Borel sets and the Borel hierarchy

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Definition 2.1.1

A subset A of a topological space $X = (X, \tau)$ is called **Borel** if it belongs to the σ-algebra on X generated by τ, i.e. if it is in the smallest collection of subsets of X containing all open sets and closed under complements and countable unions. The collection of all Borel subsets of X is denoted by $\mathbf{Bor}(X)$, possibly suppressing X from the notation if the space is clear from the context. We instead write $\mathbf{Bor}(X, \tau)$ or simply $\mathbf{Bor}(\tau)$ when we want to make explicit the topology τ we started with.

Fact 2.1.2—see [Kec95, Theorem 10.1]

$\mathbf{Bor}(X)$ is the smallest collection of subsets of X containing all open and closed sets, and closed under countable unions and countable intersections.

Proof.

Let $\mathcal{E}$ be the smallest collection of subsets of X containing all open and closed sets, and closed under countable unions and countable intersections. Notice that $\mathbf{Bor}(X)$ is also closed under countable intersections, as $\bigcap_{n\in\omega} A_n = X \setminus \left(\bigcup_{n\in\omega}(X \setminus A_n)\right)$. Since $\mathbf{Bor}(X)$ contains all open and closed sets and is closed under countable unions by definition, we obtain $\mathcal{E} \subseteq \mathbf{Bor}(X)$.

Conversely, let $\mathcal{E}' = \{A \in \mathcal{E} \mid X \setminus A \in \mathcal{E}\}$: we claim that $\mathbf{Bor}(X) \subseteq \mathcal{E}' \subseteq \mathcal{E}$, whence $\mathbf{Bor}(X) = \mathcal{E}$. Indeed, $\mathcal{E}'$ contains all open sets, so it is enough to show that it is a σ-algebra because by definition $\mathbf{Bor}(X)$ is the *smallest* σ-algebra containing the open sets. Closure of $\mathcal{E}'$ under complements directly follows from its definition. To see that it is also closed under countable unions, let $A_n \in \mathcal{E}'$. On the one hand $\bigcup_{n\in\omega} A_n \in \mathcal{E}$ because $\mathcal{E}' \subseteq \mathcal{E}$ and the latter is closed under countable unions. On the other hand $X \setminus \bigcup_{n\in\omega} A_n = \bigcap_{n\in\omega}(X \setminus A_n) \in \mathcal{E}$ because $X \setminus A_n \in \mathcal{E}$ (by definition of $\mathcal{E}'$) and the latter is closed under countable intersections. □

Definition

Let $X = (X, \tau)$ be a topological space.

$$\mathbf{\Sigma}^0_1(X) = \{U \subseteq X \mid U \text{ open}\} \qquad \mathbf{\Sigma}^0_\alpha(X) = \{\bigcup_{n\in\omega} A_n \mid A_n \in \bigcup_{1\leq\beta<\alpha} \mathbf{\Pi}^0_\beta\}$$

$$\mathbf{\Pi}^0_1(X) = \{C \subseteq X \mid C \text{ closed}\} \qquad \mathbf{\Pi}^0_\alpha(X) = \{\bigcap_{n\in\omega} A_n \mid A_n \in \bigcup_{1\leq\beta<\alpha} \mathbf{\Sigma}^0_\beta\}$$

We also set $\mathbf{\Delta}^0_\alpha(X) = \mathbf{\Sigma}^0_\alpha(X) \cap \mathbf{\Pi}^0_\alpha(X)$ for every $\alpha \geq 1$.

$\mathbf{\Sigma}^0_1(X) = \tau$ consists of all open sets, $\mathbf{\Pi}^0_1(X)$ consists of all closed sets, $\mathbf{\Delta}^0_1(X)$ consists of all clopen sets, $\mathbf{\Sigma}^0_2(X)$ is the collection of all $\mathbf{F}_\sigma$ sets, $\mathbf{\Pi}^0_2(X)$ is the collection of all $\mathbf{G}_\delta$ sets, $\mathbf{\Sigma}^0_3(X)$ is the collection of all $\mathbf{G}_{\delta\sigma}$ sets, and so on. Pointclasses of the form $\mathbf{\Sigma}^0_\alpha(X)$ (respectively: $\mathbf{\Pi}^0_\alpha(X)$, $\mathbf{\Delta}^0_\alpha(X)$) are called **additive** (respectively: **multiplicative**, **ambiguous**) **classes**.

Lemma 2.1.5

Let X be a metrizable space and $\alpha \geq 1$.

1. $\mathbf{\Pi}^0_\alpha(X) = \{X \setminus A \mid A \in \mathbf{\Sigma}^0_\alpha(X)\}$.
2. $\mathbf{\Sigma}^0_\alpha(X), \mathbf{\Pi}^0_\alpha(X) \subseteq \mathbf{\Delta}^0_{\alpha+1}(X) \subseteq \mathbf{\Sigma}^0_{\alpha+1}(X), \mathbf{\Pi}^0_{\alpha+1}(X)$.
3. $\mathbf{Bor}(X) = \bigcup_{1 \leq \alpha < \omega_1} \mathbf{\Sigma}^0_\alpha(X) = \bigcup_{1 \leq \alpha < \omega_1} \mathbf{\Pi}^0_\alpha(X) = \bigcup_{1 \leq \alpha < \omega_1} \mathbf{\Delta}^0_\alpha(X)$.
4. $\mathbf{\Sigma}^0_\alpha$ is closed under continuous preimages (i.e. it is a **boldface** pointclass): if Y is metrizable, $A \in \mathbf{\Sigma}^0_\alpha(Y)$, and $f \colon X \to Y$ is continuous, then $f^{-1}(A) \in \mathbf{\Sigma}^0_\alpha(X)$. Similarly for $\mathbf{\Pi}^0_\alpha$, $\mathbf{\Delta}^0_\alpha$, and $\mathbf{Bor}$.
5. $\mathbf{\Sigma}^0_\alpha(X)$ is closed under countable unions and finite intersections, $\mathbf{\Pi}^0_\alpha(X)$ is closed under countable intersections and finite unions, and $\mathbf{\Delta}^0_\alpha(X)$ is a Boolean algebra.
6. If $Y \subseteq X$, then $\mathbf{\Sigma}^0_\alpha(Y) = \mathbf{\Sigma}^0_\alpha(X) \restriction Y := \{A \cap Y \mid A \in \mathbf{\Sigma}^0_\alpha(X)\}$. Similarly for $\mathbf{\Pi}^0_\alpha$ and $\mathbf{Bor}$.

