

Overview degli argomenti del seminario

ChatGPT

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1 Polish groups

1.1 Overview

A **Polish group** is a topological group whose topology is separable and completely metrizable; equivalently, it is a group equipped with a Polish space topology. These groups arise naturally in analysis, algebra and logic (for example the real numbers under addition, the unitary group of a Hilbert space, or the infinite permutation group S_∞) and are fundamental in descriptive set theory. They provide the setting for rich interactions between group actions and definable sets: for instance, any Borel action of a Polish group on a Polish space yields Borel or analytic orbit equivalence relations of central interest in classification problems.

1.2 Detailed Summary

- **Definition:** A *topological group* G is **Polish** if its topology is induced by a complete metric and G is separable (equivalently, second-countable). In particular, every Polish group is a G_δ subgroup of some completely metrizable group. Classic examples include $(\mathbb{R}^n, +)$, any separable Banach space (as an additive group), the infinite symmetric group S_∞ (with the pointwise convergence topology), and all compact metrizable groups (which are complete).
- **Basic properties:** Polish groups have cardinality continuum unless trivial. They admit a compatible complete invariant metric. Notably, any Polish group can be embedded as a closed subgroup of the unitary group of a separable Hilbert space (by Uspenskiĭ's theorem), or of S_∞ , making S_∞ a universal Polish group. Topologically, Polish groups are in particular **Baire spaces** (the interior of any countable union of

nowhere dense sets is empty), which follows from the fact that they are complete metrizable.

- **Descriptive set aspects:** Any Polish group G has an underlying standard Borel structure, so one can consider Borel or analytic subsets and Borel measurable homomorphisms of G . Actions of Polish groups on Polish (or standard Borel) spaces produce equivalence relations whose complexity is studied via DST. A key result is Effros’s theorem: for a continuous action of a Polish group on a Polish space, the orbit of any point is an analytic set, and the orbit map $x \mapsto G \cdot x$ is a Borel function into the Effros Borel space of closed sets. In fact, “the condition of being Polish is essential in classification problems of mathematics and logic” – Polish groups naturally unify many examples and appear ubiquitously in DST.
- **Theorems (Kechris Ch.9):** Kechris develops the theory of Polish groups, including the fact that any Polish group is isomorphic (as a Borel group) to a closed subgroup of S_∞ . One also has the **Dichotomy Theorem**: for any Borel (or analytic) homomorphism of a Polish group, the kernel and image have regular descriptive-set-theoretic properties. More generally, results in this chapter (9.1–9.3 of Kechris) illustrate that problems about group actions (orbit equivalence, ergodicity, invariant measures) for Polish groups can often be reduced to questions about well-understood model cases. Although we do not reproduce Kechris’s proofs here, the core idea is that Polishness brings powerful Baire-category and metrization tools (like the Open Mapping and Closed Graph Theorems) into play for group-theoretic contexts.
- **Relevance:** Polish groups form the stage for many structural theorems in DST, such as the theory of Borel reducibility of equivalence relations (orbit equivalence of Polish-group actions) and the study of invariant sets. They connect descriptive set theory to areas like group representation, ergodic theory, and logic. The interplay between the algebraic structure of G and its topology (and Borel σ -algebra) is key to understanding classification up to isomorphism or orbit equivalence in a definable way.

2 Standard Borel spaces

2.1 Overview

A **standard Borel space** is a measurable space isomorphic to the Borel σ -algebra of some Polish space (Standard Borel space - Wikipedia). Equivalently, it is a set X with a σ -algebra Σ such that there is a topology on X making it Polish and Σ is exactly the collection of Borel sets for that topology (Standard Borel space - Wikipedia). Standard Borel spaces capture the most general setting for “nice” measurability theory: up to isomorphism, there is essentially only one uncountable such space (of cardinality continuum) and the discrete countable cases (Standard Borel space - Wikipedia).

