The Cantor and Baire space

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Consider $A \neq \emptyset$ with the discrete topology and $A^{\mathbb{N}}$ with the product topology.
$A^{<\mathbb{N}} = \bigcup_{n\in\mathbb{N}} A^n$ is the set of finite sequences of elements of A.
$\mathrm{lh}(s)$ is the **length** of $s \in A^{<\mathbb{N}}$, the unique $n \in \mathbb{N}$ such that $s \in A^n$.
A basis for the topology of $A^{\mathbb{N}}$ is $\{\boldsymbol{N}_s \mid s \in A^{<\mathbb{N}}\}$

$$\boldsymbol{N}_s = \{x \in A^{\mathbb{N}} \mid x \restriction \mathrm{lh}(s) = s\}.$$

If two basic open sets intersect, then one is contained in the other, and $\boldsymbol{N}_s \subseteq \boldsymbol{N}_t$ if and only if $t \subseteq s$. The space $A^{\mathbb{N}}$ is second-countable (and thus separable) if and only if A is (at most) countable.
$A^{\mathbb{N}}$ is completely metrizable, as witnessed by the complete metric

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 2^{-(n+1)} & \text{if } x \neq y \text{ and } n \text{ is least s.t. } x(n) \neq y(n) \end{cases} \tag{1.1}$$

The sets $\boldsymbol{N}_s$ are exactly the nonempty open balls with respect to d.

The metric d from (1.1) is actually an **ultrametric**: for all $x, y, z \in A^{\mathbb{N}}$

$$d(x, y) \leq \max\{d(x, z), d(z, y)\}.$$

This implies that $A^{\mathbb{N}}$ is also **zero-dimensional**, i.e. it admits a basis consisting only of clopen sets. In fact each $\boldsymbol{N}_s$ is clopen.

Here are some basic facts (EXERCISE!) on $A^{\mathbb{N}}$

- For $x_n, x \in A^{\mathbb{N}}$

$$\begin{aligned} x_n \to x &\Leftrightarrow \forall i \in \mathbb{N}\, (x_n(i) \to x(i)) \\ &\Leftrightarrow \forall i \in \mathbb{N}\, (x_n(i) = x(i) \text{ for all but finitely many } n). \end{aligned}$$

- The finite products $(A^{\mathbb{N}})^n$ (for $n \geq 1$) and the countable product $(A^{\mathbb{N}})^{\mathbb{N}}$ are all homeomorphic to $A^{\mathbb{N}}$.
- If A has more than one point, the space $A^{\mathbb{N}}$ is **perfect**, i.e. it has no isolated points.

- If $d(x, z) \neq d(y, z)$, then $d(x, y) = \max\{d(x, z), d(y, z)\}$ ("all triangles are isosceles with legs longer than or equal to the basis").
- The "open" balls $\mathrm{B}_d(x, \varepsilon) = \{y \in A^{\mathbb{N}} \mid d(x, y) < \varepsilon\}$ and the "closed" balls $\mathrm{B}^{\mathrm{cl}}_d(x, \varepsilon) = \{y \in A^{\mathbb{N}} \mid d(x, y) \leq \varepsilon\}$ are both clopen.
- If $y \in \mathrm{B}_d(x, \varepsilon)$, then $\mathrm{B}_d(y, \varepsilon) = \mathrm{B}_d(x, \varepsilon)$ ("all elements of an open ball are centers of it").
- If two open (closed) balls intersect, then one is contained in the other one.
- A sequence $(x_n)_{n \in \mathbb{N}}$ is d-Cauchy if and only if $d(x_n, x_{n+1}) \to 0$.

Definition 1.3.2

A **tree** on $A \neq \emptyset$ is a $T \subseteq A^{<\mathbb{N}}$ s.t. $s \in T \Rightarrow s \restriction n \in T$ for $n \leq \mathrm{lh}(s)$.
A tree is **pruned** if it has no **terminal nodes** (also called **leafs**).
s is terminal if there is no t such that $s \subset t$.
The **body** of T is

$$[T] = \{x \in A^{\mathbb{N}} \mid \forall n \in \mathbb{N} \, (x \restriction n \in T)\}.$$

A tree T is **well-founded** if $[T] = \emptyset$, otherwise it is **ill-founded**.

Proposition 1.3.3 [Kec95, Proposition 2.4]

The map $T \mapsto [T]$ is a bijection between pruned trees on A and closed subsets of $A^{\mathbb{N}}$. Its inverse is given by

$$F \mapsto T_F = \{x \restriction n \mid x \in F \land n \in \mathbb{N}\}.$$

We call T_F the **tree** of F.

Proof

$[T] \subseteq A^{\mathbb{N}}$ is closed

If $x \notin [T]$ then $x \restriction n \notin T$ for some $n \in \mathbb{N}$, so $x \in \boldsymbol{N}_{x \restriction n}$ is disjoint from $[T]$.

$T \mapsto [T]$ is injective.