Proof.

All proofs are by induction on $\alpha \geq 1$ (Exercise!). For the first equality of part 3 observe that $\mathbf{\Sigma}^0_\alpha(X) \subseteq \mathbf{Bor}(X)$ (this can be proved again by induction on $\alpha \geq 1$) and that $\mathbf{Bor}(X) \subseteq \bigcup_{1 \leq \alpha < \omega_1} \mathbf{\Sigma}^0_\alpha(X)$ because the latter is a σ-algebra containing all open sets of X. □

$\mathbf{\Sigma}^0_\alpha(X)$, $\mathbf{\Pi}^0_\alpha(X)$, $\mathbf{\Delta}^0_\alpha(X)$ form the **Borel hierarchy** measuring the complexity of the Borel subsets of X. The **Borel rank** of $A \in \mathbf{Bor}(X)$ is the smallest $1 \leq \alpha < \omega_1$ such that $A \in \mathbf{\Sigma}^0_\alpha(X) \cup \mathbf{\Pi}^0_\alpha(X)$. The $\subseteq$-smallest of the $\mathbf{\Sigma}^0_\alpha(X)$, $\mathbf{\Pi}^0_\alpha(X)$, and $\mathbf{\Delta}^0_\alpha(X)$ to which $A \subseteq X$ belongs is the **Borel class** of A.

$$
\begin{tikzcd}[column sep=small]
 & \mathbf{\Sigma}^0_1 \arrow[dr, phantom, sloped, "\subseteq"] & & \mathbf{\Sigma}^0_2 & & \mathbf{\Sigma}^0_\alpha \arrow[dr, phantom, sloped, "\subseteq"] & & \mathbf{\Sigma}^0_{\alpha+1} & \\
\mathbf{\Delta}^0_1 \arrow[ur, phantom, sloped, "\subseteq"] \arrow[dr, phantom, sloped, "\subseteq"] & & \mathbf{\Delta}^0_2 \arrow[ur, phantom, sloped, "\subseteq"] \arrow[dr, phantom, sloped, "\subseteq"] & & \cdots\ \mathbf{\Delta}^0_\alpha \arrow[ur, phantom, sloped, "\subseteq"] \arrow[dr, phantom, sloped, "\subseteq"] & & \mathbf{\Delta}^0_{\alpha+1} \arrow[ur, phantom, sloped, "\subseteq"] \arrow[dr, phantom, sloped, "\subseteq"] & & \cdots \\
 & \mathbf{\Pi}^0_1 \arrow[ur, phantom, sloped, "\subseteq"] & & \mathbf{\Pi}^0_2 & & \mathbf{\Pi}^0_\alpha \arrow[ur, phantom, sloped, "\subseteq"] & & \mathbf{\Pi}^0_{\alpha+1} &
\end{tikzcd}
$$

Example 2.1.7

1. Every countable subset A of a Polish space X is $\mathbf{\Sigma}^0_2(X)$, and its complement is $\mathbf{\Pi}^0_2(X)$.
2. Semi-open intervals $[a;b), (a;b] \subseteq \mathbb{R}$ are $\mathbf{\Delta}^0_2(\mathbb{R})$, but they are neither $\mathbf{\Sigma}^0_1(\mathbb{R})$ nor $\mathbf{\Pi}^0_1(\mathbb{R})$ (Exercise!).
3. Let $C_0 = c_0 \cap [0;1]^\omega = \{(x_n)_{n\in\omega} \in [0;1]^\omega \mid x_n \to 0\}$. Then $C_0 \in \mathbf{\Pi}^0_3([0;1]^\omega)$. In fact,

$$\begin{aligned}(x_n)_{n\in\omega} \in C_0 &\Leftrightarrow \forall\varepsilon \in \mathbb{R}^+ \, \exists n \in \omega \, \forall m \geq n \, (x_m \leq \varepsilon)\\ &\Leftrightarrow \forall\varepsilon \in \mathbb{Q}^+ \, \exists n \in \omega \, \forall m \geq n \, (x_m \leq \varepsilon),\end{aligned}$$

therefore $C_0 = \bigcap_{\varepsilon\in\mathbb{Q}^+} \bigcup_{n\in\omega} \bigcap_{m\geq n} A_{\varepsilon,m}$, where $A_{\varepsilon,m} = \{(x_n)_{n\in\omega} \mid x_m \leq \varepsilon\}$ is $\mathbf{\Pi}^0_1([0;1]^\omega)$, whence $C_0 \in \mathbf{\Pi}^0_3([0;1]^\omega)$ by Lemma 2.1.5.

Example 2.1.7

4. Given $f \in C([0;1])$, let $D_f = \{x \in [0;1] \mid f' \text{ exists }\}$ (at endpoints we take one-side derivatives). Then $D_f \in \mathbf{\Pi}^0_3([0;1])$ (Exercise!). *Hint*. Observe that $x \in D_f$ if and only if for all $\varepsilon \in \mathbb{Q}^+$ there is $\delta \in \mathbb{Q}^+$ such that for all $p, q \in [0;1] \cap \mathbb{Q}$

$$|p-x|, |q-x| < \delta \Rightarrow \left| \frac{f(p)-f(x)}{p-x} - \frac{f(q)-f(x)}{q-x} \right| \leq \varepsilon.$$

Notice that we could replace $\leq$ with $<$ at the end of the previous formula: however, we would then obtain only $D_f \in \mathbf{\Pi}^0_4([0;1])$ rather than $D_f \in \mathbf{\Pi}^0_3([0;1])$.

