# Managing Financial Crises *

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**Abstract**

In this paper, we revisit the question of how to manage financial crises in the workhorse framework proposed by Mendoza (2010) and Bianchi and Mendoza (2018), henceforth BM. We first show that, in this model economy, there is a multiplicity of constrained-efficient equilibria. Second, under certain conditions, there is one allocation that dominates all the others in welfare terms. This is the allocation in which there is no collateral constraint, which we call the unconstrained equilibrium (UE). Another constrained-efficient equilibrium of interest is the one selected in BM, obtained by restricting the collateral constraint multiplier to satisfy the optimality condition for the choice of intermediate input. In this allocation, the planner has only one margin left on which can operate. As a result, a tax/subsidy on debt is the only instrument available. The optimal tax/subsidy that implements the constrained-efficient allocation selected by BM has two components, an ex ante and an ex post component. We quantitatively show that both affect welfare gains, and both components yield much lower welfare gains if implemented alone. This implies that the ex ante and ex post policy interventions are complements rather than substitutes.

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# 1 Introduction

How to manage financial crises remains one of the central questions in macroeconomic policy. A time-honored perspective, dating back to Bagehot (1873), argues that crises can be simply managed ex post, by providing liquidity to solvent institutions against good collateral and at penalty rates. Early 21st-century crises proved to be very costly to manage ex post. As a result, a large literature emerged focused on policies for preventing crises or minimizing their severity, the so-called macroprudential policies.

In this paper, we study the question of how to manage financial crises within the widely used model of financial crises developed by Bianchi and Mendoza (2018)—henceforth BM—building on the framework of **?**. In this workhorse environment, crises arise when an occasionally binding borrowing constraint becomes binding, triggering a sharp contraction in economic activity. The central amplification mechanism operates through asset prices, which affect the value of collateral and thus borrowing capacity. Our focus is on the normative dimension: what should a planner do in this environment, and what policies can implement the optimal allocation?

This literature has focused on constrained efficient allocations, defined as those that maximize welfare subject to both the resource constraint and the borrowing constraint as the critical financial friction in the economy. Our first contribution is to show that, in this model, constrained-efficient allocations formulated as Markov Perfect Equilibria (MPE) exhibit multiplicity. This multiplicity stems from the forward-looking nature of asset prices, which enter the collateral constraint and influence future allocations.

Under certain regularity conditions, we also show that the unconstrained allocation emerges as the dominant constrained-efficient equilibrium (UE). The UE is the outcome that would arise in the absence of the borrowing constraint in the model economy. Given the model's structure, which extends the real business cycle framework by incorporating borrowing frictions, the unconstrained allocation is typically the best feasible outcome, conditional on the asset market structure. From a policy perspective, this implies that avoiding financial crises entirely—by eliminating the constraint—is an ideal goal.

Next, we note that one way to select a particular equilibrium from the set of constrained-efficient outcomes is to restrict the collateral constraint multiplier to satisfy the optimality condition for the choice of intermediate input, as in BM (Proposition II in Appendix). In this allocation, the planner has only one margin left on which she/he can operate. As a result, a tax/subsidy on debt is the only instrument available.

Our analysis of this particular constrained-efficient allocation, in which the planner is limited by this instrument, shows that it is generally not possible to achieve the UE, unless the working capital constraint is absent (ie, $theta = 0$), or the planner has access to a subsidy on intermediate input as a second additional instrument.

Here, we also extend the analysis in BM, which decomposes the optimal time-consistent tax on debt into two components: an ex ante, macroprudential (or crisis prevention) component that is active when the constraint is slack, and an ex post (crisis-resolution) component that operates when the constraint binds. We show that these two components are complementary and that the ex ante component is not the main source of welfare gains–if used in isolation from the ex post component, it leads to a welfare loss. The two components work in conjunction to generate the welfare gains associated with the particular constrained efficient allocation on which BM focuses, and, in general, cannot be substituted for each other. In particular, our quantitative results demonstrate that restricting policy to only one component, either the ex ante or ex post tax, leads to limited or even negative welfare gains. In contrast, the full policy, which incorporates both components, is welfare-improving.

Returning to Bagehot's principle of the late 19th century and the macro-prudential policy revolution of the early 21st century, our analysis underscores that effective financial crisis management requires macroprudential regulation and crisis resolution policies to be designed jointly, rather than in isolation. The right balance between crisis-prevention and crisis-resolution policies cannot be determined independently of the institutional and economic structure. Instead, it must reflect the specific constraints and feedback effects that shape the financial crisis dynamics.

The rest of the paper is structured as follows: Section 2 presents the model economy and its competitive equilibrium. Section 3 discusses efficiency in this model environment. Section 4 characterizes the optimal time-consistent policy that implements the constrained efficient allocation in BM. Section 5 quantitatively assesses the sources of welfare gains associated with this policy. Section 6 concludes. Proofs and computational details are reported in an Appendix.

# 2 Model

In this section, we briefly describe the model economy and define its decentralized competitive equilibrium. We refer the reader to Bianchi and Mendoza (2018) for a comprehensive description.

Consider an infinite-horizon small open economy in discrete time. The economy's exogenous state is $s_t = (z_t, R_t, \kappa_t)$, where $z_t \in [\underline{z}, \overline{z}]$ is the total factor productivity (TFP), $R_t \in [\underline{R}, \overline{R}]$ is the gross interest rate, and $\kappa_t \in [\underline{\kappa}, \overline{\kappa}]$ is the credit regime specified below. Let $S = [\underline{z}, \overline{z}] \times [\underline{R}, \overline{R}] \times [\underline{\kappa}, \overline{\kappa}]$. We assume that $S \subset \mathbb{R}^3_{++}$ is finite and $\{s_t\}_{t=0}^{\infty}$ is a stationary Markov process. We denote the histories of states as $s^t = (s_0, s_1, \dots, s_t) \in S^t$, where $S^t = S^{t-1} \times S$ for all $t > 0$ with $S^0 = \{s_0\}$. The conditional expectation operator given a specific $s^t$ is $\mathbb{E}_t$. To simplify notation, whenever possible, the history dependence is implicit.

There is a unit measure of domestic agents that are firm-households.[1] The representative agent's preferences over history-contingent sequences of consumption $c_t$ and labor $h_t$ are described by the utility function

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t - g(h_t)), \tag{1}$$

where $\beta \in (0,1)$ is the discount factor, $u : \mathbb{R}_+ \to \mathbb{R}$ is a twice continuously differentiable, strictly increasing, and strictly concave period utility function that satisfies $\lim_{x \downarrow 0} u'(x) = \infty$, and $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a twice continuously differentiable, strictly increasing, and convex labor disutility function. We assume $\beta \overline{R} < 1$, so that the equilibria we consider have well-defined stationary distributions. The GHH (Greenwood et al., 1988) preferences over a composite good $\tilde{c}_t \equiv c_t - g(h_t)$ eliminate the impact of the variations in marginal utility of consumption on the labor supply. Although this assumption can be relaxed, it significantly simplifies the theoretical analysis.

