Projective sets

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Let X be a separable metrizable space. For $n \geq 1$, let $\mathbf{\Sigma}^1_{n+1}(X)$ be the collection of all $A \subseteq X$ for which there is a Polish space Y and a continuous $f \colon Y \to X$ such that $A = f[B]$ for some $B \in \mathbf{\Pi}^1_n(Y)$. Moreover we set

$$\mathbf{\Pi}^1_{n+1}(X) = \{X \setminus A \mid A \in \mathbf{\Sigma}^1_{n+1}(X)\} = \check{\mathbf{\Sigma}}^1_{n+1}(X)$$
$$\mathbf{\Delta}^1_{n+1}(X) = \mathbf{\Sigma}^1_{n+1}(X) \cap \mathbf{\Pi}^1_{n+1}(X) = \mathbf{\Delta}_{\mathbf{\Sigma}^1_{n+1}}(X).$$

A set $A \subseteq X$ is called **projective** if it belongs to $\mathbf{\Sigma}^1_n(X)$ for some $n \geq 1$; the collection of all projective subsets of X is denoted by $\mathbf{Proj}(X)$.

Proposition 3.3.2

Let X be a Polish space. Then for every $n \geq 1$

$$\mathbf{\Sigma}^1_n(X), \mathbf{\Pi}^1_n(X) \subseteq \mathbf{\Delta}^1_{n+1}(X) \subseteq \mathbf{\Sigma}^1_{n+1}(X), \mathbf{\Pi}^1_{n+1}(X).$$

In particular, $\mathbf{Bor}(X)$ is contained in all these classes.

Proof.

By induction on $n \geq 1$. The inclusion $\mathbf{\Sigma}^1_n(X) \subseteq \mathbf{\Sigma}^1_{n+1}(X)$ follows from $\mathbf{Bor}(X) \subseteq \mathbf{\Pi}^1_1(X)$ in the case $n = 1$, and by the inductive hypothesis $\mathbf{\Pi}^1_{n-1}(X) \subseteq \mathbf{\Pi}^1_n(X)$ when $n > 1$. The inclusion $\mathbf{\Pi}^1_n(X) \subseteq \mathbf{\Sigma}^1_{n+1}(X)$ follows from the fact that every set is the image of itself under the identity function. □

Proposition 3.3.3

Let X be Polish. The pointclasses $\mathbf{\Sigma}^1_n(X)$ and $\mathbf{\Pi}^1_n(X)$ are closed under countable unions, countable intersections, and Borel preimages (hence they are boldface pointclasses). Moreover $\mathbf{\Sigma}^1_n(X)$ is also closed under Borel images. Therefore, $\mathbf{\Delta}^1_n(X)$ is a σ-algebra and $\mathbf{\Delta}^1_n$ is closed under Borel preimages (hence a boldface pointclass).

It can be shown ([Kec95, Exercise 37.8]) that if X is uncountable, then $\mathbf{\Delta}^1_{n+1}(X)$ is not the smallest σ-algebra containing $\mathbf{\Sigma}^1_n(X)$.

Proof

By induction on $n \geq 1$. The case $n = 1$ corresponds to analytic, co-analytic, and bi-analytic sets, so we may assume $n > 1$. It is enough to consider the case of $\mathbf{\Sigma}^1_n(X)$.
Let $A_k \in \mathbf{\Sigma}^1_n(X)$, let Y_k Polish, $f_k \colon Y_k \to X$ continuous such that $A_k = f[B_k]$ for some $B_k \in \mathbf{\Pi}^1_{n-1}(Y_k)$, towards proving $\bigcup_k A_k, \bigcap_k A_k \in \mathbf{\Sigma}^1_n(X)$. Let $Y = \bigoplus_k Y_k$ and $B = \bigcup_k B_k$.
Since $\mathbf{\Pi}^1_{n-1}$ is a boldface pointclass and $\mathbf{\Pi}^1_{n-1}(Y)$ is closed under countable unions, it follows that $B \in \mathbf{\Pi}^1_{n-1}(Y)$. Then $\bigcup_k A_k = f[B] \in \mathbf{\Sigma}^1_n(X)$, where $f = \bigcup_k f_k \colon Y \to X$ continuous.

