Analytic sets

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Definition 3.1.1 [Kec95, Definition 14.1]

Let X be a separable metrizable space. $A \subseteq X$ is called **analytic** if there is a Polish space Y and a continuous surjection from Y onto A, i.e. a function $f\colon Y \to X$ such that $f(Y) = A$.
$A \subseteq X$ is **coanalytic** if $X \setminus A$ is analytic, and it is bi-analytic if A and its complement are analytic.
The collection of analytic, coanalytic, bi-analytic subsets of X is denoted by $\boldsymbol{\Sigma}^1_1(X)$, $\boldsymbol{\Pi}^1_1(X)$, $\boldsymbol{\Delta}^1_1(X)$.

Proposition 3.1.2

Let X be Polish and $\emptyset \neq A \subseteq X$. The following are equivalent:

1. A is analytic;
2. A is a continuous image of ω^ω;
3. $A = \pi_X(C)$ for some nonempty $C \in \mathbf{\Pi}^0_1(X \times \omega^\omega)$, where π_X is the projection on X;
4. $A = \pi_X(C)$ for some Polish Y and nonempty $C \in \mathbf{Bor}(X \times Y)$;
5. $A = f(C)$ for some nonempty $C \in \mathbf{Bor}(Y)$ and $f \colon Y \to X$ Borel, where Y is any Polish space (or even just $Y = \omega^\omega$).

In particular, when $X = \omega^\omega$, then $A \subseteq X$ is analytic if and only it is the projection of the body of a (pruned) tree on $\omega \times \omega$.

3⇒4⇒5 are obvious.

1 A is analytic ⇒ 2 A is a continuous image of ω^ω

Let $f \colon Y \to A$ be a continuous surjection with Y Polish (necessarily nonempty). By Theorem 1.3.17, there is a continuous surjection $g \colon \omega^\omega \to Y$: thus A is the image of ω^ω under the continuous function $f \circ g$.

2⇒ 3 $A = \pi_X(C)$ for some $\emptyset \neq C \in \mathbf{\Pi}^0_1(X \times \omega^\omega)$

Let $f \colon \omega^\omega \to A$ be a continuous surjection. Then $C = (\mathrm{graph}(f))^{-1} \subseteq X \times \omega^\omega$ is closed, and $A = \pi_X(C)$.

5 $A = f(C)$ with $\emptyset \neq C \in \mathbf{Bor}(Y)$ and $f \colon Y \to X$ Borel, Y Polish ⇒ 1

We can assume $C = Y = \omega^\omega$ as every nonempty Borel subset of a Polish space is a continuous image of ω^ω by Proposition 2.2.5, so let $f \colon \omega^\omega \to X$ be a Borel function such that $A = f(\omega^\omega)$. By Proposition 2.4.12, there is a Polish topology τ' on ω^ω such that $f \colon (\omega^\omega, \tau') \to X$ is continuous, hence A is analytic.

Remark

By Corollary 3.6.11, in ❷–❸ of Proposition 3.1.2 we cannot replace ω^ω with 2^ω or with any σ-compact space (like the real line $\mathbb{R}$). However, since ω^ω is homeomorphic to a $\mathbf{G}_\delta$ subset of 2^ω part ❸ could be replaced by

- $A = \pi_X(G)$ for some nonempty $G \in \mathbf{\Pi}^0_2(X \times 2^\omega)$.

By part ❺ the notion of analytic set depends only on the Borel structure of Polish spaces rather than on their topology, and it is thus immune from changes of topologies as those considered in Sections 2.2 and 2.4. Indeed, such a formulation of analyticity can be used to define $\mathbf{\Sigma}^1_1$ subsets of standard Borel spaces. Moreover, since any Borel set is the image of itself through the identity function, we easily get

Corollary 3.1.4

$\mathbf{Bor}(X) \subseteq \mathbf{\Sigma}^1_1(X)$ for any Polish space X. Since $\mathbf{Bor}(X)$ is closed under complements, it also follows $\mathbf{Bor}(X) \subseteq \mathbf{\Delta}^1_1(X)$.