2.2 Detailed Summary

- **Definition:** Formally, (X, Σ) is **standard Borel** if there exists a Polish topology on X whose Borel σ -algebra is Σ (Standard Borel space - Wikipedia). In particular, Σ must be countably generated and the space must admit a complete separable metric compatible with Σ . This notion is independent of the specific topology chosen (up to isomorphism).
- **Kuratowski’s theorem:** A fundamental classification states that if X is an uncountable Polish space, then its Borel space is isomorphic (as measurable spaces) to the real line \mathbb{R} , or to \mathbb{Z} , or is finite (Standard Borel space - Wikipedia). In consequence, *up to Borel isomorphism every uncountable standard Borel space has the cardinality of the continuum* (Standard Borel space - Wikipedia). Thus any two uncountable standard Borel spaces are isomorphic as measurable spaces, which justifies treating “the” standard Borel structure on a continuum as unique.
- **Properties:** Standard Borel spaces enjoy many convenient properties not shared by arbitrary measurable spaces. For instance, any bijection between standard Borel spaces that is measurable in one direction is automatically an isomorphism (its inverse is measurable) (Standard Borel space - Wikipedia). Equivalently, by Suslin’s theorem, a set that is both analytic and co-analytic in a standard Borel space must be Borel, so measurability behaves well under complementation. Also, countable products or coproducts of standard Borel spaces remain standard Borel (Standard Borel space - Wikipedia).

- **Theorems (Kechris 12.A–12.C):** Kechris proves (Theorem 12.13) that any Borel set in a Polish space X is the preimage of a Borel set in \mathbb{R} under some Borel bijection from X onto \mathbb{R} (when X is uncountable). Equivalently, one can show any standard Borel space of size continuum is Borel-isomorphic to $(\mathbb{R}, \mathcal{B})$. He also establishes the “transfer” theorem: most nice properties (analyticity, etc.) are preserved when mapping one standard Borel onto another.
- **Relevance:** Standard Borel spaces are the natural universe for measurable dynamics, ergodic theory, and any area using Borel sets abstractly. In DST, one often reduces problems to standard Borel spaces since then one can choose convenient Polish topologies. They justify statements like “we may assume X is a complete separable metric space” when dealing with Borel sets abstractly. This notion underpins uniformization theorems and classification of Borel equivalence relations, among other results.

3 (Strong) Choquet games and spaces

3.1 Overview

Choquet games are two-player topological games that characterize Baire-category properties and completeness of metrics. In the *Choquet game* $G(X)$ on a nonempty space X , players I and II alternate picking nonempty open sets $U_0 \supseteq V_0 \supseteq U_1 \supseteq V_1 \supseteq \dots$, and Player II wins if the intersection $\bigcap_n U_n$ is nonempty (Choquet game - Wikipedia). A space is **Choquet** if Player II has a winning strategy in $G(X)$ (equivalently, Player I has *no* winning strategy) (Choquet game - Wikipedia). There is a stronger version, the *strong Choquet game*, where Player I names a point and a neighborhood each move. In fact, a metrizable space is **strong Choquet** if and only if it is completely metrizable (Polish) (Choquet game - Wikipedia). Choquet games thus provide a game-theoretic characterization of key topological features: Choquet spaces are Baire spaces, and **Polish spaces are exactly the strong Choquet spaces** (Choquet game - Wikipedia).

3.2 Detailed Summary

- **Definitions:** In the *Choquet game* on X (written $G(X)$), Player I first chooses any nonempty open set $U_0 \subseteq X$, then Player II chooses a nonempty open $V_0 \subseteq U_0$, then I chooses $U_1 \subseteq V_0$, and so on, always

shrinking (possibly with $U_n \supseteq V_n \supseteq U_{n+1}$). If the intersection $\bigcap_n U_n$ is empty, Player I wins; otherwise (the intersection is nonempty) Player II wins (Choquet game - Wikipedia). A space X is called a **Choquet space** if Player II has a (winning) strategy ensuring nonemptiness of the intersection. Equivalently, by Oxtoby's theorem, X is Choquet iff Player I has no winning strategy, which is in turn equivalent to X being a Baire space (every countable union of nowhere dense sets has empty interior) (Choquet game - Wikipedia).

