Universal spaces

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Theorem 1.2.1 [Kec95, Theorem 4.14]

Every separable metrizable space is homeomorphic to a subspace of the Hilbert cube $[0;1]^{\mathbb{N}}$. In particular, the Polish spaces are, up to homeomorphism, exactly the $\mathbf{G}_\delta$ subspaces of $[0;1]^{\mathbb{N}}$, and the compact metrizable spaces are, up to homeomorphism, the closed (equivalently, compact) subspaces of $[0;1]^{\mathbb{N}}$.

Thus all compact metrizable spaces are Polish. Actually any compatible metric on a compact metrizable space X is complete, and X is **totally bounded** (for every $\varepsilon > 0$, X can be covered by finitely many open balls of radius $< \varepsilon$), whence it is second-countable.

Corollary 1.2.2

Every separable metrizable space X admits a **compactification** Y, i.e. a compact metrizable space in which X can be embedded as a dense subset. If X is Polish, then X can be embedded into Y as a dense $\mathbf{G}_\delta$.

Proof of Theorem 1.2.1

$D = \{z_n \mid n \in \mathbb{N}\}$ dense in (X, d) with $d \leq 1$. The map

$$f\colon X \to [0;1]^{\mathbb{N}}, \qquad x \mapsto (d(x, z_n))_{n\in\mathbb{N}}$$

is continuous,

Proof.

Fix $U = \{y \in [0;1]^{\mathbb{N}} \mid a < y(k) < b\}$ in the sub-base of $[0;1]^{\mathbb{N}}$, with $k \in \mathbb{N}$ and $0 < a < b < 1$. Then , and $f^{-1}(U) = \mathrm{B}_d(z_k, b) \setminus \mathrm{B}^{\mathrm{cl}}_d(z_k, a)$, where $\mathrm{B}^{\mathrm{cl}}_d(z_k, a) = \{x \in X \mid d(z_k, x) \leq a\}$. □

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is continuous, and injective.

Proof.

If $d(x, y) = \varepsilon > 0$ and $z_k \in \mathrm{B}_d(x, \varepsilon/2)$, then $f(x)(k) = d(x, z_k) < \varepsilon/2$ while $f(y)(k) = d(y, z_k) \geq \varepsilon/2$, whence $f(x) \neq f(y)$. □

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is continuous, and injective.

$f^{-1}\colon f(X) \to X$ is continuous.

Proof.

If $f(x_m) \to f(x)$, then $d(x_m, z_k) \to d(x, z_k)$ for all k. Fix $\varepsilon > 0$ and let $\bar{k} \in \mathbb{N}$ be such that $d(x, z_{\bar{k}}) < \varepsilon/2$. As $d(x_m, z_{\bar{k}}) \to d(x, z_{\bar{k}})$, let M be such that $d(x_m, z_{\bar{k}}) < \varepsilon/2$ for all $m \geq M$. Then for any such m we have $d(x_m, x) \leq d(x_m, z_{\bar{k}}) + d(z_{\bar{k}}, x) < \varepsilon$. So $x_m \to x$. □

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$$f\colon X \to [0;1]^{\mathbb{N}}, \qquad x \mapsto (d(x, z_n))_{n\in\mathbb{N}}$$

is continuous, and injective.

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The part concerning Polish spaces follows from Proposition 1.1.8.
If (X, d) is compact, then it is totally bounded, hence second-countable and separable. Let $f\colon X \to [0;1]^{\mathbb{N}}$ be a topological embedding: then $f(X)$ is compact (a continuous image of a compact set is compact as well), hence closed in $[0;1]^{\mathbb{N}}$ because the latter is a Hausdorff space. For the other direction recall that a closed subset of compact space is compact: since $[0;1]^{\mathbb{N}}$ is compact by Tychonoff, we are done.

Remark 1.2.3

The compactification is not unique. For example, $\mathbb{N}^{\mathbb{N}}$ can be densely embedded in $2^{\mathbb{N}}$

$$\mathbb{N}^{\mathbb{N}} \to 2^{\mathbb{N}}, \qquad x \mapsto \underbrace{0\dots0}_{x(0)}1\underbrace{0\dots0}_{x(1)}1\underbrace{0\dots0}_{x(2)}1\dots;$$

and can be densely embedded in $[0;1]$ as $\mathbb{N}^{\mathbb{N}}$ is homeomorphic to $\mathbb{R}\setminus\mathbb{Q}$.

If X is Polish, then the set $\mathcal{K}(X) = \{K \subseteq X \mid K \text{ is compact}\}$ can be equipped with the so-called **Vietoris topology**, i.e. with the topology generated by the sets of the form

$$\{K \in \mathcal{K}(X) \mid K \subseteq U\}$$

and

$$\{K \in \mathcal{K}(X) \mid K \cap U \neq \emptyset\}$$

for U open in X.

$\mathcal{K}(X)$ is Polish — see [Kec95, Section 4.F].

The collection of finite subsets of a countable dense subset of X is countable dense in $\mathcal{K}(X)$. The **Hausdorff metric** d_H:

$$d_H(K,L) = \begin{cases} 0 & \text{if } K = L = \emptyset \\ 1 & \text{if \emph{exactly} one of } K, L \text{ is } \emptyset \\ \max\{\delta(K,L), \delta(L,K)\} & \text{if } K, L \neq \emptyset, \end{cases}$$

where $\delta(K,L) = \max_{x\in K} d(x,L)$, is a compatible complete metric.