Proposition 3.1.5

Let X be Polish. Then $\mathbf{\Sigma}^1_1(X)$ is closed under countable unions and countable intersections. Moreover, analytic sets are closed under Borel images and preimages, that is: if X, Y are Polish and $f \colon X \to Y$ is Borel, then for $A \subseteq X$ and $B \subseteq Y$ analytic, both $f(A)$ and $f^{-1}(B)$ are analytic. In particular, $\mathbf{\Sigma}^1_1$ is a boldface pointclass.

Proof

Let $f_n \colon Y_n \to X$ be continuous and let $A_n = f_n(Y_n)$. WLOG the Y_ns are pairwise disjoint. Then $\bigcup_{n\in\omega} f_n$ is a continuous surjection from $\bigoplus_{n\in\omega} Y_n$ onto $\bigcup_{n\in\omega} A_n$, so the latter set is analytic.
The set $Z = \{(y_n)_{n\in\omega} \in \prod_{n\in\omega} Y_n \mid \forall n \forall m (f_n(y_n) = f_m(y_m))\}$, Then Z is the continuous preimage of the diagonal of X^ω and thus it is closed (hence Polish). Then the map $g \colon Z \to X$ sending $(y_n)_{n\in\omega}$ to $f_0(y_0)$ is continuous and such that $g(Z) = \bigcap_{n\in\omega} A_n$, thus the intersection of the A_n's is analytic.

(continues)

Proof (continued)

Closure under Borel images follows from Proposition 3.1.2 part 5, so let us show that if $B \in \mathbf{\Sigma}^1_1(Y)$ and $f : X \to Y$ is Borel then $f^{-1}(B) \in \mathbf{\Sigma}^1_1(X)$. Consider $\mathrm{graph}(f) \subseteq X \times Y$, which is Borel by Proposition 2.4.4 and thus analytic by Proposition 2.2.5. Notice that $x \in f^{-1}(B)$ if and only if there is $y \in Y$ such that $y \in B$ and $(x, y) \in \mathrm{graph}(f)$, i.e.

$$f^{-1}(B) = \pi_X(F)$$

where $F = \{(x, y) \in X \times Y \mid y \in B\} \cap \mathrm{graph}(f)$. Let $g : Z \to Y$ be a continuous function on a Polish space Z such that $g(Z) = B$. Then the map $X \times Z \to X \times Y$ defined by $(x, z) \mapsto (x, g(z))$ is continuous, and its range is $\{(x, y) \in X \times Y \mid y \in B\}$, which is thus an analytic set. It follows that F is analytic because $\mathbf{\Sigma}^1_1(X \times Y)$ is closed under (countable) intersections, and therefore $f^{-1}(B)$ is analytic because it is the continuous image of an analytic set. □

Remark 3.1.7

It is clear that according to Definition 3.1.1, being analytic does not depend too much on the ambient space: if $Y \subseteq X$ are separable metrizable spaces and $A \subseteq Y$, then $A \in \mathbf{\Sigma}^1_1(Y)$ if and only if $A \in \mathbf{\Sigma}^1_1(X)$ (any continuous function witnessing one of the two statements, witnesses the other one as well). Using this and closure under finite intersections we easily get that if $Y \subseteq X$ are Polish, then $\mathbf{\Sigma}^1_1(Y) = \mathbf{\Sigma}^1_1(X) \restriction Y$. This equality holds even if we just have $Y \in \mathbf{\Sigma}^1_1(X)$, but it fails if Y is *not* analytic: in such case, we still have $Y \in \mathbf{\Sigma}^1_1(X) \restriction Y$ because Y can be written as $X \cap Y$ and $X \in \mathbf{\Sigma}^1_1(X)$, but $Y \notin \mathbf{\Sigma}^1_1(Y)$ because otherwise if would be the image of a Polish space, whence $Y \in \mathbf{\Sigma}^1_1(X)$.