- **Strong Choquet game:** The strong version $G^s(X)$ modifies the play so that at stage n Player I first chooses a point $x_n \in X$ and an open neighborhood U_n of x_n , then Player II chooses a nonempty open $V_n \subseteq U_n$ containing x_n , with $V_n \subseteq U_n$. Player II wins if $\{x_n\}$ converges to some point (equivalently, the neighborhoods shrink to a point). One shows **every nonempty complete metric space (and every compact Hausdorff T_2 space)** is strong Choquet (Choquet game - Wikipedia). Conversely, a *separable metrizable* (i.e. second-countable) space is Polish (complete metric) if and only if it is strong Choquet (Choquet game - Wikipedia). Thus strong Choquet is exactly the game-theoretic analog of complete metrizability.
- **Properties:** Every strong Choquet space is Choquet; but not every Choquet space is strong Choquet. Choquet spaces are always Baire: in fact X is Baire iff Player I has no winning strategy in $G(X)$ (Choquet game - Wikipedia). Many classical spaces are Choquet: e.g. any complete metric or compact metric space (strong Choquet), and any G_δ subspace of a complete metric space. The Choquet property is hereditary for G_δ subsets. In non-metrizable settings, Choquet conditions are more subtle (studied by Choquet himself and later Becker–Kechris), but in DST one mostly focuses on metrizable cases.
- **Relevance:** Choquet games connect descriptive set theory to topology: many DST arguments use these games to establish the Baire property or perfect set property for definable sets. For example, one can prove a set is comeager by showing Player I has a winning strategy in a Banach–Mazur game (a variant of Choquet game). In classification, Choquet games characterize when a definable set is “large” in the sense of category. In functional analysis, the Choquet game relates to the existence of generic points in Banach spaces. Kechris (Chapter 8.C–E) uses these games to prove equivalences like: “*A separable metrizable space is Polish if and only if it is strong Choquet*” (Choquet

game - Wikipedia), and “*every Choquet space is Baire*” (Choquet game - Wikipedia).

4 The Banach–Mazur game

4.1 Overview

The **Banach–Mazur game** is another two-player game on a topological space X , closely related to Choquet games, used to characterize the Baire property. In one common version, players alternate choosing nested nonempty open sets $U_0 \supseteq V_0 \supseteq U_1 \supseteq V_1 \supseteq \dots$ (just like the strong Choquet game, but here Player I wins if the intersection contains a point of a predetermined target set $A \subseteq X$, and Player II wins otherwise). A fundamental theorem states that Player I has a winning strategy in the Banach–Mazur game with target A if and only if A is comeager (dense G_δ) in some nonempty open set, while Player II has a winning strategy if and only if A is meager. Thus determinacy of this game is equivalent to the Baire property: in ZF+AD one deduces all sets have the Baire property, and in ZFC one uses it to show all *Borel* sets do.

4.2 Detailed Summary

- **Definition:** The classical *Banach–Mazur (B–M) game* $G^{**}(A)$ on a Polish (or any topological) space X with target $A \subseteq X$ proceeds as follows: Player I chooses any nonempty open set U_0 , Player II chooses a nonempty open $V_0 \subseteq U_0$, then I picks $U_1 \subseteq V_0$, II picks $V_1 \subseteq U_1$, etc., continuing indefinitely. After play $(U_0, V_0, U_1, V_1, \dots)$, the players look at $\bigcap_n U_n$. If this intersection meets A , then Player I wins; otherwise Player II wins. (Equivalently, II wins if $\bigcap_n U_n \subseteq X \setminus A$.)
- **Main result:** It is well-known (and used to prove the Baire Category Theorem) that

I has a winning strategy in $G^{**}(A) \iff A$ is comeager in some nonempty open set,

II has a winning strategy in $G^{**}(A) \iff A$ is meager in X .

In particular, Player I can force landing in A exactly when A contains a dense G_δ in some open set; otherwise A is “small” and Player II can avoid it. The cited result shows the strong link between this game and the notion of meager/comeager. In effect, **the Banach–Mazur game determines Baire-category** of sets.

- **Consequences:** From this it follows that **all Borel sets have the Baire property**: since Borel determinacy holds for this game in ZF, every Borel A yields one of the two cases, meaning A differs from a G_δ set by a meager set. More generally, one derives that analytic sets have the Baire property under AD. The game also illustrates the “topological determinacy” phenomenon: for many definable classes (Borel, analytic), these games are determined, linking set-theoretic axioms with regularity properties.
- **Relation to Choquet:** The Banach–Mazur game can be seen as a variant of the Choquet game where Player I tries to steer the play into a given set A . Actually, if $A = X$ this game is essentially the strong Choquet game. Kechris (Chapter 8.H and 21.C) uses it to prove that any Borel (indeed any analytic) set has the perfect set or Baire properties, and that determinacy of these games for *arbitrary* sets would imply strong axioms like AD. The core idea in proofs is to construct winning strategies by carefully choosing shrinking open sets; one often sketches it as in: e.g. to show I wins when A is comeager, I always plays a basic open in a fixed dense $G_\delta \subseteq A$.