Assume that $s \in S \setminus T$. As S is pruned, recursively define $(s_n)_{n \in \mathbb{N}}$ such that $s_0 = s$, $s_{n+1} \supset s_n$, and $s_n \in S$. Then $x = \bigcup_{n \in \mathbb{N}} s_n \in [S] \setminus [T]$.

$T \mapsto [T]$ is surjective.

$F \mapsto T_F$ is the inverse of $T \mapsto [T]$.

$F = [T_F]$

$F \subseteq [T_F]$ is obvious. For the other inclusion consider an arbitrary $x \in [T_F]$. For every $n \in \mathbb{N}$ there is $y_n \in F$ such that $x \restriction n = y_n \restriction n$. But then $y_n \to x$, whence $x \in F$ because F is closed.

Remark 1.3.4

The proof shows that if $C \subseteq A^{\mathbb{N}}$ is an arbitrary set, then the body of $T_C = \{x \upharpoonright n \mid x \in C \wedge n \in \mathbb{N}\}$ coincides with the closure $\mathrm{Cl}(C)$ of C.

The following notion of A-scheme is used to build continuous functions from the space $A^{\mathbb{N}}$ to some metrizable space X. In [Kec95], 2-schemes (respectively, $\mathbb{N}$-schemes) satisfying the hypothesis of Lemma 1.3.6(b) are called **Cantor schemes** (respectively, **Lusin schemes**) with vanishing diameters.

Definition 1.3.5

Let $A \neq \emptyset$. An A-**scheme** on a metric space (X, d) is a family $\mathcal{S} = \{B_s \mid s \in A^{<\mathbb{N}}\}$ of subsets of X such that

(i) $B_{s^\frown a} \subseteq B_s$; (Monotonicity)

(ii) $\mathrm{diam}(B_{x \upharpoonright n}) \to 0$. (Vanishing diameters)

(The latter will often be ensured by requiring that $\mathrm{diam}(B_s) \leq 2^{-\mathrm{lh}(s)}$.)
Every A-scheme induces a function. Set

$$D_{\mathcal{S}} = \{x \in A^{\mathbb{N}} \mid \bigcap_{n \in \mathbb{N}} B_{x \upharpoonright n} \neq \emptyset\}$$

and

$$f_{\mathcal{S}} \colon D_{\mathcal{S}} \to X, \qquad x \mapsto f_{\mathcal{S}}(x) \in \bigcap_{n \in \mathbb{N}} B_{x \upharpoonright n}.$$

Condition (ii) in Definition 1.3.5 implies in particular that $\bigcap_{n \in \mathbb{N}} B_{x \upharpoonright n}$ contains at most one point, thus $f_{\mathcal{S}}$ is well-defined.

Lemma 1.3.6

$\mathcal{S} = \{B_s \mid s \in A^{<\mathbb{N}}\}$ an A-scheme on (Y, d), and $f\colon D \to Y$ its induced function. Then

(a) f is continuous and $f(\boldsymbol{N}_s \cap D) \subseteq B_s \cap f(D)$.

(b) If $B_{s^\frown a} \cap B_{s^\frown a'} = \emptyset$ for $a \neq a'$, then f is injective. Moreover

$$f(\boldsymbol{N}_s \cap D) = B_s \cap f(D) \quad (1.2) \qquad f(D) = \bigcap_{n\in\mathbb{N}} \bigcup_{s\in A^n} B_s \quad (1.3)$$

(c) If $B_{s^\frown a} \cap B_{s^\frown a'} = \emptyset$ for $a \neq a'$ and every B_s is open in $f(D)$, then f is a (topological) embedding, i.e. f is a homeomorphism between D and $f(D)$.

(d) If $B_s = \bigcup_{a\in A} B_{s^\frown a}$ for all $s \in A^{<\mathbb{N}}$, then $f(\boldsymbol{N}_s \cap D) = B_s$. Thus $f(D) = B_\emptyset$ so if $B_\emptyset = Y$ then f is surjective.

(e) If (Y, d) is complete and $\mathrm{Cl}(B_{s^\frown a}) \subseteq B_s$, then $D = [T]$ with $T = \{s \in A^{<\mathbb{N}} \mid B_s \neq \emptyset\}$, thus D is a closed subset of $A^{\mathbb{N}}$. If moreover each B_s is nonempty, then $D = A^{\mathbb{N}}$.

Proof of part (a) of Lemma 1.3.6

f is continuous and $f(\boldsymbol{N}_s \cap D) \subseteq B_s \cap f(D)$.

Proof.

If $x \in \boldsymbol{N}_s \cap D$ then $f(x) \in \bigcap_{n\in\mathbb{N}} B_{x\restriction n} \subseteq B_s$, so $f(\boldsymbol{N}_s \cap D) \subseteq B_s \cap f(D)$ holds.
Let $V \subseteq Y$ open neighborhood of $f(x)$. Let $\varepsilon > 0$ such that $B_d(f(x), \varepsilon) \subseteq V$. By (ii) in Definition 1.3.5 (vanishing diameters), there is n such that $\mathrm{diam}(B_{x\restriction n}) < \varepsilon$. Since $f(x) \in B_{x\restriction n}$ by definition of f, we have $B_{x\restriction n} \subseteq B_d(f(x), \varepsilon) \subseteq V$. Thus the open neighborhood $\boldsymbol{N}_{x\restriction n} \cap D$ of x is such that $f(\boldsymbol{N}_{x\restriction n} \cap D) \subseteq B_{x\restriction n} \cap f(D) \subseteq V$. $\square$

Proof of part (b) of Lemma 1.3.6

$B_{s^\frown a} \cap B_{s^\frown a'} = \emptyset$ implies f is injective.

