Exercise sheet 1

Descriptive Set Theory

Academic year 2023-24

1. Prove that the following are Polish subspaces of the Baire space ω^ω:

$$\begin{aligned} A &= \{x \in \omega^\omega \mid x \text{ has infinite range}\} \\ B &= \{x \in \omega^\omega \mid x^{-1}(n) \text{ is infinite, for every } n \in \omega\}. \end{aligned}$$

   In contrast, show that

$$\begin{aligned} C &= \{x \in \omega^\omega \mid x \text{ is not surjective}\} \\ D &= \{x \in \omega^\omega \mid x \text{ has finite range}\} \end{aligned}$$

   are *not* Polish. (Use the fact that in a Polish space X, if $A \subseteq X$ is F_σ and both dense and codense, then A is not G_δ.)

2. Let $2^{(\omega^{<\omega})}$ be endowed with the product over the countable index set $\omega^{<\omega}$ of the discrete topology on $2 = \{0,1\}$. Let $\mathrm{Tr} \subseteq 2^{(\omega^{<\omega})}$ be the set consisting of all characteristic functions of trees on ω. Show that Tr is closed in $2^{(\omega^{<\omega})}$ and thus it is a Polish space. Show also that the set $\mathrm{PTr} \subseteq \mathrm{Tr}$ of (the characteristic functions of) pruned trees is G_δ and thus Polish as well. Finally, prove that $\mathrm{Tr} \setminus \mathrm{PTr}$ is *not* a Polish space.

3. Let
$$\mathcal{L} = \{R_i \mid i < I\} \cup \{f_j \mid j < J\} \cup \{a_k \mid k < K\}$$
   with $I, J, K \leq \omega$ be an at most countable first-order language, and let M be a countable $\mathcal{L}$-structure. Without loss of generality, we may assume that the domain of M is ω itself. Prove that the group of automorphisms $\mathrm{Aut}(M)$ of M is a Polish subgroup of S_∞.

4. Consider the Polish space $X = \omega^{\omega\times\omega} \times \omega$. Let Gp be the space of those $(f, a) \in X$ such that $\langle \omega, f, a\rangle$ is a group with operation f and neutral element a.

(a) Prove that Gp is a Polish subspace of X.

(b) Prove that the subspace of Gp consisting of Abelian groups is Polish, and similarly for the subspace of non-Abelian groups.

(c) Prove that the subspace of Gp consisting of Archimedean groups is Polish.

5. Suppose that d is an ultrametric on a space X. Prove the following statements:

(a) If $d(x, z) \neq d(y, z)$, then $d(x, y) = \max\{d(x, z), d(y, z)\}$ ("all triangles are isosceles with legs longer than or equal to the basis").

(b) The "open" balls $B_d(x, \varepsilon) = \{y \in A^\omega \mid d(x, y) < \varepsilon\}$ and the "closed" balls $B_d^{cl}(x, \varepsilon) = \{y \in A^\omega \mid d(x, y) \leq \varepsilon\}$ are both clopen.

(c) If $y \in B_d(x, \varepsilon)$, then $B_d(y, \varepsilon) = B_d(x, \varepsilon)$ ("all elements of an open ball are centers of it").

(d) If two open (closed) balls intersect, then one is contained in the other one.