5 Games for the Perfect Set Property (PSP)

5.1 Overview

A set $A \subseteq X$ in a Polish space is said to have the **Perfect Set Property (PSP)** if either A is countable or else A contains a nonempty perfect subset (hence of cardinality continuum). The PSP can be characterized by a two-player game, often called the “*-game”. In this game $G^*(A)$ (played on a perfect Polish space X), the players alternately choose two disjoint open sets, and Player II selects one of them; intuitively II is trying to stay out of A while I tries to force the play into A . The outcome of a play is a single point $x \in X$, and Player I wins if $x \in A$. A central theorem is: **Player I has a winning strategy in $G^*(A)$ if and only if A contains a perfect (Cantor) set, and Player II has a winning strategy if and only if A is countable.** Thus determinacy of this game exactly captures the PSP: one of the players wins, meaning any set is either countable or has a perfect subset.

5.2 Detailed Summary

- Definition of $G^*(A)$:** Fix a nonempty perfect Polish space X with a compatible complete metric and a basis $\{V_n\}$ of nonempty open sets. The $*$ -game $G^*(A)$ for $A \subseteq X$ is defined by transfinite “cut-and-choose” moves: First I plays two disjoint nonempty basic open sets U_0^0, U_1^0 (of diameter < 1). Then II picks one of them (say U_0^1) and I responds with two disjoint open sets U_0^1, U_1^1 of diameter $< 1/2$ contained in the chosen set. Then II picks one of U_0^1, U_1^1 , and I plays two smaller opens of diameter $< 1/4$ inside that, etc. Because X is perfect, this process can continue indefinitely. At the end there is a unique point $x \in X$ in the nested intersection. Player **I** wins if $x \in A$, and Player **II** wins otherwise.
- Main theorem:** It can be shown (cf. Kechris 21.A–21.B) that
 - *I has a winning strategy in $G^*(A)$ if and only if A contains a perfect Cantor-like subset* (hence is uncountable with a perfect part).
 - *II has a winning strategy if and only if A is countable.* In other words, A has PSP (uncountable \implies contains perfect) precisely when $G^*(A)$ is determined and one of these conditions holds. This result is Theorem 8.2 in the lecture notes. The proof uses classic strategies: if A has a perfect subset, I can “force” the play to land inside that Cantor set; conversely if A is countable, II can successively avoid enumerated points of A .
- Relevance:** This game-theoretic characterization implies immediately that *every Borel set has the PSP*: by Martin’s theorem all Borel games are determined, so for any Borel A exactly one of I or II wins $G^*(A)$, yielding one of the two outcomes. More conceptually, it shows PSP is a “second-order Borel property” and can be derived from determinacy. In DST, the PSP and related games connect to large cardinals (AD implies all sets of reals have PSP) and to classical results like the perfect set theorem for analytic sets. In practice, one often uses this game as a tool to prove specific sets are uncountable by describing a winning strategy.

6 Structural properties of the Borel hierarchy

6.1 Overview

The **Borel hierarchy** on a Polish space X consists of the pointclasses Σ_α^0 , Π_α^0 , and Δ_α^0 indexed by countable ordinals α . Two key structural features are: (1) *Strictness*: for any nonempty Polish X and any countable α , the inclusions

$$\Delta_\alpha^0 \subsetneq \Sigma_\alpha^0 \subsetneq \Delta_{\alpha+1}^0$$

are all proper (descriptive set theory - Borel hierarchy doesn't "collapse" before ω_1 - Mathematics Stack Exchange). In particular, no new Borel sets appear before reaching height ω_1 - the Borel hierarchy runs through all countable ordinals. (2) *Cardinality*: each nontrivial class has continuum many sets. In fact, every uncountable standard Borel space has cardinality 2^{\aleph_0} (Standard Borel space - Wikipedia), so each level of the Borel hierarchy also has size continuum in that case.