Remark 2.1.8

The **Tarski-Kuratowski algorithm** allows us to compute the Borel class of a set by looking at the logical form of one of its (optimal) definitions. In particular, we take advantage of the following well-known correspondence between logical symbols and set-theoretical operations:

- $\neg$ (negation) $\rightsquigarrow$ complementation $X \setminus \cdot$
- $\wedge$ (conjunction) $\rightsquigarrow$ intersection $\cap$
- $\vee$ (disjunction) $\rightsquigarrow$ union $\cup$
- $\forall n$ (universal quantification over a *countable* set) $\rightsquigarrow$ $\bigcap_n$
- $\exists n$ (existential quantification over a *countable* set) $\rightsquigarrow$ $\bigcup_n$

The implication $\Rightarrow$ and bi-implication $\Leftrightarrow$ are treated exploiting the fact that $\varphi \Rightarrow \psi$ is equivalent to $\neg\varphi \vee \psi$ and $\varphi \Leftrightarrow \psi$ is equivalent to $(\varphi \Rightarrow \psi) \wedge (\psi \Rightarrow \varphi)$.

For example, consider the set $A \subseteq 2^\omega$ defined by

$$x \in A \Leftrightarrow \varphi(x)$$

where $\varphi(x)$ is the formula $\exists n\, \forall m \geq n\, \neg(x(m) = 1)$ (i.e. A is the set of binary sequences which are eventually equal to 0). Applying the Tarski-Kuratowski algorithm to $\varphi(x)$ yields the fact that

$$A = \bigcup_{n\in\omega} \bigcap_{m \geq n} (2^\omega \setminus B_m)$$

where $B_m = \{x \in 2^\omega \mid x(m) = 1\}$. Since the latter is a clopen set (for each $m \in \omega$), we get that A is a $\mathbf{\Sigma}^0_2$ set. Similarly, one can check that

$$B = \{x \in [0;1]^\omega \mid x \text{ is eventually constant}\}$$

is a $\mathbf{\Sigma}^0_2$ subset of $[0;1]^\omega$.

When X is countable, its Borel hierarchy collapses to level 2 because all of its subsets are $\boldsymbol{\Sigma}^0_2$ by Example 2.1.71.

Fact 2.1.9

Let X be a metrizable space. If X is countable, then $\mathbf{Bor}(X) = \boldsymbol{\Delta}^0_2(X) = \mathscr{P}(X)$.

A **boldface pointclass** $\boldsymbol{\Gamma}$ is an operator assigning to each topological space X a collection $\boldsymbol{\Gamma}(X) \subseteq \mathscr{P}(X)$ which is closed under continuous preimages in the following strong sense: if $A \in \boldsymbol{\Gamma}(Y)$ and $f \colon X \to Y$ is continuous, then $f^{-1}(A) \in \boldsymbol{\Gamma}(X)$. We denote by $\check{\boldsymbol{\Gamma}}$ the **dual** of $\boldsymbol{\Gamma}$, where $\check{\boldsymbol{\Gamma}}(X) = \{X \setminus A \mid A \in \boldsymbol{\Gamma}(X)\}$, and by $\boldsymbol{\Delta}_{\boldsymbol{\Gamma}}$, where $\boldsymbol{\Delta}_{\boldsymbol{\Gamma}}(X) = \boldsymbol{\Gamma}(X) \cap \check{\boldsymbol{\Gamma}}(X)$, the **ambiguous class** associated to $\boldsymbol{\Gamma}$. (Notice that the dual of $\check{\boldsymbol{\Gamma}}$ is $\boldsymbol{\Gamma}$ itself, and that $\boldsymbol{\Delta}_{\boldsymbol{\Gamma}} = \boldsymbol{\Delta}_{\check{\boldsymbol{\Gamma}}}$.) When $\boldsymbol{\Gamma}(\omega^\omega) = \check{\boldsymbol{\Gamma}}(\omega^\omega)$ (so that $\boldsymbol{\Gamma}(\omega^\omega) = \boldsymbol{\Delta}_{\boldsymbol{\Gamma}}(\omega^\omega)$) we say that $\boldsymbol{\Gamma}$ is **selfdual**, otherwise we say that $\boldsymbol{\Gamma}$ is **nonselfdual**.

Example

By Lemma 2.1.5, all of $\boldsymbol{\Sigma}^0_\alpha$, $\boldsymbol{\Pi}^0_\alpha$, $\boldsymbol{\Delta}^0_\alpha$, and $\mathbf{Bor}$ are boldface pointclasses. The dual of $\boldsymbol{\Sigma}^0_\alpha$ is $\boldsymbol{\Pi}^0_\alpha$, and viceversa. The pointclasses $\boldsymbol{\Delta}^0_\alpha$ and $\mathbf{Bor}$ are selfdual.

Definition 2.1.11

A set $\mathcal{U} \subseteq Y \times X$ is Y**-universal for** $\boldsymbol{\Gamma}(X)$ if $\mathcal{U} \in \boldsymbol{\Gamma}(Y \times X)$ and $\boldsymbol{\Gamma}(X) = \{\mathcal{U}_y \mid y \in Y\}$, where $\mathcal{U}_y = \{x \in X \mid (y, x) \in \mathcal{U}\}$ is the **vertical section** of $\mathcal{U}$ at y.

