p$ is the completion of $(\mathbb{Q}, d_p)$ where d_p is the metric induced by $|\cdot|_p$, i.e. $d_p(q, q') = |q - q'|_p$. Each p-adic number may be written in a unique way as

$$\sum_{i=k}^{\infty} a_i p^i$$

where $k \in \mathbb{Z}$ is such that $a_k \neq 0$ and each a_i belongs to $\{0, \ldots, p-1\}$. The subring of p-**adic integers** $\mathbb{Z}_p$ consists of those p-adic numbers such that $a_i = 0$ for all $i < 0$. Both $\mathbb{Q}_p$ and $\mathbb{Z}_p$ are Polish spaces. Indeed, $\mathbb{Z}_p$ is homeomorphic to the Cantor space $2^{\mathbb{N}}$, while $\mathbb{Q}_p$ is homeomorphic to $2^{\mathbb{N}}$ minus a point.

7. All separable Banach spaces are Polish. These include
   - the ℓ^p spaces ($1 \leq p < \infty$), in particular the **Hilbert space** ℓ^2 (which can be shown to be homeomorphic to $\mathbb{R}^{\mathbb{N}}$);
   - c_0, the space of converging-to-0 sequences with the sup norm;
   - the $L^p(\mu)$ spaces ($1 \leq p < \infty$), μ a σ-finite measure on a countably generated σ-algebra;
   - $C(X)$, the space of continuous (real or complex) functions on a compact metrizable space X with the sup norm.

8. Let X, Y be separable Banach spaces, and $L(X, Y)$ be the (generally non-separable) Banach space of bounded linear operators $T\colon X \to Y$ with norm $\|T\| = \sup\{\|Tx\| \mid x \in X, \|x\| \leq 1\}$. Then the unit ball

$$L_1(X, Y) = \{T \in L(X, Y) \mid \|T\| \leq 1\}$$

endowed with the (relative) **strong topology** is Polish.
(The strong topology is the weakest topology on $L(X, Y)$ for which the maps $L(X, Y) \to Y$, $T \mapsto Tx$, are continuous, for $x \in X$.)

Proof.

Let $D \subseteq X$ be a countable dense subset of X closed under rational linear combinations. Y^D with the product topology is Polish. Consider the following closed (hence Polish) subset of Y^D:

$$F = \{f \in Y^D \mid \forall x, y \in D\, \forall p, q \in \mathbb{Q}\, (f(px + qy) = pf(x) + qf(y)) \\ \wedge \forall x \in D\, (\|f(x)\| \leq \|x\|)\}.$$

The map $L_1(X, Y) \to F$, $T \mapsto T \restriction D$, is a homeomorphism. □

Proposition 1.1.4

If X is Polish and $Y \subseteq X$ is open, then Y is Polish.

Proof

Y is second-countable, so we need to show that it is completely metrizable. Let $F = X \setminus Y$, and for any $x \in X$ set $d(x, F) \coloneqq \inf\{d(x, y) \mid y \in F\}$. Define d' on Y by

$$d'(x, y) = d(x, y) + \left|\tfrac{1}{d(x,F)} - \tfrac{1}{d(y,F)}\right|.$$

Let's assume that d and d' generate the same topology on $Y = X \setminus F$.
Any d'-Cauchy sequence $(y_i)_{i \in \mathbb{N}}$ in Y is also d-Cauchy, hence $y_i \to y$ for some $y \in X$: we claim that $y \in Y$.
$(1/d(y_i, F))_{i \in \mathbb{N}}$ is Cauchy in $\mathbb{R}$ because of the second term in the definition of d', hence it converges in $\mathbb{R}$, and thus $(d(y_i, F))_{i \in \mathbb{N}}$ is bounded away from 0. As $d(y_i, F) \to d(y, F)$ by continuity $d(y, F) \neq 0$, whence $y \notin F$, i.e. $y \in Y$.

(continues)

Proof (continued).

