Baire category

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X a topological space, $A \subseteq X$ is **nowhere dense** if one of the following equivalent conditions hold:

- there is no open set $U \subseteq X$ such that $A \cap U$ is dense in U;
- its closure $\mathrm{Cl}(A)$ has empty interior;
- it is disjoint from an open dense set.

A is nowhere dense if and only if $\mathrm{Cl}(A)$ is nowhere dense.
The collection of nowhere dense subsets of X is closed under subsets and finite unions, that is, it is an ideal of sets.

Definition 1.5.1

$A \subseteq X$ is **meager** if it can be written as a countable union of nowhere dense sets.
A is **comeager** if its complement is meager; equivalently, A is comeager iff it contains the intersection of countably many open dense sets.

Older terminology: meager sets are called "sets of the first category", non-meager sets are called "sets of the second category", while comeager sets are called "residual sets".

The collection of meager subsets of X is a σ-ideal on X, that is, it is closed under subsets and countable unions.
Dually, the collection of comeager subsets of X is a σ-filter on X, i.e. it is closed under supersets and countable intersections.

Examples 1.5.2

- If $U \subseteq X$ is open, its boundary $\mathrm{Cl}(U) \setminus U$ is nowhere dense: every nonempty open set disjoint from U is contained in $X \setminus \mathrm{Cl}(U)$.
  Every second-countable space X can be written as a disjoint union $Z \cup F$ with Z a comeager zero-dimensional $\mathbf{G}_\delta$ set and F a meager $\mathbf{F}_\sigma$ set. Indeed, it is enough to let $F = \bigcup_{n \in \omega} \mathrm{Cl}(U_n) \setminus U_n$ for $\{U_n \mid n \in \omega\}$ a countable basis for X, and $Z = X \setminus F$.
- Similarly, if $F \subseteq X$ is closed, then $F \setminus \mathrm{Int}(F)$ is nowhere dense.
- The set of rationals $\mathbb{Q}$ is a meager subsets or $\mathbb{R}$, and hence the irrationals are a comeager subset of $\mathbb{R}$. The set of positive reals is neither meager nor comeager in $\mathbb{R}$.

Examples 1.5.2

- More generally, a countable subset of a perfect space is always meager. On the other hand, if x is isolated in X then $\{x\}$ is not meager.
- Any dense $\mathbf{G}_\delta$ set $G \subseteq X$ is comeager.

Meager sets are usually regarded as "topologically small sets". However, it might happen that a space is meager in itself, as in the case of the rationals $\mathbb{Q}$: in such spaces, the Baire category notions of meager and comeager are useless — all their subsets are meager.

Definition 1.5.3

A topological space X is **Baire** if it satisfies the following equivalent conditions:

1. Every nonempty open subset of X is non-meager.
2. Every comeager set in X is dense.
3. The intersection of countably many dense open subsets of X is dense.

If X is nonempty and Baire, then the intersection of any two dense $\mathbf{G}_\delta$ subsets of X must be dense, and hence nonempty.

Lemma 1.5.4 [Kec95, Proposition 8.3]

If a topological space X is Baire, then so are all its open subspaces.

Proof.

Let $U \subseteq X$ be open, and let $\{U_n \mid n \in \omega\}$ be a family of open dense subsets of U. Then each $V_n = U_n \cup (X \setminus \mathrm{Cl}(U))$ is open dense in X. Therefore $\bigcap_{n\in\omega} V_n = (\bigcap_{n\in\omega} U_n) \cup (X \setminus \mathrm{Cl}(U))$ is dense in X, hence $\bigcap_{n\in\omega} U_n$ is dense in U. □

Theorem 1.5.5 [Kec95, Theorem 8.4]

Every completely metrizable space X is Baire.

Proof.

Let d be a compatible complete metric on X, and let $U \subseteq X$ be any nonempty open set: we want to show that $U \cap \bigcap_{n\in\omega} U_n \neq \emptyset$. To this aim, we recursively build a sequence of open balls $B_n = \mathrm{B}_d(x_n, \varepsilon_n)$ such that $\mathrm{Cl}(B_n) \subseteq \mathrm{B}^{\mathrm{cl}}_d(x_n, \varepsilon_n) \subseteq U \cap U_n \cap \mathrm{B}_{n-1}$ (where $B_{-1} = X$) and $\varepsilon_n \leq 2^{-n}$. This implies that $(x_n)_{n\in\omega}$ is a Cauchy sequence in (X, d), and thus $\lim_n x_n \in \bigcap_{n\in\omega} \mathrm{Cl}(B_n) \subseteq U \cap \bigcap_{n\in\omega} U_n$.
We define the open balls B_n by recursion on $n \in \omega$. Since U_0 is open dense, the set $U \cap U_0$ is open and nonempty: pick any $x_0 \in U \cap U_0$, and let ε_0 be small enough so that $\varepsilon_0 \leq 2^{-0}$ and $\mathrm{B}^{\mathrm{cl}}_d(x_0, \varepsilon_0) \subseteq U \cap U_0$. Set $B_0 = \mathrm{B}_d(x_0, \varepsilon_0)$. Now suppose that $n > 0$, so that by inductive hypothesis B_{n-1} is a nonempty open subset of U. By density, $B_{n-1} \cap U_n$ is open and nonempty. Pick any $x_n \in B_{n-1} \cap U_n$, let ε_n be small enough so that $\varepsilon_n \leq 2^{-n}$ and $\mathrm{B}^{\mathrm{cl}}_d(x_n, \varepsilon_n) \subseteq B_{n-1} \cap U_n = U \cap U_n \cap B_{n-1}$, and set $B_n = \mathrm{B}_d(x_n, \varepsilon_n)$. This concludes the construction and the proof. □