6.2 Detailed Summary

- **Strictness of levels:** Kechris proves (Theorem 22.4) that for any uncountable Polish space X and any countable ordinal ξ , the hierarchy does not collapse:

$$\Delta_\xi^0(X) \subsetneq \Sigma_\xi^0(X) \subsetneq \Delta_{\xi+1}^0(X).$$

Equivalently, there are sets in $\Sigma_\xi^0 \setminus \Pi_\xi^0$ and in $\Pi_\xi^0 \setminus \Sigma_\xi^0$ at every level (descriptive set theory - Borel hierarchy doesn't "collapse" before ω_1 - Mathematics Stack Exchange). This means one cannot generate the full Borel σ -algebra by fewer than ω_1 steps of alternate countable unions and intersections of open sets. The cited MathOverflow answer summarizes: "*for any uncountable Polish space, the Borel hierarchy is strict*" (descriptive set theory - Borel hierarchy doesn't "collapse" before ω_1 - Mathematics Stack Exchange).

- **Density of classes:** Another structural fact is that the Borel σ -algebra is exhausted only at level ω_1 : $\bigcup_{\alpha < \xi} \Sigma_\alpha^0 \neq \mathcal{B}(X)$ for every countable ξ (descriptive set theory - Borel hierarchy doesn't "collapse" before ω_1 - Mathematics Stack Exchange). This implies e.g. that there are arbitrarily high (countable) Borel ranks: for every countable α there exists a Borel set of exact Borel rank α .

- **Cardinality:** Since the Borel σ -algebra is generated by a countable basis, it has cardinality at most continuum. Conversely, for uncountable X there are continuum many basic opens, so each nontrivial class $\Sigma_\alpha^0(X)$ has cardinality continuum. In fact, by Kuratowski's theorem all uncountable Borel sets have cardinality 2^{\aleph_0} (Standard Borel space - Wikipedia).
- **Separation and reduction:** An important property is that disjoint Borel sets in Σ_ξ^0 can often be separated by a Δ_ξ^0 set (the Separation Theorem) and one can reduce questions about a given Borel set to canonical examples (universal Σ_ξ^0 sets). While Kechris's section 22.C focuses on these structural results (and items like closure under continuous images), the key takeaway is that the Borel pointclasses are “as complicated as possible” at each level: none of them coincides with another, and each level is closed under the natural operations (countable unions for Σ^0 , intersections for Π^0) but otherwise distinguished.

7 The difference hierarchy

7.1 Overview

The **difference hierarchy** is a refinement of the Borel hierarchy that decomposes $\Delta_{\alpha+1}^0$ sets into iterated differences of simpler sets. The classic Hausdorff–Kuratowski theorem says that any $\Delta_{\alpha+1}^0$ set can be written as a union of differences of a decreasing sequence of Π_α^0 sets. In effect, one measures the “complexity” of a $\Delta_{\alpha+1}^0$ set by how many times one needs to alternate set-differences at the Π_α^0 level. For example, every F_σ (Σ_2^0) set is a difference of two closed (Π_1^0) sets, etc.

7.2 Detailed Summary

- **Definition:** Formally, for each countable ordinal α one defines the α -th difference hierarchy $D(\Pi_\alpha^0)$ consisting of sets that can be expressed as alternating differences of Π_α^0 sets. For instance, $D_2(\Pi_\alpha^0)$ are sets of the form $C_0 \setminus C_1$ with $C_i \in \Pi_\alpha^0$, $D_3(\Pi_\alpha^0)$ are finite unions of two differences of Π_α^0 sets, etc., extending transfinitely.
- **Hausdorff theorem:** The key result (Hausdorff) is that *every* $\Delta_{\alpha+1}^0$ set arises in this way. Precisely:

Theorem: $B \subseteq X$ is $\Delta_{\alpha+1}^0$ if and only if there is a countable decreasing sequence of Π_α^0 sets $(C_\beta)_{\beta < \omega_1}$ such that

$$B = \bigcup_{\beta \text{ even}} (C_\beta \setminus C_{\beta+1}).$$

In other words, B is a countable union of disjoint “blocks” where we alternately subtract one Π_α^0 set from another. This characterization shows that the difference hierarchy *exhausts* the Δ -classes.

