Exercise sheet 2

Descriptive Set Theory

Academic year 2023-24

1. Prove that the map
$$\omega^\omega \to 2^\omega, \qquad x \mapsto \underbrace{0\dots0}_{x(0)}1\underbrace{0\dots0}_{x(1)}1\underbrace{0\dots0}_{x(2)}1\dots$$
is a (topological) embedding, and argue that this provides an alternative proof of the fact that
$$\{x \in 2^\omega \mid x(n) = 1 \text{ for infinitely many } n \in \omega\}$$
is a dense Polish subspace of 2^ω. In contrast, show that 2^ω cannot be embedded as a dense subset in ω^ω.

   [*Hint.* Use compactness.]

2. A set $A \subseteq \omega^\omega$ is **bounded** if there is $z \in \omega^\omega$ such that for all $x \in A$ we have $x(n) \leq z(n)$ for all $n \in \omega$. Prove that the following conditions are equivalent for an arbitrary $F \subseteq \omega^\omega$:

   (a) F is compact;

   (b) F is closed and bounded;

   (c) $F = [T]$ with T a finitely branching tree (i.e. every node in T has only finitely many successors).

   Conclude that $A \subseteq \omega^\omega$ is contained in a compact set (equivalently, has compact closure) if and only A is bounded, and therefore ω^ω is not locally compact.

3. A subset of a topological space is **σ-compact** (or K_σ) if it can be written as a countable union of compact spaces. (For example, finite-dimensional Euclidean spaces $\mathbb{R}^n$ are σ-compact.) A set A is **eventually bounded** if there is $z \in \omega^\omega$ such that for all $x \in A$ there is $n \in \omega$ for which $x(m) \leq z(m)$ for all $m \geq n$. Prove that the following conditions are equivalent for an arbitrary $A \subseteq \omega^\omega$:

(a) A is contained in a σ-compact set;

(b) A is eventually bounded.

Conclude that $F \subseteq \omega^\omega$ is a σ-compact set if and only if it is F_σ and eventually bounded, and that ω^ω is not σ-compact. Provide an explicit example of a subset of the Baire space which is σ-compact but not compact. Argue[1] that ω^ω cannot be embedded as an F_σ (so neither closed) set into a σ-compact Polish space like $\mathbb{R}^n$.

4. Show that for every nonempty Polish space X there is a continuous open surjection $f\colon \omega^\omega \to X$.

   [*Hint.* First show that if X is a metric space, then for every open U and every $\varepsilon \in \mathbb{R}^+$ there is a countable covering $(U_n)_{n\in\omega}$ of U such that $\mathrm{cl}(U_n) \subseteq U$ and $\mathrm{diam}(U_n) < \varepsilon$, for all $n \in \omega$. Use this to build an appropriate ω-scheme inducing the function f.]

5. Recall the notion of Cantor-Bendixson rank of a Polish space from Section 1.4 in the notes for the course. For each ordinal $\alpha < \omega_1$, provide an example of a Polish space X with Cantor-Bendixson rank α. (*Optional:* show that such an X can always be taken as a countable space, and that if α is a successor ordinal than X can be taken to be compact.)

   [*Hint.* To geometrically visualize the problem it is easier to work in $\mathbb{R}^2$. Use a construction by transfinite recursion over α. The cases $\alpha = 0, 1$ are easy. For $\alpha = 2$ consider $X = \{x\} \cup \{x_n \mid n \in \omega\}$ with $x_n \to x$ and all x_n isolated. This suggest the strategy when $\alpha = \beta + 1$ is successor: consider a sequence of spaces of Cantor-Bendixson rank β and construe them as a sequence of spaces accumulating towards a point. For limit cases, consider the (disjoint) sum of spaces with Cantor-Bendixson rank cofinal in α.]

[1] This is half of Hurewicz's Theorem 1.3.19 on the notes. Modulo the homeomorphism between ω^ω and Irr, it also provides an alternative proof of the fact that the space Irr of irrational numbers is G_δ but not F_σ as a subset of $\mathbb{R}$.