The agent produces the final good from capital $k_t$, labor, and intermediate inputs $v_t$ using a concave Cobb—Douglas production function $F : \mathbb{R}^3_+ \to \mathbb{R}_+$. The initial level of capital is $k_0 = 1$. The price of capital is $q_t$, while the price of internationally traded inputs is fixed at $p_v > 0$. The agent can invest in a one-period bond $b_{t+1}$ traded internationally at the price $1/R_t$, with $b_0 \in B = [\underline{b}, \overline{b}] \subset \mathbb{R}$ given. The agent's budget constraint is thus

$$c_t + q_t k_{t+1} + \frac{b_{t+1}}{R_t} \leq z_t F(k_t, h_t, v_t) - p_v v_t + q_t k_t + b_t. \tag{2}$$

The agent can issue debt ($b_{t+1} < 0$) and must finance a fraction $\theta \in [0,1]$ of intermediate inputs in advance with an intraperiod loan at a zero interest rate. Borrowing requires collateral in the form of the capital stock, but only a fraction $\kappa_t$ of the value of capital is pledgeable. Consequently, the agent faces a collateral constraint[2]

$$-\frac{b_{t+1}}{R_t} + \theta p_v v_t \leq \kappa_t q_t k_t. \tag{3}$$

The agent's problem is to choose $\{(c_t, h_t, v_t, b_{t+1}, k_{t+1})\}_{t=0}^{\infty}$ to maximize (1) subject to (2)

[1] Bianchi and Mendoza (2018, Appendix C) show that there exists an equivalent environment with separate households and firms.

[2] Bianchi and Mendoza (2018, Appendix A.5) provide a microfoundation through a limited enforcement problem.

and (3) for all $(t, s^t)$. The first-order conditions for this problem are

$$g'(h_t) = z_t F_h(k_t, h_t, v_t), \tag{4}$$

$$\left(1 + \theta \frac{\mu_t}{u'(\tilde{c}_t)}\right) p_v = z_t F_v(k_t, h_t, v_t), \tag{5}$$

$$u'(\tilde{c}_t) = \beta R_t \mathbb{E}_t u'(\tilde{c}_{t+1}) + \mu_t, \tag{6}$$

$$q_t u'(\tilde{c}_t) = \beta \mathbb{E}_t \left[ u'(\tilde{c}_{t+1}) \Big( z_{t+1} F_k(k_{t+1}, h_{t+1}, v_{t+1}) + q_{t+1} \Big) + \mu_{t+1} \kappa_{t+1} q_{t+1} \right], \tag{7}$$

$$0 = \mu_t \left( \kappa_t q_t k_t + \frac{b_{t+1}}{R_t} - \theta p_v v_t \right), \qquad \mu_t \geq 0, \tag{8}$$

where $\mu_t$ is the Lagrange multiplier on the collateral constraint (3). The equation (4) equates the marginal rate of substitution of leisure for consumption with the marginal product of labor. According to (5), the working capital and collateral constraints introduce a wedge between the marginal product of intermediate inputs and their price. When the collateral constraint binds, and as long as $\theta > 0$, an increase in inputs must be compensated by a decrease in borrowing, thus raising the marginal cost of inputs compared to the case of no working capital constraint ($\theta = 0$). (6) is a standard bond Euler equation: when the borrowing constraint binds, other things equal, the agent's marginal utility of consumption today is greater than in the unconstrained case. At the same time, the collateral constraint introduces an additional marginal benefit of capital, since greater capital allows to borrow more when the constraint binds. This is captured in the asset pricing condition (7): if we solve it forward, we can express the asset price $q_t$ as an expected discounted sum of dividends, where the discounting is adjusted to include the collateral value. Finally, (8) comprises the complementary slackness conditions associated with the collateral constraint (3).

The capital stock is in fixed supply normalized to one. We define a decentralized competitive equilibrium (**DE**) as follows.

**Definition 1** (Competitive equilibrium)**.** *A decentralized competitive equilibrium is an allocation $\{(c_t, h_t, v_t, b_{t+1}, k_{t+1})\}_{t=0}^{\infty}$, prices $\{q_t\}_{t=0}^{\infty}$, and Lagrange multipliers $\{\mu_t\}_{t=0}^{\infty}$, such that the following holds.*

1. *Given prices, the allocation solves the agent's problem: that is, together with Lagrange multipliers, it satisfies (2) holding with equality and (3)–(8) for all $t \geq 0$ and $s^t \in S^t$.*

2. *Prices are such that the capital market clears: $k_{t+1}(s^t) = 1$ for all $t \geq 0$ and $s^t \in S^t$.*

Going forward, it will be useful to represent the DE using the recursive notation. We denote the aggregate state as $x = (b, s) \in X = B \times S$, where $b \in B$ is aggregate bond holdings, and $s = (z, R, \kappa) \in S$ is the exogenous state. We denote the conditional expectation operator given $s \in S$ as $\mathbb{E}_s$ and use $y(x)$ interchangeably with $y_x$ to denote the value of a variable $y$ at the state $x \in X$. Let $\operatorname{int} Y$ and $\operatorname{cl} Y$ denote the interior and closure, respectively, of a generic set $Y$. Clearly, $B = [\underline{b}, \overline{b}]$ needs to be large enough, so that $b_{t+1}(s^t) \in \operatorname{int} B$ for all $t \geq 0$ and $s^t \in S^t$, although $\underline{b}$ cannot be too low due to the collateral constraint. An admissible $B$ can be found numerically, given a specific model calibration.

Let $\mathcal{F}(X)$ denote the set of all real-valued functions on $X$. Imposing the capital market clearing condition in (2)–(8), we obtain the following recursive representation of the DE.