Remark

Closure of $\mathbf{\Sigma}^1_1(X)$ under countable unions and intersections yields an alternative proof the Borel sets are (bi-)analytic. Indeed, all open and closed subsets of a Polish space X are analytic because they are Polish, and thus can be written as a continuous image of themselves via the identity function. Since $\mathbf{Bor}(X)$ is the smallest class of subsets of X containing open and closed sets and closed under countable unions and intersection, the result follows.

Example 3.1.8

$\mathrm{LO} \subseteq 2^{\omega\times\omega}$ is the Polish space of (codes for) countable linear orders, $\mathrm{WO} = \{x \in \mathrm{LO} \mid x \text{ is a well-order}\}$ and $\mathrm{NWO} = \mathrm{LO} \setminus \mathrm{WO}$.
Then $x \in \mathrm{NWO} \Leftrightarrow \exists z \in \omega^\omega\, \forall n < m\, (z(n) \neq z(m) \wedge x(z(m), z(n)) = 1)$.
Therefore $\mathrm{NWO} = \pi_{\mathrm{LO}}\left(\bigcap_{n<m} C_{n,m}\right)$, where
$C_{n,m} = \{(x, z) \in \mathrm{LO} \times \omega^\omega \mid z(n) \neq z(m) \wedge x(z(m), z(n)) = 1\}$. Since each $C_{n,m}$ is clopen, NWO is analytic, whence WO is coanalytic.

Example 3.1.9

$\mathrm{Tr} \subseteq 2^{(\omega^{<\omega})}$, the set of (characteristic functions of) trees on ω, is closed in $2^{(\omega^{<\omega})}$ (Exercise!), hence a Polish space. Let $\mathrm{IF} \subseteq \mathrm{Tr}$ be the set of (characteristic functions of) ill-founded trees, i.e. of Ts such that $[T] \neq \emptyset$. Fix a bijection $h \colon \omega \to \omega^{<\omega}$. Then

$$x \in \mathrm{IF} \Leftrightarrow \exists z \in \omega^\omega\, \forall n \leq m\, (x(h(z(n))) = 1 \wedge h(z(n)) \subset h(z(m))).$$

Therefore $\mathrm{IF} = \pi_{\mathrm{Tr}}\left(\bigcap_{n\leq m}(A_n \cap B_{n,m})\right)$, where
$A_n = \{(x, z) \in \mathrm{Tr} \times \omega^\omega \mid x(h(z(n))) = 1\}$ and
$B_{n,m} = \{(x, z) \in \mathrm{Tr} \times \omega^\omega \mid h(z(n)) \subseteq h(z(m))\}$. Since all the A_n and $B_{n,m}$ are clopen, if follows that IF is the projection of a closed subset of $\mathrm{Tr} \times \omega^\omega$, hence $\mathrm{IF} \in \mathbf{\Sigma}^1_1(\mathrm{Tr})$.

Remark 3.1.10

We used the Tarski-Kuratowski algorithm (Remark 2.1.8): the existential quantification $\exists x$ with x varying over an uncountable Polish (or even just standard Borel) space corresponds to a projection. Thus, if a subset of a Polish space X is defined by a formula $\psi(x)$ of the form

$$\exists y_0 \ldots \exists y_n \, \varphi(x, y_0, \ldots, y_n)$$

with each y_i varying over an uncountable Polish space Y_i and $\varphi(x, y_0, \ldots, y_n)$ a formula involving only quantifications over countable sets, connectives, and "atomic formulas" defining Borel subsets of $X \times Y_0 \times \ldots \times Y_n$, then $\varphi(x, y_0, \ldots, y_n)$ defines a Borel set $B \subseteq X \times Y_0 \times \ldots \times Y_n$ by Remark 2.1.8, whence $\psi(x)$ defines the $\mathbf{\Sigma}^1_1(X)$ set $A = \pi_X(B)$.