Such a universal set $\mathcal{U}$ provides a **parametrization** (or **coding**) of the sets in $\boldsymbol{\Gamma}(X)$, where we view y as a parameter (or code) of $\mathcal{U}_y$. (Notice however that the code of a set in $\boldsymbol{\Gamma}(X)$ is in general not unique.)

Remark 2.1.12

If $\mathcal{U}$ is Y-universal for $\boldsymbol{\Gamma}(X)$, then its complement $\mathcal{U}^\complement = (Y \times X) \setminus \mathcal{U}$ is Y universal for $\check{\boldsymbol{\Gamma}}(X)$.

Remark 2.1.13

To show that $\mathcal{U} \subseteq Y \times X$ is Y-universal for $\mathbf{\Gamma}(X)$ it is enough to show that $\mathcal{U} \in \mathbf{\Gamma}(Y \times X)$ and that for every $A \in \mathbf{\Gamma}(X)$, $A = \mathcal{U}_y$ for some $y \in Y$. Indeed, the remaining condition "$\mathcal{U}_y \in \mathbf{\Gamma}(X)$ for all $y \in Y$" already follows from the fact that each $\mathcal{U}_y$ is the continuous preimage of $\mathcal{U}$ via the map $x \mapsto (y, x)$.

Theorem 2.1.14 [Kec95, Theorem 22.3]

Let X be a separable metrizable space. Then for each $1 \leq \alpha < \omega_1$ there is a 2^ω-universal set for $\mathbf{\Sigma}^0_\alpha(X)$, and similarly for $\mathbf{\Pi}^0_\alpha(X)$. Moreover, we can replace 2^ω with any uncountable Polish space.

Proof

By (simultaneous) induction on $\alpha \geq 1$.
$\{V_n \mid n \in \omega\}$ is a countable basis for X, and let $\mathcal{U} \subseteq 2^\omega \times X$

$$(y, x) \in \mathcal{U} \Leftrightarrow x \in \bigcup\{V_n \mid y(n) = 1\}.$$

- $\mathcal{U} \in \mathbf{\Sigma}^0_1(2^\omega \times X)$. In fact if $(y, x) \in \mathcal{U}$ and $n \in \omega$ is such that $x \in V_n$ and $y(n) = 1$, then $(y, x) \in \boldsymbol{N}_{y\restriction(n+1)} \times V_n \subseteq \mathcal{U}$.
- Given $U \subseteq X$ open, let $y \in 2^\omega$ be such that $y(n) = 1 \Leftrightarrow V_n \subseteq U$, so that $U = \bigcup\{V_n \mid y(n) = 1\}$; then y is a code for U, i.e. $\mathcal{U}_y = U$.

By Remark 2.1.13, this shows that $\mathcal{U}$ is 2^ω-universal for $\mathbf{\Sigma}^0_1(X)$, whence $\mathcal{U}^\complement$ is 2^ω-universal for $\mathbf{\Pi}^0_1(X)$ by Remark 2.1.12.

(continues)

Proof (continued)

Let $\alpha > 1$, and fix $(\eta_n)_{n\in\omega}$ such that $1 \leq \eta_n < \alpha$, $\eta_n \leq \eta_{n+1}$, and $\sup\{\eta_n + 1 \mid n \in \omega\} = \alpha$. When $\alpha = \beta + 1$ set $\eta_n = \beta$.
Let $\mathcal{U}_n$ be 2^ω-universal for $\mathbf{\Pi}^0_{\eta_n}(X)$ and let $\rho\colon 2^\omega \to (2^\omega)^\omega$ be a homeomorphism, and let $\pi_k\colon (2^\omega)^\omega \to 2^\omega$ be the projection on the k-th coordinate, so that $\pi_k((y_n)_{n\in\omega}) = y_k$. Each $f_k = \pi_k \circ \rho$ is continuous, and if $y = \rho^{-1}((y_k)_{k\in\omega})$ then $f_k(y) = y_k$. Define $\mathcal{U} \subseteq 2^\omega \times X$ by setting

$$(y, x) \in \mathcal{U} \Leftrightarrow \exists k \in \omega\, [(f_k(y), x) \in \mathcal{U}_k].$$

$\mathcal{U}$ is 2^ω-universal for $\mathbf{\Sigma}^0_\alpha(X)$ and hence $\mathcal{U}^\complement$ is 2^ω-universal for $\mathbf{\Pi}^0_\alpha(X)$.

$A_k = \{(y, x) \in 2^\omega \times X \mid (f_k(y), x) \in \mathcal{U}_k\}$ is in $\mathbf{\Pi}^0_{\eta_k}(2^\omega \times X)$ as it is the preimage of $\mathcal{U}_k$ under $(y, x) \mapsto (f_k(y), x)$. So $\mathcal{U}$ is in $\mathbf{\Sigma}^0_\alpha(2^\omega \times X)$.

(continues)

Proof (continued)

$\rho\colon 2^\omega \to (2^\omega)^\omega$ is a homeomorphism.
$\mathcal{U} \subseteq 2^\omega \times X$ is defined by $(y, x) \in \mathcal{U} \Leftrightarrow \exists k \in \omega\, [(f_k(y), x) \in \mathcal{U}_k]$, and it is in $\mathbf{\Sigma}^0_\alpha(2^\omega \times X)$.
Let $A = \bigcup_{n\in\omega} A_n$ with each $A_n \in \bigcup_{1\leq\beta<\alpha} \mathbf{\Pi}^0_\beta(X)$. W.L.O.G $A_n \in \mathbf{\Pi}^0_{\eta_n}$.
For every $k \in \omega$, let $y_k \in 2^\omega$ be such that $(\mathcal{U}_k)_{y_k} = A_k$. Finally, let $y = \rho^{-1}((y_k)_{k\in\omega})$. Then $\mathcal{U}_y = A$. Indeed,

$$\begin{aligned} x \in A &\Leftrightarrow \exists k \in \omega\, [x \in A_k] \\ &\Leftrightarrow \exists k \in \omega\, [(y_k, x) \in \mathcal{U}_k] \\ &\Leftrightarrow \exists k \in \omega\, [(f_k(y), x) \in \mathcal{U}_k] \\ &\Leftrightarrow (y, x) \in \mathcal{U}. \end{aligned}$$

Therefore $\mathcal{U}$ is 2^ω-universal for $\mathbf{\Sigma}^0_\alpha(X)$.
Next we want to replace 2^ω with an uncountable Polish space Y.