Similarly, every locally compact T_2 space is Baire [Kec95, Theorem 8.4].

Corollary 1.5.6

Let X be a nonempty completely metrizable space. If an $\mathbf{F}_\sigma$ set $A \subseteq X$ is both dense and codense, then it is not $\mathbf{G}_\delta$.

Proof.

If A, $X \setminus A$ were dense $\mathbf{G}_\delta$, their intersection should be dense because X is Baire, a contradiction. □

Given $A, B \subseteq X$, write $A =^* B$ if $A \triangle B$ is meager in X.
The relation $=^*$ is reflexive, symmetric, and transitive:

If $A, B, C \subseteq X$ then $A \setminus C \subseteq (A \setminus B) \cup (B \setminus C)$ and $C \setminus A \subseteq (C \setminus B) \cup (B \setminus A)$: thus if $A =^* B$ and $B =^* C$, then also $A =^* C$.

Thus

$=^*$ is an equivalence relation.

Definition 1.5.7

Let X be a topological space. A set $A \subseteq X$ has the **Baire Property** BP if $A =^* U$ for some open set $U \subseteq X$.
BP(X) be the collection of all subsets of X with the Baire property.

If $A \in \mathrm{BP}(X)$ and $A =^* B$, then $B \in \mathrm{BP}(X)$.

Remark

$\mathcal{I}$ a σ-ideal on X, its elements are called $\mathcal{I}$-**small**.
$A \subseteq X$ is $\mathcal{I}$-**regular** if $A \triangle U \in \mathcal{I}$ for some open $U \subseteq X$.
For example, fix a measure μ on X, and consider the σ-ideal $\mathcal{I}$ of all subsets of X with measure 0.

Proposition 1.5.9 [Kec95, Proposition 8.22]

Let X be a topological space. Then BP(X) is the smallest σ-algebra containing all open sets and all meager sets.

Proof.

Every open and every meager is in BP(X).

Suppose $A =^* U$ and $U \subseteq X$ open.
Then $(X \setminus A) \setminus (X \setminus U) = U \setminus A$ and $(X \setminus U) \setminus (X \setminus A) = A \setminus U$, so $X \setminus A =^* X \setminus U$. But $X \setminus U$ is closed, hence $X \setminus U =^* \mathrm{Int}(X \setminus U)$. Thus $X \setminus A =^* \mathrm{Int}(X \setminus U)$. Thus BP$(X)$ is closed under complements.

Suppose that $A_n =^* U_n$. Since $\bigcup_n A_n \setminus \bigcup_n U_n \subseteq \bigcup_n (A_n \setminus U_n)$ and $\bigcup_n U_n \setminus \bigcup_n A_n \subseteq \bigcup_n (U_n \setminus A_n)$, it follows that $\bigcup_n A_n =^* V := \bigcup_n U_n$. Thus BP$(X)$ is also closed under countable unions.

Conversely, every set A with the BP is a Boolean combination of an open set and meager set: if $A =^* U$ for $U \subseteq X$ open and $M = A \triangle U$, so that M is meager, then $A = M \triangle U$. $\square$

Remark

The above proof actually shows that the smallest algebra on X containing all open sets and all meager sets is actually a σ-algebra, and it coincides with BP(X).

Sets with the Baire property can be approximated internally by a $\mathbf{G}_\delta$ set, and externally by an $\mathbf{F}_\sigma$ set (modulo meager sets).

Proposition 1.5.11 [Kec95, Proposition 8.23]

Let X be a topological space and $A \subseteq X$. Then the following are equivalent:

1. $A \in$ BP(X);
2. $A = G \cup M$ for some $\mathbf{G}_\delta$ set G and some meager set M;
3. $A = F \setminus M$ for some $\mathbf{F}_\sigma$ set F and some meager set M.