Remark 3.1.10, (continued)

The universal quantification $\forall x$, with x varying over an uncountable Polish or standard Borel space, is equivalent to $\neg\exists x \neg$. Therefore, if $\varphi(x, y_0, \ldots, y_n)$ defines a Borel subset of $X \times Y_0 \times \ldots \times Y_n$, then

$$\forall y_0 \ldots \forall y_n \, \varphi(x, y_0, \ldots, y_n)$$

defines a coanalytic subset of X.
By the closure properties of $\mathbf{\Sigma}^1_1$ and $\mathbf{\Pi}^1_1$:

- if φ defines a set in $\mathbf{\Sigma}^1_1(X)$ (respectively, $\mathbf{\Pi}^1_1(X)$), then both $\exists n \, \varphi$ and $\forall n \, \varphi$, with n varying over a countable set, define an analytic (respectively, coanalytic) subset of X;
- if φ defines a set in $\mathbf{\Sigma}^1_1(X)$ (respectively, $\mathbf{\Pi}^1_1(X)$), then $\neg\varphi$ defines a coanalytic (respectively, analytic) subset of X.

Exercise

Let X be a Polish space. Show that

$$A = \{(x_n)_{n\in\omega} \in X^\omega \mid (x_n)_{n\in\omega} \text{ has a convergent subsequence}\}$$

is analytic.

Solution.

Let d be a compatible metric on X. Notice that

$$\begin{aligned}(x_n)_{n\in\omega} \in A \Leftrightarrow \exists y \in \omega^\omega \, (\forall n, m \in \omega \, (y(n) \neq y(m)) \wedge \\ \forall \varepsilon \in \mathbb{Q}^+ \, \exists N \in \omega \, \forall n, m \geq N \, (d(x_{y(n)}, x_{y(m)}) < \varepsilon)),\end{aligned}$$

and then use the Tarski-Kuratowski algorithm. Alternatively, observe that

$$(x_n)_{n\in\omega} \in A \Leftrightarrow \exists y \in X \, \forall \varepsilon \in \mathbb{Q}^+ \, \exists n \in \omega \, (d(x_n, y) < \varepsilon).$$

Exercise

Show that

$$\begin{aligned}\mathrm{CN} &= \{(f_n)_{n\in\omega} \in C([0;1])^\omega \mid (f_n)_{n\in\omega} \text{ converges pointwise}\} \\ \mathrm{CN}_0 &= \{(f_n)_{n\in\omega} \in C([0;1])^\omega \mid f_n \to 0 \text{ pointwise}\}\end{aligned}$$

are coanalytic subsets of $C([0;1])^\omega$.

Solution

$$\begin{aligned}(f_n)_{n\in\omega} \in \mathrm{CN} \Leftrightarrow \forall x \in [0;1] \, \forall \varepsilon \in \mathbb{Q}^+ \, \exists N \in \omega \\ \forall n, m \geq N \, (d(f_n(x), f_m(x)) < \varepsilon) \\ (f_n)_{n\in\omega} \in \mathrm{CN}_0 \Leftrightarrow \forall x \in [0;1] \, \forall \varepsilon \in \mathbb{Q}^+ \, \exists N \in \omega \\ \forall n \geq N \, (d(f_n(x), 0) < \varepsilon).\end{aligned}$$

Exercise

Let A be the set of sequences $(x_n)_{n\in\omega} \in \mathbb{R}^\omega$ such that all the accumulation points of $(x_n)_{n\in\omega}$ are multiple of the same integer $k \in \mathbb{Z}$ (where such k may vary depending on the sequence $(x_n)_{n\in\omega}$ under consideration). Show that $A \in \mathbf{\Pi}^1_1(\mathbb{R}^\omega)$.

Solution

$(x_n)_{n\in\omega} \in A$ if and only if

$$\exists k \in \mathbb{Z}\,\forall y \in \mathbb{R}\,(\forall \varepsilon \in \mathbb{Q}^+\,\exists n \in \omega\,(d(y, x_n) < \varepsilon) \to \exists j \in \mathbb{Z}\,(y = k \cdot j)).$$

Exercise

Show that the collection of sequences $(x_n)_{n\in\omega} \in [0;1]^\omega$ having an irrational among their accumulation points is analytic.

