

# Simple Linear Regression: Estimation

EC 320: Introduction to Econometrics

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# Prologue

# Housekeeping

**Problem Set 3** released tonight (due Jan 31st)

- Will be focused on simple linear regressions
- This will leave you with ample time for Midterm revision

## **Late Submissions**

- Accepting until Wednesday 11:59pm, 5pp penalty per day
- Come to office hours for any help with work

## **Labs**

# Navigating Metrics

## Where are we?

- Reviewed core ideas from statistics
- Developed a framework for thinking about causality
- Dabbled in regression analysis.

Also, **R**.

# Navigating Metrics

## Where we're going

- Learn the mechanics of *how* OLS works
- Interpret regression results (mechanically and critically)
- Extend ideas about causality to a regression context
- Think more deeply about statistical inference
- Lay a foundation for more-sophisticated regression techniques.

Also, **more R**.

# Simple Linear Regression

# Addressing Questions

## Example: Effect of police on crime

**Policy Question:** Do on-campus police reduce crime on campus?

- **Empirical Question:** Does the number of on-campus police officers affect campus crime rates? If so, by how much?

How can we answer these questions?

- Prior beliefs.
- Theory.
- **Data!**

# Let's "Look" at Data

## Example: Effect of police on crime

Search:

	Police per 1000 Students ▾	Crimes per 1000 students ▾
1	20.42	1.1
2	0.15	2
3	0.47	1.41
4	14.68	2.06
5	23.75	1.52
6	7.68	2.76

Showing 1 to 6 of 96 entries

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# Take 2

## Example: Effect of police on crime

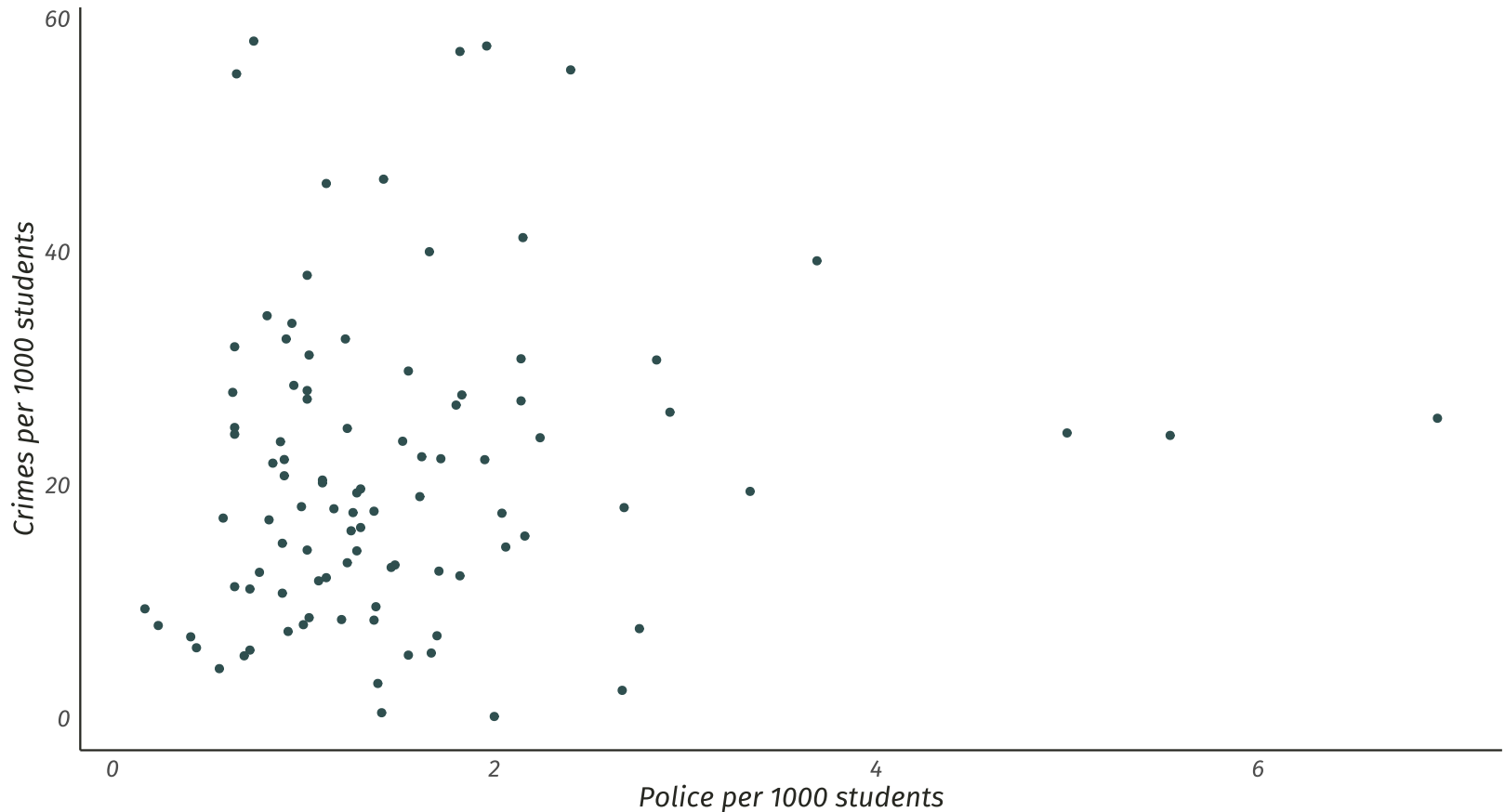
*"Looking"* at data wasn't especially helpful.

Let's try using a scatter plot.

- Plot each data point in  $(X, Y)$ -space.
- Police on the  $X$ -axis.
- Crime on the  $Y$ -axis.

# Take 2

## Example: Effect of police on crime



# Take 2

## Example: Effect of police on crime

The scatter plot tells us more than the spreadsheet.

- Somewhat weak *positive* relationship.
- Sample correlation coefficient of 0.14 confirms this.

But our question was

Does the number of on-campus police officers affect campus crime rates? If so, by how much?

- The scatter plot and correlation coefficient provide only a partial answer.

# Take 3

## Example: Effect of police on crime

Our next step is to estimate a **statistical model**.

To keep it simple, we will relate an **explained variable**  $Y$  to an **explanatory variable**  $X$  in a linear model.

