Changes of topologies

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Lemma 2.2.1 [Kec95, Lemma 13.3]

$X = (X, \tau)$ Polish and $(\tau_n)_{n\in\omega}$ be a sequence of Polish topologies on X with $\tau \subseteq \tau_n$ for every $n \in \omega$. Then the topology τ_∞ generated by $\bigcup_{n\in\omega} \tau_n$ is Polish. Moreover, if $\tau_n \subseteq \boldsymbol{\Sigma}^0_\alpha(X, \tau)$ for every $n \in \omega$, then $\tau_\infty \subseteq \boldsymbol{\Sigma}^0_\alpha(X, \tau)$.

Notice that if all τ_n are zero-dimensional, then so is τ_∞.

Proof.

Let $X_n = (X, \tau_n)$, and let $f\colon X \to \prod_n X_n$, $x \mapsto (x, x, x, \dots)$.
We claim that $f(X)$ is closed in $\prod_n X_n$: if $(x_n)_n \notin f(X)$, then $x_i \neq x_j$ for some $i < j$, so if $U, V \in \tau$ are disjoint and $x_i \in U \in \tau_i$ and $x_j \in V \in \tau_j$, so $(x_n)_n \in X_0 \times \ldots \times X_{i-1} \times U \times X_{i+1} \times \ldots \times X_{j-1} \times V \times X_{j+1} \times \ldots$ is open and disjoint from $f(X)$.
Thus $f(X)$ is Polish and $f\colon (X, \tau_\infty) \to f(X)$ is a homeomorphism.
If $\mathcal{B}_n$ is a countable basis for τ_n then $\bigcup_n \mathcal{B}_n \subseteq \boldsymbol{\Sigma}^0_\alpha(X, \tau)$ is a countable subbasis for τ_∞, hence $\tau_\infty \subseteq \boldsymbol{\Sigma}^0_\alpha(X, \tau)$. □

Theorem 2.2.2 (Kuratowski) [Kec95, Theorem 22.18 and Exercises 22.19 and 22.20]

$X = (X, \tau)$ Polish, $1 \leq \alpha < \omega_1$, and $A_n \subseteq X$ in $\mathbf{\Delta}^0_\alpha(X, \tau)$ for all n. Then there is a Polish topology $\tau' \supseteq \tau$ on X such that $\tau' \subseteq \mathbf{\Sigma}^0_\alpha(X, \tau)$ and $A_n \in \mathbf{\Delta}^0_1(X, \tau')$ for all $n \in \omega$.
When $\alpha > 1$ we can require τ' to be zero-dimensional, and if $\alpha > 1$ is a successor ordinal and $\forall n(A_n = A)$, we may require $\tau' \subseteq \mathbf{\Delta}^0_\alpha(X, \tau)$ (dropping zero-dimensionality, unless τ was already zero-dimensional).

Proof

If $\alpha = 1$, set $\tau' = \tau$.
If $\alpha = 2$, then each A_n and its complement $X \setminus A_n$ are $\mathbf{G}_\delta$, hence Polish. Let τ_n be the direct sum of the relative topologies on A_n and $X \setminus A_n$: then $\tau_n \supseteq \tau$ is still Polish, $A_n \in \mathbf{\Delta}^0_1(X, \tau_n)$, and $\tau_n \subseteq \mathbf{\Sigma}^0_2(X, \tau)$ because it consists of the sets of the form $(U \cap A_n) \cup (V \setminus A_n)$ for $U, V \in \tau$. Letting τ' be the topology generated by $\bigcup_{n \in \omega} \tau_n$, by Lemma 2.2.1 we get the desired result.

