Souslin's theorem

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X a Polish space. $P, Q \subseteq X$ are **Borel separable** if there is R Borel **separating** P from Q, i.e. $P \subseteq R$ and $R \cap Q = \emptyset$.

The Lusin separation theorem 3.2.1 [Kec95, Theorem 14.7]

X Polish space. If $A, B \subseteq X$ are disjoint analytic sets, then they are Borel separable.

Souslin's theorem 3.2.3 [Kec95, Theorem 14.11]

If X is Polish, then $\mathbf{Bor}(X) = \mathbf{\Delta}^1_1(X)$.

Corollary 3.2.2 [Kec95, Corollary 14.9]

X Polish space, and $(A_n)_{n\in\mathbb{N}}$ pairwise disjoint analytic sets. Then there are pairwise disjoint Borel sets B_n with $B_n \supseteq A_n$.

Proof.

For $n \neq m$ let $C_{n,m}$ be Borel separating A_n from A_m. Inductively set $B_0 = \bigcap_{m>0} C_{0,m}$ and $B_{n+1} = \bigcap_{m\neq n+1} C_{n+1,m} \setminus \bigcup_{i\leq n} B_i$. Then $A_n \subseteq B_n$, as $B_n \cap A_m = \emptyset$ for $m \neq n$. $\square$

Proof of Lusin's separation theorem

Observation: if $P = \bigcup_{m \in \omega} P_m$, $Q = \bigcup_{n \in \omega} Q_n$, and $R_{m,n}$ is Borel separating P_m and Q_n, then $\bigcup_{m \in \omega} \bigcap_{n \in \omega} R_{m,n}$ is a Borel separating P, Q.

WLOG $A \neq \emptyset \neq B$ and let $f \colon \mathbb{N}^{\mathbb{N}} \to A$ and $g \colon \mathbb{N}^{\mathbb{N}} \to B$ be continuous surjections. For $s \in \mathbb{N}^{<\omega}$, set $A_s = f[\boldsymbol{N}_s]$ and $B_s = g[\boldsymbol{N}_s]$, so that $A = A_\emptyset$, $B = B_\emptyset$, $A_s = \bigcup_{m \in \mathbb{N}} A_{s^\frown m}$, and $B_s = \bigcup_{n \in \mathbb{N}} B_{s^\frown n}$. Let

$$T = \{(s,t) \in \mathbb{N}^{<\omega} \times \mathbb{N}^{<\omega} \mid \mathrm{lh}(s) = \mathrm{lh}(t) \wedge A_s, B_t \text{ not Borel separable}\}.$$

By the observation T is a pruned tree. Towards a contradiction, suppose that A and B are not Borel separable. Then T is nonempty, hence $[T] \neq \emptyset$. Pick any $(x,y) \in [T]$: since $A \cap B = \emptyset$, $f(x) \in A$, and $g(y) \in B$, we get $f(x) \neq g(y)$. Let U, V be disjoint open neighborhoods of $f(x)$ and $g(y)$, respectively, and use the continuity of f and g to find $n \in \omega$ large enough so that $\boldsymbol{N}_{x \restriction n} \subseteq f^{-1}[U]$ and $\boldsymbol{N}_{y \restriction n} \subseteq g^{-1}[V]$, i.e. $A_{x \restriction n} \subseteq U$ and $B_{y \restriction n} \subseteq V$. Then U would be a Borel set separating $A_{x \restriction n}$ from $B_{y \restriction n}$, contradicting $(x \restriction n, y \restriction n) \in T$.

The following is a partial converse to Proposition 2.4.4.

Theorem 3.2.4 [Kec95, Theorem 14.12]

Let $f \colon X \to Y$ be a function between the Polish spaces. The following are equivalent:

- f is a Borel function;
- $\mathrm{graph}(f)$ is Borel;
- $\mathrm{graph}(f)$ is analytic.

In particular, if f is a Borel bijection, then f is a Borel isomorphism (i.e. f^{-1} is also Borel).

Proof.

It is enough to show that if $\mathrm{graph}(f)$ is analytic then f is a Borel function. Given any Borel $B \subseteq Y$, for every $x \in X$

$$x \in f^{-1}[B] \Leftrightarrow \exists y \, ((x,y) \in \mathrm{graph}(f) \wedge y \in B),$$

hence $f^{-1}[B]$ is analytic. Since $\mathrm{Bor}(Y)$ is closed under complements, it follows that $f^{-1}[B] \in \boldsymbol{\Delta}^1_1(X) = \mathrm{Bor}(X)$ for every $B \in \mathbf{Bor}(Y)$. $\square$

Theorem 3.2.5 (Lusin-Souslin) [Kec95, Theorem 15.1]

Let X, Y be Polish spaces and $f \colon X \to Y$ be continuous.
If $A \subseteq X$ is Borel and $f \restriction A$ is injective, then $f[A]$ is Borel.

Proof.