# Simple Linear Regression Model

We express the relationship between a **explained variable** and an **explanatory variable** as linear:

$$Y_i = \beta_1 + \beta_2 X_i + u_i.$$

- $\beta_1$  is the **intercept** or constant.
- $\beta_2$  is the **slope coefficient**.
- $u_i$  is an **error term** or disturbance term.

*Simple* = Only one explanatory variable.

# Simple Linear Regression Model

The **intercept** tells us the expected value of  $Y_i$  when  $X_i = 0$ .

$$Y_i = \beta_1 + \beta_2 X_i + u_i$$

Usually not the focus of an analysis.

# Simple Linear Regression Model

The **slope coefficient** tells us the expected change in  $Y_i$  when  $X_i$  increases by one.

$$Y_i = \beta_1 + \beta_2 X_i + u_i$$

"A one-unit increase in  $X_i$  is associated with a  $\beta_2$ -unit increase in  $Y_i$ ."

Under certain (strong) assumptions about the error term,  $\beta_2$  is the *effect of  $X_i$  on  $Y_i$* .

- Otherwise, it's the *association of  $X_i$  with  $Y_i$* .

# Simple Linear Regression Model

The **error term** reminds us that  $X_i$  does not perfectly explain  $Y_i$ .

$$Y_i = \beta_1 + \beta_2 X_i + u_i$$

Represents all other factors that explain  $Y_i$ .

- Useful mnemonic: pretend that  $u$  stands for "*unobserved*" or "*unexplained*."



# Take 3, continued

## Example: Effect of police on crime

How might we apply the simple linear regression model to our question about the effect of on-campus police on campus crime?

- Which variable is  $X$ ? Which is  $Y$ ?

$$\text{Crime}_i = \beta_1 + \beta_2 \text{Police}_i + u_i.$$

- $\beta_1$  is the crime rate for colleges without police.
- $\beta_2$  is the increase in the crime rate for an additional police officer per 1000 students.

# Take 3, continued

## Example: Effect of police on crime

How might we apply the simple linear regression model to our question?

$$\text{Crime}_i = \beta_1 + \beta_2 \text{Police}_i + u_i$$

$\beta_1$  and  $\beta_2$  are the population parameters we want, but we cannot observe them.

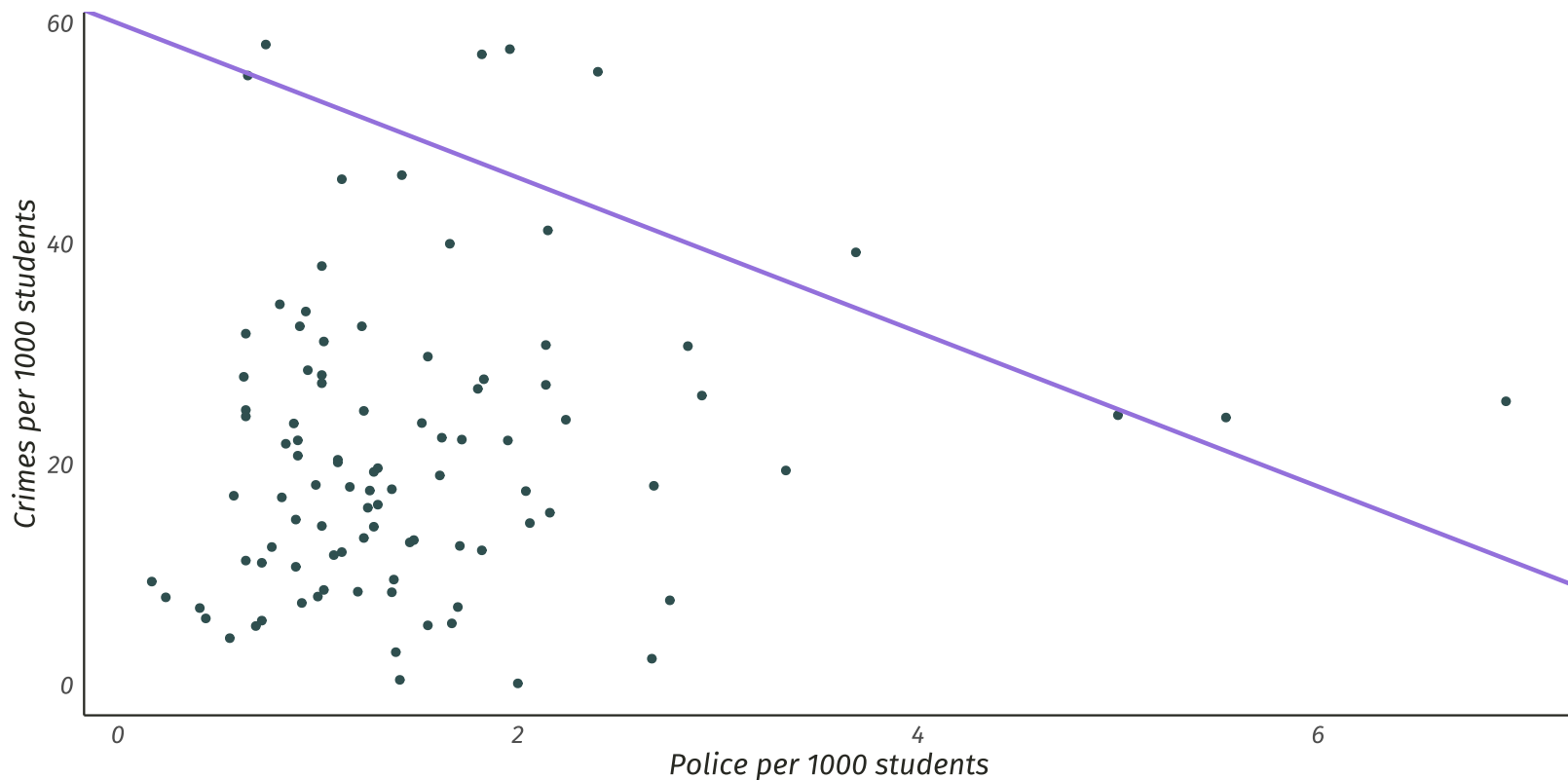
Instead, we must estimate the population parameters.

