

# Perfect Polish spaces

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## Definition

- A point  $x$  of a topological space  $X$  is **isolated** if there is an open neighborhood  $U$  of it such that  $U = \{x\}$ .
- A space is **perfect** if it has no isolated point.
- A subset  $P \subseteq X$  is **perfect in  $X$**  if it is closed and perfect with respect to the relative topology.

## Remark

If  $x$  is not isolated in  $(X, d)$ , then every open neighborhood  $U$  of  $x$  is infinite, as we can recursively define  $(x_n)_{n \in \omega}$  distinct points in  $U$ : fix  $\varepsilon > 0$  such that  $B_d(x, \varepsilon) \subseteq U$  and pick any  $x_0 \in B_d(x, \varepsilon) \setminus \{x\}$ . Then let  $x_{n+1}$  be any point in  $B_d(x, d(x, x_n)) \setminus \{x\}$  (such  $x_n$ 's exist because  $x$  is not isolated in  $X$ ).

$\mathbb{R}^n$ ,  $\mathbb{R}^\omega$ ,  $\mathbb{C}^n$ ,  $\mathbb{C}^\omega$ ,  $[0; 1]^n$ ,  $[0; 1]^\omega$ ,  $2^\omega$ ,  $\omega^\omega$ ,  $C(X)$  with  $X$  compact metrizable, ... are perfect Polish spaces.

### Theorem 1.4.3 [Kec95, Theorem 6.2]

Let  $X$  be a nonempty perfect completely metrizable space. Then  $2^\omega$  embeds into  $X$ , that is: there is a closed  $C \subseteq X$  homeomorphic to  $2^\omega$ .

#### Proof.

Fix a complete compatible metric  $d \leq 1$  on  $X$ . By Lemma 1.3.6, it is enough to build a 2-scheme  $\{B_s \mid s \in 2^{<\omega}\}$  such that

- ①  $B_{s \smallfrown i} \cap B_{s \smallfrown j} = \emptyset$  if  $i \neq j$ ;
- ②  $B_s$  is open and nonempty;
- ③  $\text{Cl}(B_{s \smallfrown i}) \subseteq B_s$ .

Let  $B_\emptyset$  be nonempty open. Given  $B_s$ , define  $B_{s \smallfrown 0}$  and  $B_{s \smallfrown 1}$  as follows. Choose two distinct points  $x_0, x_1 \in B_s$ , let  $\varepsilon_i > 0$  be small enough so that  $B_d(x_i, \varepsilon_i) \subseteq B_s$ , and set  $B_{s \smallfrown i} = B_d(x_i, \varepsilon)$  where  $\varepsilon = \frac{1}{2} \min\{2^{-(\text{lh}(s)+1)}, \varepsilon_0, \varepsilon_1, d(x_0, x_1)\}$ . It is easy to check that such  $B_{s \smallfrown i}$  has the required properties.  $\square$

### Corollary 1.4.4 [Kec95, Corollary 6.3]

Every nonempty perfect Polish space has the cardinality of the continuum  $2^{\aleph_0}$ . The same is true for nonempty perfect subsets of a Polish space.

#### Proof.

By Theorem 1.4.3, a nonempty perfect (subset of a) Polish space  $X$  contains a copy of  $2^\omega$ , and thus has cardinality  $\geq 2^{\aleph_0}$ . The fact that  $|X| \leq 2^{\aleph_0}$  follows from the fact that by Theorem 1.3.17 the space  $\omega^\omega$  surjects onto  $X$ , together with the fact that  $\omega^\omega$  has cardinality  $2^{\aleph_0}$ .  $\square$

### Remark 1.4.5

One can directly show that  $\omega^\omega$  surjects onto  $X$  as follows. Let  $D = \{x_n \mid n \in \omega\}$  be a countable dense subset of  $X$ . Then the map  $f: \omega^\omega \rightarrow X$  defined by

$$f(y) = \begin{cases} \lim_{n \rightarrow \infty} x_{y(n)} & \text{if } (x_{y(n)})_{n \in \omega} \text{ converges in } X \\ x_0 & \text{otherwise} \end{cases}$$

is clearly surjective. More generally, this argument shows that if a metrizable space  $X$  has a dense subset of cardinality  $\kappa$ , then there is a surjection of  $\kappa^\omega$  onto  $X$  and thus  $X$  has cardinality  $\leq \kappa^{\aleph_0}$ .