Proof.

Given distinct $x, x' \in D \subseteq A^{\mathbb{N}}$, let $n \in \mathbb{N}$ be least such that $x(n) \neq x'(n)$. Then setting $s = x \restriction n = x' \restriction n$ one has $B_{s^\frown\langle x(n)\rangle} \cap B_{s^\frown\langle x'(n)\rangle} = \emptyset$ by our assumption on the scheme: since by definition of f we have $f(x) \in B_{x\restriction(n+1)} = B_{s^\frown\langle x(n)\rangle}$ and $f(x') \in B_{x'\restriction(n+1)} = B_{s^\frown\langle x'(n)\rangle}$, it follows that $f(x) \neq f(x')$. □

Proof of part (b) of Lemma 1.3.6

$f(\boldsymbol{N}_s \cap D) = B_s \cap f(D)$ (1.2)

Proof.

$f(\boldsymbol{N}_s \cap D) \subseteq B_s \cap f(D)$ follows from part (a). Suppose $x \in D$ is such that $f(x) \in B_s$, and consider $t = x \restriction \mathrm{lh}(s)$, so that $f(x) \in B_t$. Since they have the same length, if $s \neq t$ then $B_s \cap B_t = \emptyset$ by the previous argument, against $f(x) \in B_s \cap B_t$. Therefore $s = t = x \restriction \mathrm{lh}(s)$, whence $x \in \boldsymbol{N}_s$ and $f(x) \in f(\boldsymbol{N}_s \cap D)$. □

Proof of part (b) of Lemma 1.3.6

$f(D) = \bigcap_{n\in\mathbb{N}} \bigcup_{s\in A^n} B_s$ (1.3)

Proof.

$f(D) \subseteq \bigcap_{n\in\mathbb{N}} \bigcup_{s\in A^n} B_s$ holds, so it is enough to prove the reverse inclusion. Let $y \in \bigcap_{n\in\mathbb{N}} \bigcup_{s\in A^n} B_s$. By this and our hypothesis on the scheme, for each $n \in \mathbb{N}$ there is a *unique* $s_n \in A^n$ such that $y \in B_{s_n}$, so that $s_n \subseteq s_m$ whenever $n \leq m$. Thus $x = \bigcup_{n\in\mathbb{N}} s_n$ is such that $y \in \bigcap_{n\in\mathbb{N}} B_{x\restriction n}$ which entails both $x \in D$ and $f(x) = y$, i.e. $y \in f(D)$. □

Proof of part (c) of Lemma 1.3.6

If $B_{s^\frown a} \cap B_{s^\frown a'} = \emptyset$ for $a \neq a'$ and every B_s is open in $f(D)$, then f is a (topological) embedding, i.e. f is a homeomorphism between D and $f(D)$.

Proof.

Let $U = \boldsymbol{N}_s \cap D$ (with $s \in A^{<\mathbb{N}}$) be an arbitrary basic open set of D. Since the hypothesis of part (b) is satisfied, by (1.2) we have $f(U) = B_s \cap f(D)$: by our hypothesis on B_s, the set $f(U)$ is then open in $f(D)$. □

Proof of part (d) of Lemma 1.3.6

If $B_s = \bigcup_{a \in A} B_{s^\frown a}$ for all $s \in A^{<\mathbb{N}}$, then $f(\boldsymbol{N}_s \cap D) = B_s$. Thus $f(D) = B_\emptyset$ so if $B_\emptyset = Y$ then f is surjective.

Proof.

It is enough to show that $B_s \subseteq f(\boldsymbol{N}_s \cap D)$. Fix any $y \in B_s$ and set $n = \mathrm{lh}(s)$. For $i < n$, set also $x_i = s(i)$. Since $B_s = \bigcup_{a \in A} B_{s^\frown a}$, by hypothesis there is $x_n \in A$ such that $y \in B_{s^\frown \langle x_n \rangle} = B_{\langle x_0, \ldots, x_n \rangle}$. Since $B_{s^\frown \langle x_n \rangle} = \bigcup_{a \in A} B_{s^\frown \langle x_n \rangle ^\frown a}$ we then get that $y \in B_{s^\frown \langle x_n, x_{n+1} \rangle} = B_{\langle x_0, \ldots, x_{n+1} \rangle}$ for some $x_{n+1} \in A$. Continuing this process, we recursively construct a sequence $x = (x_k)_{k \in \mathbb{N}} \in A^{\mathbb{N}}$ such that $s \subseteq x$ and $y \in B_{\langle x_0, \ldots, x_{k-1} \rangle} = B_{x \restriction k}$ for all $k \in \mathbb{N}$. Therefore y witnesses that $x \in D$, and $f(x) = y$ by definition of f. Thus x witnesses $y \in f(\boldsymbol{N}_s \cap D)$. □

Proof of part (e) of Lemma 1.3.6

If (Y, d) is complete and $\mathrm{Cl}(B_{s^\frown a}) \subseteq B_s$, then $D = [T]$ with $T = \{s \in A^{<\mathbb{N}} \mid B_s \neq \emptyset\}$, thus D is a closed subset of $A^{\mathbb{N}}$.