- **Examples:** Concretely, any F_σ set (a Σ_2^0 set) can be written as $C_0 \setminus C_1$ for closed sets $C_1 \subseteq C_0$. Similarly, any Boolean combination of G_δ sets (a Δ_3^0 set) is a finite union of differences of two G_δ ’s, etc. The construction in Kechris and related notes uses transfinite recursion and the completeness of the metric to peel off “layers” of a Borel set.
- **Relevance:** The difference hierarchy gives a finer measure of Borel complexity than mere class rank. It is fundamental in proofs (via transfinite induction) that analyze Borel sets: for example, in Wadge theory one often needs to know how to decompose sets. In descriptive set theory, it also appears in connection with hierarchies of equivalence relations (like the Δ -hierarchy of equivalence relations) and in determinacy: each level of the difference hierarchy corresponds to determinacy of a certain type of game. In summary, the difference hierarchy theorem (Hausdorff’s theorem) shows exactly how $\Delta_{\alpha+1}^0$ sets are built from Π_α^0 sets by countably many differences.

8 The Baire hierarchy

8.1 Overview

The **Baire hierarchy** classifies real-valued functions on a Polish space by successive pointwise limits of simpler functions. The *Baire class 0* functions are the continuous ones, and in general a function is of *Baire class α* if it can be obtained as a pointwise limit of a sequence of functions from lower classes (Baire function - Wikipedia). Thus Baire class 1 consists of all pointwise limits of continuous functions. Classical results (going back to Lebesgue) show that this hierarchy is strict: for each countable α there are functions in class $\alpha+1$ not in any lower class, and moreover there exist functions (without AC) not in any Baire class (Baire function - Wikipedia). In fact, Baire-measurable functions correspond exactly to pointwise limits of continuous functions, and this hierarchy parallels the Borel hierarchy of level sets.

8.2 Detailed Summary

- **Definition:** A real-valued function $f : X \rightarrow \mathbb{R}$ on a topological space X is said to be *Baire class 0* if it is continuous. For a countable ordinal $\alpha > 0$, f is *Baire class α* if there is a sequence (f_n) of functions of class $< \alpha$ that converge pointwise to f (Baire function - Wikipedia). Equivalently, f is Baire class α if f can be obtained by α many iterated pointwise limits starting from continuous functions.
- **Characterizations:** In metric spaces one has classical characterizations: for example f is Baire class 1 iff for every open set $U \subseteq \mathbb{R}$ the preimage $f^{-1}(U)$ is an F_σ set (a countable union of closed sets). Higher classes correspond to more complicated preimages (e.g. f is Baire 2 if $f^{-1}(U)$ is a countable union of G_δ 's, etc.). These facts are discussed in §24.A–24.B of Kechris (with proofs).
- **Strictness and non-Baire sets:** A theorem of Lebesgue (cited in Kechris, or see Wikipedia) says that for each countable α , there are functions of Baire class α not in any lower class. Moreover, there exist functions (on $[0, 1]$ for example) that are not of Baire class α for any $\alpha < \omega_1$ (so some Borel functions are not Baire-measurable in the pointwise limit sense) (Baire function - Wikipedia). In fact, the Baire hierarchy *does not exhaust* all Borel functions: it was shown that under AD every set of reals is Baire-measurable, but under ZFC there are pathological Borel functions with no countable pointwise approximations.
- **Relation to Borel:** Every Baire class function is Borel-measurable (since continuous functions are Borel and pointwise limits of Borel functions remain Borel). The converse fails in general, but one has the **Lusin theorem**: every Borel function from \mathbb{R} to \mathbb{R} can be made continuous on a large set, showing it is *almost* Baire of small class. Kechris's exposition (24.A–24.B) includes proofs that the Baire classes are closed under natural operations (sums, products, etc.) and that if f is Borel then f belongs to some countable Baire class (though this requires additional set theory).
- **Summary:** The Baire hierarchy provides a fine gradation of measurability for functions:
 - Class 0 = continuous.

- Class 1 = pointwise limits of continuous (characterized by F_σ preimages).
- In general, class α = limits of lower classes (Baire function - Wikipedia).
- Each class properly extends the previous (Baire function - Wikipedia), and many classical pathological functions live at high levels. This hierarchy is fundamental in real analysis and DST, especially in effective descriptive set theory and the study of Polish group representations.