### Definition

A point  $x$  in a topological space  $X$  is a **condensation point** if every open neighborhood of  $x$  is uncountable.

### Theorem 1.4.7 (Cantor-Bendixson) [Kec95, Theorem 6.4]

A separable metrizable  $X$  can be written as a disjoint union  $X = P \cup C$  with  $P$  perfect and  $C$  a countable open set. If  $X$  is Polish, then such a decomposition is unique.

If  $X = \mathbb{Q}$ , then we could set  $P = \mathbb{Q}$  and  $C = \emptyset$ , or  $P = \emptyset$  and  $C = \mathbb{Q}$ . When  $X$  is Polish, the perfect subset  $P$  above is the **perfect kernel** of  $X$ .

### Proof

Let  $X^* = \{x \in X \mid x \text{ is a condensation point of } X\}$ . Set  $P = X^*$  and  $C = X \setminus P$ . We claim that  $P$  and  $C$  are as required. First observe that if  $\mathcal{B}$  is a countable basis for  $X$  then  $C = \bigcup \{U \in \mathcal{B} \mid U \text{ is countable}\}$ , thus  $C$  is open and countable and  $P$  is closed. To show that  $P$  has no isolated point (with respect to its relative topology!), let  $x \in P$  and  $U$  be an open neighborhood of  $x$ . Then  $U$  is uncountable because  $x$  is a condensation point, and since  $C$  is countable this implies that there is some (in fact, uncountably many)  $y \in U \cap P$  witnessing that  $U \cap P \neq \{x\}$ .

(continues)

### Proof (continued).

To prove uniqueness, suppose that  $X$  is Polish and that  $X = P_1 \cup C_1$  is another decomposition as in the statement of the theorem. Notice that if  $Y$  is a perfect Polish space then  $Y^* = Y$ . Indeed, if  $y \in Y$  and  $U$  is an open neighborhood of  $y$ , then  $U \cap Y$  is a perfect nonempty Polish space, and thus has cardinality  $2^{\aleph_0}$  by Corollary 1.4.4. Thus  $P_1^* = P_1$ , whence  $P_1 = P_1^* \subseteq X^* = P$  because  $Y \subseteq Z$  implies  $Y^* \subseteq Z^*$ . Moreover, if  $x \in C_1$  then  $x \in C$ , as witnessed by the countable open set  $C_1$  itself: therefore  $C_1 \subseteq C$ . It follows that  $P_1 = P$  and  $C_1 = C$ .  $\square$

### Remark

In the proof above we actually showed that if  $P_1$  is a perfect subset of a Polish space  $X$ , then  $P_1 \subseteq P$ , where  $P = X^*$  is the perfect kernel of  $X$ . Thus the perfect kernel of a Polish space can also be characterized as the largest (with respect to inclusion) perfect subset of it.

### Corollary 1.4.9 [Kec95, Corollary 6.5]

Any uncountable Polish space contains a (necessarily closed) homeomorphic copy of  $2^\omega$  and has cardinality  $2^{\aleph_0}$ .

### Proof.

Let  $P$  be the perfect kernel of the Polish space  $X$ . Since  $C = X \setminus P$  is a countable open set, if  $X$  is uncountable then  $P \neq \emptyset$ , and thus  $P$  itself is a nonempty perfect Polish space. Thus the result follows from Theorem 1.4.3 and Remark 1.4.5.  $\square$

There is an algorithmic way to find the perfect kernel of a Polish space  $X$ . The following construction is due to Cantor and it is the reason behind the introduction of ordinals.

## Definition

For any topological space  $X$ , let

$$X' = \{x \in X \mid x \text{ is not isolated in } X\}.$$

We call  $X'$  the **Cantor-Bendixson derivative** of  $X$ . Clearly,  $X'$  is closed and  $X$  is perfect if and only if  $X = X'$ .