Proof.

Enough to show that $\forall x \in A^{\mathbb{N}} [x \in D \Leftrightarrow \forall n \in \mathbb{N} (B_{x \restriction n} \neq \emptyset)]$.
The $\Rightarrow$ direction follows from definition of D. Assume that $y_n \in B_{x \restriction n} \neq \emptyset$. We claim that $(y_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in (Y, d). Given $\varepsilon > 0$, pick $N \in \mathbb{N}$ such that $\mathrm{diam}(B_{x \restriction N}) < \varepsilon$. For all $n, m \geq N$, we have $y_n \in B_{x \restriction n} \subseteq B_{x \restriction N}$ and $y_m \in B_{x \restriction m} \subseteq B_{x \restriction N}$, therefore $d(y_n, y_m) < \varepsilon$.
Let $y = \lim_n y_n$: we claim that $y \in \bigcap_{n \in \mathbb{N}} B_{x \restriction n}$, whence $x \in D$. Given any $n \in \mathbb{N}$, the sequence $(y_m)_{m > n}$ is contained in $B_{x \restriction (n+1)}$, and thus

$$y = \lim_n y_n = \lim_{m > n} y_m \in \mathrm{Cl}(B_{x \restriction (n+1)}) \subseteq B_{x \restriction n}.$$

Since $n \in \mathbb{N}$ was arbitrary, this shows $y \in \bigcap_{n \in \mathbb{N}} B_{x \restriction n}$, as desired. □

Given two families $\mathcal{S} = \{B_s \mid s \in A^{<\mathbb{N}}\}$ and $\mathcal{S}' = \{B'_s \mid s \in A^{<\mathbb{N}}\}$ of subsets of a metric space (X, d), we write $\mathcal{S} \sqsubseteq \mathcal{S}'$ if $B_s \subseteq B'_s$ for all $s \in A^{<\mathbb{N}}$.
Moreover, we set $\mathrm{Cl}(\mathcal{S}) = \{\mathrm{Cl}(B_s) \mid s \in A^{<\mathbb{N}}\}$. Obviously, $\mathcal{S} \sqsubseteq \mathrm{Cl}(\mathcal{S})$.

Lemma 1.3.7

Let $\mathcal{S} = \{B_s \mid s \in A^{<\mathbb{N}}\}$ and $\mathcal{S}' = \{B'_s \mid s \in A^{<\mathbb{N}}\}$ be two A-schemes on the same metric space (X, d).

1. If $\mathcal{S} \sqsubseteq \mathcal{S}'$, then $f_{\mathcal{S}'}$ extends $f_{\mathcal{S}}$, that is, $D_{\mathcal{S}} \subseteq D_{\mathcal{S}'}$ and $f_{\mathcal{S}'}(x) = f_{\mathcal{S}}(x)$ for every $x \in D_{\mathcal{S}}$.
2. The family $\mathrm{Cl}(\mathcal{S})$ is an A-scheme, so $f_{\mathrm{Cl}(\mathcal{S})}$ extends $f_{\mathcal{S}}$. If moreover (X, d) is complete, then $D_{\mathrm{Cl}(\mathcal{S})}$ is closed and hence contains $\mathrm{Cl}(D_{\mathcal{S}})$.

Proof.

1 The inclusion $D_{\mathcal{S}} \subseteq D_{\mathcal{S}'}$ immediately follows that for every $x \in A^{\mathbb{N}}$ we have $\bigcap_{n \in \mathbb{N}} B_{x \restriction n} \subseteq \bigcap_{n \in \mathbb{N}} B'_{x \restriction n}$. It follows that if $x \in D_{\mathcal{S}}$, then $f_{\mathcal{S}}(x) \in \bigcap_{n \in \mathbb{N}} B_{x \restriction n} \subseteq \bigcap_{n \in \mathbb{N}} B'_{x \restriction n} = \{f_{\mathcal{S}'}(x)\}$, thus $f_{\mathcal{S}}(x) = f_{\mathcal{S}'}(x)$.
2 Since $B_{s^\frown a} \subseteq B_s$ implies $\mathrm{Cl}(B_{s^\frown a}) \subseteq \mathrm{Cl}(B_s)$ and $\mathrm{diam}(B_s) = \mathrm{diam}(\mathrm{Cl}(B_s))$, the family $\mathrm{Cl}(\mathcal{S})$ is an A-scheme. Such an A-scheme automatically satisfies the condition in Lemma 1.3.6(e) because its elements are closed, thus $D_{\mathrm{Cl}(\mathcal{S})}$ is closed. □