(continues)

Proof (continued)

Consider the closed subspace $F \subseteq \prod_{k \in \omega} Y_k$ defined by

$$(y_k)_k \in F \Leftrightarrow \forall k, k' \left(f_k(y_k) = f_{k'}(y_{k'}) \right).$$

Let $B = F \cap \prod_k B_k = \bigcap_k \pi_k^{-1}[B_k]$, where $\pi_k \colon F \to Y_k$ is the restriction to F of the projection. By the closure properties of $\mathbf{\Pi}^1_{n-1}$, we have $B \in \mathbf{\Pi}^1_{n-1}(F)$. Finally, the map $f = f_0 \circ \pi_0$ is continuous and such that $\bigcap_k A_k = f[B] \in \mathbf{\Sigma}^1_n(X)$.

Suppose $f \colon X \to Y$ is Borel, X and Y Polish, and $A \in \mathbf{\Sigma}^1_n(X)$ and $B \in \mathbf{\Sigma}^1_n(Y)$, towards proving $f^{-1}[B] \in \mathbf{\Sigma}^1_n(X)$ and $f[A] \in \mathbf{\Sigma}^1_n(Y)$. Let us prove the former. The set

$$f^{-1}[B] = \{x \in X \mid \exists y \in Y \left((x, y) \in \mathrm{graph}(f) \cap X \times B\right)\}$$

is in $\mathbf{\Sigma}^1_n(X)$, since $\mathrm{graph}(f)$ is $\mathbf{\Delta}^1_1(X \times Y) \subseteq \mathbf{\Sigma}^1_n(X \times Y)$, $\mathbf{\Sigma}^1_n$ is closed under projections and intersections, and $X \times B \in \mathbf{\Sigma}^1_n(X \times Y)$.

(continues)

Proof (continued).

We must prove that $X \times B \in \mathbf{\Sigma}^1_n(X \times Y)$, where $B \in \mathbf{\Sigma}^1_n(Y)$.
Let $C \in \mathbf{\Pi}^1_{n-1}(Z)$ with Z Polish and $f \colon Z \to Y$ is continuous and $f[C] = B$. Then $\mathrm{id}_X \times f \colon X \times Z \to X \times Y$ maps $X \times C$ onto $X \times B$. But $X \times C = \pi_Z^{-1}[C]$ is in $\mathbf{\Pi}^0_{n-1}$ because by inductive hypothesis $\mathbf{\Pi}^0_{n-1}$ is closed under continuous preimages, hence we are done.
The proof that if $A \in \mathbf{\Sigma}^1_n(X)$ then $f[A] \in \mathbf{\Sigma}^1_n(Y)$ is similar. □

Remark 3.3.4

As for analytic sets, if $Y \subseteq X$ are Polish, then $\mathbf{\Sigma}^1_n(Y) = \mathbf{\Sigma}^1_n(X) \upharpoonright Y$ (for any $n \geq 1$), and similarly for $\mathbf{\Pi}^1_n$ and $\mathbf{\Delta}^1_n$. Indeed, the same if true when $Y \in \mathbf{\Sigma}^1_n(X)$ with X Polish. It follows that if Y is a Polish subspace (or even just a projective subset) of the Polish space X, then $\mathbf{Proj}(Y) = \mathbf{Proj}(X) \upharpoonright Y$.

Proposition 3.3.5

Let X be a Polish space. For every nonzero $n \in \omega$ and $A \subseteq X$ the following are equivalent:

1. $A \in \mathbf{\Sigma}^1_{n+1}(X)$;
2. $A = f[B]$, where $f \colon Y \to X$ is a Borel function with Y Polish and $B \in \mathbf{\Pi}^1_n(Y)$;
3. $A = f[B]$, where $f \colon \mathbb{N}^\mathbb{N} \to X$ is a Borel function and $B \in \mathbf{\Pi}^1_n(\mathbb{N}^\mathbb{N})$;
4. $A = \pi_X[C]$ for some $C \in \mathbf{\Pi}^1_n(X \times \mathbb{N}^\mathbb{N})$;
5. $A = \pi_X[C]$ for some Polish space Y and $C \in \mathbf{\Pi}^1_n(X \times Y)$.

Moreover, in parts ❸–❹ we can replace $\mathbb{N}^\mathbb{N}$ by any uncountable Polish space Z.

Proof.