Theorem 3.1.15 (Souslin, see [Kec95, Theorem 14.2])

If X is Polish then for every uncountable Polish Y there is a Y-universal set for $\mathbf{\Sigma}^1_1(X)$. So if X is infinite then $|\mathbf{\Sigma}^1_1(X)| = 2^{\aleph_0}$. Moreover, if X is uncountable then $\mathbf{\Sigma}^1_1(X) \neq \mathbf{\Pi}^1_1(X)$, $\mathbf{\Sigma}^1_1(X)$ is not closed under complements, and $\mathbf{Bor}(X) \subseteq \mathbf{\Delta}^1_1(X) \subset \mathbf{\Sigma}^1_1(X)$.

Proof.

The additional parts follow from the first one by Lemma 2.1.15 and using standard arguments (see Section 2.1), so let us show that there is a Y-universal set for $\mathbf{\Sigma}^1_1(X)$ for any uncountable Polish space Y. Let $\mathcal{V}$ be Y-universal for $\mathbf{\Pi}^0_1(X \times \omega^\omega)$, and let $\mathcal{U} = \pi_{Y\times X}(\mathcal{V})$. Then $\mathcal{U}$ and all its vertical sections are clearly analytic. Conversely, given $A \in \mathbf{\Sigma}^1_1(X)$ let $C \in \mathbf{\Pi}^0_1(X \times \omega^\omega)$ be such that $A = \pi_X(C)$, and let $y \in Y$ be such that $C = \mathcal{V}_{(y)}$. Then $A = \mathcal{U}_{(y)}$. Indeed, for every $x \in X$

$$x \in A \Leftrightarrow \exists z\,((x,z) \in C) \Leftrightarrow \exists z\,((y,x,z) \in \mathcal{V}) \Leftrightarrow (y,x) \in \mathcal{U} \Leftrightarrow x \in \mathcal{U}_{(y)}.$$

□

Corollary 3.1.16

The collection $\mathbf{\Pi}^1_1$ and $\mathbf{\Delta}^1_1$ are boldface pointclasses, and $\mathbf{Bor}(X) \subseteq \mathbf{\Delta}^1_1(X) \subseteq \mathbf{\Pi}^1_1(X)$ for any Polish space X; if moreover X is uncountable, then $\mathbf{\Delta}^1_1(X) \subset \mathbf{\Pi}^1_1(X)$. Finally, $\mathbf{\Pi}^1_1(X)$ is closed under countable unions and countable intersections but not under complements, while $\mathbf{\Delta}^1_1(X)$ is a σ-algebra; moreover both $\mathbf{\Pi}^1_1$ and $\mathbf{\Delta}^1_1$ are closed under Borel preimages but, if X is uncountable, they are not closed under continuous (hence neither Borel) images.

Proof.

All the statements easily follow from the analogous closure properties of $\mathbf{\Sigma}^1_1(X)$ given in Proposition 3.1.5 and from Theorem 3.1.15. To see that when X is uncountable $\mathbf{\Pi}^1_1$ is not closed under continuous images, observe that if this were true then $\mathbf{\Sigma}^1_1(X) \subseteq \mathbf{\Pi}^1_1(X)$ by $\mathbf{Bor}(X) \subseteq \mathbf{\Pi}^1_1(X)$ and Proposition 3.1.2; thus it would follow that $\mathbf{\Sigma}^1_1(X) = \mathbf{\Delta}^1_1(X)$, contradicting Theorem 3.1.15. The case of $\mathbf{\Delta}^1_1(X)$ is similar. □

Example—see [Kec95, Theorem 27.1]