Let us prove that d and d' generate the same topology on $Y = X \setminus F$. As $\mathrm{B}_{d'}(x,\varepsilon) \subseteq \mathrm{B}_d(x,\varepsilon) \cap Y$ for all $x \in Y$ and $\varepsilon > 0$, it suffices to show that for all $x \in Y$ and $\varepsilon > 0$ there is $\varepsilon' > 0$ such that $\mathrm{B}_d(x,\varepsilon') \cap Y \subseteq \mathrm{B}_{d'}(x,\varepsilon)$. Choose $0 < \varepsilon' < \frac{\varepsilon}{2}$ such that for all $y \in \mathrm{B}_d(x,\varepsilon') \cap Y$

$$\left| \frac{1}{d(x,F)} - \frac{1}{d(y,F)} \right| < \frac{\varepsilon}{2}.$$

Then for all $y \in \mathrm{B}_d(x,\varepsilon') \cap Y$ one has

$$d'(x,y) = d(x,y) + \left| \frac{1}{d(x,F)} - \frac{1}{d(y,F)} \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Thus $\mathrm{B}_d(x,\varepsilon') \cap Y \subseteq \mathrm{B}_{d'}(x,\varepsilon)$, as desired. □

It follows that countable intersections of open subsets of a given Polish space are Polish as well.

Example

Consider the symmetric group S_∞ of all permutations of $\mathbb{N}$. Formally, S_∞ is the subspace of $\mathbb{N}^{\mathbb{N}}$ consisting of all bijections from $\mathbb{N}$ into itself. Thus S_∞ is the (countable) intersection of the following open sets, where n, m vary over *distinct* natural numbers:

- $\{x \in \mathbb{N}^{\mathbb{N}} \mid x(n) \neq x(m)\}$
- $\bigcup_{k \in \mathbb{N}} \{x \in \mathbb{N}^{\mathbb{N}} \mid x(k) = n\}$.

Thus S_∞ is a Polish space. Indeed, it is even a **Polish group**, i.e. a topological group (i.e. a group equipped with a topology turning its operations into continuous functions) whose topology is Polish.

Definition

A subset A of a topological space X is $\mathbf{G}_\delta$ if it can be written as a countable intersection of open subsets of X, and it is $\mathbf{F}_\sigma$ if it can be written as a countable union of closed sets (equivalently: if its complement is $\mathbf{G}_\delta$).

The collection of $\mathbf{G}_\delta$ subsets is closed under countable intersections and finite unions, while the collection of all $\mathbf{F}_\sigma$ subsets is closed under countable unions and finite intersections. It can be shown that in a Polish space, the intersection of two dense $\mathbf{G}_\delta$ sets is dense.

Example (1.1.7)

The rationals $\mathbb{Q} = \bigcup_{q \in \mathbb{Q}} \{q\}$ form an $\mathbf{F}_\sigma$ subset of $\mathbb{R}$, hence the irrationals $\mathrm{Irr} = \mathbb{R} \setminus \mathbb{Q}$ form a $\mathbf{G}_\delta$ set. Since $\mathbb{Q}$ and Irr are both dense, $\mathbb{Q}$ is not $\mathbf{G}_\delta$ and hence Irr is not $\mathbf{F}_\sigma$.
$[0;1) = \bigcup_{n \in \mathbb{N}} [0; 1 - 2^{-n}] = \bigcap_{n \in \mathbb{N}} (-2^{-n}; 1)$ is an example of a subset of $\mathbb{R}$ which is both $\mathbf{F}_\sigma$ and $\mathbf{G}_\delta$, but neither open nor closed.

If X is metrizable, then any closed F is $\mathbf{G}_\delta$, as $F = \bigcap_{n \in \mathbb{N}} U_n$ where $U_n = \bigcup \{ \mathrm{B}_d(x, 2^{-n}) \mid x \in F \}$ with d any compatible metric on X.

Proposition 1.1.4 [Kec95, Theorem 3.11]

X Polish and $Y \subseteq X$. The following are equivalent:

1. Y is Polish (with the induced topology);
2. Y is a $\mathbf{G}_\delta$ subset of X.

Z_1 a topological space, $Z_2 = (Z_2, d')$ metric, $A \subseteq Z_1$, and $f \colon A \to Z_2$. The **oscillation** of f at $z \in Z_1$ is

$$\mathrm{osc}_f(z) = \inf\{\mathrm{diam}(f(U \cap A)) \mid U \subseteq Z_1 \text{ open}, z \in U\},$$

where $\mathrm{diam}(B) = \sup\{d'(x,y) \mid x, y \in B\}$ if $B \neq \emptyset$ and $\mathrm{diam}(\emptyset) = 0$.

$$\operatorname{osc}_f(z) = \inf\{\operatorname{diam}(f(U \cap A)) \mid U \subseteq Z_1 \text{ open}, z \in U\}$$

If $z \in Z_1 \setminus \mathrm{Cl}(A)$ then $\operatorname{osc}_f(z) = 0$, while if $z \in A$ then $\operatorname{osc}_f(z) = 0$ if and only if z is a continuity point of f. Moreover $A_\varepsilon = \{z \in Z_1 \mid \operatorname{osc}_f(z) < \varepsilon\}$ is open, whence $\{z \in Z_1 \mid \operatorname{osc}_f(z) = 0\} = \bigcap_{n \in \mathbb{N}} A_{2^{-n}}$ is $\mathbf{G}_\delta$.
In particular

If Z_1, Z_2 are topological spaces with Z_2 metrizable, then the set of points of continuity of $f \colon Z_1 \to Z_2$ is $\mathbf{G}_\delta$ [Kec95, Proposition 3.6].