**Remark 1** (Recursive equilibrium)**.** *A DE of Definition 1 is, equivalently, a set of allocation functions $\{\tilde{c}, h, v, b'\} \subset \mathcal{F}(X)$, an asset price function $q \in \mathcal{F}(X)$, and a Lagrange multiplier function $\mu \in \mathcal{F}(X)$ that satisfy*

$$\tilde{c}_x + \frac{b'_x}{R} = zF(1, h_x, v_x) - p_v v_x - g(h_x) + b, \tag{9}$$

$$-\frac{b'_x}{R} + \theta p_v v_x \leq \kappa q_x, \tag{10}$$

$$g'(h_x) = zF_h(1, h_x, v_x), \tag{11}$$

$$\left(1 + \theta \frac{\mu_x}{u'(\tilde{c}_x)}\right) p_v = zF_v(1, h_x, v_x), \tag{12}$$

$$u'(\tilde{c}_x) = \beta R \mathbb{E}_s u'(\tilde{c}_{x'}) + \mu_x \qquad \text{if } b'_x \in \operatorname{int} B, \tag{13}$$

$$q_x u'(\tilde{c}_x) = \beta \mathbb{E}_s \left[ u'(\tilde{c}_{x'}) \Big( z' F_k(1, h_{x'}, v_{x'}) + q_{x'} \Big) + \mu_{x'} \kappa' q_{x'} \right], \tag{14}$$

$$0 = \mu_x \left( \kappa q_x + \frac{b'_x}{R} - \theta p_v v_x \right), \qquad \mu_x \geq 0, \tag{15}$$

*with $x' = (b'_x, (z', R', \kappa'))$, for all $x = (b, s) = (b, (z, R, \kappa)) \in X$.*

# 3 Efficiency Analysis

This section explores the properties of efficient and constrained-efficient allocations in the model environment described in Section 2.

We begin by defining the unconstrained allocation (UE) chosen by a benevolent social planner that makes decisions subject to the economy's resource constraint only. Throughout the paper, we also call the UE allocation "first-best". We show that the DE has two distortions relative to the UE that arise due to the collateral constraint. First, provided the collateral constraint may bind in some states, the DE features inferior consumption smoothing. Second, under the working capital constraint, the DE entails an inefficiently low level of intermediate inputs. If there were no collateral constraint, the DE would coincide with the UE.

We then define a time-consistent constrained-efficient allocation as part of a Markov perfect equilibrium of a noncooperative game between successive benevolent social planners who face the same constraints as the representative agent but internalize the impact of allocations on the market price of capital. We argue that constrained-efficient equilibria are generally not unique in this model economy, and there may exist a Markov perfect equilibrium that entails the unconstrained allocation. Finally, if we drop the Markov perfection requirement and allow the planner's decisions to be history-contingent, we find that, under certain conditions, the unique constrained-efficient allocation is the unconstrained allocation, and any constrained-efficient plan is time consistent.

## 3.1 Unconstrained Allocation

The efficient ("unconstrained") allocation is the allocation chosen by a benevolent social planner maximizing the household's lifetime utility subject to resource constraints.[3] We define it in recursive form as follows.

**Definition 2** (Unconstrained Allocation)**.** *The unconstrained allocation is the set of functions $\{\tilde{c}, h, v, b'\} \subset \mathcal{F}(X)$ generated by the solution to the Bellman equation*

$$V(b,s) = \max_{\hat{c},\hat{h},\hat{v},\hat{b}} \Big[ u(\hat{c}) + \beta \mathbb{E}_s V(\hat{b}, s') \Big],$$

*subject to*

$$\hat{c} + \frac{\hat{b}}{R} \leq zF(1,\hat{h},\hat{v}) - p_v \hat{v} - g(\hat{h}) + b,$$

*for all $(b,s) = (b,(z,R,\kappa)) \in X$.*

Importantly, the unconstrained problem in Definition 2 is not affected by the collateral constraint. In particular, in this case, $\kappa$ is a redundant exogenous state. We obtain the following characterization of the unconstrained allocation.

**Proposition 1** (Unconstrained allocation)**.** *Let $f(z) = \max_{\hat{h},\hat{v}}\{zF(1,\hat{h},\hat{v}) - p_v\hat{v} - g(\hat{h})\}$. Suppose $\underline{b} > -\frac{\overline{R}}{\overline{R}-1} f(\underline{z})$ if $\overline{R} > 1$ and $\underline{b} < \frac{\underline{R}}{1-\underline{R}} f(\underline{z})$ if $\underline{R} < 1$. Then there exists a unique*

[3] Since financial markets are incomplete, this allocation may be considered a *constrained Pareto optimum* (Diamond, 1967). We refer to it as efficient because we are going to define a *constrained-efficient* allocation as an allocation chosen by a planner who is subject to the resource, collateral, and asset pricing constraints. If the government could introduce additional assets, welfare could be improved. In particular, if the domestic economy had access to a complete set of Arrow—Debreu securities, the efficient allocation would entail the perfect consumption risk sharing between the domestic economy and the rest of the world.

*solution to the Bellman equation in Definition 2, and the unconstrained allocation functions* $\{\tilde{c}, h, v, b'\} \subset \mathcal{F}(X)$ *are continuous and satisfy (9), (11),*

$$p_v = zF_v(1, h_x, v_x), \tag{16}$$

$$u'(\tilde{c}_x) = \beta R \mathbb{E}_s u'(\tilde{c}_{x'}) \qquad \text{if } b'_x \in \operatorname{int} B, \tag{17}$$

*with* $x' = (b'_x, (z', R', \kappa'))$, *for all* $x = (b, s) = (b, (z, R, \kappa)) \in X$. *The function* $\tilde{c}$ *is strictly increasing in* $b$, *and* $b'$ *is strictly increasing in* $b$ *whenever* $b'_x \in \operatorname{int} B$. *The functions* $h$ *and* $v$ *depend on* $z$ *only, and* $h$, $v$, *and* $f$ *are strictly increasing in* $z$.

*Proof.* See Appendix A.1. ■

Proposition 1 first provides restrictions on the admissible set of bond holdings $B = [\underline{b}, \overline{b}]$. In particular, if $\overline{R} > 1$, the lower bound $\underline{b}$ cannot be below than minus the natural borrowing limit $\frac{\overline{R}}{\overline{R}-1} f(\underline{z})$. Since the choice of labor and intermediate inputs is static, the planner's problem is similar to an income fluctuation problem (Schechtman and Escudero, 1977) with a stochastic endowment $f(z)$, where $f$ is strictly increasing, and a stochastic interest rate $R$. Our assumption $\beta\overline{R} < 1$ ensures the existence of a well-defined ergodic distribution of bond holdings, which can be shown analytically under further assumptions on $u$, $R$, and $z$, or otherwise verified numerically.[4] This means that the bond holdings upper bound $\overline{b}$ can be set to a sufficiently big number, such that it never binds.