- $\hat{\beta}_1$  and  $\hat{\beta}_2$  generate predictions of  $\text{Crime}_i$  called  $\text{Cr}\hat{\text{ime}}_i$ .
- We call the predictions of the dependent variable **fitted values**.
- Together, these trace a line:  $\text{Cr}\hat{\text{ime}}_i = \hat{\beta}_1 + \hat{\beta}_2 \text{Police}_i$ .

# Take 3, attempted

## Example: Effect of police on crime

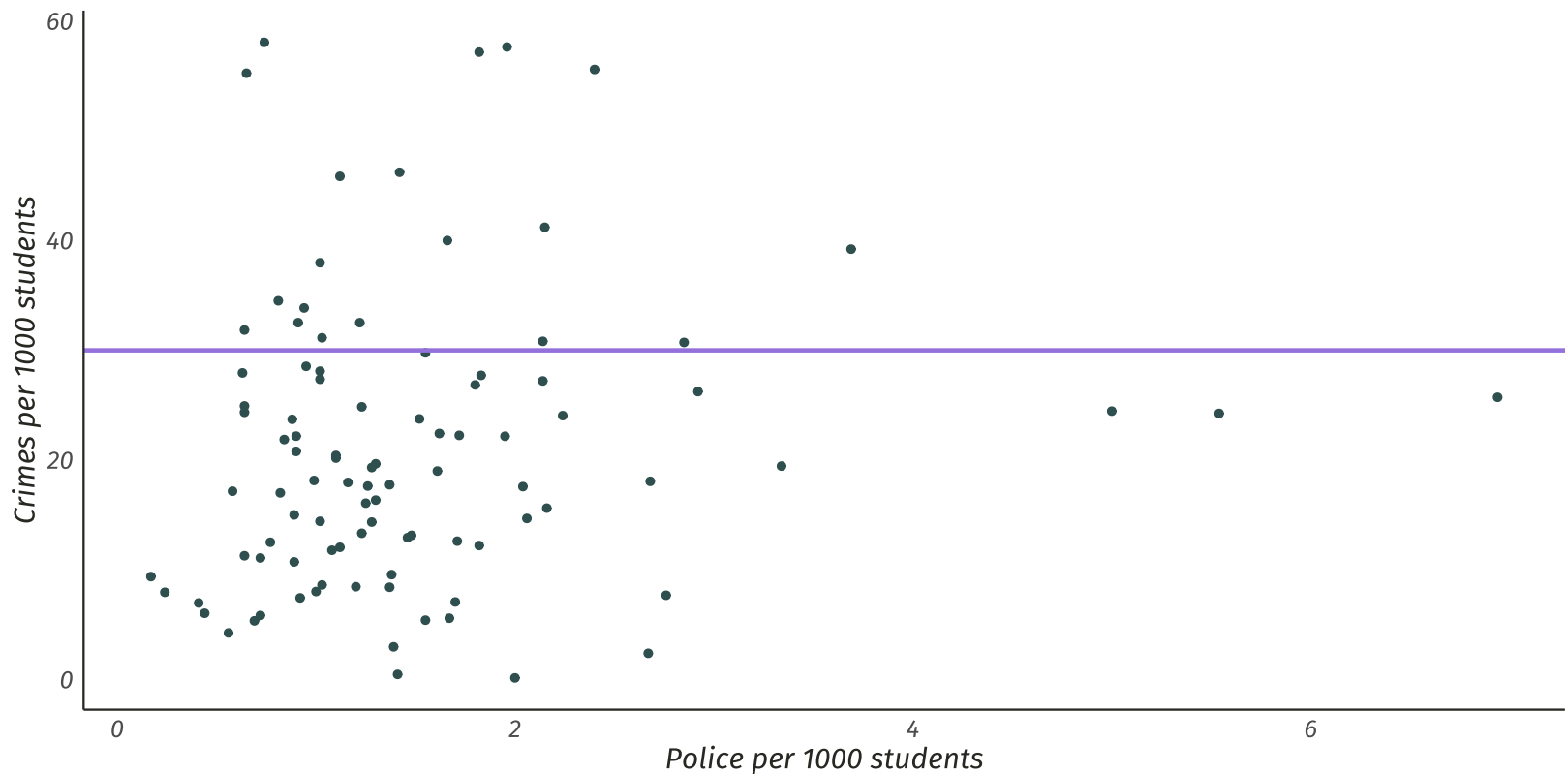
Guess:  $\hat{\beta}_1 = 60$  and  $\hat{\beta}_2 = -7$ .