(continues)

Proof (continued)

If Y is an uncountable Polish space, let $f\colon 2^\omega \to Y$ be an embedding. It is enough to show that for every $\alpha \geq 1$ there is a Y-universal set $\mathcal{U}$ for $\mathbf{\Pi}^0_\alpha(X)$. Let $\mathcal{U}'$ be 2^ω-universal for $\mathbf{\Pi}^0_\alpha(X)$, and let $\mathcal{U} \subseteq Y \times X$ be defined by

$$(y, x) \in \mathcal{U} \Leftrightarrow y \in \operatorname{rng}(f) \wedge (f^{-1}(y), x) \in \mathcal{U}'.$$

Then

$$\begin{aligned}\mathbf{\Pi}^0_\alpha(X) = \{\mathcal{U}'_z \mid z \in 2^\omega\} &= \{\mathcal{U}_{f(z)} \mid z \in 2^\omega\} \\ &= \{\mathcal{U}_y \mid y \in \operatorname{rng}(f)\} = \{\mathcal{U}_y \mid y \in Y\},\end{aligned}$$

where the last equality holds as $\mathcal{U}_y = \emptyset$ when $y \notin \operatorname{rng}(f)$, and $\emptyset \in \mathbf{\Pi}^0_\alpha(X)$. Moreover, $\mathcal{U}$ is the image of $\mathcal{U}'$ under the homeomorphism between $2^\omega \times X$ and $\operatorname{rng}(f) \times X$ given by $(y, x) \mapsto (f(y), x)$, whence it is in $\mathbf{\Pi}^0_\alpha(\operatorname{rng}(f) \times X)$. Since $\operatorname{rng}(f) \times X$ is closed in $Y \times X$, it follows that $\mathcal{U}$ is in $\mathbf{\Pi}^0_\alpha(Y \times X)$ as well, hence it is as required. □

Lemma 2.1.15 [Kec95, Exercise 22.7]

X metrizable, $\mathbf{\Gamma}$ a boldface pointclass such that $\mathbf{\Gamma}(X) = \check{\mathbf{\Gamma}}(X)$ (equivalently: $\mathbf{\Gamma}(X)$ is closed under complements). Then there is no X-universal set for $\mathbf{\Gamma}(X)$.

Proof.

Suppose $\mathcal{U} \subseteq X \times X$ is X-universal for $\mathbf{\Gamma}(X)$. Let $f\colon X \to X \times X$, $x \mapsto (x, x)$, and set

$$D = \{x \in X \mid f(x) \notin \mathcal{U}\} = f^{-1}((X \times X) \setminus \mathcal{U}) = X \setminus f^{-1}(\mathcal{U}).$$

As $D \in \mathbf{\Gamma}(X)$, let y_0 be such that $D = \mathcal{U}_{y_0}$. Then we reach a contradiction: $(y_0, y_0) \in \mathcal{U} \Leftrightarrow y_0 \in D \Leftrightarrow (y_0, y_0) \notin \mathcal{U}$. □

Corollary 2.1.16

Let X be a separable metrizable space. Then there is no X-universal set for $\mathbf{\Delta}^0_\alpha(X)$, $1 \leq \alpha < \omega_1$, and for $\mathbf{Bor}(X)$. If moreover X is Polish and uncountable, then there is no 2^ω-universal set (and, more generally, no Y universal set with Y Polish uncountable) for $\mathbf{\Delta}^0_\alpha(X)$, $\alpha \geq 2$, and for $\mathbf{Bor}(X)$.

Proof.

By Theorem 2.1.14 and Lemma 2.1.15. The additional part concerning uncountable Polish spaces follows from the argument used at the end of the proof of Theorem 2.1.14. □

Theorem 2.1.17 [Kec95, Theorem 22.4]

Let X be an uncountable Polish space. Then $\mathbf{\Sigma}^0_\alpha(X) \neq \mathbf{\Pi}^0_\alpha(X)$ for each $1 \leq \alpha < \omega_1$.

Proof.

Assume towards a contradiction that for some α as in the statement it holds $\mathbf{\Sigma}^0_\alpha(X) = \mathbf{\Pi}^0_\alpha(X)$. Then $\mathbf{\Sigma}^0_\alpha(X)$ would be closed under complements: since there is an X-universal set for $\mathbf{\Sigma}^0_\alpha(X)$ by Theorem 2.1.14, this contradicts Lemma 2.1.15. □

Corollary [Kec95, Theorem 22.4, Exercise 22.5, and Exercise 22.8]

X uncountable Polish and $1 \leq \alpha < \omega_1$.

1. $\mathbf{\Delta}^0_\alpha(X) \subset \mathbf{\Sigma}^0_\alpha(X) \subset \mathbf{\Delta}^0_{\alpha+1}(X)$, and similarly for $\mathbf{\Pi}^0_\alpha(X)$.
2. If α is limit, then
$$\bigcup_{1\leq\beta<\alpha} \mathbf{\Sigma}^0_\beta(X) = \bigcup_{1\leq\beta<\alpha} \mathbf{\Pi}^0_\beta(X) = \bigcup_{1\leq\beta<\alpha} \mathbf{\Delta}^0_\beta(X) \subset \mathbf{\Delta}^0_\alpha(X).$$
3. $\mathbf{\Sigma}^0_\alpha(X)$ is not closed under either complements or countable intersections.
$\mathbf{\Pi}^0_\alpha(X)$ is not closed under either complements or countable unions.
For $\alpha \geq 2$ or $\alpha = 1$ and X zero-dimensional, $\mathbf{\Delta}^0_\alpha(X)$ is not closed under either countable unions or intersections.