Claim 1.1.8.1 [Kec95, Theorem 3.8]

Let Z_1 be metrizable and Z_2 be completely metrizable, $A \subseteq Z_1$, and $f \colon A \to Z_2$ be continuous. Then there is a $\mathbf{G}_\delta$ set G with $A \subseteq G \subseteq \mathrm{Cl}(A)$ and a continuous function $g \colon G \to Z_2$ with $g \restriction A = f$.

Proof of the claim.

Let d' be a compatible complete metric on Z_2. Let $G = \mathrm{Cl}(A) \cap \{z \in Z_1 \mid \operatorname{osc}_f(z) = 0\}$. This is a $\mathbf{G}_\delta$ set, and since f is continuous on A we have $A \subseteq G \subseteq \mathrm{Cl}(A)$. Let $z \in G$, and fix a sequence $(z_n)_{n \in \mathbb{N}}$ of points of A converging to $z \in \mathrm{Cl}(A)$. Then $(f(z_n))_{n \in \mathbb{N}}$ is a d'-Cauchy sequence. Indeed, for every $\varepsilon > 0$ there is an open neighborhood U of z with $\operatorname{diam}(f(U \cap A)) < \varepsilon$ because $\operatorname{osc}_f(z) = 0$, and since $z_n \in U$ for all but finitely many n's (because $z_n \to z$) it follows that there is $N \in \mathbb{N}$ such that $d'(f(z_n), f(z_m)) \le \operatorname{diam}(f(U \cap A)) < \varepsilon$ for all $n, m \ge N$. Thus $(f(z_n))_{n \in \mathbb{N}}$ converges in Z_2, and we can set $g(z) = \lim_{n \to \infty} f(z_n)$.
Clearly g is well-defined (i.e. the value of $g(z)$ is independent of the choice of the sequence $z_n \to z$) and extends f. Finally, to see that g is continuous we have to show that $\operatorname{osc}_g(z) = 0$ for all $z \in G$. But given any open $U \subseteq Z_1$, $g(G \cap U) \subseteq \mathrm{Cl}(f(A \cap U))$ by definition of g, thus $\operatorname{diam}(g(G \cap U)) \le \operatorname{diam}(f(A \cap U))$, whence $\operatorname{osc}_g(z) \le \operatorname{osc}_f(z) = 0$. □

Proof of Proposition 1.1.4.

Assume Y is a Polish subspace of X, and let us prove that Y is $\mathbf{G}_\delta$. By Claim 1.1.8.1 with $Z_1 = X$, $Z_2 = A = Y$, and $f = \mathrm{id}_Y$ the identity function on Y, we get that there is a $\mathbf{G}_\delta$ subset of X and a continuous function $g\colon G \to Y$ such that $Y \subseteq G \subseteq \mathrm{Cl}(Y)$ and $g \upharpoonright Y = \mathrm{id}_Y$. Since Y is dense in G and g is continuous, we have that $g = \mathrm{id}_G$. On the other hand, $\mathrm{rng}(g) \subseteq Y$, hence $G \subseteq Y$. It follows that $Y = G$ is $\mathbf{G}_\delta$ in X. □

$\mathbb{R} \setminus \mathbb{Q}$ is Polish (it is homeomorphic to $\mathbb{N}^\mathbb{N}$). By Example 1.1.7 $\mathbb{Q}$ is not a $\mathbf{G}_\delta$ subspace of $\mathbb{R}$, so it is not a Polish space.
Using Claim 1.1.8.1 one can prove

Theorem (Lavrentiev) [Kec95, Theorem 3.9]

Z_1, Z_2 completely metrizable, $A \subseteq Z_1$, $B \subseteq Z_2$, and $f\colon A \to B$ a homeomorphism. Then f can be extended to a homeomorphism $h\colon G \to H$ where $G \supseteq A$, $H \supseteq B$ are $\mathbf{G}_\delta$. In particular, a homeomorphism between dense subsets of Z_1, Z_2 can be extended to a homeomorphism between dense $\mathbf{G}_\delta$ sets.