Remark 1.3.8

It follows that not all continuous partial functions from $A^{\mathbb{N}}$ into a (complete) metric space X are induced by a scheme. For example, set $A = 2$, $X = \mathbb{N}^{\mathbb{N}}$, and consider the inverse f of the embedding from Remark 1.2.3. If it where induced by a 2-scheme $\mathcal{S}$, then f could be extended to a total continuous function $g\colon 2^{\mathbb{N}} \to \mathbb{N}^{\mathbb{N}}$ by Lemma 1.3.7(2) and the fact that the domain of f is dense in $2^{\mathbb{N}}$. But this would imply that $g(2^{\mathbb{N}})$ is compact, and since f was already onto $\mathbb{N}^{\mathbb{N}}$ and the latter is not compact we get a contradiction.

In the opposite direction, given a continuous function $f\colon C \to X$ with $C \subseteq A^{\mathbb{N}}$ and X a metric space one can canonically reconstruct a family $\mathcal{S}_f = \{B_s \mid s \in A^{<\mathbb{N}}\}$ by setting $B_s = f(C \cap \boldsymbol{N}_s)$. It turns out that when C is closed, the family $\mathcal{S}_f$ is an A-scheme inducing exactly the function f, and that the properties of f translate to properties of the scheme.

Lemma 1.3.9

Let $f\colon C \to X$ be a continuous function from a closed set $C \subseteq A^{\mathbb{N}}$ to a metric space (X, d).

1. The family $\mathcal{S}_f$ is an A-scheme inducing f, i.e. it is such that $D_{\mathcal{S}_f} = C$ and $f_{\mathcal{S}_f}(x) = f(x)$ for all $x \in C$.
2. The function f is injective if and only if $B_{s^\frown a} \cap B_{s^\frown a'} = \emptyset$ for all $s \in A^{<\mathbb{N}}$ and distinct $a, a' \in A$.
3. The function f is a (topological) embedding if and only if $B_{s^\frown a} \cap B_{s^\frown a'} = \emptyset$ for all $s \in A^{<\mathbb{N}}$ and distinct $a, a' \in A$ and each B_s is open relatively to $f(C)$.
4. For all $s \in A^{<\mathbb{N}}$ it holds $B_s = \bigcup_{a \in A} B_{s^\frown a}$, and $B_\emptyset = X$ if and only if f is surjective.
5. Every A-scheme $\mathcal{S}'$ such that $\mathcal{S}_f \sqsubseteq \mathcal{S}' \sqsubseteq \mathrm{Cl}(\mathcal{S}_f)$ induce the function f, i.e. $f_{\mathcal{S}'} = f$.

Proof of ❶ of Lemma 1.3.9

$\mathcal{S}_f$ is an A-scheme inducing f, i.e. $D_{\mathcal{S}_f} = C$ and $\forall x \in C(nf_{\mathcal{S}_f}(x) = f(x))$

As $\boldsymbol{N}_{s^\frown a} \subseteq \boldsymbol{N}_s$ then $B_{s^\frown a} = f(C \cap \boldsymbol{N}_{s^\frown a}) \subseteq f(C \cap \boldsymbol{N}_s) = B_s$. Moreover as f is continuous and with closed domain then $\operatorname{osc}_f(x) = 0$ for all $x \in A^{<\mathbb{N}}$. Since $\operatorname{diam}(B_{x \restriction n})$ decreases when n gets larger and the $\boldsymbol{N}_{x \restriction n}$ form a neighborhood basis of x, it follows that $\operatorname{diam}(B_{x \restriction n}) \to 0$. By construction, if $x \in C$ then $f(x) \in \bigcap_{n \in \mathbb{N}} B_{x \restriction n}$, thus $C \subseteq D_{\mathcal{S}_f}$ and $f_{\mathcal{S}_f}$ extends f. Finally, assume that $x \in D_{\mathcal{S}_f}$, so that, in particular, $B_{x \restriction n} \neq \emptyset$ for all n. Then for each n there is $y_n \in C \cap \boldsymbol{N}_{x \restriction n}$, hence $y_n \to x$ and so $x \in C$ because C is closed.

Proof of ❷ and ❸ of Lemma 1.3.9

f is injective iff $B_{s^\frown a} \cap B_{s^\frown a'} = \emptyset$ for all $s \in A^{<\mathbb{N}}$ and distinct $a, a' \in A$.

One direction follows from part (a) and Lemma 1.3.6(b). The other direction easily follows from the fact that $\boldsymbol{N}_{s^\frown a} \cap \boldsymbol{N}_{s^\frown a'} = \emptyset$ if $a \neq a'$ and the fact that by definition $B_s = f(C \cap \boldsymbol{N}_s)$.

The function f is a (topological) embedding if and only if $B_{s^\frown a} \cap B_{s^\frown a'} = \emptyset$ for all $s \in A^{<\mathbb{N}}$ and distinct $a, a' \in A$ and each B_s is open relatively to $f(C)$.

The backward direction follows from part ❶ and Lemma 1.3.6(c), the forward direction follows from part ❷ and the definition of B_s.