According to Proposition 1, labor $h$ and intermediate inputs $v$ are jointly defined by (11) and (16). Hence, $h$ and $v$ are independent of bond holdings $b$ and only vary with the TFP $z$. Moreover, both $h$ and $v$, and thus output $zF(1, h(z), v(z))$, are strictly increasing in $z$. In the unconstrained allocation, next-period bond holdings $b'$ are pinned down by the Euler equation (17) whenever $b'_x > \underline{b}$, given that $\overline{b}$ can be chosen such that it never binds. [5]

As long as bond holdings follow the Euler equation (17) in the unconstrained allocation, next-period bond holdings $b'$ are strictly increasing in current bond holdings $b$. The absence of the collateral constraint is, of course, crucial for this fact. In particular, the DE policy function $b'$ can be shown to be strictly increasing only in the region where the collateral constraint is slack but strictly *decreasing* in the binding region. In contrast, the net consumption function $\tilde{c}$ is strictly increasing in both the unconstrained (for all $x \in X$, as shown in Proposition 1) and in the DE (can be shown numerically).

[4]If $u$ has constant relative risk aversion form and $R$ is deterministic, a sufficient condition is that $z$ is either independent and identically distributed (Schechtman and Escudero, 1977) or $z \in \{\underline{z}, \overline{z}\}$ with $\Pr(z = \overline{z} \mid z = \overline{z}) \geq \Pr(z = \overline{z} \mid z = \underline{z})$ (Huggett, 1993).

[5]Specifically, we can verify numerically that if $\underline{b}$ is sufficiently close to minus the natural borrowing limit, we have $b'_x > \underline{b}$ for all $x \in X$ except $x = (\underline{b}, (\underline{z}, \overline{R}, \cdot))$, given an arbitrarily fine grid for $B$. Moreover, $b_t > \underline{b}$ for all $t \geq 0$ over a 100,000-period stochastic simulation after a 1,000-period burn-in.

Proposition 1 implies that the DE of Remark 1 has two distortions, both due to the collateral constraint. Note that both the DE and the unconstrained allocation share the same labor optimality condition (11). Comparing (12) and (16), we observe that, if there is a working capital constraint ($\theta > 0$), intermediate inputs are inefficiently low in the DE whenever the collateral constraint is strictly binding ($\mu_x > 0$). In turn, lower inputs lead to lower labor and output. At the same time, if the collateral constraint is slack, labor and inputs in the DE are efficient, since they are jointly defined by (11) and (16), as in the unconstrained allocation. Moreover, (13) and (17) imply that net consumption is inefficiently low in the DE whenever $\mu_x > 0$, indicating inferior consumption smoothing. If the collateral constraint is *never* binding in the DE, its allocation is efficient, since in that case (12) is equivalent to (16) and (13) is equivalent to (17).

**Remark 2** (Unconstrained equilibrium)**.** *Consider a DE of Remark 1. If $\mu_x = 0$ for all $x \in X$, the DE allocation is efficient in the sense defined above.*

## 3.2 Constrained-Efficient equilibria

In general, policy constraints may prevent the possibility of implementing the unconstrained allocation in a competitive equilibrium. For this reason, an alternative concept of *constrained* efficiency has received significant attention. Constrained efficiency in environments with price-dependent borrowing constraints has been studied, for instance, by Kehoe and Levine (1993), Caballero and Krishnamurthy (2001), Lorenzoni (2008), and Dávila and Korinek (2018). In turn, this theory is related to the earlier analysis of constrained efficiency in incomplete-market economies by Diamond (1967), Hart (1975), Stiglitz (1982), and Geanakoplos and Polemarchakis (1985). A unified treatment was given by Farhi and Werning (2016).

A constrained-efficient allocation is an allocation chosen by a social planner (**SP**) that faces the same constraints as the representative agent but internalizes the impact of allocations on the market price of capital. Hence, such a social planner maximizes the agent's utility subject to the aggregate budget constraint (9), collateral constraint (10), and the asset pricing constraint (14). In addition, since the planner faces the same collateral constraint as the agent, and the market price of capital depends on whether the collateral constraint is binding in the future, as reflected by the Lagrange multiplier $\mu_{x'}$ affecting the next-period payoff on capital in (14), any SP allocation that can be implemented in a competitive equilibrium with some government policies must satisfy the representative agent's complementary slackness conditions (15).

Since the asset pricing constraint (14) is forward-looking, the issue of *time consistency* arises. We next define a time-consistent constrained-efficient allocation as part of a *Markov* *perfect equilibrium* (**MPE**) of a noncooperative policy game between successive benevolent social planners (Maskin and Tirole, 1988, 2001; Krusell et al., 1996; Klein et al., 2008).

**Definition 3** (Constrained-efficient MPE)**.** *A constrained-efficient MPE is a set of allocation functions* $\{\tilde{c}, h, v, b'\} \subset \mathcal{F}(X)$*, an asset price function* $q \in \mathcal{F}(X)$*, and a Lagrange multiplier function* $\mu \in \mathcal{F}(X)$ *that satisfy the following.*

1. *Given a value function* $V \in \mathcal{F}(X)$*,*

$$(\tilde{c}_x, h_x, v_x, b'_x, q_x, \mu_x) \in \underset{\hat{c},\hat{h},\hat{v},\hat{b},\hat{q},\hat{\mu}}{\arg\max} \Big[u(\hat{c}) + \beta \mathbb{E}_s V(\hat{b}, s')\Big]$$

*subject to*

$$\begin{aligned} \hat{c} + \frac{\hat{b}}{R} &\leq zF(1, \hat{h}, \hat{v}) - p_v \hat{v} - g(\hat{h}) + b, \\ -\frac{\hat{b}}{R} + \theta p_v \hat{v} &\leq \kappa \hat{q}, \\ \hat{q} u'(\hat{c}) &= \beta \mathbb{E}_s \left[ u'(\tilde{c}_{x'}) \Big( z' F_k(1, h_{x'}, v_{x'}) + q_{x'} \Big) + \mu_{x'} \kappa' q_{x'} \right], \\ 0 &= \hat{\mu} \left( \kappa \hat{q} + \frac{\hat{b}}{R} - \theta p_v \hat{v} \right), \qquad \hat{\mu} \geq 0, \end{aligned}$$

*with* $x' = (\hat{b}, (z', R', \kappa'))$*, for all* $x = (b, s) = (b, (z, R, \kappa)) \in X$*.*

2. *The value function satisfies* $V(b, s) = u(\tilde{c}_x) + \beta \mathbb{E}_s V(b'_x, s')$ *for all* $x = (b, s) \in X$*.*

The two conditions in Definition 3 jointly define an MPE in $\{\tilde{c}, h, v, b', q, \mu\} \subset \mathcal{F}(X)$ with the associated value $V \in \mathcal{F}(X)$. The first condition postulates that it is suboptimal to deviate from $\{\tilde{c}, h, v, b', q, \mu\}$—the functions used to evaluate the conditional expectation in the asset pricing constraint (14)—in any state $x \in X$. The second condition postulates that the value function $V$ solves the Bellman equation.

