Borel functions and the Baire stratification

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Definition [Kec95, Definition 24.2]

Let X, Y be metrizable spaces, and $\mathbf{\Gamma}$ be a boldface pointcass. A function $f \colon X \to Y$ if **$\mathbf{\Gamma}$-measurable** if $f^{-1}(U) \in \mathbf{\Gamma}(X)$ for every open $U \subseteq Y$.

Notice that if $\mathbf{\Gamma}(X)$ is closed under countable unions and finite intersections and Y is second-countable, then in the previous definition it is enough to restrict U to any countable subbasis for Y. Clearly, the notion of $\mathbf{\Sigma}^0_1$-measurability coincides with continuity.

Definition

Let X, Y be metrizable spaces. A function $f \colon X \to Y$ is called **Borel** if it is $\mathbf{Bor}$-measurable. Equivalently, f is Borel if and only if $f^{-1}(B) \in \mathbf{Bor}(X)$ for every $B \in \mathbf{Bor}(Y)$, i.e. if it is Borel as a function between the Borel spaces $(X, \mathbf{Bor}(X))$ and $(Y, \mathbf{Bor}(Y))$.
$f \colon X \to Y$ is a **Borel isomorphism** if it is a bijection and both f and f^{-1} are Borel; when such a function exists, we say that X and Y are **Borel isomorphic**.

Remark 2.4.3

If Y is second-countable, then $f\colon X \to Y$ is Borel if and only if there is $1 \leq \alpha < \omega_1$ such that f is $\boldsymbol{\Sigma}^0_\alpha$-measurable. (Indeed, it is enough to set $\alpha = \sup\{\alpha_n \mid n \in \omega\}$, where the α_n are such that $f^{-1}(U_n) \in \boldsymbol{\Sigma}^0_{\alpha_n}(X)$ for $\{U_n \mid n \in \omega\}$ a countable basis for Y.) Since $\boldsymbol{\Sigma}^0_\alpha(X) \subseteq \boldsymbol{\Sigma}^0_\beta(X)$ when $\alpha \leq \beta$, $\boldsymbol{\Sigma}^0_\alpha$-measurability gives a stratification of the Borel functions in at most ω_1-many levels. If X is an uncountable Polish space and $|Y| \geq 2$, this hierarchy does not collapse before ω_1. To see this, fix distinct $y_0, y_1 \in Y$, and given $1 \leq \alpha < \omega_1$ pick $A \in \boldsymbol{\Delta}^0_\alpha(X) \setminus \bigcup_{1 \leq \beta < \alpha}(\boldsymbol{\Sigma}^0_\beta(X) \cup \boldsymbol{\Pi}^0_\beta(X))$: then the function defined by $f(x) = y_0$ if $x \in A$ and $f(x) = y_1$ if $x \notin A$ is $\boldsymbol{\Sigma}^0_\alpha$-measurable but not $\boldsymbol{\Sigma}^0_\beta$-measurable for any $1 \leq \beta < \alpha$.

Here are some basic facts (EXERCISE!) concerning $\boldsymbol{\Sigma}^0_\alpha$-measurable functions.

- If $f\colon X \to Y$ is $\boldsymbol{\Sigma}^0_\alpha$-measurable, then for every $\beta < \omega_1$: if $A \in \boldsymbol{\Sigma}^0_{1+\beta}(Y)/\boldsymbol{\Pi}^0_{1+\beta}(Y)/\boldsymbol{\Delta}^0_{1+\beta}(Y)$ then $f^{-1}(A) \in \boldsymbol{\Sigma}^0_{\alpha+\beta}(X)/\boldsymbol{\Pi}^0_{\alpha+\beta}(X)/\boldsymbol{\Delta}^0_{\alpha+\beta}(X)$.
  Thus if $f\colon X \to Y$ is $\boldsymbol{\Sigma}^0_\alpha$-measurable and $g\colon Y \to Z$ is $\boldsymbol{\Sigma}^0_{1+\beta}$-measurable, then $g \circ f\colon X \to Z$ is $\boldsymbol{\Sigma}^0_{\alpha+\beta}$-measurable.
- If X_i and Y_i are metrizable, Y_i second-countable, $i < I \leq \omega$, and each $f_i\colon X_i \to Y_i$ is $\boldsymbol{\Sigma}^0_\alpha$-measurable, then the product function $\prod_{i<I} f_i\colon \prod_{i<I} X_i \to \prod_{i<I} Y_i$ is $\boldsymbol{\Sigma}^0_\alpha$-measurable as well.
- If X, Y_i are metrizable with Y_i second-countable, $i < I \leq \omega$, and each $f_i\colon X \to Y_i$ is $\boldsymbol{\Sigma}^0_\alpha$-measurable, then $g\colon X \to \prod_{i<I} Y_i$ sending $x \in X$ to $(f_i(x))_{i<I}$ is $\boldsymbol{\Sigma}^0_\alpha$-measurable.

