The PSP in the projective hierarchy
Projective functions

Alessandro Andretta

Dipartimento di Matematica
Università di Torino

Theorem 3.4.1

Let X be a Polish space. Then every analytic $A \subseteq X$ has the PSP.

Proof

$A = \pi_X[C]$ with $C \subseteq X \times \mathbb{N}^{\mathbb{N}}$ closed, and $C' := C \setminus \bigcup\{U \in \mathcal{B} \mid \pi_X[C \cap U] \text{ is countable}\}$, where $\mathcal{B}$ is a countable base for $X \times \mathbb{N}^{\mathbb{N}}$. Notice that $A' = \pi_X[C']$ is an analytic subset of A, and that $A \setminus A'$ is countable. If A' is empty, this means that A is countable and we are done; otherwise, $C' \neq \emptyset$ and by construction we have that for every $U \subseteq X \times \mathbb{N}^{\mathbb{N}}$ such that $U \cap C' \neq \emptyset$, the set $\pi_X[U \cap C']$ is uncountable. Fix a complete compatible metric d on C' and build a 2-scheme $\{B_s \mid s \in 2^{<\omega}\}$ on C' such that for all $s \in 2^{<\omega}$ and $i \in \{0, 1\}$:

1. B_s is open and nonempty;
2. $\mathrm{Cl}(B_{s^\frown i}) \subseteq B_s$;
3. $\pi_X[B_{s^\frown 0}] \cap \pi_X[B_{s^\frown 1}] = \emptyset$;
4. $\mathrm{diam}(B_s) \leq 2^{-\mathrm{lh}(s)}$.

Let $f \colon 2^{\mathbb{N}} \to C' \subseteq X \times \mathbb{N}^{\mathbb{N}}$ be the induced embedding. Then $\pi_X \circ f$ embeds $2^{\mathbb{N}}$ into $A' \subseteq A$: indeed, it is continuous, injective 3, and such that $\mathrm{ran}(\pi_X \circ f) \subseteq A' \subseteq A$, as $\pi_X[C'] = A'$.

(continues)

Proof (continued).

1. B_s is open and nonempty;
2. $\mathrm{Cl}(B_{s^\frown i}) \subseteq B_s$;
3. $\pi_X[B_{s^\frown 0}] \cap \pi_X[B_{s^\frown 1}] = \emptyset$;
4. $\mathrm{diam}(B_s) \leq 2^{-\mathrm{lh}(s)}$.

The scheme is constructed by induction on $\mathrm{lh}(s)$.
Choose $\emptyset \neq B_\emptyset \in \boldsymbol{\Sigma}^0_1(C')$ with diameter ≤ 1.
Suppose B_s has been defined. Pick $z_0 = (x_0, y_0)$ and $z_1 = (x_1, y_1)$ in B_s such that $x_0 \neq x_1$ — this is possible because $\pi_X[B_s]$ is uncountable by definition of C'. Let $U_0, U_1 \subseteq X$ be disjoint open sets such that $x_i \in U_i$, for $i = 0, 1$. Choose $\varepsilon > 0$ small enough so that $\mathrm{B}^{\mathrm{cl}}_d(z_i, \varepsilon) \subseteq B_s \cap \pi_X^{-1}(U_i)$ and $\varepsilon < 2^{-(\mathrm{lh}(2)+2)}$: then setting $B_{s^\frown i} = \mathrm{B}_d(z_i, \varepsilon)$ we have that 1–4 are satisfied. □