(continues)

Proof (continued)

$\alpha = 2$.
If $A_n = A$, for all n then $\tau' = \{(U \cap A) \cup (V \setminus A) \mid U, V \in \tau\}$ so $\tau' \subseteq \mathbf{\Delta}^0_2(X, \tau)$. To see that τ' can be zero-dimensional, it's enough to observe that w.l.o.g. $\{A_n \mid n \in \omega\} \supseteq \{X \setminus U_k \mid k \in \omega\}$, where $\{U_k \mid k \in \omega\}$ is any countable basis for τ: it follows that $\{U_k \cap A_n, U_k \setminus A_n \mid n, k \in \omega\}$ consists of τ'-clopen sets.
α a limit ordinal. Then $A_n = \bigcup_{i \in \omega} A_{n,i} = \bigcap_{i \in \omega} B_{n,i}$, with $A_{n,i}, B_{n,i} \in \mathbf{\Delta}^0_{\alpha_{n,i}}(X, \tau)$ for some $1 < \alpha_{n,i} < \alpha$. By inductive hypothesis, let $\tau'_{n,i}$ and $\tau''_{n,i}$ be (zero-dimensional) Polish topologies for $A_{n,i}$ and $B_{n,i}$, respectively. In particular $A_{n,i} \in \mathbf{\Delta}^0_1(X, \tau'_{n,i})$ and $\tau \subseteq \tau'_{n,i} \subseteq \mathbf{\Sigma}^0_{\alpha_{n,i}}(X, \tau) \subseteq \mathbf{\Sigma}^0_\alpha(X, \tau)$, and similarly for $B_{n,i}$ and $\tau''_{n,i}$. Then letting τ' be the topology generated by $\bigcup_{n,i}(\tau'_{n,i} \cup \tau''_{n,i})$, we get from Lemma 2.2.1 that $\tau' \supseteq \tau$ is a Polish topology such that $\tau' \subseteq \mathbf{\Sigma}^0_\alpha(X, \tau)$. Moreover, $A_n = \bigcup_i A_{n,i} = \bigcap_i B_{n,i} \in \mathbf{\Delta}^0_1(X, \tau')$ because all the $A_{n,i}$ and $B_{n,i}$ are τ'-clopen.

(continues)

Proof (continued).

Let $\alpha = \beta + 1 \geq 3$ be a successor ordinal. Then $A_n = \bigcup_i \bigcap_j A_{n,i,j} = \bigcap_i \bigcup_j B_{n,i,j}$ with $A_{n,i,j}, B_{n,i,j} \in \mathbf{\Delta}^0_\beta(X,\tau)$. By inductive hypothesis, for each $n, i, j \in \omega$ there are Polish topologies $\tau'_{n,i,j}$ and $\tau''_{n,i,j}$ refining τ such that $A_{n,i,j}$ is $\tau'_{n,i,j}$-clopen, $B_{n,i,j}$ is $\tau''_{n,i,j}$-clopen, and $\tau'_{n,i,j}, \tau''_{n,i,j} \subseteq \mathbf{\Sigma}^0_\beta(X,\tau)$. Let τ_∞ be the topology generated by $\bigcup_{n,i,j}(\tau'_{n,i,j} \cup \tau''_{n,i,j})$, so that all $A_{n,i,j}$ and $B_{n,i,j}$ are τ_∞-clopen, $\tau \subseteq \tau_\infty \subseteq \mathbf{\Sigma}^0_\beta(X,\tau)$, and τ_∞ is Polish by Lemma 2.2.1. It follows that $A_n \in \mathbf{\Delta}^0_2(X,\tau_\infty)$. By case $\alpha = 2$ applied to the A_n's viewed as subsets of (X,τ_∞), we get that there is a (zero-dimensional) Polish topology $\tau' \supseteq \tau_\infty \supseteq \tau$ such that each A_n is τ'-clopen and $\tau' \subseteq \mathbf{\Sigma}^0_2(X,\tau_\infty) \subseteq \mathbf{\Sigma}^0_{\beta+1}(X,\tau)$, whence τ' is as desired because $\beta + 1 = \alpha$.
Finally, if $\forall n(A_n = A)$ then we require $\tau' \subseteq \mathbf{\Delta}^0_2(X,\tau_\infty) \subseteq \mathbf{\Delta}^0_\alpha(X,\tau)$. $\square$

Corollary 2.2.3, essentially [Kec95, Theorem 13.1 and Exercise 13.5]

Let $X = (X,\tau)$ be a Polish space and $A_n \in \mathbf{Bor}(X,\tau)$ for every $n \in \omega$. Then there is a Polish topology $\tau' \supseteq \tau$ such that $\mathbf{Bor}(X,\tau') = \mathbf{Bor}(X,\tau)$ and each A_n is clopen with respect to τ'. Moreover, τ' can be taken to be zero-dimensional.