Proof of parts 4 and 5 of Lemma 1.3.9

$B_s = \bigcup_{a\in A} B_{s^\frown a}$, and $B_\emptyset = X$ iff f is surjective.

If $y \in B_s = f(C \cap \boldsymbol{N}_s)$ then there is $s \subseteq x \in A^{\mathbb{N}}$ such that $x \in C$ and $f(x) = y$. But then $y = f(x) \in f(C \cap N_{s^\frown x(\mathrm{lh}(s))})$. So $B_s \subseteq \bigcup_{a\in A} B_{s^\frown a}$. The remaining parts are trivial.

Any $\mathcal{S}'$ such that $\mathcal{S}_f \sqsubseteq \mathcal{S}' \sqsubseteq \mathrm{Cl}(\mathcal{S}_f)$ induces f, i.e. $f_{\mathcal{S}'} = f$.

By Lemma 1.3.7 it is enough to consider the case $\mathcal{S}' = \mathrm{Cl}(\mathcal{S}_f)$ and prove that $D_{\mathrm{Cl}(\mathcal{S}_f)} = C$. This easily follows from $C = D_{\mathcal{S}_f} \subseteq D_{\mathrm{Cl}(\mathcal{S}_f)}$ and by $\mathrm{Cl}(B_s) \neq \emptyset \Leftrightarrow B_s \neq \emptyset$ we get

$$D_{\mathrm{Cl}(\mathcal{S}_f)} \subseteq \{x \in A^{\mathbb{N}} \mid \forall n\, (\mathrm{Cl}(B_{x\restriction n}) \neq \emptyset)\}$$
$$= \{x \in A^{\mathbb{N}} \mid \forall n\, (B_{x\restriction n} \neq \emptyset)\} \subseteq C,$$

where the last inclusion follows from the final part of the proof of part 1.

Definition 1.3.10

A closed set F in a topological space X is a **retract** of X if there is a continuous function $f\colon X \to F$ (called **retraction**) such that $f(x) = x$ for all $x \in F$ (in particular, f is surjective).

Proposition 1.3.11 [Kec95, Proposition 2.8]

Every nonempty closed subset F of $A^{\mathbb{N}}$ is a retract of it.

Corollary

Let (X, d) be a complete metric space. Let A be a nonempty set, $C \subseteq A^{\mathbb{N}}$, and $f\colon C \to X$. Then f is continuous if and only if there is a total continuous function $g\colon A^{\mathbb{N}} \to X$ such that $g \restriction C = f$.

Proof of Proposition 1.3.11

Let T_F be the tree of F. Define $\varphi\colon A^{<\mathbb{N}} \to A^{<\mathbb{N}}$ by recursion on $\mathrm{lh}(s)$:

1. $\mathrm{lh}(\varphi(s)) = \mathrm{lh}(s)$;
2. if $s \subseteq t$ then $\varphi(s) \subseteq \varphi(t)$ (φ is **monotone**);
3. $\varphi(s) \in T_F$;
4. if $s \in T_F$, then $\varphi(s) = s$.

1 and 2 imply that φ is **increasing**.
Set $\varphi(\emptyset) = \emptyset$. Let $s = t^\frown a$ and assume that $\varphi(t)$ has been already defined. Define $\varphi(s)$ as follows: if $s \in T_F$, then set $\varphi(s) = s$. If $s \notin T_F$, then let $\varphi(s)$ be any sequence $\varphi(t)^\frown b \in T_F$, which exists since T_F is pruned and $\varphi(t) \in T_F$ by condition 3.

(continues)

Proof of Proposition 1.3.11 (continued)

$\varphi\colon A^{<\mathbb{N}} \to T_F$ is increasing, and it is the identity on T_F:

1. $\mathrm{lh}(\varphi(s)) = \mathrm{lh}(s)$;
2. if $s \subseteq t$ then $\varphi(s) \subseteq \varphi(t)$;
3. $\varphi(s) \in T_F$;
4. if $s \in T_F$, then $\varphi(s) = s$.

Equip F with the restriction of the metric on $A^{\mathbb{N}}$ and observe that it is still a complete metric on F. Consider the A-scheme $\{B_s \mid s \in A^{<\mathbb{N}}\}$ on F defined by $B_s = \boldsymbol{N}_{\varphi(s)} \cap F$: we claim that the induced map f is the desired retraction of $A^{\mathbb{N}}$ onto F. Conditions 1–2 guarantee that the definition of the B_s's yields to an A-scheme (whence f is a continuous map), condition 3 and the fact that B_s is clopen guarantee that f is defined on the whole $A^{\mathbb{N}}$ (in fact, $\varphi(s) \in T_F$ ensures $B_s \neq \emptyset$), while condition 4 guarantees that $f(x) = x$ for every $x \in F$. □

Theorem 1.3.13 [Kec95, Theorem 7.8]

Every zero-dimensional separable metrizable space can be embedded into both $\mathbb{N}^{\mathbb{N}}$ and $2^{\mathbb{N}}$. Every zero-dimensional Polish space is homeomorphic to a closed subspace of $\mathbb{N}^{\mathbb{N}}$ and to a $\mathbf{G}_\delta$ subspace of $2^{\mathbb{N}}$.