$\mathbf{\Delta}^0_\alpha(X) \subset \mathbf{\Sigma}^0_\alpha(X) \subset \mathbf{\Delta}^0_{\alpha+1}(X)$, and similarly for $\mathbf{\Pi}^0_\alpha(X)$

If one of the inclusion were not strict, then $\mathbf{\Sigma}^0_\alpha(X)$ would be closed under complements, whence $\mathbf{\Sigma}^0_\alpha(X) = \mathbf{\Pi}^0_\alpha(X)$, contradicting Theorem 2.1.17.

If α is limit $\bigcup_{1\leq\beta<\alpha} \mathbf{\Sigma}^0_\beta(X) \subset \mathbf{\Delta}^0_\alpha(X)$.

Let $(C_n)_{n\in\omega}$ be pairwise disjoint uncountable closed subsets of X. (E.g. $C_n = f(\mathbf{N}_{0^{(n)}{}^\frown 1})$, where f is an embedding of 2^ω into X.) Fix $(\alpha_n)_{n\in\omega}$ cofinal in α, and pick $A_n \in \mathbf{\Pi}^0_{\alpha_n}(C_n) \setminus \mathbf{\Sigma}^0_{\alpha_n}(C_n)$. Let $A = \bigcup_{n\in\omega} A_n$. Then A is in $\mathbf{\Sigma}^0_\alpha(X)$, and the same is true for
$$X \setminus A = \bigcup_{n\in\omega}(C_n \setminus A_n) \cup \Big(X \setminus \bigcup_{n\in\omega} C_n\Big)$$
as $\alpha > 2$, therefore $A \in \mathbf{\Delta}^0_\alpha(X)$. If A where in $\mathbf{\Pi}^0_\beta(X)$ for some $1 \leq \beta < \alpha$, then $A_n = A \cap C_n \in \mathbf{\Pi}^0_\beta(X) \restriction C_n = \mathbf{\Pi}^0_\beta(C_n)$ for every $n \in \omega$. But when $n \in \omega$ is such that $\beta < \alpha_n$, this contradicts the choice of A_n.

$\mathbf{\Sigma}^0_\alpha(X)$ is not closed under either complements or countable intersections.

If $\mathbf{\Sigma}^0_\alpha(X)$ were closed under complements, then $\mathbf{\Sigma}^0_\alpha(X) = \mathbf{\Pi}^0_\alpha(X)$, contradicting Theorem 2.1.17. If $\mathbf{\Sigma}^0_\alpha(X)$ were closed under countable intersections, then $\mathbf{\Pi}^0_{\alpha+1}(X) = \mathbf{\Sigma}^0_\alpha(X)$. But then $\mathbf{\Pi}^0_\alpha(X) \subseteq \mathbf{\Pi}^0_{\alpha+1}(X) = \mathbf{\Sigma}^0_\alpha(X)$, whence $\mathbf{\Sigma}^0_{\alpha+1}(X) = \mathbf{\Sigma}^0_\alpha(X) = \mathbf{\Pi}^0_{\alpha+1}(X)$, contradicting Theorem 2.1.17 again.

$\mathbf{\Pi}^0_\alpha(X)$ is not closed under either complements or countable unions.

Follows from the case of $\mathbf{\Sigma}^0_\alpha(X)$.

For $\alpha \geq 2$ or $\alpha = 1$ and X zero-dimensional, $\mathbf{\Delta}^0_\alpha(X)$ is not closed under either countable unions or intersections.

If $\mathbf{\Delta}^0_\alpha(X)$ were closed under countable unions, then $\mathbf{\Sigma}^0_\alpha(X) = \mathbf{\Delta}^0_\alpha(X)$, whence $\mathbf{\Pi}^0_\alpha(X) = \mathbf{\Delta}^0_\alpha(X)$ against Theorem 2.1.17. The case of countable intersections is similar but with the role of $\mathbf{\Sigma}^0_\alpha(X)$ and $\mathbf{\Pi}^0_\alpha(X)$ interchanged.

Proposition 2.1.19

Let X be an infinite Polish space. For every $1 \leq \alpha < \omega_1$

$$|\mathbf{\Sigma}^0_\alpha(X)| = |\mathbf{\Pi}^0_\alpha(X)| = |\mathbf{Bor}(X)| = 2^{\aleph_0}.$$

Thus if X is uncountable then there is a non-Borel subset of X. Moreover, $2 \leq |\mathbf{\Delta}^0_\alpha(X)| \leq 2^{\aleph_0}$ for all $1 \leq \alpha < \omega_1$, and $|\mathbf{\Delta}^0_\alpha(X)| = 2^{\aleph_0}$ if $\alpha \geq 2$.

Proof

$|\mathbf{\Sigma}^0_\alpha(X)| = |\mathbf{\Pi}^0_\alpha(X)|$ is witnessed by $A \mapsto X \setminus A$, so we only consider $\mathbf{\Sigma}^0_\alpha(X)$ and $\mathbf{Bor}(X)$.
For a lower bound enough to show $|\mathbf{\Sigma}^0_1(X)| \geq 2^{\aleph_0}$.

- If X is uncountable, then $|X| = 2^{\aleph_0}$ hence the sets $X \setminus \{x\}$ form a family of size $2^{\aleph_0}$ of distinct open sets.
- If X is countable, then $Y = X \setminus X'$ is an infinite open set carrying the discrete topology, so $\mathscr{P}(Y)$ is a family of size $2^{\aleph_0}$ of distinct open sets.