Proof.

Let $\alpha > 1$ be such that $A_n \in \mathbf{\Sigma}^0_\alpha(X,\tau)$ for every $n \in \omega$, and let τ' be the (zero-dimensional) topology given by Theorem 2.2.2. By induction on $1 \leq \beta < \omega_1$, one can easily show that

$$\mathbf{\Sigma}^0_\beta(X,\tau) \subseteq \mathbf{\Sigma}^0_\beta(X,\tau') \subseteq \mathbf{\Sigma}^0_{\alpha+\beta}(X,\tau),$$

whence $\mathbf{Bor}(X,\tau') = \bigcup_{1\leq\beta<\omega_1} \mathbf{\Sigma}^0_\beta(X,\tau') = \bigcup_{1\leq\beta<\omega_1} \mathbf{\Sigma}^0_\beta(X,\tau) = \mathbf{Bor}(X,\tau)$. $\square$

There is no analogue of Corollary 2.2.3 for $\aleph_1$-many Borel sets $\mathcal{A} = \{A_\alpha \mid \alpha < \omega_1\}$, even if we require that they have Borel rank bounded by the same $1 < \beta < \omega_1$: if X is uncountable Polish space and $A \subseteq X$ of size $\aleph_1$, then $\mathcal{A} = \{\{x\} \mid x \in A\}$ is a family of $\aleph_1$-many closed sets. Then any τ' such that $\mathcal{A} \subseteq \tau'$ cannot be separable, as all points in A would be τ'-isolated.

Also the possibility of having a "good" change of topology turning a non-Borel set into an open (or even just Borel) one is hopeless.

Theorem 3.2.5 shows that if f is an injective continuous function between Polish spaces X and Y, then $f(A) \in \mathbf{Bor}(Y)$ for every $A \in \mathbf{Bor}(X)$.

Now suppose towards a contradiction that there is a set $A \subseteq X$ which is not Borel (with respect to the Polish topology τ on X), but it is such that there is a Polish topology $\tau' \supseteq \tau$ such that $A \in \mathbf{Bor}(X, \tau')$. Then the identity function $\mathrm{id}_X \colon (X, \tau') \to (X, \tau)$ would be continuous and injective, and since $A \in \mathbf{Bor}(X, \tau')$ then $\mathrm{id}_X(A) = A$ would be in $\mathbf{Bor}(X, \tau)$, a contradiction.

Theorem (Alexandrov, Hausdorff) [Kec95, Theorem 13.6]

(X, τ) Polish and $A \in \mathbf{Bor}(X)$. Then A has the PSP. In particular, every uncountable Borel subset of a Polish space has cardinality $2^{\aleph_0}$.

Proof.

By Corollary 2.2.3, let $\tau' \supseteq \tau$ be a Polish topology on X such that $A \in \mathbf{\Delta}^0_1(X, \tau')$, so that $(A, \tau' \restriction A)$ is Polish (where $\tau' \restriction A$ is the relative topology of τ' on the set A). If A is uncountable, then by Corollary 1.4.9 there is a continuous injection $f \colon 2^\omega \to (A, \tau' \restriction A) \subseteq (X, \tau')$. As $\tau \subseteq \tau'$, the function f is continuous also as a function from 2^ω to (X, τ), and thus it is an embedding of 2^ω into A (with respect to the original topology τ of X). $\square$

Proposition 2.2.5 (Lusin-Souslin) [Kec95, Theorem 13.7]

Let X be Polish and $A \subseteq X$ be Borel. There is a closed set $F \subseteq \omega^\omega$ and a continuous bijection $f\colon F \to A$. In particular, if $A \neq \emptyset$, there is also a continuous surjection $g\colon \omega^\omega \to A$ extending f.

Proof.

Apply Corollary 2.2.3 to get a Polish topology τ' refining the topology τ of X such that A is τ'-clopen (hence Polish with respect to the relative topology of τ'). Then by Theorem 1.3.17 there are F and f (or even g if $A \neq \emptyset$) as in the statement, except that f (respectively, g) is continuous as a function between F (respectively, ω^ω) and (X, τ'). But since $\tau \subseteq \tau'$, the function remains continuous when equipping X with its original topology τ, hence we are done. $\square$