Theorem 1.2.1 shows that $\mathcal{K}([0;1]^{\mathbb{N}})$ is a Polish (hyper)space which contains, up to homeomorphism, all compact metrizable spaces.

Theorem 1.2.4 [Kec95, Theorem 4.17]

Every Polish space G is homeomorphic to a closed subspace of $\mathbb{R}^{\mathbb{N}}$.

Proof

By Theorem 1.2.1 w.l.o.g. $G \subseteq [0;1]^{\mathbb{N}}$ is $\mathbf{G}_\delta$, so $G = \bigcap_{n \in \mathbb{N}} U_n$ with $U_n \subseteq [0;1]^{\mathbb{N}}$ open. Let $F_n = [0;1]^{\mathbb{N}} \setminus U_n$. Define $f\colon G \to \mathbb{R}^{\mathbb{N}}$ by

$$f(x)(n) = \begin{cases} x(i) & \text{if } n = 2i+1 \\ \frac{1}{d(x,F_i)} & \text{if } n = 2i. \end{cases}$$

where d be a compatible complete metric on $[0;1]^{\mathbb{N}}$. Then f is injective, continuous (as in Theorem 1.2.1), and $f^{-1}\colon f(G) \to G$ is continuous.

(continues)

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Proof (continued).

$$f(x)(n) = \begin{cases} x(i) & \text{if } n = 2i+1 \\ \frac{1}{d(x,F_i)} & \text{if } n = 2i. \end{cases}$$

$f^{-1}\colon f(G) \to G$ is continuous, that is: if $f(x_n) \to y \in \mathbb{R}^{\mathbb{N}}$, then $y \in f(G)$ and $x_n \to x$ such that $f(x) = y$.
In fact by the odd coordinates, $x_n \to x$ where $x(i) = y(2i+1)$. Moreover, $f(x_n) \to y$ implies that $(1/d(x_n, F_i))_{n \in \mathbb{N}}$ converges (in $\mathbb{R}$) for each $i \in \mathbb{N}$. As in Proposition 1.1.4, $d(x, F_i) = \lim_{n \to \infty} d(x_n, F_i) > 0$. As $i \in \mathbb{N}$ is arbitrary, $x \notin F_i$ for all i, so $x \in G = [0;1]^{\mathbb{N}} \setminus \bigcup_{i \in \mathbb{N}} F_i$. That $f(x) = y$ is obvious. $\square$

Given a Polish space X

$$\mathbf{F}(X) = \{C \subseteq X \mid C \text{ is closed}\}$$

can be endowed with a Polish topology, so that $\mathbf{F}(\mathbb{R}^{\mathbb{N}})$ will be the Polish space of all Polish spaces (Section 2.3). Let

$$\mathbf{F}_h(X) = \{A \subseteq X \mid A \text{ is homeomorphic to some } C \in \mathbf{F}(X)\}$$

of all homeomorphic copies of closed subsets of X. Clearly $\mathbf{F}_h(X) \supseteq \mathbf{F}(X)$, and by Proposition 1.1.8 all $A \in \mathbf{F}_h(X)$ are $\mathbf{G}_\delta$. If X is compact, we have that $\mathbf{F}_h(X) = \mathbf{F}(X) = \mathcal{K}(X)$.
If X is $\mathbf{K}_\sigma$, i.e. X can be written as a countable union of compact sets, then each continuous image of a closed set is $\mathbf{F}_\sigma$: it follows that each $A \in \mathbf{F}_h(X)$ is both $\mathbf{F}_\sigma$ and $\mathbf{G}_\delta$, thus $\mathbf{F}_h(X)$ does not in general contain all $\mathbf{G}_\delta$ subsets of X. This applies e.g. to $X = \mathbb{R}^n$ for any $n \in \mathbb{N}$. E.g. if $X = \mathbb{R}$, then $A = \{2^{-n} \mid n \in \mathbb{N}\}$ is neither open nor closed (but it is $\mathbf{F}_\sigma$ and $\mathbf{G}_\delta$) and belongs to $\mathbf{F}_h(\mathbb{R})$ because it is homeomorphic to $C = \mathbb{N} \subseteq \mathbb{R}$. So $\mathbf{F}_h(\mathbb{R}) \supset \mathbf{F}(\mathbb{R})$.

Corollary

Every $\mathbf{G}_\delta$ subset of $\mathbb{R}^{\mathbb{N}}$ is homeomorphic to a closed subset of $\mathbb{R}^{\mathbb{N}}$. Thus $\mathbf{F}_h(\mathbb{R}^{\mathbb{N}})$ coincides with the collection of all $\mathbf{G}_\delta$ subsets of $\mathbb{R}^{\mathbb{N}}$.

Proof.

If $A \subseteq \mathbb{R}^{\mathbb{N}}$ is $\mathbf{G}_\delta$ then it is Polish by Proposition 1.1.8, and the result follows from Theorem 1.2.4. $\square$

By Corollary 1.3.14, a similar result holds for the Baire space $\mathbb{N}^{\mathbb{N}}$, that is, $\mathbf{F}_h(\mathbb{N}^{\mathbb{N}})$ consists of all $\mathbf{G}_\delta$ subsets of $\mathbb{N}^{\mathbb{N}}$.