The next result generalizes to all countable α's the following well-know topological fact (when $\alpha = 1$): if X, Y are topological spaces with Y Hausdorff and $f \colon X \to Y$ is continuous, then its graph is a closed set.

Proposition 2.4.4

Let X, Y be metrizable spaces with Y separable. If $f \colon X \to Y$ is $\boldsymbol{\Sigma}^0_\alpha$-measurable, then its graph

$$\operatorname{graph}(f) = \{(x, y) \in X \times Y \mid f(x) = y\}$$

is in $\boldsymbol{\Pi}^0_\alpha(X \times Y)$. In particular, the graph of a Borel function is Borel.

Proof.

The set $\operatorname{graph}(f)$ is the preimage of the diagonal of Y, which is a closed set, via the $\boldsymbol{\Sigma}^0_\alpha$-measurable function $f \times \mathrm{id}_Y \colon X \times Y \to Y \times Y$. □

When X and Y are both Polish, then a partial converse is true: if $f \colon X \to Y$ has a Borel graph, then f is Borel (Theorem 3.2.4).

Theorem 2.4.5 (Lebesgue, Hausdorff, Banach, see [Kec95, Theorem 24.3])

Let X, Y be metrizable spaces, with Y separable. Let $1 < \alpha < \omega_1$.

1. If α is a successor ordinal, then f is $\boldsymbol{\Sigma}^0_{\alpha+1}$-measurable if and only if $f = \lim_{n\to\infty} f_n$, where each $f_n \colon X \to Y$ is $\boldsymbol{\Sigma}^0_\alpha$-measurable.
2. If α is a limit ordinal, then f is $\boldsymbol{\Sigma}^0_{\alpha+1}$-measurable if and only if $f = \lim_{n\to\infty} f_n$, where each $f_n \colon X \to Y$ is $\boldsymbol{\Sigma}^0_{\beta_n}$-measurable for some $1 \leq \beta_n < \alpha$.

Remark

When λ is a (countable) limit ordinal, there is no natural way to obtain the collection of $\boldsymbol{\Sigma}^0_\lambda$-measurable functions as pointwise limits of simpler functions. Indeed, by part 2 of the previous theorem the closure under pointwise limits of all the preceding classes (that is, of the collection of all functions which are $\boldsymbol{\Sigma}^0_\beta$-measurable for some $1 \leq \beta < \lambda$) already coincide with the collection of all $\boldsymbol{\Sigma}^0_{\lambda+1}$-measurable, a class which is in general strictly larger than the class of $\boldsymbol{\Sigma}^0_\lambda$-measurable functions by Remark 2.4.3.

The situation when $\alpha = 1$ is more delicate. It is still true that a limit of $\boldsymbol{\Sigma}^0_1$-measurable (i.e. continuous) functions is $\boldsymbol{\Sigma}^0_2$-measurable, but the converse may fail. Indeed, if X is connected and Y is totally disconnected, then any continuous functions $f \colon X \to Y$ is constant, and therefore also a limit of continuous functions must be constant; however, if both X and Y contain at least two points, then there are non-constant $\boldsymbol{\Sigma}^0_2$-measurable functions (for example, we can let $f = \chi_{\{0\}} \colon \mathbb{R} \to \{0, 1\}$ be the characteristic function of the singleton $\{0\}$.) This problem can be overcome by requiring that either Y is a well-behaved space, or X is far from being connected.

Theorem 2.4.7 (Lebesgue, Hausdorff, Banach, see [Kec95, Theorem 24.10])

Let X, Y be separable metrizable and $f \colon X \to Y$ be $\boldsymbol{\Sigma}^0_2$-measurable. If either $Y = \mathbb{R}$ (or $Y = \mathbb{R}^n$, $Y = \mathbb{C}^n$, Y is an interval in $\mathbb{R}$, and so on), or else X is zero-dimensional, then f is the (pointwise) limit of a sequence of continuous functions.

Definition

Suppose that X and Y satisfy the hypothesis of Theorem 2.4.7. Let $\mathcal{B}_0(X, Y)$ be the collection of all continuous functions $f \colon X \to Y$, and for $\alpha < \omega_1$ inductively define $\mathcal{B}_\alpha(X, Y) = \{\lim_{n\to\infty} f_n \mid f_n \in \bigcup_{\nu<\alpha} \mathcal{B}_\nu(X, Y)\}$. Functions in $\mathcal{B}_\alpha(X, Y)$ are called **Baire class** α functions.