Arguing as in Corollary 1.2.5

Corollary 1.3.14

Every $\mathbf{G}_\delta$ of $\mathbb{N}^{\mathbb{N}}$ is homeomorphic to a closed set, $\mathbf{F}_h(\mathbb{N}^{\mathbb{N}}) = \mathbf{G}_\delta(\mathbb{N}^{\mathbb{N}})$.

Corollary

Every closed subset of a zero-dimensional Polish X is a retract of it.

Proof.

Let F be closed in X. By Theorem 1.3.13, W.L.O.G. X and F are closed in $\mathbb{N}^{\mathbb{N}}$. By Proposition 1.3.11 there is a retraction f of $\mathbb{N}^{\mathbb{N}}$ onto F. It follows that $f \upharpoonright X$ is a retraction of X onto F. □

Proof of Theorem 1.3.13

$\mathbb{N}^{\mathbb{N}}$ is homeomorphic to a $\mathbf{G}_\delta$ subset of $2^{\mathbb{N}}$, so it is enough to show that every zero-dimensional metric space (X, d) with $d \leq 1$ can be embedded in $\mathbb{N}^{\mathbb{N}}$.
Construct an $\mathbb{N}$-scheme $\{B_s \mid s \in \mathbb{N}^{<\mathbb{N}}\}$ on (X, d) such that
B_s is clopen, $\operatorname{diam}(B_s) \leq 2^{-\operatorname{lh}(s)}$, $B_{s^\frown i} \cap B_{s^\frown j} = \emptyset$ for $i \neq j$,
$B_\emptyset = X$ and $B_s = \bigcup_{i \in \mathbb{N}} B_{s^\frown i}$.
By Lemma 1.3.6, this yields a homeomorphism between $D \subseteq \mathbb{N}^{\mathbb{N}}$ and X; moreover, when d is complete (i.e. X is Polish) then D is closed.
Set $B_\emptyset = X$.
Cover B_s with clopen sets $B'_{s^\frown i}$ in the countable basis of X so that $\operatorname{diam}(B'_{s^\frown i}) \leq 2^{-(\operatorname{lh}(s)+1)}$, and then recursively set $B_{s^\frown 0} = B'_{s^\frown 0} \cap B_s$ and $B_{s^\frown (i+1)} = (B'_{s^\frown (i+1)} \setminus \bigcup_{j \leq i} B'_{s^\frown j}) \cap B_s = (B'_{s^\frown (i+1)} \setminus \bigcup_{j \leq i} B_{s^\frown j}) \cap B_s$. □

The Cantor and the Baire space are surjectively universal.

Theorem [Kec95, Theorem 4.18]

Every nonempty compact metrizable space is a continuous image of $2^{\mathbb{N}}$.

Proof.

$f(x) = \sum_{n=0}^{\infty} x(n)2^{-(n+1)}$ maps $2^{\mathbb{N}}$ continuously onto $[0;1]$, so $y \mapsto (f(y(i)))_{i\in\mathbb{N}}$ is a continuous surjection $g\colon (2^{\mathbb{N}})^{\mathbb{N}} \to [0;1]^{\mathbb{N}}$. As $(2^{\mathbb{N}})^{\mathbb{N}}$ and $2^{\mathbb{N}}$ are homeomorphic we have a continuous surjection $2^{\mathbb{N}} \twoheadrightarrow [0;1]^{\mathbb{N}}$. As every compact metrizable space X is homeomorphic to a $K \in \mathcal{K}([0;1]^{\mathbb{N}})$, $F = g^{-1}(K) \subseteq 2^{\mathbb{N}}$ continuously surjects onto X. Compose this surjection with a retraction of $2^{\mathbb{N}}$ onto F. □

Theorem 1.3.17 [Kec95, Theorem 7.9]

Let X be a Polish space. Then there is a closed set $F \subseteq \mathbb{N}^{\mathbb{N}}$ and a continuous bijection $f\colon F \to X$. In particular, if X is nonempty, then there is a continuous surjection $g\colon \mathbb{N}^{\mathbb{N}} \to X$ (extending f).

Proof of Theorem 1.3.17

Every Polish space is homeomorphic to a closed subspace of $\mathbb{R}^{\mathbb{N}}$, so it is enough to show that there is a continuous bijection $f\colon \mathbb{N}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$.

Claim 1.3.17.1

For $a < b$ in $\mathbb{R}$ there is a continuous bijection between $\mathbb{N}^{\mathbb{N}}$ and $[a;b)$.

Assume the Claim for now. Fix a bijection $\varphi\colon \mathbb{N} \to \mathbb{Z}$ and continuous bijections $f_k\colon \boldsymbol{N}_{\langle k\rangle} \to [\varphi(k);\varphi(k)+1)$.

$f = \bigcup_{k\in\mathbb{N}} f_k\colon \mathbb{N}^{\mathbb{N}} \to \mathbb{R}$ is a continuous bijection.