9 Uniformization theorems

9.1 Overview

A **uniformization** of a relation $P \subseteq X \times Y$ is a subset $P^* \subseteq P$ that is the graph of a (partial) function whose domain is $\text{proj}_X(P)$, picking exactly one y -value for each x in the projection (Uniformization Theorems | SpringerLink). Uniformization theorems give conditions under which one can choose such definable selections. Kechris’s Chapter 18 presents classic results: e.g. **Lusin’s and Novikov’s theorems** for analytic and Borel relations, and the **Kondo–Novikov–Addison theorem** in the projective hierarchy. Roughly speaking, if P is Borel (or analytic) and its vertical sections P_x are “nice” (for instance, all countable or all σ -compact), then there exists a *Borel* function $f : \text{proj}_X(P) \rightarrow Y$ with graph inside P . For example, **Novikov’s theorem** states: if $P \subseteq X \times Y$ is Borel and each section P_x is at most countable, then P admits a Borel uniformization. **Arsenin–Kunugui** extended this by allowing sections that are σ -compact. These theorems are indispensable in DST for constructing measurable selections and studying equivalence relations.

9.2 Detailed Summary

- **Uniformization (Definition):** For $P \subseteq X \times Y$, a *uniformization* is any subset $P^* \subseteq P$ such that each $x \in \text{proj}_X(P)$ appears exactly once; equivalently, P^* is the graph of a function f with $\text{dom}(f) = \text{proj}_X(P)$ and $(x, f(x)) \in P$ for all x (Uniformization Theorems | SpringerLink). In other words, $f(x) \in P_x$ is a “choice” of a y -coordinate for each x . The question is: when can f be chosen to be Borel (or analytic, etc.) if P itself is Borel (or analytic)?

- **Key results:** Kechris’s sections 18.A–18.D include the following prototypical theorems:
 - **Novikov’s Uniformization Theorem:** If $P \subseteq X \times Y$ is Borel (with X, Y Polish) and each section P_x is countable (or more generally uniformly countable), then there is a Borel uniformizing function f on $\text{proj}_X(P)$. Thus any countable-to-one Borel relation can be resolved by a Borel selector.
 - **Arsenin–Kunugui Theorem:** If P is Borel and each section P_x is σ -compact (in Y), then there is again a Borel uniformization. This covers situations where each P_x is, say, a countable union of compact sets.
 - **Lusin’s Theorem:** For analytic P with projections covering a Polish space, there exists an *analytic* uniformization. In fact, Lusin showed that any analytic relation can be uniformized by an analytic function on a co-analytic domain. Novikov’s theorem is often proved first for F_σ relations by transfinite induction and then extended to analytic.
 - **Kondo–Novikov–Addison Theorem:** In the projective hierarchy, every Π^1_{2n+1} set admits a Σ^1_{2n+2} uniformization, and similarly for higher levels (using scales). This implies, for instance, that any co-analytic set $P \subseteq \mathbb{R}^2$ has a Σ^1_2 (analytic) selector on a co-analytic domain.
- **Techniques and relevance:** The proofs combine topology (like selection theorems for complete metric spaces) with effective descriptive set theory (scales, pointclasses) and transfinite recursion. The upshot is that many selection problems admit solutions of the same or only slightly higher definability level. Uniformization theorems are used throughout DST: for example, to reduce classification problems to single-valued functions, to prove Silver’s dichotomy for equivalence relations, and to analyze the structure of Borel equivalence classes. Kechris’s exposition emphasizes theorems 18.10–18.18 (including those by Arsenin–Kunugui and Novikov) and points out how they follow from or imply separation results. Intuitively, they say “if the relation P is not too wild (e.g. has small sections), one can choose a measurable section”.