The set IF of (characteristic functions of) ill-founded trees on ω is a $\mathbf{\Sigma}^1_1$-complete subset of Tr (see Example 3.1.9). Indeed, let $A \subseteq \omega^\omega$ be an arbitrary analytic set, and let T be a (pruned) tree on $\omega \times \omega$ such that A is the projection of $[T]$. Consider the continuous map $f \colon \omega^\omega \to \mathrm{Tr}$ sending $x \in \omega^\omega$ to the (characteristic function of the) tree $T(x) = \{t \in \omega^{<\omega} \mid (x \restriction \mathrm{lh}(t), t) \in T\}$. Then $x \in A \Leftrightarrow T(x) \in \mathrm{IF}$, so that $A = f^{-1}(\mathrm{IF})$.

It follows that the set

$$\mathrm{WF} = \{x \in \mathrm{Tr} \mid x \text{ is (the characteristic function of) a well-founded tree}\}$$

is $\mathbf{\Pi}^1_1$-complete.

Example 3.1.18 (Lusin-Sierpiński, [Kec95, Thm 27.12])

The set NWO of (codes for) non-well-founded linear orders of ω is a $\boldsymbol{\Sigma}^1_1$-complete subset of LO, so WO is $\boldsymbol{\Pi}^1_1$-complete.
It suffices to show that there is a continuous $f\colon \mathrm{Tr} \to \mathrm{LO}$ such that $\mathrm{IF} = f^{-1}(\mathrm{NWO})$. Order $\omega^{<\omega}$ with the **Kleene-Brouwer ordering**

$$s <_{\mathrm{KB}} t \Leftrightarrow s \supset t \vee \exists i < \mathrm{lh}(s), \mathrm{lh}(t) \, (s \restriction i = t \restriction i \wedge s(i) < t(i)).$$

A tree T on ω is well-founded iff $<_{\mathrm{KB}}$ on T is a well-order [Kec95, Prop 2.12]. Let $h\colon \omega \to \omega^{<\omega}$ be any bijection. Given $x \in \mathrm{Tr}$, let T_x be the corresponding tree on ω and define $f(T_x) \in \mathrm{LO}$:

$$\begin{aligned} f(T_x)(n,m) = 1 \Leftrightarrow {} & (h(n), h(m) \in T_x \wedge h(n) <_{\mathrm{KB}} h(m)) \vee \\ & (h(n) \in T_x \wedge h(m) \notin T_x) \vee (h(n), h(m) \notin T_x \wedge n < m). \end{aligned}$$

$f(T_x)$ is a linear ordering isomorphic (via h) to the ordering of $\omega^{<\omega}$ in which all elements of T_x precede those in $\omega^{<\omega} \setminus T_x$, T_x is ordered by $<_{\mathrm{KB}}$, $\omega^{<\omega} \setminus T_x$ is ordered using the bijection h in order type $\leq \omega$. Then $x \in \mathrm{IF} \Leftrightarrow f(x) \in \mathrm{NWO}$.

More examples of $\boldsymbol{\Sigma}^1_1$- and $\boldsymbol{\Pi}^1_1$-complete sets

1. The set of compact $K \subseteq [0;1]$ such that K contains an irrational number is a $\boldsymbol{\Sigma}^1_1$-complete subset of $\mathbf{K}([0;1])$ (Hurewicz, see [Kec95, Exercise 27.4]).
2. If X is an uncountable Polish space, then the set of all countable compact subsets of X is a $\boldsymbol{\Pi}^1_1$-complete subset of $\mathbf{K}(X)$ (Hurewicz, see [Kec95, Theorem 27.5]).
3. Let X be a Polish space which is not σ-compact. Then the set of all sequences $(x_n)_{n\in\omega} \in X^\omega$ having a convergent subsequence is $\boldsymbol{\Sigma}^1_1$-complete (see [Kec95, Exercise 27.15]).
4. The set of all differentiable $f \in C([0;1])$ is $\boldsymbol{\Pi}^1_1$-complete (Mazurkiewicz, see [Kec95, Theorem 33.9]).
5. The set of all $f \in C([0;1])$ satisfying Rolle's theorem (i.e. those f for which for all $a < b$ in $[0;1]$, if $f(a) = f(b)$, there is $c \in (a;b)$ with $f'(c) = 0$) is $\boldsymbol{\Sigma}^1_1$-complete (Woodin).