It can be shown that if all coanalytic subsets of $\mathbb{N}^\mathbb{N}$ have the PSP, then $\aleph_1$ is inaccessible in Gödel's constructible universe L, and hence, in particular, $\mathrm{V} \neq \mathrm{L}$. Thus it is consistent with ZFC that there is a coanalytic subset of $\mathbb{N}^\mathbb{N}$ without the PSP (this happens e.g. in any model of ZFC + $\mathrm{V} = \mathrm{L}$), and the consistency strength of the statement "all $\boldsymbol{\Pi}^1_1$ sets have the PSP" is at least (in fact, equal to) that of the existence of an inaccessible cardinal.
Conversely, if we assume the existence of an inaccessible cardinal λ, then by collapsing λ to ω_1 with the Lévy collapsing forcing we get a model of ZFC in which all projective sets have the PSP. Furthermore, it can be shown using game-theoretic techniques that large cardinal assumptions (e.g. the existence of infinitely many Woodin cardinals with a measurable above) directly imply that all projective sets have the PSP, and thus that there is no projective counterexample to the continuous hypothesis. The same is true if one assumes strong forcing axioms, like the Proper Forcing Axiom PFA.
The above discussion shows in particular that it is independent of ZFC whether all coanalytic (i.e. $\boldsymbol{\Pi}^1_1$) sets have the PSP, thus Theorem 3.4.1 is optimal when working in ZFC alone.

Definition 3.5.1

A function $f\colon X \to Y$ between Polish spaces is **projective** if it is $\mathbf{\Sigma}^1_n$-measurable for some $n \geq 1$, i.e. if $f^{-1}[U] \in \mathbf{\Sigma}^1_n(X)$ for every open set $U \subseteq Y$; equivalently for every basic open U.

Definition 3.5.1 induces a stratification of projective functions according to their level of measurability: $\mathbf{\Sigma}^1_n$-measurable functions are $\mathbf{\Sigma}^1_m$-measurable for all $m \geq n$.
If X is uncountable and $|Y| \geq 2$, then the hierarchy does not collapse before ω.

For each $n \geq 1$ there are $\mathbf{\Sigma}^1_{n+1}$-measurable functions which are not $\mathbf{\Sigma}^1_n$-measurable.

Proof.

Pick any $A \in \mathbf{\Delta}^1_{n+1} \setminus (\mathbf{\Sigma}^1_n(X) \cup \mathbf{\Pi}^1_n(X))$ and fix distinct points $y_0, y_1 \in Y$: then the function $f\colon X \to Y$ defined by $f(x) = y_0$ if $x \in A$ and $f(x) = y_1$ otherwise is as required. □

Finally, we notice that this hierarchy extends the Baire stratification of all Borel functions, in the sense that its first level coincides with the collection of Borel functions.

Lemma 3.5.2

A function $f\colon X \to Y$ between Polish spaces is $\mathbf{\Sigma}^1_1$-measurable if and only if it is Borel.

Proof.

Since Borel sets are analytic, the backward direction is obvious. For the forward direction, since f is $\mathbf{\Sigma}^1_1$-measurable, then for every $\mathbf{G}_\delta$ set $A = \bigcap_{n\in\omega} U_n$ we have $f^{-1}(A) = \bigcap_{n\in\omega} f^{-1}[U_n] \in \mathbf{\Sigma}^1_1(X)$ because analytic sets are closed under countable intersections. In particular, this applies to closed sets, hence $f^{-1}[U] \in \mathbf{\Delta}^1_1(X)$ for all open sets $U \subseteq Y$. The result thus follows from Souslin's Theorem 3.2.3. □

Proposition 3.5.3

X, Y Polish spaces, $n > 0$ and $f\colon X \to Y$. The following are equivalent:

1. f is $\mathbf{\Sigma}^1_n$-measurable;
2. $\mathrm{graph}(f) \in \mathbf{\Delta}^1_n(X \times Y)$;
3. $\mathrm{graph}(f) \in \mathbf{\Sigma}^1_n(X \times Y)$;
4. $f^{-1}[A] \in \mathbf{\Sigma}^1_n(X)$ for every $A \in \mathbf{\Sigma}^1_n(Y)$;
5. $f^{-1}[A] \in \mathbf{\Sigma}^1_n(X)$ for every $A \in \mathbf{\Delta}^1_n(Y)$;
6. $f^{-1}[A] \in \mathbf{\Sigma}^1_n(X)$ for every $A \in \mathbf{Bor}(Y)$;

In particular, f is a projective function if and only if has a projective graph, i.e. $\mathrm{graph}(f) \in \mathbf{Proj}(X \times Y)$.