Define the **iterated Cantor-Bendixson derivative** by recursion on the ordinals as follows:

$$\begin{aligned} X^{(0)} &= X \\ X^{(\alpha+1)} &= (X^{(\alpha)})' \\ X^{(\lambda)} &= \bigcap_{\alpha < \lambda} X^{(\alpha)}, \quad \text{if } \lambda \text{ is limit.} \end{aligned}$$

Arguing by induction on  $\alpha$ , one can check that the  $X^{(\alpha)}$ 's form a decreasing sequence of closed subsets of  $X$ , and that if  $X$  is second-countable and  $\alpha < \omega_1$  then  $X \setminus X^{(\alpha)}$  is a countable open set.

## Lemma [Kec95, Theorem 6.9]

Let  $X$  be a second countable topological space and  $(F_\alpha)_{\alpha < \rho}$  be a strictly decreasing transfinite sequence of closed sets (i.e.  $F_\beta \subset F_\alpha$  for all  $\alpha < \beta$ ). Then  $\rho$  is a countable ordinal.

This holds similarly for strictly increasing transfinite sequences of closed sets (and thus for strictly decreasing or increasing transfinite families of open sets).

## Proof.

Let  $\mathcal{B} = \{U_n \mid n \in \omega\}$  be a countable basis for  $X$ . For  $F \subseteq X$  closed, let  $N(F) = \{n \in \omega \mid U_n \cap F \neq \emptyset\}$ . Notice that if  $F \subseteq G$  then  $N(F) \subseteq N(G)$ , and that  $F \subset G$  implies  $N(G) \setminus N(F) \neq \emptyset$ . (Indeed, if  $x \in G \setminus F$  then  $x \in U_n \subseteq X \setminus F$  for some  $n \in \omega$ , so that  $n \in N(G) \setminus N(F)$ .) For each  $\alpha < \rho$ , pick some  $n_\alpha \in N(F_\alpha) \setminus N(F_{\alpha+1})$ : since the map  $\alpha \mapsto n_\alpha$  is an injection between  $\rho$  and  $\omega$ , we must conclude that  $\rho$  is a countable ordinal.  $\square$

Thus if  $X$  is Polish there is  $\alpha_0 < \omega_1$  such that  $X^{(\alpha)} = X^{(\alpha_0)}$  for all  $\alpha \geq \alpha_0$  (the countable ordinal  $\alpha_0$  is called the **Cantor-Bendixson rank** of  $X$ ). Then  $X^{(\alpha_0)}$ , also denoted by  $X^\infty$ , is the perfect kernel of  $X$ , and  $X$  is countable if and only if  $X^\infty = \emptyset$ .

### Remark

Notice that the decomposition given by the proof of Theorem 1.4.7 and the one given by the iterated Cantor-Bendixson derivatives can give different decompositions if applied to non-Polish separable metrizable spaces: indeed, if  $X = \mathbb{Q}$  then the former gives  $P = \emptyset$  and  $C = \mathbb{Q}$ , while the latter gives  $P = \mathbb{Q}$  and  $C = \emptyset$ .

The results above imply that there is no simple counterexample to the Continuum Hypothesis, but they actually show more.

### Definition

A subset  $A$  of a topological space  $X$  has the **Perfect Set Property** (PSP for short) if either it is countable or there is an embedding of  $2^\omega$  into  $A$ .

Clearly, if  $X$  is separable and  $A$  has the PSP then  $A$  satisfies the Continuum Hypothesis: either  $A$  is countable, or has cardinality  $2^{\aleph_0}$ . However, the PSP is a stronger property: while it is independent of ZFC that all subsets of  $\mathbb{R}$  are either countable or of size  $2^{\aleph_0}$  (i.e. that the Continuum Hypothesis CH holds), it can be proved in ZFC that there is a set without the PSP.

### Proposition 1.4.14 [Kec95, Example 8.24]

If  $X$  is an uncountable Polish space, then there is  $A \subseteq X$  without the PSP.

Sets  $A$  as above are called **Bernstein sets**.

## Proof of Proposition 1.4.14

There are exactly  $2^{\aleph_0}$  perfect subsets of  $X$ .