A brief inspection of the planner's constraints suggests that a time-consistent SP allocation need not be unique. When the collateral constraint is binding, any $\hat{\mu} \geq 0$ is a feasible choice for the current planner, because $\hat{\mu}$ affects only the representative agent's complementary slackness conditions. Since $\hat{\mu}$ does not directly affect the planner's payoff, the planner is indifferent between any $\hat{\mu} \geq 0$. However, in an MPE, $\hat{\mu} = \mu$, and $\mu$ affects the SP asset price function $q$, and thus—through its effect on the value of collateral—the SP allocation functions $\{\tilde{c}, h, v, b'\}$. Consequently, there generally exist multiple welfare-ranked MPE. In particular, if there exists an MPE that entails $\mu_x = 0$ for all $x \in X$, it is the worst MPE, since it implies the lowest asset prices.

**Remark 3** (MPE multiplicity)**.** *There generally exist multiple welfare ranked constrained-efficient MPE of Definition 3.*

Bianchi and Mendoza (2018) studied an MPE of Definition 3 under the assumption that the Lagrange multiplier function $\mu$ is restricted to satisfy the DE input optimality condition (12). With $\theta > 0$, this additional constraint (i.e. (12)) selects a *specific* MPE by leaving the social planner with a single degree of freedom relative to the DE outcome—namely, improving the allocation of bond holdings.[6] We refer the reader to their in-depth analysis of the resulting constrained-efficient allocation.

As we show next, however, there may exist a constrained-efficient MPE that entails the *unconstrained* allocation.

## 3.3 The Unconstrained Allocation as a Constrained-Efficient Plan

In this subsection, we begin by characterizing the unconstrained equilibrium as a Markov Perfect, constrained-efficient allocation. We then identify the necessary conditions for the existence of such an allocation. Following this, we turn to the time-consistent (non-Markovian) allocation plan and derive the conditions under which the unconstrained allocation can also be supported as a time-consistent equilibrium.

Let $\left\{\tilde{c}^{\mathrm{FB}}, h^{\mathrm{FB}}, v^{\mathrm{FB}}, b^{\mathrm{FB}}\right\} \subset \mathcal{F}(X)$ be the unconstrained allocation functions of Definition 2. The best possible (i.e., welfare dominant) constrained-efficient MPE is an MPE that entails the unconstrained allocation functions.

**Definition 4** (Unconstrained MPE)**.** *A constrained-efficient MPE of Definition 3 is an Unconstrained MPE if* $\{\tilde{c}, h, v, b'\} = \left\{\tilde{c}^{FB}, h^{FB}, v^{FB}, b^{FB}\right\}$.

According to Remark 2, the unconstrained allocation functions are part of the DE in the economy without the collateral constraint. Let us denote the asset price function in this economy as $q^{\mathrm{FB}} \in \mathcal{F}(X)$. It is defined by (14) evaluated at the unconstrained allocation

[6] In the analysis of Bianchi and Mendoza (2018), the planner is, moreover, subject to the DE labor optimality condition (11). They show in Appendix A.1 (Proposition II) that (11), (12), and (15) are slack constraints in the sense that (11) is satisfied at the SP allocation and, if $\theta > 0$, (12) can be used to construct $\mu$ that satisfies (15) at the SP allocation. This construction selects a specific MPE of Definition 3, since, for a given allocation, many functions $\mu$ satisfy (15), but only one of them satisfies (12) if $\theta > 0$.

with $\mu_x = 0$ for all $x \in X$:

$$
\begin{aligned}
q_x^{\mathrm{FB}} &= \beta \mathbb{E}_s \left[ \frac{u'\left(\tilde{c}_{x'}^{\mathrm{FB}}\right)}{u'\left(\tilde{c}_x^{\mathrm{FB}}\right)} \left( z' F_k \left(1, h_{x'}^{\mathrm{FB}}, v_{x'}^{\mathrm{FB}}\right) + q_{x'}^{\mathrm{FB}} \right) \right] \\
&= \sum_{t=1}^{\infty} \beta^t \sum_{s^t \in S^t} \Pr\bigl(s^t \mid s\bigr) \frac{u'\left(\tilde{c}_{x(s^t)}^{\mathrm{FB}}\right)}{u'\left(\tilde{c}_x^{\mathrm{FB}}\right)} z_t F_k \left(1, h_{x(s^t)}^{\mathrm{FB}}, v_{x(s^t)}^{\mathrm{FB}}\right), \qquad (18)
\end{aligned}
$$

where $x(s^t) = \bigl(b^{\mathrm{FB}}(x(s^{t-1})), s_t\bigr)$ for all $t \geq 1$ and $s^t \in S^t$, with $x(s^0) = x = (b, s) \in X$. Hence, $q^{\mathrm{FB}}$ is the present discounted value of dividends evaluated at the unconstrained allocation.

Let us define the set of states $A \subset X$ at which the collateral constraint would be violated if it were imposed in the unconstrained DE:

$$
A = \left\{ x \in X \;\middle|\; q_x^{\mathrm{FB}} < q_x^A \right\}, \qquad (19)
$$

where $q^A \in \mathcal{F}(X)$ is the asset price at which the collateral constraint binds:

$$
q_x^A = \frac{1}{\kappa} \left( -\frac{b_x^{\mathrm{FB}}}{R} + \theta p_v v_x^{\mathrm{FB}} \right). \qquad (20)
$$

Let $A^c = X \setminus A$ denote the complement of $A$. If $A = \emptyset$ (equivalently, $A^c = X$), the collateral constraint is irrelevant in the DE, and the DE allocation coincides with the unconstrained allocation (Remark 2). Of course, the interesting (and empirically relevant) case is $A \neq \emptyset$.

Let $\mathbf{1}_Y$ denote the indicator function of a generic set $Y$. If the social planner could control the asset price directly, it would be sufficient to set $q_x = \mathbf{1}_A(x) q_x^A + \mathbf{1}_{A^c}(x) q_x^{\mathrm{FB}}$ to satisfy the collateral constraint for all $x \in X$ at the unconstrained allocation. (One could interpret the unconstrained allocation of Definition 2 as precisely this planning arrangement.) A constrained-efficient social planner of Definition 3, however, does not have such powers, and must abide by the asset pricing constraint (14). Inspecting the latter, we obtain the following necessary conditions for the existence of a unconstrained MPE.