If $U \subseteq \mathbb{R}$ is open, then $f^{-1}(U) = \bigcup_{k\in\mathbb{N}} f_k^{-1}(U)$, and since $f_k^{-1}(U)$ is open in the clopen set $\boldsymbol{N}_{\langle k\rangle}$, then $f^{-1}(U)$ is open in $\mathbb{N}^{\mathbb{N}}$.
Thus there is a continuous bijection $\mathbb{N}^{\mathbb{N}} \approx (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ so we are done, modulo the Claim.

Proof of the claim

Let $\{B_s \mid s \in \mathbb{N}^{<\mathbb{N}}\}$ be an $\mathbb{N}$-scheme on $[a; b]$ such that:

- $B_s = [a_s; b_s)$ for some $a \leq a_s < b_s \leq b$;
- $a_\emptyset = a$ and $b_\emptyset = b$ (i.e. $B_\emptyset = [a; b)$);
- $\mathrm{Cl}(B_{s^\frown n}) = [a_{s^\frown n}; b_{s^\frown n}] \subseteq [a_s; b_s) = B_s$ with $a_s \leq a_{s^\frown n} < b_{s^\frown n} < b_s$;
- $a_{s^\frown 0 = a_s}$ and $b_{s^\frown n} = a_{s^\frown (n+1)}$;
- $\lim_n b_{s^\frown n} = b_s$.

The induced map h is a continuous injection $\mathbb{N}^\mathbb{N} \to [a; b]$. As $B_s \subseteq [a; b)$ then $\mathrm{rng}(h) \subseteq [a; b)$. Moreover, for the same reason the $\mathbb{N}$-scheme above can also be construed as an $\mathbb{N}$-scheme on $[a; b)$, and the above conditions ensure that, when construed in this way, the scheme satisfies also condition (d) of Lemma 1.3.6, whence the induced map h is onto $[a; b)$.

Remark

If X perfect then we can take $F = \mathbb{N}^\mathbb{N}$, that is, there is a continuous bijection $f\colon \mathbb{N}^\mathbb{N} \to X$. The converse holds as well: if $f\colon \mathbb{N}^\mathbb{N} \to X$ is a continuous bijection, then no $x \in X$ can be isolated in it, otherwise $f^{-1}(x)$ would be isolated in $\mathbb{N}^\mathbb{N}$, which is clearly impossible.

Remark 1.3.19

In general, the inverse of the continuous bijection f is not continuous. However, the continuous bijection $f\colon \mathbb{N}^\mathbb{N} \to \mathbb{R}$ constructed in the previous proof is such that $f(\boldsymbol{N}_s)$ is an half-open interval. Thus the resulting continuous bijection $\mathbb{N}^\mathbb{N} \to \mathbb{R}^\mathbb{N}$ maps open sets to $\mathbf{F}_\sigma$ sets, and therefore we can conclude that the same is true for arbitrary Polish spaces, that is: For any Polish space X there is a continuous bijection $f\colon F \to X$ with $F \subseteq \mathbb{N}^\mathbb{N}$ closed such that f maps open sets to $\mathbf{F}_\sigma$ sets.

Exercise

Observe that the continuous bijection constructed in the proof of Claim 1.3.17.1 is actually an isomorphism between the linear orders $(\mathbb{N}^{\mathbb{N}}, \leq_{\text{lex}})$, where $\leq_{\text{lex}}$ is the lexicographic ordering, and $([0;1), \leq)$, where $\leq$ is the usual ordering on $\mathbb{R}$. Infer that $(\mathbb{N}^{\mathbb{N}}, \tau_{\text{lex}})$, where τ_{lex} is the topology generated by the open $\leq_{\text{lex}}$-intervals, is homeomorphic to $[0;1)$, so it is a Polish space and it is of dimension 1. Argue that τ_{lex} is coarser than the usual product topology on $\mathbb{N}^{\mathbb{N}}$, and that each basic open set $\boldsymbol{N}_s$ is both $\mathbf{F}_\sigma$ and $\mathbf{G}_\delta$ with respect to τ_{lex}, so that every open set in the usual product topology on $\mathbb{N}^{\mathbb{N}}$ is $\mathbf{F}_\sigma$ with respect to τ_{lex}.

More applications of schemes.

Theorem (Brouwer) [Kec95, Theorem 7.4]

The Cantor space $2^{\mathbb{N}}$ is the unique, up to homeomorphism, nonempty, compact metrizable (hence Polish) zero-dimensional space without isolated points.

Theorem (Alexandrov-Urysohn) [Kec95, Theorem 7.7]

The Baire space $\mathbb{N}^{\mathbb{N}}$ is the unique, up to homeomorphism, nonempty Polish zero-dimensional space for which all compact subsets have empty interior.

Theorem 1.3.26 (Hurewicz) [Kec95, Theorem 7.10]

Let X be Polish. Then X contains a closed subspace homeomorphic to $\mathbb{N}^{\mathbb{N}}$ iff and only if X is not K_σ, i.e. X cannot be written as a countable union of compact sets.