❶ $A \in \mathrm{BP}(X)$ implies ❷ $A = G \cup M$ with $G \in \mathbf{G}_\delta$ and M meager

Pick any open set $U \subseteq X$ such that $A =^* U$. Write $A \,\Delta\, U$ as $\bigcup_{n \in \omega} D_n$ with each D_n nowhere dense, and set $F = \bigcup_{n \in \omega} \mathrm{Cl}(D_n)$. Then F is a meager $\mathbf{F}_\sigma$ set with $A \,\Delta\, U \subseteq F$. Set $G = U \setminus F$ and $M = A \setminus G$. Since $G \subseteq A$, we have $A = G \cup M$. Moreover, the set G is $\mathbf{G}_\delta$ by definition, and M is meager because $M \subseteq F$, hence we are done.

❶ $A \in \mathrm{BP}(X)$ implies ❸ $A = F \setminus M$ with $F \in \mathbf{F}_\sigma$ and M meager

The proof in the previous paragraph works for every set with the BP: in particular, it can be applied to $X \setminus A$ because $X \setminus A \in \mathrm{BP}(X)$ by 1 and Proposition 1.5.9. Therefore $X \setminus A = G \cup M$ with M meager and G a $\mathbf{G}_\delta$ set. Then $F := X \setminus G \in \mathbf{F}_\sigma$ so $A = F \cap (X \setminus M) = F \setminus M$.

The implications ❷ $\Rightarrow$ ❶ and ❸ $\Rightarrow$ ❶ follow from Proposition 1.5.9.

Corollary 1.5.12

Let X be a nonempty perfect Polish space, and $A \in \mathrm{BP}(X)$ non-meager. Then there is an embedding of 2^ω into A.

Proof.

By Proposition 1.5.11, A contains a non-meager $\mathbf{G}_\delta$ set G. In particular, G is Polish, and it is uncountable (Exercise 1.5.24).
Therefore 2^ω embeds into $G \subseteq A$ by Corollary 1.4.9. $\square$

Working in ZFC, it is possible to show that there are sets without the Baire property.

Proposition see [Kec95, Example 8.24]

If X is a nonempty perfect Polish space, then $\mathrm{BP}(X) \neq \mathscr{P}(X)$.

Proof.

Let $A \subseteq X$ be a Bernstein set (Proposition 1.4.14). We claim that $A \notin \mathrm{BP}(X)$. If not, since either A or $X \setminus A$ is non-meager (because X is Baire by Theorem 1.5.5), the Cantor space would embed in one of A or $X \setminus A$ by Corollary 1.5.12, contradicting the fact that A is a Bernstein set. □

$D \subseteq U$ is nowhere dense in U iff it is nowhere dense in X.

Proof.

If $V \subseteq U$ is open dense in U and such that $V \cap D = \emptyset$, then $W = V \cup (X \setminus \mathrm{Cl}(U))$ is open dense in X and such that $W \cap D = \emptyset$. Conversely, if $W \subseteq X$ is open dense in X and such that $W \cap D = \emptyset$, then $V = W \cap U$ is open dense in U and such that $V \cap D = \emptyset$. □

Therefore

$M \subseteq U$ is meager in U iff it is meager in X.

Definition

X a topological space, $U \subseteq X$ open.
$A \subseteq X$ is **meager in** U if $A \cap U$ is meager in U or, equivalently, in X.
A is **comeager in** U if $U \setminus A$ is meager.

If A is (co)meager in some open $U \subseteq X$, then A is (co)meager in every open $V \subseteq U$.

Proposition 1.5.15 [Kec95, Proposition 8.26]

If $A \in \mathrm{BP}(X)$, then either A is meager, or else it is comeager in some nonempty open $U \subseteq X$. If X is Baire, then exactly one of these holds.

Proof.

Let $U \subseteq X$ be an open set such that $A =^* U$. If $U = \emptyset$, then A is meager, otherwise, it is comeager in U. □

Recall that if $A \subseteq X \times Y$ and $\bar{x} \in X$ and $\bar{y} \in Y$ then

$$A_{(\bar{x})} = \{y \in Y \mid (\bar{x}, y) \in A\} \qquad A^{(\bar{y})} = \{x \in X \mid (x, \bar{y}) \in A\}$$

Theorem [Kec95, Theorem 8.4]

Let X, Y be second countable topological spaces, and suppose $A \in \mathrm{BP}(X \times Y)$.

1. $\{x \in X \mid A_{(x)} \in \mathrm{BP}(Y)\}$ is comeager in X. $\{y \in Y \mid A^{(y)} \in \mathrm{BP}(X)\}$ is comeager in Y.
2. A is meager iff $\{x \in X \mid A_{(x)}$ is meager in $Y\}$ is comeager in X, iff $\{y \in Y \mid A^{(y)}$ is meager in $X\}$ is comeager in Y.
3. A is comeager iff $\{x \in X \mid A_{(x)}$ is comeager in $Y\}$ is comeager in X, iff $\{y \in Y \mid A^{(y)}$ is comeager in $X\}$ is comeager in Y.

If X, Y are second-countable and Baire, then $X \times Y$ is Baire.
By the Kuratowski-Ulam Theorem no well-ordering of X, a non-empty perfect Polish space, has the BP as a subset of X^2.