# Take 4

## Example: Effect of police on crime

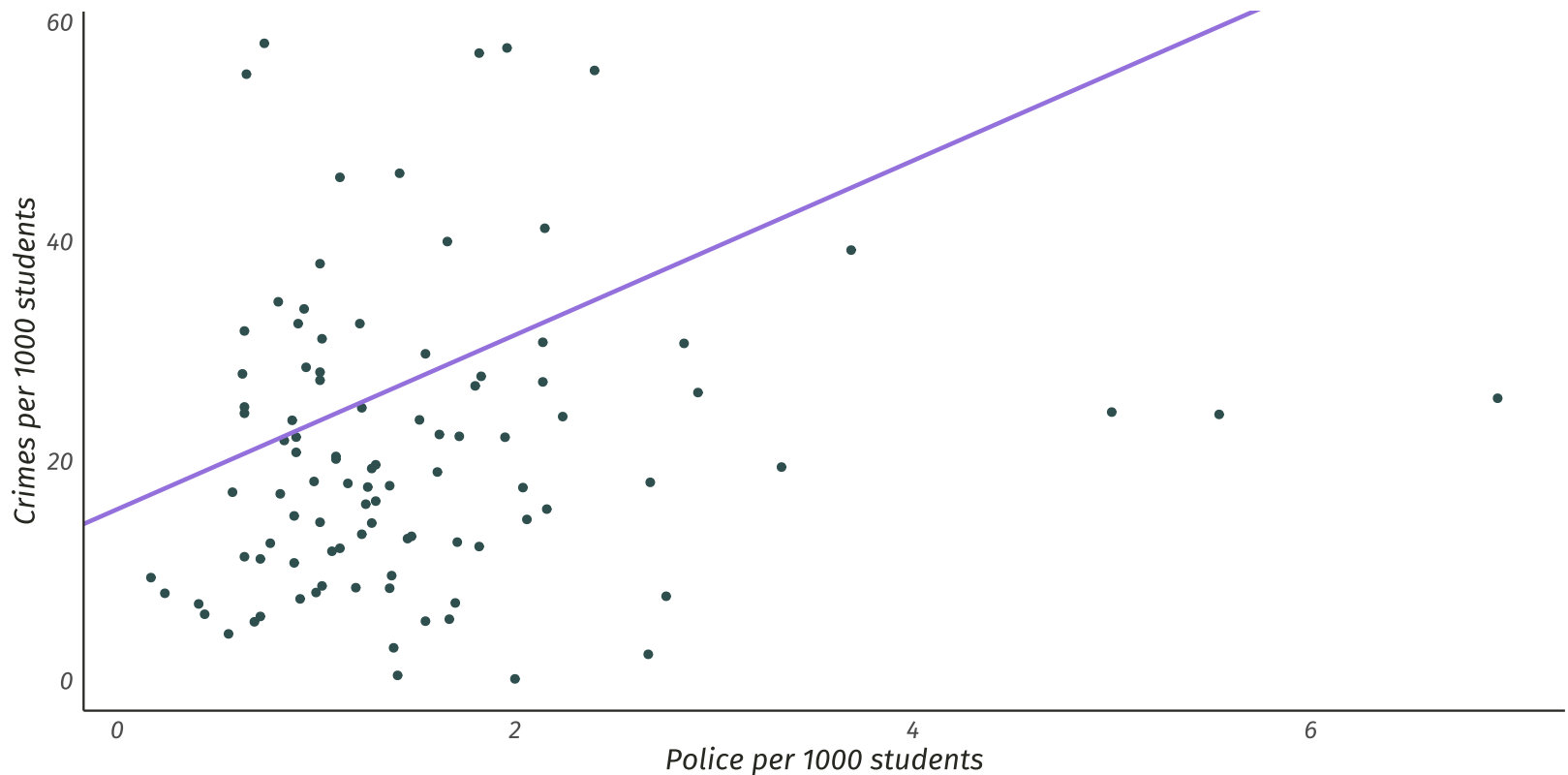
Guess:  $\hat{\beta}_1 = 30$  and  $\hat{\beta}_2 = 0$ .



# Take 5

## Example: Effect of police on crime

Guess:  $\hat{\beta}_1 = 15.6$  and  $\hat{\beta}_2 = 7.94$ .



# Residuals

Using  $\hat{\beta}_1$  and  $\hat{\beta}_2$  to make  $\hat{Y}_i$  generates misses called **residuals**:

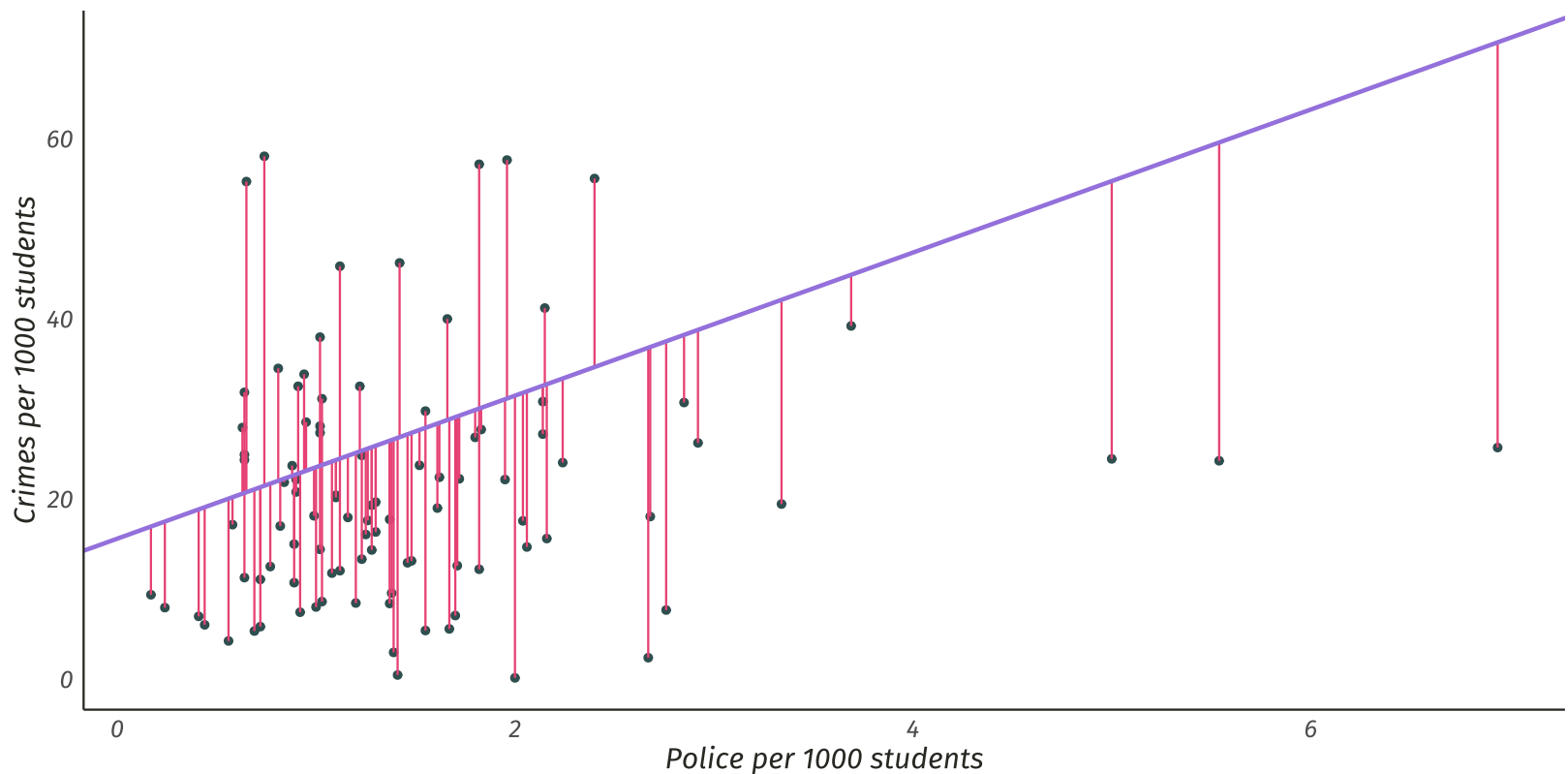
$$\hat{u}_i = Y_i - \hat{Y}_i.$$

- Sometimes called  $e_i$ .