10 Partition theorems

10.1 Overview

Partition theorems in descriptive set theory are analogues of classical Ramsey-theoretic results, asserting that certain “nice” colorings (Borel or analytic) of infinite structures admit large homogeneous subsets. Two fundamental examples are **Silver’s theorem (dichotomy)** for equivalence relations and the **Galvin–Prikry theorem** for colorings of infinite subsets of \mathbb{N} . Silver’s theorem says that any co-analytic equivalence relation on a Polish space either has only countably many classes, or else there is a perfect set of pairwise inequivalent points (Silver’s dichotomy - Wikipedia). Galvin–Prikry proved that any Borel coloring of the space $[\mathbb{N}]^\omega$ (all infinite subsets of \mathbb{N}) admits an infinite monochromatic subset; Silver extended this to analytic colorings (co.combinatorics - Is there Ramsey Theorem for infinitary tuples? - MathOverflow). These results (covered in Kechris 19.A–19.E) show that for definable partitions, one always finds either a “small” homogeneous structure or a large one of perfect size.

10.2 Detailed Summary

- **Galvin–Prikry theorem:** Consider the space $[\mathbb{N}]^\omega$ of infinite subsets of \mathbb{N} (with the topology inherited from Cantor space). If this space is partitioned (colored) into finitely many Borel pieces, then one of the pieces contains a homeomorphic copy of $[\mathbb{N}]^\omega$ itself (in particular, it contains all infinite subsets of some infinite $X \subseteq \mathbb{N}$). Equivalently, any Borel coloring of $[\mathbb{N}]^\omega$ has an infinite monochromatic set (co.combinatorics - Is there Ramsey Theorem for infinitary tuples? - MathOverflow). Ali Enayat’s MathOverflow answer summarizes: “*for Borel colorings of $[\mathbb{N}]^\omega$, an infinite monochromatic subset always exists*” (co.combinatorics - Is there Ramsey Theorem for infinitary tuples? - MathOverflow). Silver (1970) showed the same conclusion holds when the coloring is merely analytic. This is a descriptive version of Ramsey’s theorem for infinite subsets.
- **Silver’s theorem (dichotomy):** Let E be a Borel (even co-analytic) equivalence relation on a Polish space X . Silver’s dichotomy asserts that either E has countably many equivalence classes or there are continuum many. More strongly, *if E is co-analytic, then either E has only countably many classes, or there is a perfect set of reals which are pairwise inequivalent under E* (Silver’s dichotomy - Wikipedia). In

the latter case E has continuum-many classes. Thus no intermediate cardinalities occur for definable equivalence relations. In effect, one gets a perfect homogeneous set for the “not E ” relation, analogous to Galvin–Prikry.

- **Other partition results:** Kechris’s chapter also discusses related theorems, such as Mycielski’s theorem, which guarantees a perfect independent set in certain situations (for example, if E is an equivalence relation all of whose classes are meager, one can find a perfect set of mutually inequivalent points). There are also Ramsey-type results for Borel graphs and trees. All these theorems typically use the perfect set property and determinacy of appropriate games to build perfect homogeneous sets.
- **Relevance:** Partition theorems like Silver’s and Galvin–Prikry’s are cornerstones of modern DST and invariant descriptive set theory. They imply that any definable attempt to “color” or classify a perfect Polish space must either fail to distinguish continuum many points (producing a perfect homogeneous set) or be essentially countable. This dichotomy underpins many classification results: for instance, **Silver’s dichotomy** implies that any Borel or analytic equivalence relation E either has only countably many or continuum-many classes, ruling out a medium-size classification. These theorems also frequently combine with other principles (like AD or large cardinals) to yield structural insights about higher-level sets. In summary, partition theorems ensure that definable partitions on Polish spaces either admit a perfect homogeneous substructure or collapse to a small case, greatly constraining the possible complexity of Borel and analytic relations (Silver’s dichotomy - Wikipedia) (co.combinatorics - Is there Ramsey Theorem for infinitary tuples? - MathOverflow).

11 Sources

- Kechris, *Classical Descriptive Set Theory*, Chapters 8, 12,

18–19; lecture notes from Kechris’s class (where available); plus references such as

- (Standard Borel space - Wikipedia)
- (Choquet game - Wikipedia)

- (descriptive set theory - Borel hierarchy doesn't “collapse” before ω_1 - Mathematics Stack Exchange)
- (Baire function - Wikipedia)
- (Baire function - Wikipedia)
- (Uniformization Theorems | SpringerLink)
- (Silver's dichotomy - Wikipedia)
- (co.combinatorics - Is there Ramsey Theorem for infinitary tuples? - MathOverflow)