More examples of $\mathbf{\Sigma}^1_1$- and $\mathbf{\Pi}^1_1$-complete sets

6. Both
$$\mathrm{CN} = \{(f_n)_{n\in\omega} \in C([0;1])^\omega \mid (f_n)_{n\in\omega} \text{ converges pointwise}\}$$
and
$$\mathrm{CN}_0 = \{(f_n)_{n\in\omega} \in C([0;1])^\omega \mid f_n \to 0 \text{ pointwise}\}$$
are $\mathbf{\Pi}^1_1$-complete subsets of $C([0;1])^\omega$ (see [Kec95, Theorem 33.11]).

The descriptive set theoretic complexity can be used to show that two classes of objects are different.
Given a class $\mathcal{F}$ of separable Banach spaces, a separable Banach space Y is called **universal** for $\mathcal{F}$ if all its closed subspace are in $\mathcal{F}$ and, conversely, every $X \in \mathcal{F}$ is isomorphic to some closed subspace of Y. For example, $C(2^\omega)$ is universal for the class of all separable Banach spaces.
Problem 49 in the Scottish Book, due to Banach and Mazur, asks whether there is a separable Banach space with separable dual, which is universal for the class of separable Banach spaces with separable dual. Wojtaszczyk answered this negatively using methods of Szlenk. Here we sketch an easier proof due to Bourgain involving the computation of the complexity of the relevant classes of Banach spaces.

Theorem [Kec95, Section 33.K]

There is no universal space for the class of separable Banach spaces having separable dual.

Sketch of the proof

Given $K \in \mathbf{K}(2^\omega) \setminus \{\emptyset\}$, consider the separable Banach space $C(K)$. The dual $C^*(K)$ is the space of signed Borel measures on K (see [Kec95, Exercise 17.32]). If K is countable, then $C^*(K)$ is isomorphic to ℓ^1 if K is infinite, and to $\mathbb{R}^n$ if K has size $n \in \omega$. Therefore $C^*(K)$ is separable when K is countable. On the other hand, if K is uncountable then $C^*(K)$ is non-separable. (Consider, for example, the Dirac measures δ_x for $x \in K$: if $x \neq y$, then $\|\delta_x - \delta_y\| = 2$.)
As in the proof of [Kec95, Theorem 33.24], the map $\mathbf{K}(2^\omega) \setminus \{\emptyset\} \to \mathrm{Subs}(C(2^\omega))$, $K \mapsto C(K)$ is Borel. Thus

$$\mathcal{F} = \{X \in \mathrm{Subs}(C(2^\omega)) \mid X^* \text{ is separable}\}$$

is not analytic as $\{K \in \mathbf{K}(2^\omega) \mid K \neq \emptyset \text{ and } K \text{ countable}\}$ is $\mathbf{\Pi}^1_1$-complete in $\mathbf{K}(2^\omega)$.

(continues)

Sketch of the proof (continued)

Assume towards a contradiction that there is $Y \in \mathrm{Subs}(C(2^\omega))$ which is universal for $\mathcal{F}$. Using Theorem 2.3.6, one can see that the relation $\sqsubseteq$ of embedding (i.e. being isomorphic to a closed subspace) between separable Banach spaces is analytic, i.e. it is a $\mathbf{\Sigma}^1_1$ subset of the square $\mathrm{Subs}(C(2^\omega)) \times \mathrm{Subs}(C(2^\omega))$ (see [Kec95, Exercise 33.26]). It would then follow that

$$\mathcal{F} = \{X \in \mathrm{Subs}(C(2^\omega)) \mid X \sqsubseteq Y\}$$

is an analytic subset of $\mathrm{Subs}(C(2^\omega))$, a contradiction.