# Residuals

## Example: Effect of police on crime

Using  $\hat{\beta}_1 = 15.6$  and  $\hat{\beta}_2 = 7.94$  to make  $\text{Crime}_i$  generates **residuals**.



# Residuals

We want an estimator that makes fewer big misses.

Why not minimize  $\sum_{i=1}^n \hat{u}_i$ ?

- There are positive *and* negative residuals  $\implies$  no solution (can always find a line with more negative residuals).

**Alternative:** Minimize the sum of squared residuals a.k.a. the **residual sum of squares (RSS)**.

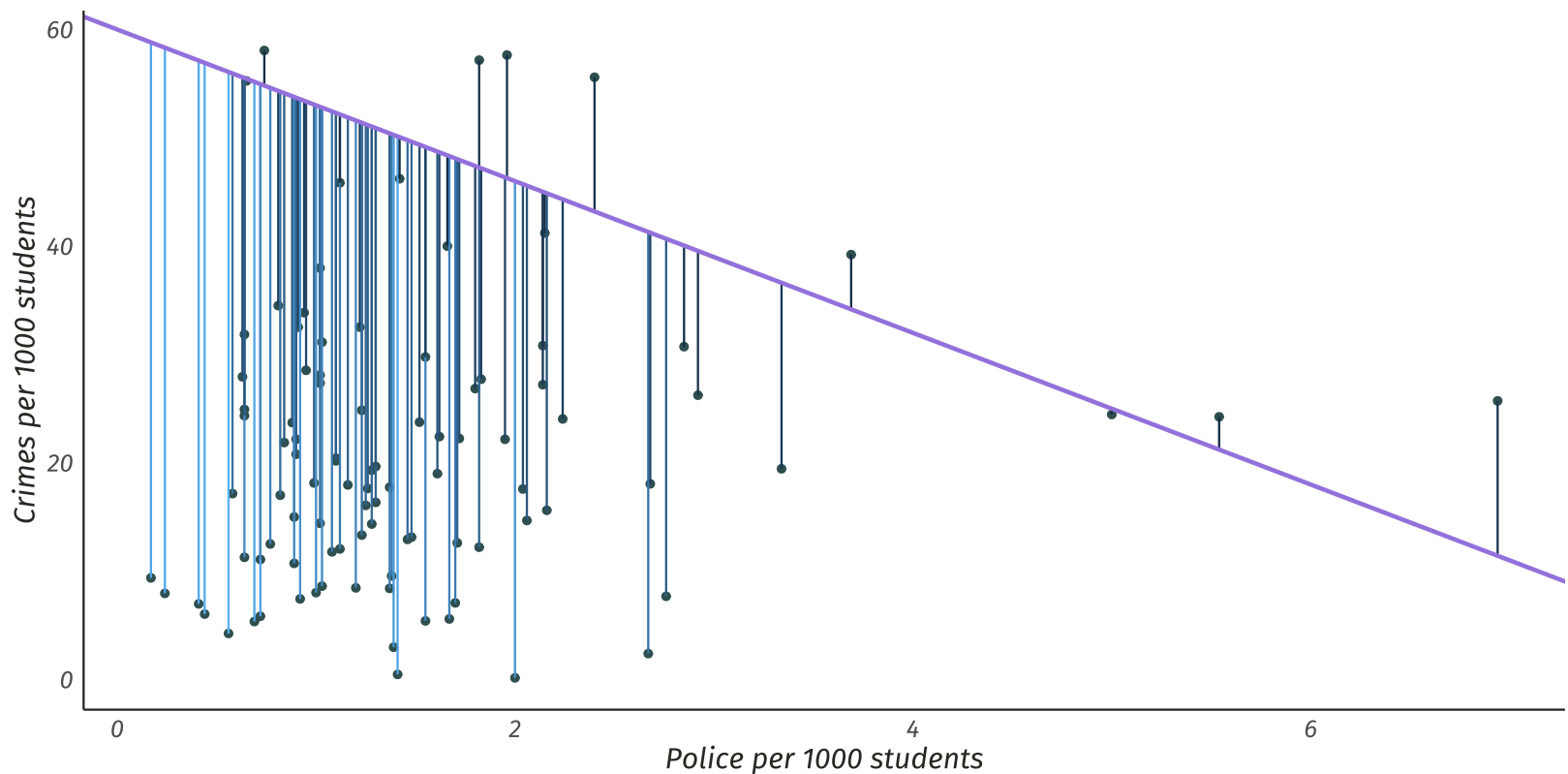
- Squared numbers are never negative.



# Residuals

## Example: Effect of police on crime

**RSS** gives bigger penalties to bigger residuals.



# Residuals

## Minimizing RSS

We could test thousands of guesses of  $\hat{\beta}_1$  and  $\hat{\beta}_2$  and pick the pair that minimizes RSS.

- Or we just do a little math and derive some useful formulas that give us RSS-minimizing coefficients without the guesswork.

# Ordinary Least Squares (OLS)

# OLS

The **OLS estimator** chooses the parameters  $\hat{\beta}_1$  and  $\hat{\beta}_2$  that minimize the **residual sum of squares (RSS)**:

$$\min_{\hat{\beta}_1, \hat{\beta}_2} \sum_{i=1}^n \hat{u}_i^2$$

This is why we call the estimator ordinary **least squares**.

# OLS Formulas

For details, see the [handout](#) posted on Canvas.

