Exercise sheet 5

Descriptive Set Theory

Academic year 2023-24

1. Let E be an equivalence relation on a Polish space X. A set $A \subseteq X$ is called E-invariant if $x \in A$ and $y\ E\ x$ implies $y \in A$, for all $x, y \in X$. Suppose that E is analytic, that is, $E \in \mathbf{\Sigma}^1_1(X^2)$. Show that if $A, B \subseteq X$ are disjoint analytic E-invariant sets, then there is a Borel E-invariant $C \subseteq X$ separating A from B, that is, $A \subseteq C$ and $C \cap B = \emptyset$.

   [*Hint.* Recursively define sets $A_n, C_n \subseteq X$ so that $A_0 = A$, C_n is a Borel set separating A_n from B, and $A_{n+1} \supseteq C_n$ is E-saturated, analytic, and disjoint from B.]

2. Let E be an equivalence relation on a Polish space X. A partial transversal for E is a set $T \subseteq X$ meeting each E-equivalence class in at most one point. Show that the following are equivalent:

   (a) E admits an uncountable analytic partial transversal;

   (b) E admits an uncountable Borel partial transversal;

   (c) there is a Borel function $f \colon \mathbb{R} \to X$ such that $f(r_0) \not\!E\ f(r_1)$ for all distinct $r_0, r_1 \in \mathbb{R}$.

3. Let E be an equivalence relation on a Polish space X. A transversal for E is a set $T \subseteq X$ meeting every E-equivalence class in exactly one point. A selector for E is a map $s \colon X \to X$ selecting one element from each E-equivalence class, that is, $s(x) \in [x]_E$ and $s(x) = s(y)$ if $x\ E\ y$. Show that if E is analytic, then the following are equivalent:

   (a) E admits an analytic transversal;

   (b) E admits a Borel transversal;

   (c) E admits a Borel selector.

4. Prove the following theorem

> Let X be a Polish space. Then every $A \in \mathbf{\Pi}^1_1(X)$ can be written as $A = \bigcup_{\xi<\omega_1} A_\xi$, where A_ξ is Borel for every $\xi < \omega_1$.

by completing the details of the following steps:

(a) First prove the theorem for $X = \mathrm{LO}$ and $A = \mathrm{WO}$ as follows:

- Given $\omega \leq \xi < \omega_1$, let WO_ξ be the set of codes for well-orders of ω with order type $\leq \xi$. Show that each WO_ξ is analytic.
- Argue that there is a Borel set A_ξ such that $\mathrm{WO}_\xi \subseteq A_\xi \subseteq \mathrm{WO}$.

  [*Optional.* Show that WO_ξ itself is Borel by showing that its complement is analytic as well.]

- Conclude that $\mathrm{WO} = \bigcup_{\xi<\omega_1} A_\xi$.

(b) Use the fact that WO is $\mathbf{\Pi}^1_1$-complete to prove the theorem for $X = \omega^\omega$ and an arbitrary $A \in \mathbf{\Pi}^1_1(\omega^\omega)$.

(c) Use the Borel isomorphism theorem for Polish spaces to transfer the result to an arbitrary uncountable Polish space X.

(d) What happens if X is a countable Polish space?