❶ $\Rightarrow$ ❷, ❹ $\Rightarrow$ ❺, and ❺ $\Rightarrow$ ❶ are obvious. Moreover, ❷ $\Rightarrow$ ❸ because every nonempty Polish Y is a continuous image of $\mathbb{N}^\mathbb{N}$ and $\mathbf{\Pi}^1_n$ is a boldface pointclass. ❸ $\Rightarrow$ ❹: If $A, f\ B$ are as in ❸, let $C = \{(x, y) \in X \times \mathbb{N}^\mathbb{N} \mid (y, x) \in \mathrm{graph}(f) \wedge y \in B\}$.
The additional part follows from the fact that Z and $\mathbb{N}^\mathbb{N}$ are Borel isomorphic by and that $\mathbf{\Pi}^1_n$

If $\mathbf{\Gamma}$ is a boldface pointclass, then $\mathrm{proj}(\mathbf{\Gamma})(X) = \{\pi_X[C] \mid C \in \mathbf{\Gamma}(X \times \mathbb{N}^\mathbb{N})\}$.

Since every Polish space is a continuous image of $\mathbb{N}^\mathbb{N}$, then $A \in \mathrm{proj}(\mathbf{\Gamma})(X)$ iff there is some Polish space Z and some $C \in \mathbf{\Gamma}(X \times Z)$ such that $A = \pi_X[C]$.

$\mathrm{proj}(\mathbf{\Gamma})$ is a boldface pointclass

Proof.

If $A = \pi_Y[C]$ for some $C \in \mathbf{\Gamma}(Y \times \mathbb{N}^\mathbb{N})$ and $f \colon X \to Y$ is continuous, then $f^{-1}[A] = \pi_X[C']$ where $C' = (f \times \mathrm{id}_{\mathbb{N}^\mathbb{N}})^{-1}[C] \in \mathbf{\Gamma}(X \times \mathbb{N}^\mathbb{N})$. □

Lemma 3.3.7

Let $\boldsymbol{\Gamma} \supseteq \boldsymbol{\Pi}^0_1$ be a boldface pointclass closed under intersections. Let X be a Polish space, and assume that there is a Y-universal set $\mathcal{V}$ for $\boldsymbol{\Gamma}(X \times \mathbb{N}^{\mathbb{N}})$ (where Y is any Polish space). Then there is a Y-universal set $\mathcal{U}$ for $\operatorname{proj}(\boldsymbol{\Gamma})$.

Proof.

Let $\mathcal{U} = \pi_{Y \times X}(\mathcal{V})$. Then $\mathcal{U} \in \operatorname{proj}(\boldsymbol{\Gamma})(Y \times X)$, and $\mathcal{U}_{(y)} \in \operatorname{proj}(\boldsymbol{\Gamma})(X)$ for all $y \in Y$, since $\operatorname{proj}(X)$ is a boldface pointclass and $\mathcal{U}_{(y)}$ is the preimage of $\mathcal{U}$ under the continuous function $x \mapsto (y, x)$. Conversely, given $A \in \operatorname{proj}(\boldsymbol{\Gamma})(X)$ let $C \in \boldsymbol{\Gamma}(X \times \mathbb{N}^{\mathbb{N}})$ be such that $A = \pi_X[C]$, and let $y \in Y$ be such that $C = \mathcal{V}_{(y)}$. Then $A = \mathcal{U}_{(y)}$, since for every $x \in X$

$$x \in A \Leftrightarrow \exists z\, ((x, z) \in C) \Leftrightarrow \exists z\, ((y, x, z) \in \mathcal{V}) \Leftrightarrow (y, x) \in \mathcal{U} \Leftrightarrow x \in \mathcal{U}_{(y)}.$$

□

Theorem

Let X be a Polish space. Then for every $n \geq 1$ and every uncountable Polish space Y there is a Y-universal set for $\boldsymbol{\Sigma}^1_n(X)$ (and hence also for $\boldsymbol{\Pi}^1_n(X)$). Theorefore, if X is infinite then

$$|\boldsymbol{\Sigma}^1_n(X)| = |\boldsymbol{\Pi}^1_n(X)| = |\boldsymbol{\Delta}^1_n(X)| = |\mathbf{Proj}(X)| = 2^{\aleph_0}.$$

Moreover, if X is uncountable then $\boldsymbol{\Sigma}^1_n(X) \neq \boldsymbol{\Pi}^1_n(X)$, neither $\boldsymbol{\Sigma}^1_n(X)$ nor $\boldsymbol{\Pi}^1_n(X)$ are closed under complements, $\boldsymbol{\Pi}^1_n(X)$ is not closed under projections (equivalently, under continuous or Borel images),

$$\boldsymbol{\Sigma}^1_n(X) \subset \boldsymbol{\Delta}^1_{n+1}(X) \subset \boldsymbol{\Sigma}^1_{n+1}(X),$$

and the same for $\boldsymbol{\Pi}^1_{n(+1)}(X)$ in place of $\boldsymbol{\Sigma}^1_{n(+1)}(X)$.