## Slope coefficient

$$\hat{\beta}_2 = \frac{\sum_{i=1}^n (Y_i - \bar{Y})(X_i - \bar{X})}{\sum_{i=1}^n (X_i - \bar{X})^2}$$

## Intercept

$$\hat{\beta}_1 = \bar{Y} - \hat{\beta}_2 \bar{X}$$

# Slope coefficient

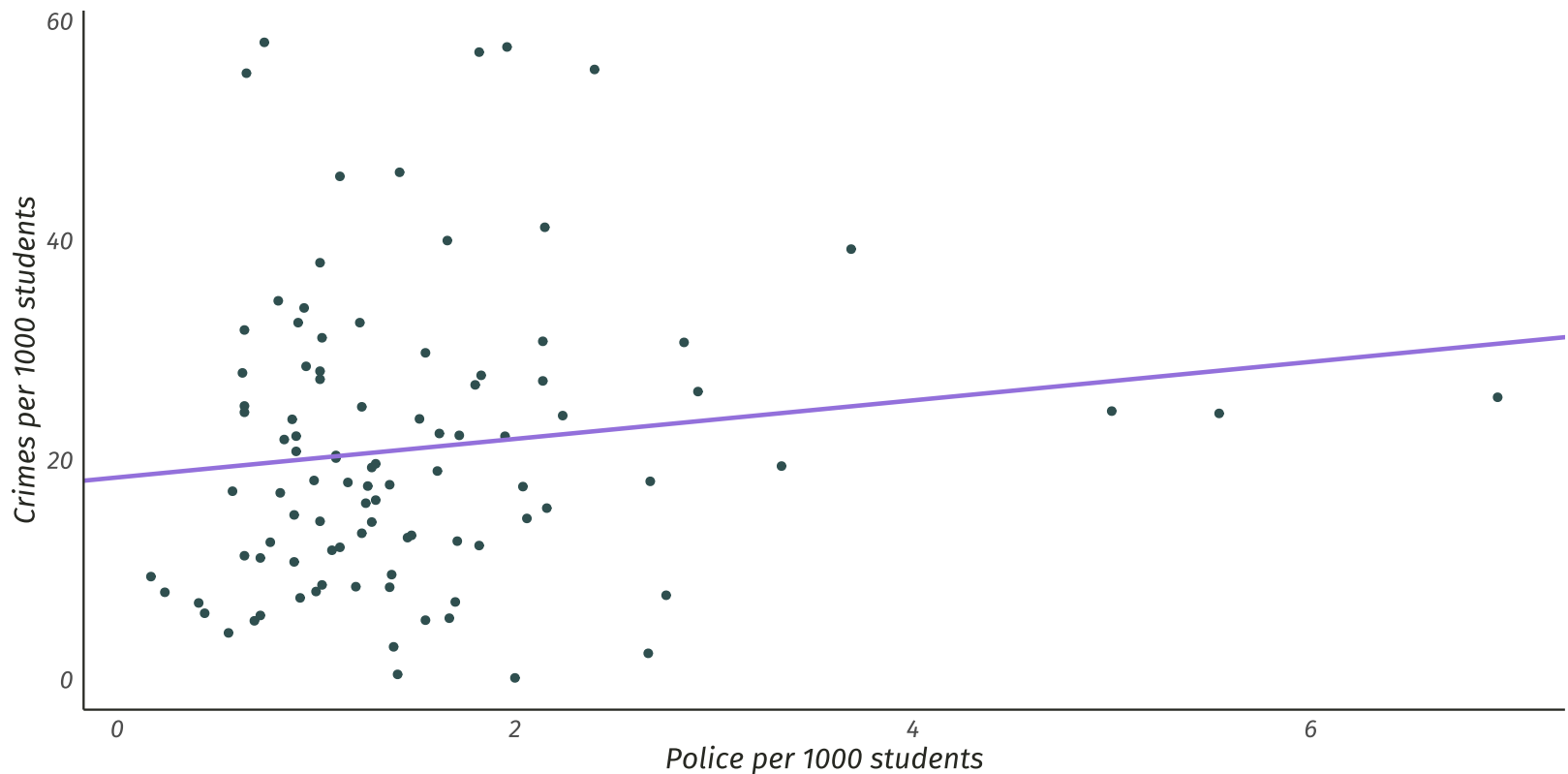
The slope estimator is equal to the sample covariance divided by the sample variance of  $X$ :

$$\begin{aligned}\hat{\beta}_2 &= \frac{\sum_{i=1}^n (Y_i - \bar{Y})(X_i - \bar{X})}{\sum_{i=1}^n (X_i - \bar{X})^2} \\&= \frac{\frac{1}{n-1} \sum_{i=1}^n (Y_i - \bar{Y})(X_i - \bar{X})}{\frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2} \\&= \frac{S_{XY}}{S_X^2}.\end{aligned}$$

# Take 6

## Example: Effect of police on crime

Using the OLS formulas, we get  $\hat{\beta}_1 = 18.41$  and  $\hat{\beta}_2 = 1.76$ .



# Coefficient Interpretation

## Example: Effect of police on crime

Using OLS gives us the fitted line

$$\text{Crime}_i = \hat{\beta}_1 + \hat{\beta}_2 \text{Police}_i.$$

What does  $\hat{\beta}_1 = 18.41$  tell us?

What does  $\hat{\beta}_2 = 1.76$  tell us?