**Proposition 2** (Unconstrained MPE existence)**.** *There exists an unconstrained MPE of Definition 4 only if* $q_x \geq \max\left\{q_x^{FB}, q_x^A\right\}$ *and* $\mu_x \left(q_x - q_x^A\right) = 0$ *for all* $x \in X$*, and for all* $x = (b, s) \in A$*, there exist* $t \geq 1$ *and* $s^t \in S^t$ *such that* $x(s^t) \in \operatorname{cl} A$*, with* $x(s^t)$ *defined as in* (18)*.*

*Proof.* See Appendix A.2. ■

The first set of necessary conditions restricts an MPE asset price $q$ in a rather intuitive way. The additional collateral value component implies that $q$ is greater than or equal to the unconstrained DE asset price $q^{\mathrm{FB}}$. Moreover, since the collateral constraint evaluated at the unconstrained allocation binds when the asset price is $q^A$, the MPE asset price $q$ cannot be lower than the former. In fact, the MPE collateral constraint is equivalent to the condition $q_x \geq q_x^A$ for all $x \in X$. The final necessary condition in Proposition 2 is more subtle and requires the following: if the MPE visits the set $A$ in which the collateral constraint is violated in the unconstrained DE, then the closure of $A$ must be visited at some point in the future. If $x \in A$, the unconstrained DE asset price is too low. Since the (unconstrained) allocation is given, the MPE asset price can be high enough only if the representative agent's Lagrange multipliers in the *next* periods are high enough. Hence, the collateral constraint must bind at some point in the future, which can happen only if the constraint binds or will be violated in the unconstrained DE in the next periods.

If $A$ is in the ergodic set defined by the unconstrained allocation, the last necessary condition in Proposition 2 will be satisfied. We assume that there exists a unique ergodic set.

**Assumption 1** (Unconstrained ergodic set)**.** *Let $b^{FB}$ denote the unconstrained policy function for next-period bond holdings, and let $\mathcal{B}(X)$ denote the Borel $\sigma$-algebra on $X$. Define the transition function $P : X \times \mathcal{B}(X) \to [0,1]$ as $P((b,s), \hat{B} \times \hat{S}) = \mathbf{1}_{\hat{B}}(b^{FB}(b,s)) \Pr\Big(s' \in \hat{S} \mid s\Big)$. Then $P$ generates a unique invariant probability measure $\lambda : \mathcal{B}(X) \to [0,1]$ that satisfies $\lambda(\hat{X}) = \int_X P(x, \hat{X})\lambda(\mathrm{d}x)$ for all $\hat{X} \in \mathcal{B}(X)$, with the associated ergodic set $X^{FB} \subset X$.*

As discussed previously, as long as $\beta\overline{R} < 1$, Assumption 1 will be satisfied and can be verified numerically. If $\lambda(A) = 0$, the collateral constraint is irrelevant almost everywhere (a.e.) on the unconstrained ergodic set $X^{\mathrm{FB}}$, and there exists an unconstrained MPE a.e. on $X^{\mathrm{FB}}$ with $q = q^{\mathrm{FB}}$ and $\mu_x = 0$ for all $x \in X^{\mathrm{FB}}$. A more interesting case is $\lambda(A) > 0$, so that the constraint would be violated in the unconstrained DE on an infinite subset of the ergodic set. The next proposition suggests that one can generally construct a candidate MPE that satisfies almost all conditions of an unconstrained MPE.

**Proposition 3** (Unconstrained MPE candidate)**.** *Suppose Assumption 1 holds and $\lambda(A) > 0$. There generally exist $\{q, \mu\} \subset \mathcal{F}(X)$ such that $\left\{\tilde{c}^{FB}, h^{FB}, v^{FB}, b^{FB}, q, \mu\right\}$ satisfy all conditions of Definition 3, except (14), possibly, holds approximately on $A$. If $\lambda(A) = 1$, then $q = q^A$ and $\mu_x = \max\{\mu_x^q, 0\}$ for all $x \in X$, where $\mu^q \in \mathcal{F}(X)$ is defined such that it satisfies*

$$q_x^A u'\left(\tilde{c}_x^{FB}\right) = \beta \mathbb{E}_s \left[u'\left(\tilde{c}_{x'}^{FB}\right)\left(z' F_k\left(1, h_{x'}^{FB}, v_{x'}^{FB}\right) + q_{x'}^A\right) + \mu_{x'}^q \kappa' q_{x'}^A\right], \tag{21}$$

*with $x' = (b_x^{FB}, (z', R', \kappa'))$, for all $x = (b,s) \in X^{FB}$, and $\mu^q(\cdot) = 0$ otherwise. If $\mu_x^q \geq 0$ on $X^{FB}$, then $\left\{\tilde{c}^{FB}, h^{FB}, v^{FB}, b^{FB}, q, \mu\right\}$ is an unconstrained MPE of Definition 4 a.e. on $X^{FB}$.*

*Proof.* See Appendix A.3. ■

The proof of Proposition 3 essentially describes a numerical strategy for constructing an unconstrained MPE of Definition 4. If $\lambda(A) \in (0, 1)$, the construction involves finding a fixed point in the asset price function $q \in \mathcal{F}(X)$ and, at each iteration and for all $x \in A$, solving a system of linear equations in next-period Lagrange multipliers, defining $\mu \in \mathcal{F}(X)$. The solution to the linear system may or may not involve nonnegative multipliers. If it always does, the construction results in an unconstrained MPE.

A significant simplification is achieved if $\lambda(A) = 1$, so that the collateral constraint is violated in the unconstrained DE a.e. on $X^{\text{FB}}$. We find this to be true numerically under the baseline calibration. In this case, we don't need to find a fixed point in the asset price, rather we can simply set $q = q^A$, which ensures that the collateral constraint holds with equality at the unconstrained allocation. We do need to solve for the multipliers such that (21) holds on $X^{\text{FB}}$. A necessary condition for the multipliers to be nonnegative follows from (21).

**Corollary 1.** *Suppose Assumption 1 holds and $\lambda(A) = 1$. There exists an unconstrained MPE of Definition 4 a.e. on $X^{FB}$ only if*

$$q_x^A u'\left(\tilde{c}_x^{FB}\right) \geq \beta \mathbb{E}_s \left[ u'\left(\tilde{c}_{x'}^{FB}\right) \left( z' F_k\left(1, h_{x'}^{FB}, v_{x'}^{FB}\right) + q_{x'}^A \right) \right], \tag{22}$$

*with $x' = (b_x^{FB}, (z', R', \kappa'))$, for all $x = (b, s) \in X^{FB}$.*

The condition (22) can be verified numerically based on the first-best allocation only. This condition is necessary but not sufficient because for each $x \in A$, there may be several states $\hat{x} \in A$ that imply identical next-period bond holdings, i.e., $b^{\text{FB}}(\hat{x}) = b^{\text{FB}}(x)$. Solving the system of equations composed of (21) for each $\hat{x} \in \left\{\hat{x} \in A \mid b^{\text{FB}}(\hat{x}) = b^{\text{FB}}(x)\right\}$ for $\{\mu_{x'}^q\}$ may result in $\mu^q(b_x^{\text{FB}}, s')$ being negative for some $s' \in S$. The difficulty in providing *sufficient* conditions for existence of a first-best MPE is related to the essence of the Markov perfection requirement that forces the planner's decision rules to be functions of the payoff-relevant state variables only. The value of the Lagrange multiplier at a state $(b_x^{\text{FB}}, s') \in X$ must be the same independently of the possible previous states $\left\{\hat{x} \in A \mid b^{\text{FB}}(\hat{x}) = b^{\text{FB}}(x)\right\}$.