(continues)

Proof (cotninued).

For the upper bound, we first consider $\mathbf{\Sigma}^0_\alpha(X)$. Let $\mathcal{U}$ be 2^ω-universal for $\mathbf{\Sigma}^0_\alpha(X)$: then $y \mapsto \mathcal{U}_y$ is a surjection of 2^ω onto $\mathbf{\Sigma}^0_\alpha(X)$, whence $|\mathbf{\Sigma}^0_\alpha(X)| \leq 2^{\aleph_0}$. As for $\mathbf{Bor}(X)$, by Lemma 2.1.5 again we have

$$|\mathbf{Bor}(X)| = \left|\bigcup_{1\leq\alpha<\omega_1} \mathbf{\Sigma}^0_\alpha(X)\right| \leq 2^{\aleph_0} \cdot \aleph_1 = 2^{\aleph_0}.$$

The existence of a non-Borel set when X is uncountable follows from a cardinality argument, as

$$|\mathbf{Bor}(X)| = 2^{\aleph_0} < 2^{(2^{\aleph_0})} = |\mathscr{P}(X)|.$$

Finally, the cardinality (in)equalities concerning the classes $\mathbf{\Delta}^0_\alpha(X)$ follow from the fact that $\{\emptyset, X\} \subseteq \mathbf{\Delta}^0_1(X) \subseteq \mathbf{Bor}(X)$, and that $\mathbf{\Sigma}^0_1(X) \subseteq \mathbf{\Delta}^0_\alpha(X)$ when $\alpha \geq 2$. □

Corollary

If τ is a Polish topology on a set X, then either $|\tau| = 2^n$ for some $n \in \omega$, or else $|\tau| = 2^{\aleph_0}$. In particular, there is no countably infinite Polish topology. The same applies to $\mathbf{\Sigma}^0_\alpha(X,\tau)$, $\mathbf{\Pi}^0_\alpha(X,\tau)$, $\mathbf{\Delta}^0_\alpha(X,\tau)$, and $\mathbf{Bor}(X,\tau)$.

Proof.

If X is finite of cardinality $n \in \omega$, then τ is discrete because so is any Hausdorff topology on a finite space. It follows that $\tau = \mathscr{P}(X)$, whence $|\tau| = 2^n$. If instead X is infinite, then $|\tau| = 2^{\aleph_0}$ by Proposition 2.1.19. □

Definition [Kec95, Definition 22.9]

Let $\mathbf{\Gamma}$ be a boldface pointclass, and X be a Polish space. A set $A \subseteq X$ is called **$\mathbf{\Gamma}$-hard** if for all $B \in \mathbf{\Gamma}(\omega^\omega)$ there is a continuous $f\colon \omega^\omega \to X$ such that $f^{-1}(A) = B$. The set A is **$\mathbf{\Gamma}$-complete** if it is $\mathbf{\Gamma}$-hard and $A \in \mathbf{\Gamma}(X)$.

Lemma 2.1.23

Let $\boldsymbol{\Gamma}$ be a boldface pointclass, and X be any Polish space.

1. If $\boldsymbol{\Gamma}$ is nonselfdual, no $\boldsymbol{\Gamma}$-hard set $A \subseteq X$ is in $\check{\boldsymbol{\Gamma}}(X)$.
2. A set $A \subseteq X$ is $\boldsymbol{\Gamma}$-hard (respectively, $\boldsymbol{\Gamma}$-complete) if and only if $X \setminus A$ is $\check{\boldsymbol{\Gamma}}$-hard (respectively, $\check{\boldsymbol{\Gamma}}$-complete).
3. If $\mathcal{U}$ is Y-universal for $\boldsymbol{\Gamma}(\omega^\omega)$, then $\mathcal{U}$ is $\boldsymbol{\Gamma}$-complete. In particular, there are $\boldsymbol{\Sigma}^0_\alpha$-complete and $\boldsymbol{\Pi}^0_\alpha$-complete sets.
4. If A is $\boldsymbol{\Gamma}$-hard and $A = f^{-1}(A')$ for some continuous $f \colon X \to X'$ and $A' \subseteq X'$ with X' Polish, then A' is $\boldsymbol{\Gamma}$-hard as well.

The proof is immediate. (EXERCISE!)

Remark

Since universality implies completeness, one may wonder whether there are 2^ω-universal sets for $\boldsymbol{\Gamma}(X)$ as soon as there is a $\boldsymbol{\Gamma}$-complete subset of X. This is not true in general: every nontrivial clopen subset of ω^ω is $\boldsymbol{\Delta}^0_1$-complete, but there is no 2^ω-universal set for $\boldsymbol{\Delta}^0_1(\omega^\omega)$ by Corollary 2.1.16. However, it can be shown that for every $\boldsymbol{\Gamma} \neq \boldsymbol{\Delta}^0_1$ consisting of Borel sets, there is a 2^ω-universal set for $\boldsymbol{\Gamma}(\omega^\omega)$ if and only if there is a $\boldsymbol{\Gamma}$-complete subset of ω^ω. In particular, this is the case for all *nonselfdual* pointclasses $\boldsymbol{\Delta}^0_1 \subseteq \boldsymbol{\Gamma} \subseteq \mathbf{Bor}$.

A method to show that $A \in \boldsymbol{\Gamma}(X) \setminus \check{\boldsymbol{\Gamma}}(X)$:

- show that A is $\boldsymbol{\Gamma}$-hard by constructing, for an arbitrary $B \in \boldsymbol{\Gamma}(\omega^\omega)$, a continuous function $f \colon \omega^\omega \to X$ such that $B = f^{-1}(A)$,
- if we know that some $C \subseteq Y$ is $\boldsymbol{\Gamma}$-hard, then it is enough to show that there is a continuous $f \colon Y \to X$ such that $C = f^{-1}(A)$.

