Perfect Polish spaces

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Definition

- A point x of a topological space X is **isolated** if there is an open neighborhood U of it such that $U = \{x\}$.
- A space is **perfect** if it has no isolated point.
- A subset $P \subseteq X$ is **perfect in** X if it is closed and perfect with respect to the relative topology.

Remark

If x is not isolated in (X, d), then every open neighborhood U of x is infinite, as we can recursively define $(x_n)_{n\in\mathbb{N}}$ distinct points in U: fix $\varepsilon > 0$ such that $\mathrm{B}_d(x, \varepsilon) \subseteq U$ and pick any $x_0 \in \mathrm{B}_d(x, \varepsilon) \setminus \{x\}$. Then let x_{n+1} be any point in $\mathrm{B}_d(x, d(x, x_n)) \setminus \{x\}$ (such x_n's exist because x is not isolated in X).

$\mathbb{R}^n$, $\mathbb{R}^\mathbb{N}$, $\mathbb{C}^n$, $\mathbb{C}^\mathbb{N}$, $[0; 1]^n$, $[0; 1]^\mathbb{N}$, $2^\mathbb{N}$, $\mathbb{N}^\mathbb{N}$, $C(X)$ with X compact metrizable, ... are perfect Polish spaces.

Theorem 1.4.3 [Kec95, Theorem 6.2]

Let X be a nonempty perfect completely metrizable space. Then $2^{\mathbb{N}}$ embeds into X, that is: there is a closed $C \subseteq X$ homeomorphic to $2^{\mathbb{N}}$.

Proof.

Fix a complete compatible metric $d \leq 1$ on X. By Lemma 1.3.6, it is enough to build a 2-scheme $\{B_s \mid s \in 2^{<\omega}\}$ such that

1. $B_{s^\frown i} \cap B_{s^\frown j} = \emptyset$ if $i \neq j$;
2. B_s is open and nonempty;
3. $\mathrm{Cl}(B_{s^\frown i}) \subseteq B_s$.

Let $B_\emptyset$ be nonempty open. Given B_s, define $B_{s^\frown 0}$ and $B_{s^\frown 1}$ as follows. Choose two distinct points $x_0, x_1 \in B_s$, let $\varepsilon_i > 0$ be small enough so that $\mathrm{B}_d(x_i, \varepsilon_i) \subseteq B_s$, and set $B_{s^\frown i} = \mathrm{B}_d(x_i, \varepsilon)$ where $\varepsilon = \frac{1}{2}\min\{2^{-(\mathrm{lh}(s)+1)}, \varepsilon_0, \varepsilon_1, d(x_0, x_1)\}$. It is easy to check that such $B_{s^\frown i}$ has the required properties. □

Corollary 1.4.4 [Kec95, Corollary 6.3]

Every nonempty perfect Polish space has the cardinality of the continuum $2^{\aleph_0}$. The same is true for nonempty perfect subsets of a Polish space.

Proof.

By Theorem 1.4.3, a nonempty perfect (subset of a) Polish space X contains a copy of $2^{\mathbb{N}}$, and thus has cardinality $\geq 2^{\aleph_0}$. The fact that $|X| \leq 2^{\aleph_0}$ follows from the fact that by Theorem 1.3.17 the space $\mathbb{N}^{\mathbb{N}}$ surjects onto X, together with the fact that $\mathbb{N}^{\mathbb{N}}$ has has cardinality $2^{\aleph_0}$. □

Remark 1.4.5

One can directly show that $\mathbb{N}^{\mathbb{N}}$ surjects onto X as follows. Let $D = \{x_n \mid n \in \mathbb{N}\}$ be a countable dense subset of X. Then the map $f \colon \mathbb{N}^{\mathbb{N}} \to X$ defined by

$$f(y) = \begin{cases} \lim_{n\to\infty} x_{y(n)} & \text{if } (x_{y(n)})_{n\in\mathbb{N}} \text{ converges in } X \\ x_0 & \text{otherwise} \end{cases}$$

is clearly surjective. More generally, this argument shows that if a metrizable space X has a dense subset of cardinality κ, then there is a surjection of κ^ω onto X and thus X has cardinality $\leq \kappa^{\aleph_0}$.

Definition

A point x in a topological space X is a **condensation point** if every open neighborhood of x is uncountable.

Theorem 1.4.7 (Cantor-Bendixson) [Kec95, Theorem 6.4]

A separable metrizable X can be written as a disjoint union $X = P \cup C$ with P perfect and C a countable open set. If X is Polish, then such a decomposition is unique.

If $X = \mathbb{Q}$, then we could set $P = \mathbb{Q}$ and $C = \emptyset$, or $P = \emptyset$ and $C = \mathbb{Q}$. When X is Polish, the perfect subset P above is the **perfect kernel** of X.