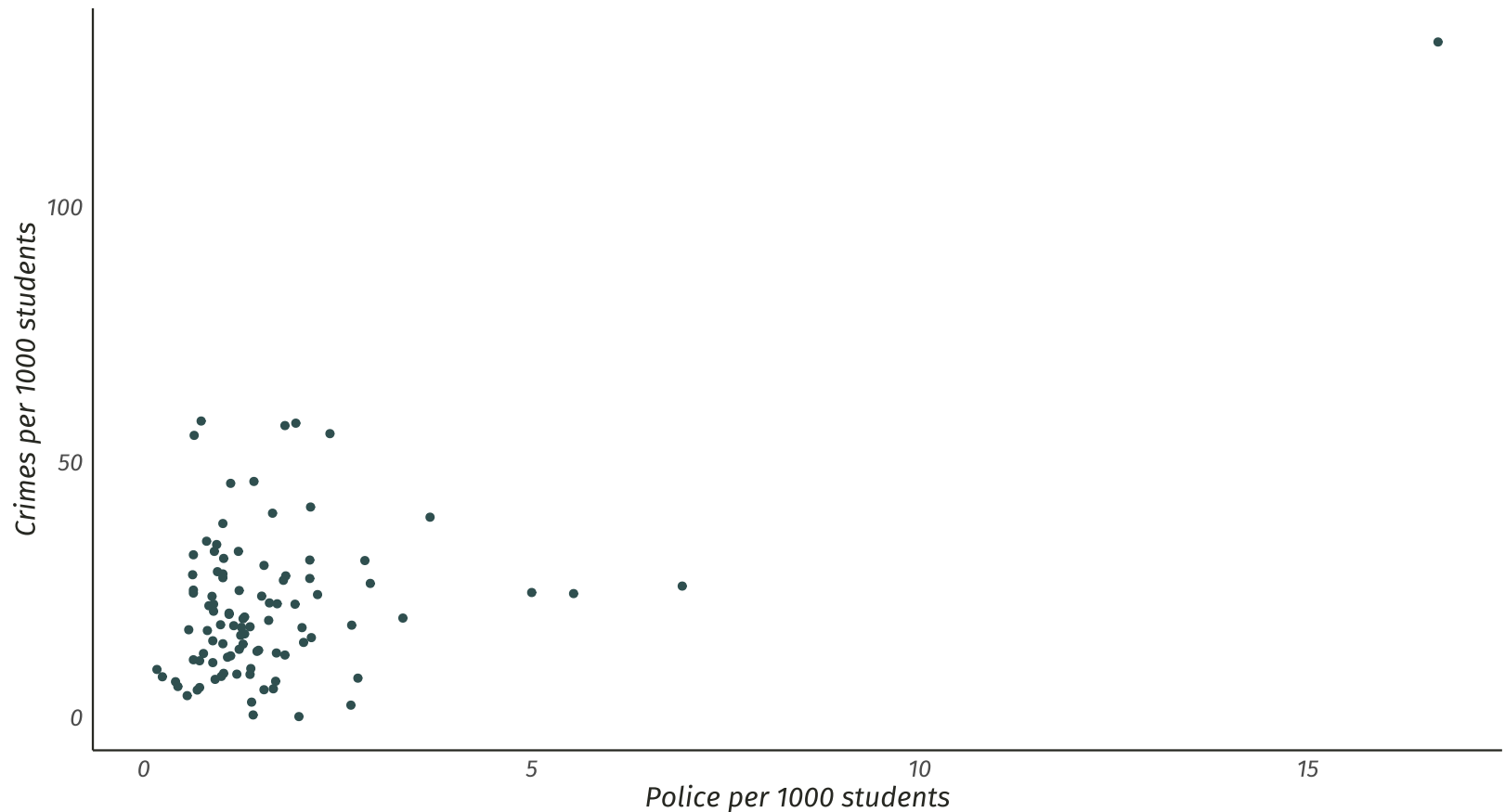
**Gut check:** Does this mean that police *cause* crime?

- Probably not. **Why?**



# Outliers

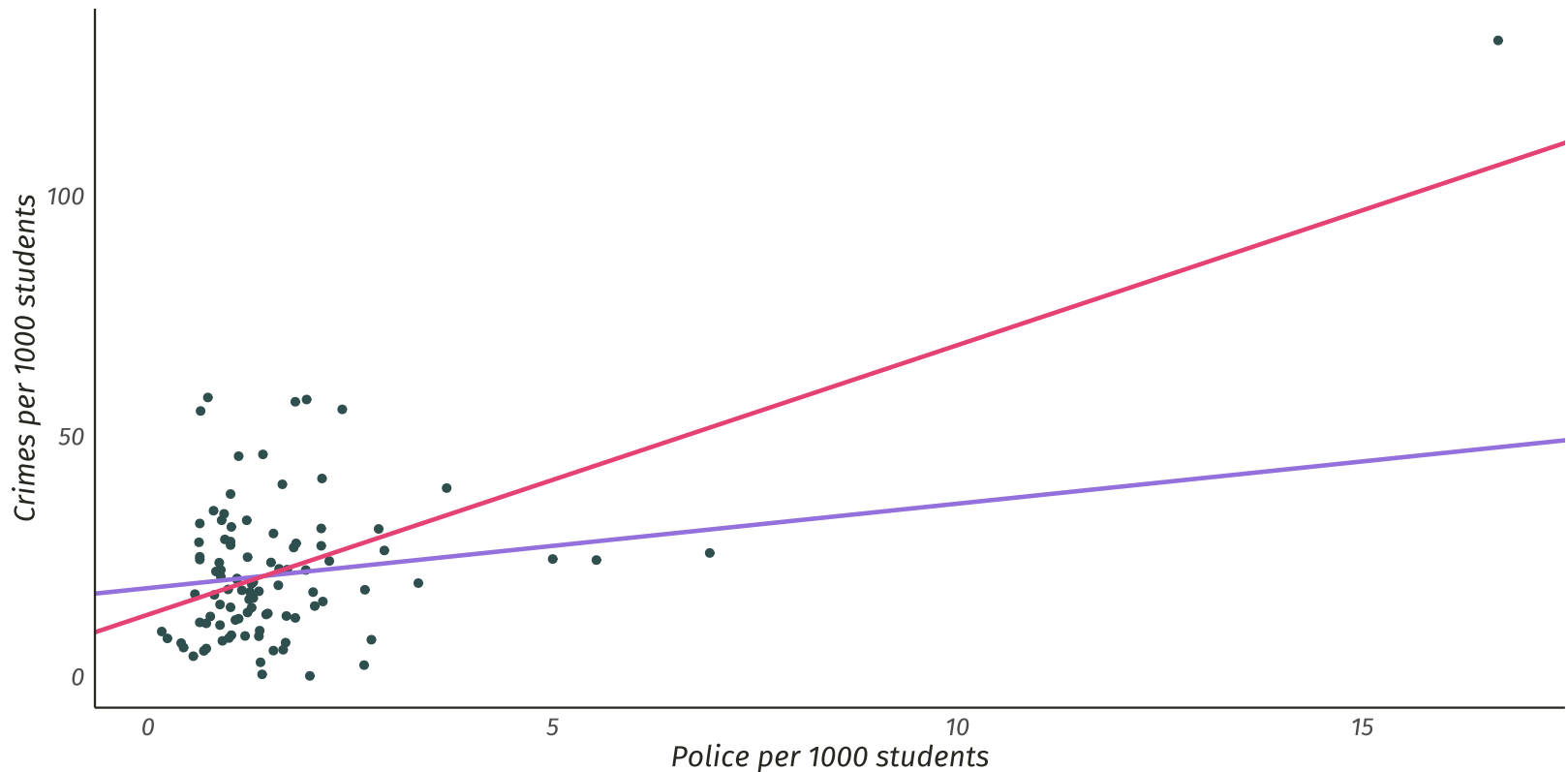
## Example: Association of police with crime



# Outliers

## Example: Association of police with crime

**Fitted line** without outlier. **Fitted line** with outlier.



# OLS Application

Suppose we do not yet have an empirical question, but wish to observe the mechanics involved in generating parameter estimates.