If we drop the Markov perfection requirement, the condition (22) is both necessary and sufficient for the existence of a time-consistent unconstrained equilibrium under Assumption 1 and $\lambda(A) = 1$. Recall that the classical definition of time consistency due to Strotz (1955), Kydland and Prescott (1977), and Calvo (1978) is the following: a plan $\{\{\pi_t(s^t \mid s^0)\}_{s^t \in S^t}\}_{t=0}^{\infty}$ is time consistent if for all $\tau > 0$ and $s^\tau \in S^\tau$, an optimal plan chosen at $s^\tau$, $\{\{\pi_t(s^t \mid s^\tau)\}_{s^t \in S^t \mid s^\tau}\}_{t=\tau}^{\infty}$, satisfies $\pi_t(s^t \mid s^0) = \pi_t(s^t \mid s^\tau)$ for all $t \geq \tau$ and $s^t \in S^t \mid s^\tau$, where

$S^t \mid s^\tau$ is the set of all histories $s^t$ that continue from $s^\tau$. Hence, any future reoptimization results in following the original plan chosen at $t = 0$. Such a plan is an outcome of a *subgame perfect equilibrium* of a game played by successive planners, but unlike in the case of an MPE, planners' strategies are not restricted to be Markovian. There is no incentive to deviate from a plan that entails the unconstrained allocation, so any such plan is going to be time consistent according to the definition above.

Consider a plan chosen by a benevolent social planner that makes decisions once and for all at $t = 0$ subject to the sequential versions of the constraints in Definition 3.

**Definition 5** (Constrained-efficient plan)**.** *An allocation* $\{(\tilde{c}_t, h_t, v_t, b_{t+1})\}_{t=0}^{\infty}$*, prices* $\{q_t\}_{t=0}^{\infty}$*, and Lagrange multipliers* $\{\mu_t\}_{t=0}^{\infty}$ *are a constrained-efficient plan if they solve*

$$\max_{\{(\tilde{c}_t, h_t, v_t, b_{t+1}, q_t, \mu_t)\}_{t=0}^{\infty}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(\tilde{c}_t)$$

*subject to*

$$\tilde{c}_t + \frac{b_{t+1}}{R_t} \leq z_t F(1, h_t, v_t) - p_v v_t - g(h_t) + b_t, \tag{23}$$

$$-\frac{b_{t+1}}{R_t} + \theta p_v v_t \leq \kappa_t q_t, \tag{24}$$

$$q_t u'(\tilde{c}_t) = \beta \mathbb{E}_t \left[ u'(\tilde{c}_{t+1}) \Big( z_{t+1} F_k(1, h_{t+1}, v_{t+1}) + q_{t+1} \Big) + \mu_{t+1} \kappa_{t+1} q_{t+1} \right], \tag{25}$$

$$0 = \mu_t \left( \kappa_t q_t + \frac{b_{t+1}}{R_t} - \theta p_v v_t \right), \qquad \mu_t \geq 0, \tag{26}$$

*for all* $t \geq 0$ *and* $s^t \in S^t$*, given* $(b_0, s_0) = (b_0, (z_0, R_0, \kappa_0)) \in X$*.*

Let $\left\{ \left( \tilde{c}_t^{\mathrm{FB}}, h_t^{\mathrm{FB}}, v_t^{\mathrm{FB}}, b_{t+1}^{\mathrm{FB}} \right) \right\}_{t=0}^{\infty}$ be generated by the unconstrained allocation functions of Definition 2 and $\left\{ q_t^A \right\}_{t=0}^{\infty}$ by $q^A$ defined in (20). We then have the following proposition.

**Proposition 4** ( Unconstrained allocation as a constrained-efficient plan)**.** *Suppose Assumption 1 holds,* $\lambda(A) = 1$*, and* (22) *holds for all* $x \in X^{FB}$*. Let* $x_0 = (b_0, s_0) \in X^{FB}$*. Then any constrained-efficient plan of Definition 5 entails the first-best allocation* $\left\{ \left( \tilde{c}_t^{FB}, h_t^{FB}, v_t^{FB}, b_{t+1}^{FB} \right) \right\}_{t=0}^{\infty}$ *and is thus time consistent. An optimal* $\{(q_t, \mu_t)\}_{t=0}^{\infty}$ *is* $\{q_t\}_{t=0}^{\infty} = \left\{ q_t^A \right\}_{t=0}^{\infty}$*,* $\mu_0 = 0$*, and*

$$\mu_{t+1}((s^t, s_{t+1})) = \frac{q_t^A(s^t) u'\left(\tilde{c}_t^{FB}(s^t)\right) - \beta \mathbb{E}_t \left[ u'\left(\tilde{c}_{t+1}^{FB}\right) \left( z_{t+1} F_k\left(1, h_{t+1}^{FB}, v_{t+1}^{FB}\right) + q_{t+1}^A \right) \right]}{\beta \mathbb{E}_t \left( \kappa_{t+1} q_{t+1}^A \right)}, \tag{27}$$

*for all* $t \geq 0$*,* $s^t \in S^t$*, and* $s_{t+1} \in S$*.*

*Proof.* See Appendix A.4. ■

Proposition 4 is a consequence of Proposition 3 and Corollary 1. A sequential planning problem of Definition 5 involves (25) for each history $s^t \in S^t$, and the simplest way to satisfy the former given the unconstrained allocation and $\{q_t\}_{t=0}^{\infty} = \left\{q_t^A\right\}_{t=0}^{\infty}$ is making $\mu_{t+1}((s^t, s_{t+1}))$ constant over $s_{t+1} \in S$, which gives (27). Given (22), the multipliers are guaranteed to be nonnegative. Note that $\{\mu_t\}_{t=0}^{\infty}$ constructed according to (27) is not Markovian since $\mu_{t+1}(s^{t+1})$ depends on the allocation and prices at $s^t$.

The possible existence of first-best equilibria has important implications for the design of optimal policy that we now explore.

# 4 Optimal policy

In this section, we characterize the policies that can implement constrained-efficient equilibria as regulated competitive equilibria. As a policy tool, we focus on a tax on debt rebated lump sum—the instrument suggested in the existing literature (Bianchi, 2011; Bianchi and Mendoza, 2018; Jeanne and Korinek, 2019).