Proof.

By induction on $n \geq 1$, using Theorem 3.1.15, Lemma 3.3.7, and the fact that $\boldsymbol{\Sigma}^1_{n+1} = \operatorname{proj}(\boldsymbol{\Pi}^1_n)$ by Proposition 3.3.5. □

Using the fact that existential quantifications correspond to projections we can extend the Tarski-Kuratowski algorithm by adding the following "rules":

- if $\psi(x,y)$ defines a Borel subset of $X \times Y$, then $\exists y\, \psi(x,y)$ and $\forall y\, \psi(x,y)$ define, respectively, a $\boldsymbol{\Sigma}^1_1$ and a $\boldsymbol{\Pi}^1_1$ subset of X;
- if $\psi(x,y)$ defines a $\boldsymbol{\Sigma}^1_n$ subset of $X \times Y$ for some $n \geq 1$, then $\exists y\, \psi(x,y)$ and $\forall y\, \psi(x,y)$ define, respectively, a $\boldsymbol{\Sigma}^1_n$ and a $\boldsymbol{\Pi}^1_{n+1}$ subset of X;
- if $\psi(x,y)$ defines a $\boldsymbol{\Pi}^1_n$ subset of $X \times Y$ for some $n \geq 1$, then $\exists y\, \psi(x,y)$ and $\forall y\, \psi(x,y)$ define, respectively, a $\boldsymbol{\Sigma}^1_{n+1}$ and a $\boldsymbol{\Pi}^1_n$ subset of X;
- if $\psi(x,y)$ defines a $\boldsymbol{\Delta}^1_n$ subset of $X \times Y$ for some $n \geq 1$, then $\exists y\, \psi(x,y)$ and $\forall y\, \psi(x,y)$ define, respectively, a $\boldsymbol{\Sigma}^1_{n+1}$ and a $\boldsymbol{\Pi}^1_{n+1}$ subset of X.

A subset A of a Polish space is projective if and only if it can be defined by a formula φ whose bounded variables range over Polish (or standard Borel) spaces and whose atomic formulas define Borel (or even just) projective sets. In fact, the Tarski-Kuratowski algorithm shows that there is a level-by-level correspondence between the topological complexity of A and the complexity of the formula φ which defines it.

Example (Woodin)

$\mathrm{MV} = \{f \in C([0;1]) \mid f$ satisfies the Mean Value Theorem$\}$, where f satisfies the Mean Value Theorem if for all $a < b$ in $[0;1]$ there is c with $a < c < b$ such that $f'(c)$ exists and $f'(c) = \frac{f(b)-f(a)}{b-a}$. Then MV is a $\boldsymbol{\Pi}^1_2$ subset of $C([0;1])$ (Exercise!), and it can be shown that indeed it is $\boldsymbol{\Pi}^1_2$-complete (hence it is not a $\boldsymbol{\Sigma}^1_2$ set).

Example (Becker)

Let $\mathcal{U} \subseteq C([0;1])^{\mathbb{N}} \times C([0;1])$ be given by: $((f_n)_n, f) \in \mathcal{U}$ iff there is a subsequence $(f_{n_i})_i$ converging pointwise to f.
Then $\mathcal{U}$ is $C([0;1])^{\mathbb{N}}$-universal for $\boldsymbol{\Sigma}^1_2(C([0;1]))$, and therefore it is a $\boldsymbol{\Sigma}^1_2$-complete set.

Example

Say that $(f_n)_n \in C([0;1])^{\mathbb{N}}$ is **quasidense** in $C([0;1])$ if every $h \in C([0;1])$ is the pointwise limit of a subsequence of $(f_n)_n$. Then the set of quasidense $(f_n)_n \in C([0;1])^{\mathbb{N}}$ is $\boldsymbol{\Pi}^1_3$-complete.