Remark

By definition, $\mathcal{B}_\alpha(X, Y) \subseteq \mathcal{B}_\beta(X, Y)$ whenever $\alpha \leq \beta < \omega_1$, and $\bigcup_{\alpha<\omega_1} \mathcal{B}_\alpha(X, Y)$ is the smallest collection of functions containing the continuous ones and closed under (pointwise) limits.

Baire class 1 functions are ubiquitous in analysis and in mathematics.

Example

- Upper semicontinuous and lower semicontinuous functions $f \colon X \to \mathbb{R}$ (where X is an arbitrary Polish space) are Baire class 1 functions.
- If X is Polish and $f \colon X \to \mathbb{R}$ has only countably many discontinuities, then f is of Baire class 1. In particular, all $f \colon [0;1] \to \mathbb{R}$ which are monotone or of bounded variation are of Baire class 1.
- Let $F \colon [0;1] \to \mathbb{R}$ be differentiable (at endpoints we take one-sided derivatives). Then its derivative F' is a Baire class 1 function.

The following result is a corollary of Theorems 2.4.5 and 2.4.7.

Theorem 2.4.11 (Lebesgue, Hausdorff, see [Kec95, Theorem 11.6])

Let X, Y be spaces satisfying the hypotheses of Theorem 2.4.7. Then $f \colon X \to Y$ is of Baire class α (for $\alpha < \omega_1$) if and only if it is $\mathbf{\Sigma}^0_{\alpha+1}$-measurable. Moreover, $\mathcal{B}_\alpha(X,Y) \subset \mathcal{B}_\beta(X,Y)$ for any $\alpha < \beta < \omega_1$, and the class of all Borel functions between X and Y is the smallest collection of functions containing the continuous ones and closed under (pointwise) limits.

Proposition 2.4.12 [Kec95, Exercise 24.5]

Let (X,τ) be a Polish space, Y be a separable metrizable space, and $1 \leq \alpha < \omega_1$. Then $f\colon (X,\tau) \to Y$ is $\mathbf{\Sigma}^0_\alpha$-measurable if and only if there is a Polish topology $\tau' \supseteq \tau$ on X such that $\tau' \subseteq \mathbf{\Sigma}^0_\alpha(X,\tau)$ and $f\colon (X,\tau') \to Y$ is continuous.
In particular, $f\colon (X,\tau) \to Y$ is Borel if and only if there is a Polish topology $\tau' \supseteq \tau$ on X such that $\mathbf{Bor}(X,\tau') = \mathbf{Bor}(X,\tau)$ and $f\colon (X,\tau') \to Y$ is continuous.

Proof.

One direction is obvious, so let us assume that f is $\mathbf{\Sigma}^0_\alpha$-measurable for some $\alpha > 1$ (the case $\alpha = 1$ is trivial). Let $\{U_n \mid n \in \omega\}$ be a countable basis for Y, and let $B_{n,i} \in \mathbf{\Delta}^0_\alpha(X,\tau)$ be such that $f^{-1}(U_n) = \bigcup_{i\in\omega} B_{n,i}$. Apply Theorem 2.2.2 to these $B_{n,i}$: then the resulting topology τ' is as required. □

Theorem

Let X be a Polish space and $\alpha < \omega_1$. Then $A \subseteq X$ is $\mathbf{\Pi}^0_{\alpha+1}$ if and only if it is the zero-set of a real-valued Baire class α function.

Proof.

If $f\colon X \to \mathbb{R}$ is of Baire class α, then f is $\mathbf{\Sigma}^0_{\alpha+1}$-measurable (Theorem 2.4.11) so $f^{-1}(0) \in \mathbf{\Pi}^0_{\alpha+1}(X)$ as $\{0\}$ is closed.
For the converse consider first $\alpha = 0$. Let d be a compatible metric on X. Given $A \subseteq X$, the function $f\colon X \to \mathbb{R}$, $f(x) = d(x,A)$ is continuous, and $A \subseteq f^{-1}(0)$. On the other hand, $f(x) = 0$ implies that x is a limit point of A, hence if A is closed $x \in A$: therefore in this case $A = f^{-1}(0)$.
Let now $\alpha \geq 1$. Let $A \in \mathbf{\Pi}^0_{\alpha+1}(X)$, so that $A = \bigcap_{n\in\omega} A_n$ with $A_n \in \mathbf{\Delta}^0_{\alpha+1}(X)$. By Theorem 2.2.2, there is a Polish topology τ' refining the topology τ of X such that each A_n is τ'-clopen and $\tau' \subseteq \mathbf{\Sigma}^0_{\alpha+1}(X,\tau)$. Then A is τ'-closed, so there if a continuous $f\colon (X,\tau') \to \mathbb{R}$ such that $f^{-1}(0) = A$. But then $f\colon (X,\tau) \to \mathbb{R}$ is $\mathbf{\Sigma}^0_{\alpha+1}$-measurable, whence f is of Baire class α by Theorem 2.4.11. □