Exercise 2.1.25

X a perfect Polish space. Show that every countable (infinite) dense subset D of X is $\mathbf{\Sigma}^0_2$-complete, whence it is an $\mathbf{F}_\sigma$ set which is not $\mathbf{G}_\delta$. Conclude that the same is true if D is **somewhere dense**, i.e. if there is some open set $U \subseteq X$ such that D is dense in U.

In particular $Q_2 = \{x \in 2^\omega \mid \exists n \in \omega \, \forall k \geq n \, (x(k) = 0)\}$ is $\mathbf{\Sigma}^0_2$-complete. Also $\mathbb{Q}$ is a $\mathbf{\Sigma}^0_2$-complete whence $\mathbb{Q}$ is not $\mathbf{G}_\delta$ and Irr is not $\mathbf{F}_\sigma$.

Exercise

P_3 the set of infinite binary matrices with all rows eventually 0, :

$$\begin{aligned} P_3 &= \{x \in 2^{\omega \times \omega} \mid \forall m \, [(x(m,n))_{n \in \omega} \in Q_2]\} \\ &= \{x \in 2^{\omega \times \omega} \mid \forall m \, \exists n \, \forall k \geq n \, (x(m,k) = 0)\}. \end{aligned}$$

Show that P_3 is $\mathbf{\Pi}^0_3$-complete.

Exercise 2.1.27

$C_3 = \{x \in \omega^\omega \mid \lim_{n \to \infty} x(n) = \infty\}$ is $\mathbf{\Pi}^0_3$-complete.

Exercise [Ki-Linton]

$A \subseteq \omega$ has **density** 0 if $\lim_{n \to \infty} \frac{|A \cap \{0, \ldots, n-1\}|}{n} = 0$. Show that the set of all characteristic functions $y \in 2^\omega$ of sets with density 0 is $\mathbf{\Pi}^0_3$-complete.

Example 2.1.30 [Stern]

Let LO be the Polish space of (codes for) countable linear orders. For $\omega < \alpha < \omega_1$, let

$$\mathrm{WO}^{<\alpha} = \{x \in 2^{\omega \times \omega} \mid L_x \text{ is a well-order of order type } < \alpha\}.$$

Then $\mathrm{WO}^{<\omega^\alpha}$ is $\mathbf{\Sigma}^0_{2 \cdot \alpha}$-complete (for every $\alpha \geq 2$). If instead $\omega^\alpha < \beta < \omega^{\alpha+1}$, then $\mathrm{WO}^\beta \in \mathbf{\Delta}^0_{2 \cdot \alpha + 2}(\mathrm{LO}) \setminus \mathbf{\Sigma}^0_{2 \cdot \alpha + 1}(\mathrm{LO})$.

Proposition [Kec95, Exercise 23.3]

$\rho\colon 2^\omega \to (2^\omega)^\omega$ be an homeomorphism, and $\rho_n = \pi_n \circ \rho$, with π_n is the nth projection. Let

$$\begin{aligned} C_1 &= \{x \in 2^\omega \mid \exists n\, (x(n) = 0)\} \\ C_{\alpha+1} &= \{x \in 2^\omega \mid \exists n\, (\rho_n(x) \notin C_\alpha)\} \\ C_\lambda &= \{x \in 2^\omega \mid \exists n\, (\rho_n(x) \notin C_{\alpha_n}\} \qquad \text{if } \lambda \text{ is limit,} \end{aligned}$$

where $(\alpha_n)_{n\in\omega}$ is increasing and cofinal in λ. Then C_α is $\mathbf{\Sigma}^0_\alpha$-complete, for $1 \leq \alpha$.

The proof is an easy induction on $1 \leq \alpha < \omega_1$ using the following claim.

Claim

Let $1 \leq \alpha < \omega_1$, $(X_n)_{n\in\omega}$ Polish, and $\alpha = \limsup_n \alpha_n + 1$. If $A_n \subseteq X_n$ is $\mathbf{\Sigma}^0_{\alpha_n}$-complete then $\prod_{n\in\omega} A_n \subseteq \prod_{n\in\omega} X_n$ is $\mathbf{\Pi}^0_\alpha$-complete.

Proof.

$x \in \prod_n A_n \Leftrightarrow \forall n\, (\pi_n(x) \in A_n)$, where π_n is the n-th projection. As π_n is continuous, $\prod_n A_n \in \mathbf{\Pi}^0_\alpha(\prod_n X_n)$. We must show that given $B = \bigcap_n B_n \subseteq \omega^\omega$ with $B_n \in \bigcup_{1\leq\beta<\alpha} \mathbf{\Sigma}^0_\beta(\omega^\omega)$, there is a continuous $g\colon \omega^\omega \to \prod_n X_n$ such that $B = g^{-1}(\prod_n A_n)$.
By the $\limsup$ assumption, define $\iota\colon \omega \to \omega$ such that $n \leq \iota(n)$ and $B_n \in \mathbf{\Sigma}^0_{\alpha_{\iota(n)}}(\omega^\omega)$. For all n, let $f_n\colon \omega^\omega \to X_{\iota(n)}$ be continuous and such that $f_n^{-1}(A_{\iota(n)}) = B_n$, and fix $y_n \in A_n$. Let g be the function sending $x \in \omega^\omega$ to the unique $(x_n)_n \in \prod_n X_n$ such that $x_{\iota(n)} = f_n(x)$ and $x_n = y_n$ if $n \notin \operatorname{rng}(\iota)$. Then g is continuous and such that $x \in B = \bigcap_n B_n \Leftrightarrow g(x) \in \prod_n A_n$. $\square$