Exercise sheet 3

Descriptive Set Theory

Academic year 2023-24

1. Prove that for every topological space X and every $A \subseteq X$ the following are equivalent:

   (a) The set A is nowhere dense, i.e. there is no open set $U \subseteq X$ such that $A \cap U$ is dense in U.

   (b) The closure of A has empty interior.

   (c) There is an open dense set $V \subseteq X$ such that $A \cap V = \emptyset$.

   Conclude that $B \subseteq X$ is comeager if and only if it contains a countable intersection of dense open sets.

2. Prove that for every topological space X, the following are equivalent:

   (a) Every nonempty open subset of X is non-meager.

   (b) Every comeager set in X is dense.

   (c) The intersection of countably many dense open subsets of X is dense.

3. Prove by induction on $1 \leq \alpha < \omega_1$ that

   (a) $\boldsymbol{\Sigma}^0_\alpha(X)$ is closed under countable unions and finite intersections;

   (b) $\boldsymbol{\Pi}^0_\alpha(X)$ is closed under countable intersections and finite unions;

   (c) $\boldsymbol{\Delta}^0_\alpha(X)$ is a Boolean algebra, i.e. it is closed under complements, finite unions, and finite intersections.

4. Let $Y \subseteq X$ be Polish spaces. Show that for every $\alpha \geq 3$

$$\boldsymbol{\Delta}^0_\alpha(Y) = \boldsymbol{\Delta}^0_\alpha(X) \restriction Y,$$

   where as usual $\boldsymbol{\Delta}^0_\alpha(X) \restriction Y = \{A \cap Y \mid A \in \boldsymbol{\Delta}^0_\alpha(X)\}$.

5. Given a continuous function $f\colon [0;1] \to \mathbb{R}$, let

$$D_f = \{x \in [0;1] \mid f' \text{ exists}\}.$$

(At endpoints we take one-side derivatives.) Prove that $D_f \in \mathbf{\Pi}^0_3([0;1])$.