Standard Borel spaces

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Definition

A **Borel space** is a pair $X = (X, \mathcal{S})$ where $\mathcal{S}$ is a σ-algebra on X such that $\mathcal{S} = \mathbf{Bor}(X, \tau)$ for some separable metrizable topology τ on X.
A function $f \colon X \to X'$ between two Borel spaces $X = (X, \mathcal{S})$ and $X' = (X', \mathcal{S}')$ is **Borel** if $f^{-1}(B) \in \mathcal{S}$ for every $B \in \mathcal{S}'$. The function f is a **Borel isomorphism** if it is bijective and both f and f^{-1} are Borel, i.e. $A \in \mathcal{S} \Leftrightarrow f(A) \in \mathcal{S}'$ for every $A \subseteq X$. In this case we say that X and X' are **Borel isomorphic**.

A σ-algebra $\mathcal{S}$ on X is **countably generated** if there is a countable $\mathcal{A} \subseteq \mathscr{P}(X)$ such that $\mathcal{S}$ is the smallest σ-algebra containing $\mathcal{A}$, and it **separates points** if for two distinct points in X there is $A \in \mathcal{S}$ which contains exactly one of them. If $\mathcal{A}$ generates $\mathcal{S}$, then $\mathcal{S}$ separates points iff for $x \neq y$ there is a generator $A \in \mathcal{A}$ separating one from the other. This can be proved observing that the collection of all $A \in \mathcal{S}$ that *do not* separate a given pair of points is a σ-algebra: therefore, if all of $\mathcal{A}$ were contained in such σ-algebra, then $\mathcal{S}$ itself could not separate those two points.

Proposition 2.3.2 [Kec95, Proposition 12.2]

Let $\mathcal{S}$ be a σ-algebra on a set X. Then the following are equivalent:

1. $(X, \mathcal{S})$ is a Borel space;
2. $(X, \mathcal{S})$ is Borel isomorphic to $(Y, \mathbf{Bor}(Y))$ for some $Y \subseteq 2^\omega$ (and thus to some $Y \subseteq Z$ for any uncountable Polish space Z);
3. $\mathcal{S}$ is countably generated and separates points.

Proof.

2 $\Rightarrow$ 1 and 1 $\Rightarrow$ 3 are obvious, so let us show that 3 $\Rightarrow$ 2. Let $\{A_n \mid n \in \omega\}$ generate $\mathcal{S}$. Define $f \colon X \to 2^\omega$ by $f(x)(n) = 1 \Leftrightarrow x \in A_n$, i.e. $f(x) = (\chi_{A_n})_n$ where χ_{A_n} is the characteristic function of A_n. Then f is injective because $\mathcal{S}$ separates points, and is Borel by definition (use that $\mathbf{Bor}(2^\omega)$ is generated by the the canonical base). Let $Y = f(X) \subseteq 2^\omega$, so that f is a bijection between X and Y. Then $(f^{-1})^{-1}(A_n) = f(A_n) = \{y \in 2^\omega \mid y(n) = 1\} \cap Y$, whence also f^{-1} is Borel, whence f is a Borel isomorphism. □

Definition [Kec95, Definition 12.5]

A Borel space $X = (X, \mathcal{S})$ is called **standard** if there is a Polish topology τ on X such that $\mathcal{S} = \mathbf{Bor}(X, \tau)$. Equivalently, X is standard if and only if it is Borel isomorphic to $(Y, \mathbf{Bor}(Y))$ for some Polish space Y.

Proposition

$X = (X, \mathcal{S})$ is standard iff there is $A \in \mathbf{Bor}(Y)$ with Y Polish such that X is Borel isomorphic to $(A, \mathbf{Bor}(Y) \restriction A)$.
If $(X, \mathcal{S})$ is standard Borel and $A \in \mathcal{S}$, then $(A, \mathcal{S} \restriction A)$ is standard Borel, where $\mathcal{S} \restriction A = \{C \cap A \mid C \in \mathcal{S}\}$.

Proof.

Let $A \subseteq Y$ be as in the statement, and let $f \colon X \to A$ be a Borel isomorphism. By Corollary 2.2.3 there is a Polish topology τ' on Y such that the Borel sets remain the same but A becomes τ'-clopen, hence Polish: since f is a Borel isomorphism also with respect to this new topology, we are done. □

Given a Polish space X equip the set

$$\mathbf{F}(X) = \{F \subseteq X \mid F \text{ is closed}\}$$

with the σ-algebra $\mathbf{B}_{\mathbf{F}(X)}$ (called the **Effros Borel structure** on $\mathbf{F}(X)$) generated by the sets

$$\{F \in \mathbf{F}(X) \mid F \cap U \neq \emptyset\}$$

where U varies over the open subsets of X. The resulting space $\mathbf{F}(X) = (\mathbf{F}(X), \mathbf{B}_{\mathbf{F}(X)})$ is called **Effros Borel space**.

Theorem [Kec95, Theorem 12.6]

Let X be a Polish space. Then the Effros Borel space $\mathbf{F}(X)$ is a standard Borel space.

In particular, let us consider the space $\mathbb{R}^\omega$. Since its closed subspaces coincide, up to homeomorphism, with the collection of all Polish spaces, we can regard $\mathbf{F}(\mathbb{R}^\omega)$ as the standard Borel space of all Polish spaces, and the same is true for $\mathbf{F}(\ell_2)$ (recall that $\mathbb{R}^\omega$ and ℓ_2 are homeomorphic). Similarly, a basic result of Banach space theory shows that every separable Banach space is isometrically isomorphic to a closed subspace of $C(2^\omega)$, i.e. there is a linear isometry between the given space and a closed subspace of $C(2^\omega)$. Since one can easily show (using Theorem 2.3.6 below) that the collection $\mathrm{Subs}(C(2^\omega))$ of all closed linear subspaces of $C(2^\omega)$ is a Borel set in $\mathbf{F}(C(2^\omega))$, we can regard $\mathrm{Subs}(C(2^\omega))$ as the standard Borel space of all separable Banach spaces. Similarly, one can form the standard Borel space of all Polish groups, the standard Borel space of all von Neumann algebras, and so on. All these spaces can be dealt with the techniques and methods of descriptive set theory.

A basic yet fundamental result concerning the Effros Borel spaces is the following selection theorem for $\mathbf{F}(X)$.

Theorem 2.3.6 (Kuratowski-Ryll-Nardzewski) [Kec95, Theorem 12.13]

Let X be Polish. Then there is a sequence of Borel functions $d_n \colon \mathbf{F}(X) \to X$, $n \in \omega$, such that for every nonempty $F \in \mathbf{F}(X)$, the set $\{d_n(F) \mid n \in \omega\}$ is a dense subset of F.