Consider the following **mini sample**  $\{X, Y\}$  data points:

Example:  $n = 4$

$i$	$x_i$	$y_i$
1	1	4
2	2	3
3	3	5
4	4	8

Regression Model:

$$Y_i = \beta_1 + \beta_2 X_i + u_i$$

$$\text{Fitted Line: } \hat{Y}_i = b_1 + b_2 X_i$$

Lets calculate the estimated parameters  $b_1$  and  $b_2$  using the OLS estimator

# OLS Application

Recall that OLS focuses on minimizing the RSS. We will take four steps.

1. Calculate the residuals,  $\hat{u}_i = Y_i - \hat{Y}_i$
2. Summate the squared residuals,  $RSS = \sum_{i=1}^n \hat{u}_i^2$
3. Differentiate for  $\frac{\partial RSS}{\partial b_j}$  such that our number of unknown parameters is equal to the number of partial differentiation equations
4. Solve for the unknown parameters

We'll use the **mini sample** to get an idea of the mechanics involved. Given larger datasets and more covariates, **R** comes to the rescue.

**Warning:** Check the second derivatives to ensure minimization of the functions, where all the second-order partial derivatives are greater than zero.

# OLS Application

## Step 1: Calculate the residuals

$$\hat{u}_1 = Y_1 - \hat{Y}_1 = Y_1 - b_1 - b_2 X_1$$

$$\hat{u}_2 = Y_2 - \hat{Y}_2 = Y_2 - b_1 - b_2 X_2$$

$$\hat{u}_3 = Y_3 - \hat{Y}_3 = Y_3 - b_1 - b_2 X_3$$

$$\hat{u}_4 = Y_4 - \hat{Y}_4 = Y_4 - b_1 - b_2 X_4$$

Plug in values from our given data for  $\{X, Y\}$

$$\hat{u}_1 = 4 - b_1 - 1 * b_2$$

$$\hat{u}_2 = 3 - b_1 - 2 * b_2$$

$$\hat{u}_3 = 5 - b_1 - 3 * b_2$$

$$\hat{u}_4 = 8 - b_1 - 4 * b_2$$

Next we'll square each of these terms and summate for RSS

# OLS Application

## Step 2: Calculate the RSS

$$\begin{aligned}RSS &= \sum_{i=1}^n \hat{u}_i^2 = \hat{u}_1^2 + \hat{u}_2^2 + \hat{u}_3^2 + \hat{u}_4^2 \\&= (4 - b_1 - b_2)^2 + (3 - b_1 - 2b_2)^2 + (5 - b_1 - 3b_2)^2 + (8 - b_1 - 4b_2)^2 \\&= 114 + 4b_1^2 + 30b_2^2 - 40b_1 - 114b_2 + 20b_1b_2\end{aligned}$$

Recall that OLS minimizes the RSS expression with respect to the specific parameters involved.

To find the values that minimize a particular expression, we need to apply differentiation.

# OLS Application

## Step 3: Differentiate RSS by parameters

To differentiate by a particular variable, multiply each term by its power value and subtract 1 from the power of each of its terms.

e.g. for  $y = 2x^3$ ,  $\partial y / \partial x = 2 * 3x^{3-1} = 6x^2$

$$\begin{aligned}\frac{\partial RSS}{\partial b_1} = 0 &\implies (4 * 2)b_1^{2-1} - (40 * 1)b_1^{1-1} + (20 * 1)b_1^{1-1}b_2 = 0 \\ &\implies 8b_1 - 40 + 20b_2 = 0 \qquad Eq(1)\end{aligned}$$

$$\begin{aligned}\frac{\partial RSS}{\partial b_2} = 0 &\implies (30 * 2)b_2^{2-1} - (114 * 1)b_2^{1-1} + (20 * 1)b_1b_2^{1-1} = 0 \\ &\implies 60b_2 - 114 + 20b_1 = 0 \qquad Eq(2)\end{aligned}$$

# OLS Application

## Step 4: Solve for parameters

With two unknowns  $\{b_1, b_2\}$  and two equations in which these unknowns satisfied the first order conditions  $\left\{ \frac{\partial RSS}{\partial b_1}, \frac{\partial RSS}{\partial b_2} \right\}$ , we can solve for our parameters.

How? Substitute one expression into the other.

$$20b_2 = 40 - 8b_1 \implies 60b_2 = 120 - 24b_1$$

substitute into second equation

$$Eq(2) : 120 - 24b_1 - 114 + 20b_1 = 0$$

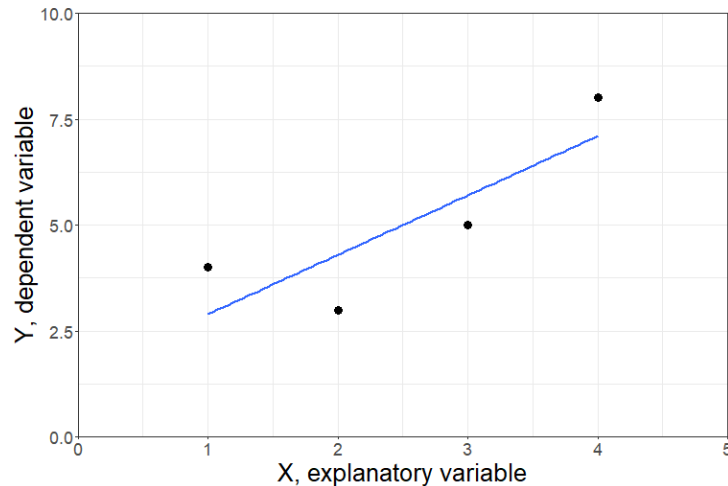
$$6 = 4b_1 \implies b_1 = 1.5$$

$$Eq(1) : 20b_2 = 40 - 8 \times 1.5 = 28 \implies b_2 = 1.4$$

OLS would prescribe  $\{1.5, 1.4\}$  for our set of parameter estimates.



# OLS Application



Fitting a line through the data points, with the aim of minimizing the RSS, results in the same implied parameters

- Such parameters will always be estimated computationally
- We will perform an exercise by hand in **PBS3** to understand the mechanics underlying the values we hang our hats on