Proof

Let $X^* = \{x \in X \mid x \text{ is a condensation point of } X\}$. Set $P = X^*$ and $C = X \setminus P$. We claim that P and C are as required. First observe that if $\mathcal{B}$ is a countable basis for X then $C = \bigcup\{U \in \mathcal{B} \mid U \text{ is countable}\}$, thus C is open and countable and P is closed. To show that P has no isolated point (with respect to its relative topology!), let $x \in P$ and U be an open neighborhood of x. Then U is uncountable because x is a condensation point, and since C is countable this implies that there is some (in fact, uncountably many) $y \in U \cap P$ witnessing that $U \cap P \neq \{x\}$.

(continues)

Proof (continued).

To prove uniqueness, suppose that X is Polish and that $X = P_1 \cup C_1$ is another decomposition as in the statement of the theorem. Notice that if Y is a perfect Polish space then $Y^* = Y$. Indeed, if $y \in Y$ and U is an open neighborhood of y, then $U \cap Y$ is a perfect nonempty Polish space, and thus has cardinality $2^{\aleph_0}$ by Corollary 1.4.4. Thus $P_1^* = P_1$, whence $P_1 = P_1^* \subseteq X^* = P$ because $Y \subseteq Z$ implies $Y^* \subseteq Z^*$. Moreover, if $x \in C_1$ then $x \in C$, as witnessed by the countable open set C_1 itself: therefore $C_1 \subseteq C$. It follows that $P_1 = P$ and $C_1 = C$. □

Remark

In the proof above we actually showed that if P_1 is a perfect subset of a Polish space X, then $P_1 \subseteq P$, where $P = X^*$ is the perfect kernel of X. Thus the perfect kernel of a Polish space can also be characterized as the largest (with respect to inclusion) perfect subset of it.

Corollary 1.4.9 [Kec95, Corollary 6.5]

Any uncountable Polish space contains a (necessarily closed) homeomorphic copy of $2^{\mathbb{N}}$ and has cardinality $2^{\aleph_0}$.

Proof.

Let P be the perfect kernel of the Polish space X. Since $C = X \setminus P$ is a countable open set, if X is uncountable then $P \neq \emptyset$, and thus P itself is a nonempty perfect Polish space. Thus the result follows from Theorem 1.4.3 and Remark 1.4.5. □

There is an algorithmic way to find the perfect kernel of a Polish space X. The following construction is due to Cantor and it is the reason behind the introduction of ordinals.

Definition

For any topological space X, let

$$X' = \{x \in X \mid x \text{ is not isolated in } X\}.$$

We call X' the **Cantor-Bendixson derivative** of X. Clearly, X' is closed and X is perfect if and only if $X = X'$.

Define the **iterated Cantor-Bendixson derivative** by recursion on the ordinals as follows:

$$\begin{aligned} X^{(0)} &= X \\ X^{(\alpha+1)} &= (X^{(\alpha)})' \\ X^{(\lambda)} &= \bigcap_{\alpha<\lambda} X^{(\alpha)}, \qquad \text{if } \lambda \text{ is limit.} \end{aligned}$$

Arguing by induction on α, one can check that the $X^{(\alpha)}$'s form a decreasing sequence of closed subsets of X, and that if X is second-countable and $\alpha < \omega_1$ then $X \setminus X^{(\alpha)}$ is a countable open set.

Lemma [Kec95, Theorem 6.9]

Let X be a second countable topological space and $(F_\alpha)_{\alpha<\rho}$ be a strictly decreasing transfinite sequence of closed sets (i.e. $F_\beta \subset F_\alpha$ for all $\alpha < \beta$). Then ρ is a countable ordinal.
This holds similarly for strictly increasing transfinite sequences of closed sets (and thus for strictly decreasing or incresing transfinite families of opens sets).

Proof.

Let $\mathcal{B} = \{U_n \mid n \in \mathbb{N}\}$ be a countable basis for X. For $F \subseteq X$ closed, let $N(F) = \{n \in \mathbb{N} \mid U_n \cap F \neq \emptyset\}$. Notice that if $F \subseteq G$ then $N(F) \subseteq N(G)$, and that $F \subset G$ implies $N(G) \setminus N(F) \neq \emptyset$. (Indeed, if $x \in G \setminus F$ then $x \in U_n \subseteq X \setminus F$ for some $n \in \mathbb{N}$, so that $n \in N(G) \setminus N(F)$.) For each $\alpha < \rho$, pick some $n_\alpha \in N(F_\alpha) \setminus N(F_{\alpha+1})$: since the map $\alpha \mapsto n_\alpha$ is an injection between ρ and $\mathbb{N}$, we must conclude that ρ is a countable ordinal. □

Thus if X is Polish there is $\alpha_0 < \omega_1$ such that $X^{(\alpha)} = X^{(\alpha_0)}$ for all $\alpha \geq \alpha_0$ (the countable ordinal α_0 is called the **Cantor-Bendixson rank** of X). Then $X^{(\alpha_0)}$, also denoted by X^∞, is the perfect kernel of X, and X is countable if and only if $X^\infty = \emptyset$.

Remark

Notice that the decomposition given by the proof of Theorem 1.4.7 and the one given by the iterated Cantor-Bendixson derivatives can give different decompositions if applied to non-Polish separable metrizable spaces: indeed, if $X = \mathbb{Q}$ then the former gives $P = \emptyset$ and $C = \mathbb{Q}$, while the latter gives $P = \mathbb{Q}$ and $C = \emptyset$.