1 f is $\mathbf{\Sigma}^1_n$-measurable $\Rightarrow$ 2 $\mathrm{graph}(f) \in \mathbf{\Delta}^1_n(X \times Y)$.

Proof.

Fix a basis $\{U_n \mid n \in \omega\}$ for Y. Then for every $x \in X$ and $y \in Y$

$$\begin{aligned}(x, y) \in \mathrm{graph}(f) &\Leftrightarrow \forall n\,(y \in U_n \Rightarrow f(x) \in U_n) \Leftrightarrow \forall n\,(f(x) \in U_n \Rightarrow y \in U_n)\\ &\Leftrightarrow \forall n\big(\underbrace{y \notin U_n \vee x \in f^{-1}[U_n]}_{\mathbf{\Sigma}^1_n}\big) \Leftrightarrow \forall n\big(\underbrace{x \notin f^{-1}[U_n] \vee y \in U_n}_{\mathbf{\Pi}^1_n}\big)\end{aligned}$$

Using the Tarski-Kuratowski algorithm, one sees that the first formulation shows that $\mathrm{graph}(f) \in \mathbf{\Sigma}^1_n(X \times Y)$, while the second one witnesses $\mathrm{graph}(f) \in \mathbf{\Pi}^1_n(X \times Y)$. □

2 $\mathrm{graph}(f) \in \mathbf{\Delta}^1_n(X \times Y) \Rightarrow$ 3 $\mathrm{graph}(f) \in \mathbf{\Sigma}^1_n(X \times Y)$.

Obvious.

(3) $\operatorname{graph}(f) \in \mathbf{\Sigma}^1_n(X \times Y) \Rightarrow$ (4) $f^{-1}[A] \in \mathbf{\Sigma}^1_n(X)$ for every $A \in \mathbf{\Sigma}^1_n(Y)$.

Proof.

For every $x \in X$, $x \in f^{-1}[A] \Leftrightarrow \exists y\, ((x,y) \in \operatorname{graph}(f) \wedge y \in A)$.
Using again the Tarski-Kuratowski algorithm, we see that this witnesses $f^{-1}[A] \in \mathbf{\Sigma}^1_n(X)$. □

(4) $f^{-1}[A] \in \mathbf{\Sigma}^1_n(X)$ for every $A \in \mathbf{\Sigma}^1_n(Y) \Rightarrow$ (5) $f^{-1}[A] \in \mathbf{\Sigma}^1_n(X)$ for every $A \in \mathbf{\Delta}^1_n(Y) \Rightarrow$
$\Rightarrow$ (6) $f^{-1}[A] \in \mathbf{\Sigma}^1_n(X)$ for every $A \in \mathbf{Bor}(Y) \Rightarrow$ (1) f is $\mathbf{\Sigma}^1_n$-measurable.

Proof.

By $\mathbf{\Sigma}^0_1(Y) \subseteq \mathbf{Bor}(Y) \subseteq \mathbf{\Delta}^1_n(Y) \subseteq \mathbf{\Sigma}^1_n(Y)$. □

Corollary

All the pointclasses $\mathbf{\Sigma}^1_n$, $\mathbf{\Pi}^1_n$, and $\mathbf{\Delta}^1_n$ (for $n \geq 1$) are closed under preimages via $\mathbf{\Sigma}^1_n$-measurable functions. Moreover, $\mathbf{\Sigma}^1_n$ is also closed under images by $\mathbf{\Sigma}^1_n$-measurable functions.

Proof.

The first part follows from Proposition 3.5.3. For the second one, argue as in the proof of Proposition 3.3.3 using Proposition 3.5.3. □