**Proof.**

If  $\mathcal{B} = \{U_n \mid n \in \omega\}$  is a countable basis for  $X$ , then the map sending  $x \in \omega^\omega$  to  $X \setminus \bigcup_{n \in \omega} U_{x(n)}$  is a surjection of  $\omega^\omega$  onto the closed subsets of  $X$ , whence there are at most  $2^{\aleph_0}$  perfect subset of  $X$ .

To show that there are at least  $2^{\aleph_0}$  such sets, it is enough to consider the case  $X = 2^\omega$  (the general case easily follows from the fact that every uncountable Polish space  $X$  contains a closed set homeomorphic to  $2^\omega$  by Corollary 1.4.9). For each  $x \in 2^\omega$ , let  $P_x = \{x\} \times 2^\omega \subseteq 2^\omega \times 2^\omega$ . It is immediate to check that  $P_x$  is perfect in  $2^\omega \times 2^\omega$  and that the map  $x \mapsto P_x$  is injective. Since  $2^\omega \times 2^\omega$  is homeomorphic to  $2^\omega$  we are done.  $\square$

(continues)

## Proof of Proposition 1.4.14

Fix a transfinite enumeration  $(P_\xi)_{\xi < 2^{\aleph_0}}$  of the nonempty perfect subsets of  $X$ . Find by transfinite recursion on  $\xi < 2^{\aleph_0}$  distinct point  $a_\xi, b_\xi \in P_\xi$ : this is possible because each  $P_\xi$ , being a perfect Polish space, has cardinality  $2^{\aleph_0}$  by Corollary 1.4.4, while the collection of points  $\{a_\nu, b_\nu \mid \nu < \xi\}$  constructed so far has cardinality  $|\xi| < 2^{\aleph_0}$ . Setting  $A = \{a_\xi \mid \xi < 2^{\aleph_0}\}$  we obtain an uncountable set that does not contain any nonempty perfect subset  $P$  (and thus  $2^\omega$  cannot be embedded into  $A$ . Indeed, if  $P$  is perfect nonempty then  $P = P_\xi$  for some  $\xi < 2^{\aleph_0}$ , whence  $b_\xi \in P \setminus A$  by construction.  $\square$

### Remark

The proof of Proposition 1.4.14 heavily uses the Axiom of Choice AC. In contrast, it can be shown via a forcing argument that it is consistent with  $\text{ZF} + \text{DC}$  (where DC is the Axiom of Dependent Choice) that all subsets of  $\mathbb{R}$  have the PSP.

The existence of a Bernstein set under ZFC naturally leads to the problem of understanding how much complicated such a set must be: the last result of this section shows that simple sets are immune from this “pathological” behaviour.

### Theorem 1.4.16

Every  $\mathbf{F}_\sigma$  or  $\mathbf{G}_\delta$  subset  $A$  of a Polish space  $X$  has the PSP, and thus satisfies the Continuum Hypothesis. The same is true for  $\mathbf{G}_{\delta\sigma}$  subsets (i.e. countable unions of  $\mathbf{G}_\delta$  sets) of  $X$ .

### Proof.

If  $A$  is  $\mathbf{G}_\delta$ , then it is Polish by Proposition 1.1.8. If it is uncountable, then its perfect kernel  $P$  is nonempty, and thus  $2^\omega$  can be embedded into  $P \subseteq A$  by Theorem 1.4.3.

If now  $A = \bigcup_{n \in \omega} A_n$  with each  $A_n$  a  $\mathbf{G}_\delta$  set,<sup>a</sup> we distinguish two cases:

- Every  $A_n$  is countable. Then  $A$  is countable as well and we are done.
- There is  $n \in \omega$  such that  $A_n$  is uncountable. Then  $2^\omega$  embeds into  $A_n \subseteq A$  by the first part of this proof, and we are done again.  $\square$

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<sup>a</sup>Observe that this includes the case of an  $\mathbf{F}_\sigma$  set  $A$  because in a metrizable space all closed sets are  $\mathbf{G}_\delta$  — see Proposition 3.6.2.