We can assume that $X = \mathbb{N}^{\mathbb{N}}$ and A closed. The $\mathbb{N}$-scheme $\mathcal{S}_f = \{B_s \mid s \in \mathbb{N}^{<\omega}\}$, where $B_s = f[A \cap \boldsymbol{N}_s]$, induces f, and moreover $B_{s^\frown n} \cap B_{s^\frown m} = \emptyset$ for all $s \in \mathbb{N}^{<\omega}$ and distinct n, m because f is injective. Each B_s is analytic so there are $\{B'_s \mid s \in \mathbb{N}^{<\omega}\}$ with B'_s Borel and $B_s \subseteq B'_s$ and $B'_{s^\frown n} \cap B'_{s^\frown m} = \emptyset$. Construct an $\mathbb{N}$-scheme $\mathcal{S}^*$ with $\mathcal{S}_f \sqsubseteq \mathcal{S}^* \sqsubseteq \mathrm{Cl}(\mathcal{S}_f)$:

$$B^*_\emptyset = B'_\emptyset \cap \mathrm{Cl}(B_\emptyset) \quad B^*_{s^\frown n} = B'_{s^\frown n} \cap B^*_s \cap \mathrm{Cl}(B_{s^\frown n}).$$

($B_s \subseteq B^*_s$ is proved by induction on $\mathrm{lh}(s)$, where in the inductive step we use in particular that $B_{s^\frown n} \subseteq B_s \subseteq B^*_s$.) $\mathcal{S}^* = \{B^*_s \mid s \in \mathbb{N}^{<\omega}\}$ still induces f by Lemma 1.3.9(e), and $B^*_{s^\frown n} \cap B^*_{s^\frown m} = \emptyset$ for all distinct n, m because $B^*_s \subseteq B'_s$ for all $s \in \mathbb{N}^{<\omega}$. Thus by Lemma 1.3.6(b) we have $f[A] = \bigcap_n \bigcup_{s \in \mathbb{N}^n} B^*_s$, and since the B^*_s are all Borel sets by construction, so is $f[A]$. □

Corollary 3.2.6 [Kec95, Corollary 15.2]

Let $f \colon X \to Y$ be a Borel function between the Polish spaces. If $A \subseteq X$ is Borel and $f \restriction A$ is injective, then $f[A]$ is Borel and f is a Borel isomorphism between A and $f[A]$.

Proof.

Let τ be the topology of X. Get a Polish topology $\tau' \supseteq \tau$ such that $\mathbf{Bor}(X, \tau') = \mathbf{Bor}(X, \tau)$ and $f \colon (X, \tau') \to Y$ is continuous. Then $A \in \mathbf{Bor}(X, \tau')$, hence $f[A]$ is Borel. Moreover, for every τ-open $U \subseteq X$, we have $U \cap A \in \mathbf{Bor}(X, \tau) = \mathbf{Bor}(X, \tau')$, hence $f[U \cap A]$ is Borel. Therefore $(f \restriction A)^{-1}$ is Borel and we are done. □

Corollary 3.2.7 [Kec95, Exercise 15.3]

Then the following are equivalent for a set $A \subseteq X$, and X Polish:

1. A is Borel;
2. A is a continuous injective image of a closed subset of $\mathbb{N}^{\mathbb{N}}$;
3. A is a Borel injective image of a Borel subset of a Polish space.

The Borel Schröder-Bernstein theorem 3.2.8 [Kec95, Theorem 15.7]

Let X, Y be Polish spaces and $f\colon X \to Y$ and $g\colon Y \to X$ be Borel injections. Then X and Y are Borel isomorphic.

Proof.

Enough to show that there are Borel $A \subseteq X$, $B \subseteq Y$ such that $f[A] = Y \setminus B$, $g[B] = X \setminus A$, because then $(f \restriction A) \cup (g^{-1} \restriction [X \setminus A])$ is a Borel isomorphism between X and Y.
Recursively define X_n, Y_n by $X_0 = X$, $Y_0 = Y$, $X_{n+1} = (g \circ f)[X_n]$, and $Y_{n+1} = (f \circ g)[Y_n]$.
Let $X_\infty = \bigcap_n X_n$ and $Y_\infty = \bigcap_n Y_n$. Then $f[X_\infty] = Y_\infty$, $f[X_n \setminus g[Y_n]] = f[X_n] \setminus Y_{n+1}$, and $g[Y_n \setminus f[X_n]] = g[Y_n] \setminus X_{n+1}$. Finally, let $A = X_\infty \cup \bigcup_n (X_n \setminus g[Y_n])$ and $B = \bigcup_n (Y_n \setminus f[X_n])$. All these sets are Borel by Corollary 3.2.6. □

The Borel isomorphism theorem 3.2.9 [Kec95, Theorem 15.6]

Two Polish spaces X and Y are Borel isomorphic if and only if they have the same cardinality. In particular, two uncountable Polish spaces are Borel isomorphic.

Proof.

If X and Y are countable, then any bijection $f\colon X \to Y$ is a Borel isomorphism, so it is enough to show that every uncountable Polish space X is Borel isomorphic to $2^{\mathbb{N}}$. By Theorem 1.3.17 and Corollary 3.2.6, X is Borel isomorphic to a closed subset of $\mathbb{N}^{\mathbb{N}}$, and hence to a $\mathbf{\Pi}^0_2$ subset of $2^{\mathbb{N}}$. On the other hand, $2^{\mathbb{N}}$ is homeomorphic to a closed subset of X by Corollary 1.4.9. Therefore the result follows from Theorem 3.2.8. □

Corollary

Let $X = (X, \tau_X)$ and $Y = (Y, \tau_Y)$ be Polish, $A \in \mathbf{Bor}(X)$ and $B \in \mathbf{Bor}(Y)$. There is a Borel isomorphism $f \colon A \to B$ iff A and B have the same cardinality. If A and B are both uncountable then they are Borel isomorphic.

Proof.

Fix Polish topologies τ_A on X and τ_B on Y such that $\mathbf{Bor}(X, \tau_X) = \mathbf{Bor}(X, \tau_A)$, A is τ_A-clopen, $\mathbf{Bor}(Y, \tau_Y) = \mathbf{Bor}(Y, \tau_B)$, and B is τ_B-clopen. In particular, A and B are Polish spaces when endowed with the relative topologies of τ_A and τ_B, respectively. Therefore the result follows from Theorem 3.2.9 and the fact that our new topologies do not change the notion of Borelness in the corresponding spaces. $\square$