The results above imply that there is no simple counterexample to the Continuum Hypothesis, but they actually show more.

Definition

A subset A of a topological space X has the **Perfect Set Property** (PSP for short) if either it is countable or there is an embedding of $2^{\mathbb{N}}$ into A.

Clearly, if X is separable and A has the PSP then A satisfies the Continuum Hypothesis: either A is countable, or has cardinality $2^{\aleph_0}$. However, the PSP is a stronger property: while it is independent of ZFC that all subsets of $\mathbb{R}$ are either countable or of size $2^{\aleph_0}$ (i.e. that the Continuum Hypothesis CH holds), it can be proved in ZFC that there is a set without the PSP.

Proposition 1.4.14 [Kec95, Example 8.24]

If X is an uncountable Polish space, then there is $A \subseteq X$ without the PSP.

Sets A as above are called **Bernstein sets**.

Proof of Proposition 1.4.14

There are exactly $2^{\aleph_0}$ perfect subsets of X.

Proof.

If $\mathcal{B} = \{U_n \mid n \in \mathbb{N}\}$ is a countable basis for X, then the map sending $x \in \mathbb{N}^{\mathbb{N}}$ to $X \setminus \bigcup_{n\in\mathbb{N}} U_{x(n)}$ is a surjection of $\mathbb{N}^{\mathbb{N}}$ onto the closed subsets of X, whence there are at most $2^{\aleph_0}$ perfect subset of X.
To show that there are at least $2^{\aleph_0}$ such sets, it is enough to consider the case $X = 2^{\mathbb{N}}$ (the general case easily follows from the fact that every uncountable Polish space X contains a closed set homeomorphic to $2^{\mathbb{N}}$ by Corollary 1.4.9). For each $x \in 2^{\mathbb{N}}$, let $P_x = \{x\} \times 2^{\mathbb{N}} \subseteq 2^{\mathbb{N}} \times 2^{\mathbb{N}}$. It is immediate to check that P_x is perfect in $2^{\mathbb{N}} \times 2^{\mathbb{N}}$ and that the map $x \mapsto P_x$ is injective. Since $2^{\mathbb{N}} \times 2^{\mathbb{N}}$ is homeomorphic to $2^{\mathbb{N}}$ we are done. □

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Proof of Proposition 1.4.14

Fix a transfinite enumeration $(P_\xi)_{\xi<2^{\aleph_0}}$ of the nonempty perfect subsets of X. Find by transfinite recursion on $\xi < 2^{\aleph_0}$ distinct point $a_\xi, b_\xi \in P_\xi$: this is possible because each P_ξ, being a perfect Polish space, has cardinality $2^{\aleph_0}$ by Corollary 1.4.4, while the collection of points $\{a_\nu, b_\nu \mid \nu < \xi\}$ constructed so far has cardinality $|\xi| < 2^{\aleph_0}$. Setting $A = \{a_\xi \mid \xi < 2^{\aleph_0}\}$ we obtain an uncountable set that does not contain any nonempty perfect subset P (and thus $2^{\mathbb{N}}$ cannot be embedded into A). Indeed, if P is perfect nonempty then $P = P_\xi$ for some $\xi < 2^{\aleph_0}$, whence $b_\xi \in P \setminus A$ by construction. □

Remark

The proof of Proposition 1.4.14 heavily uses the Axiom of Choice AC. In contrast, it can be shown via a forcing argument that it is consistent with ZF + DC (where DC is the Axiom of Dependent Choice) that all subsets of $\mathbb{R}$ have the PSP.

The existence of a Bernstein set under ZFC naturally leads to the problem of understanding how much complicated such a set must be: the last result of this section shows that simple sets are immune from this "pathological" behaviour.

Theorem 1.4.16

Every $\mathbf{F}_\sigma$ or $\mathbf{G}_\delta$ subset A of a Polish space X has the PSP, and thus satisfies the Continuum Hypothesis. The same is true for $\mathbf{G}_{\delta\sigma}$ subsets (i.e. countable unions of $\mathbf{G}_\delta$ sets) of X.

Proof.

If A is $\mathbf{G}_\delta$, then it is Polish by Proposition 1.1.8. If it is uncountable, then its perfect kernel P is nonempty, and thus $2^{\mathbb{N}}$ can be emebedded into $P \subseteq A$ by Theorem 1.4.3.

If now $A = \bigcup_{n\in\mathbb{N}} A_n$ with each A_n a $\mathbf{G}_\delta$ set,[a] we distinguish two cases:

- Every A_n is countable. Then A is countable as well and we are done.
- There is $n \in \mathbb{N}$ such that A_n is uncountable. Then $2^{\mathbb{N}}$ embeds into $A_n \subseteq A$ by the first part of this proof, and we are done again. □

[a] Observe that this includes the case of an $\mathbf{F}_\sigma$ set A because in a metrizable space all closed sets are $\mathbf{G}_\delta$ — see Proposition 3.6.2.