Although a positive debt tax generally discourages borrowing for given asset prices, the tax tends to decrease asset prices, making the collateral constraint more likely to bind for a given level of debt, and thus having an ambiguous effect on welfare and the probability of financial crises. As a consequence of this tradeoff, in the presence of the working capital constraint ($\theta > 0$), the optimal time-consistent policy generally requires both taxing debt when the collateral constraint is slack (good times) and *subsidizing* debt when the collateral constraint binds (bad times). In this case, the optimal policy induces *a* constrained-efficient MPE, but cannot achieve an unconstrained MPE. Implementing the latter requires, in addition, subsidizing intermediate inputs.

In the absence of the working capital constraint ($\theta = 0$), the tax on debt can induce an unconstrained MPE. Under the premise of Proposition 4, the optimal time-consistent policy requires only subsidizing (but not taxing) debt. Therefore, there is an important discontinuity in the optimal time-consistent policy at $\theta = 0$.

## 4.1 Regulated competitive equilibrium

Suppose the government imposes a tax on debt $\tau_t$ and provides a lump-sum transfer $T_t$. The agent's budget constraint (2) becomes

$$c_t + q_t k_{t+1} + \frac{b_{t+1}}{R_t(1+\tau_t)} \leq z_t F(k_t, h_t, v_t) - p_v v_t + q_t k_t + b_t + T_t.$$

A positive $\tau_t$ increases the effective interest rate, subsidizing saving and taxing borrowing. Conversely, a negative $\tau_t$ subsidizes borrowing. The government budget constraint is $T_t = \left(\frac{1}{1+\tau_t} - 1\right) \frac{b_{t+1}}{R_t}$, so the tax income is rebated back to the agent. Consequently, the role of $\tau_t$ is solely to affect the agent's borrowing decision but not to distort the allocation otherwise.

Given Definition 1 and Remark 1, we define a regulated competitive equilibrium (**CE**) in recursive form as follows.

**Definition 6** (Regulated competitive equilibrium)**.** *Given a tax function $\tau \in \mathcal{F}(X)$, a regulated competitive equilibrium is a set of allocation functions $\{\tilde{c}, h, v, b'\} \subset \mathcal{F}(X)$, an asset price function $q \in \mathcal{F}(X)$, and a Lagrange multiplier function $\mu \in \mathcal{F}(X)$ that satisfy (9)–(12), (14), (15), and*

$$u'(\tilde{c}_x) = (1 + \tau_x)\Big(\beta R \mathbb{E}_s u'(\tilde{c}_{x'}) + \mu_x\Big) \qquad \textit{if } b'_x \in \operatorname{int} B, \tag{28}$$

*with $x' = (b'_x, (z', R', \kappa'))$, for all $x = (b, s) = (b, (z, R, \kappa)) \in X$.*

Note that the only difference between the DE conditions in Remark 1 and the regulated CE conditions in Definition 6 is the bond Euler equation (28) that has replaced (13). Since $u'(\cdot) > 0$, the regulated CE exists only if $\tau_x > -1$ for all $x \in X$.

According to (28), an increase in $\tau_x$, other things equal, leads to a decrease in net consumption $\tilde{c}_x$, and thus to an increase in saving $b'_x$ for a given level of resources in (9). The precise equilibrium adjustment is more complicated, but the intuition regarding the positive relationship between $\tau_x$ and $b'_x$ mostly goes through. In the appendix, we explore the equilibrium response to a one-shot change in the tax rate more accurately. More precisely, let $X^\tau_u = \{x \in X \mid \mu_x = 0\}$ and $X^\tau_c = X \setminus X^\tau_u$. With $\mathbf{1}_{X^\tau_c}$, we denote the indicator function of $X^\tau_c$. In developing the intuition on how this policy tool works, in what follows, we consider a one-shot change $\mathrm{d}\tau_x$ in the tax rate at the state $x \in \operatorname{int} X^\tau_u \cup X^\tau_c$.

The general expressions for the derivatives $\mathrm{d}b'_x/\mathrm{d}\tau_x$ and $\mathrm{d}q_x/\mathrm{d}\tau_x$ in are complicated, and their signs are ambiguous. A significant simplification is achieved if $x \in \operatorname{int} X^\tau_u$, so that the collateral constraint is strictly nonbinding at $x$. In this case, the tax rate $\tau_x$ has an interpretation of a *macroprudential* tax—that is, a tax applied in "good times." Since $\mathrm{d}\mu_x = \mu_x = 0$, the equilibrium labor $h_x$ and intermediate inputs $v_x$ are pinned down by (11) and (12) independently of $\tau$ and remain constant. Hence, the aggregate resources—the right-hand side in the resource constraint (9)—also remain constant, which implies that net consumption $\tilde{c}_x$ and next-period bond holdings $b'_x$ must adjust in the opposite directions. The response of next-period bond holdings then depends on how the current and future marginal utility of consumption adjust with $b'_x$ in (28), and $\mathrm{d}b'_x/\mathrm{d}\tau_x$ simplifies to

$$\frac{\mathrm{d}b'_x}{\mathrm{d}\tau_x} = \left[-\frac{u''(\tilde{c}_x)}{R}\frac{1}{1+\tau_x} - \beta R \mathbb{E}_s\left(u''(\tilde{c}_{x'})\frac{\partial \tilde{c}_{x'}}{\partial b'_x}\right)\right]^{-1}\frac{u'(\tilde{c}_x)}{(1+\tau_x)^2}. \tag{29}$$

We then have the following corollary.

**Corollary 2.** *If $\tilde{c}(\cdot, s)$ is nondecreasing for all $s \in S$, then $\mathrm{d}b'_x/\mathrm{d}\tau_x > 0$ for all $x \in \operatorname{int} X_u^\tau$.*

We can verify numerically that $\tilde{c}(\cdot, s)$ is strictly increasing for all $s \in S$ in the DE. Hence, a one-shot positive marginal macroprudential tax must cause an increase in next-period bond holdings $b'_x$ by Corollary 2. Other things equal, this increase moves the economy further away from the region where the collateral constraint binds. However, the equilibrium asset price $q_x$ also adjusts in response to $\mathrm{d}\tau_x$. According to (14), the adjustment in $q_x$ depends on how the current and future marginal utility of consumption, the future marginal product of capital, asset price, and Lagrange multiplier adjust with $b'_x$. The derivative $\mathrm{d}q_x/\mathrm{d}\tau_x$ simplifies to

$$\frac{\mathrm{d}q_x}{\mathrm{d}\tau_x} = \frac{1}{u'(\tilde{c}_x)}\left\{\frac{1}{R}q_x u''(\tilde{c}_x) + \beta \mathbb{E}_s \frac{\partial}{\partial b'_x}\left[u'(\tilde{c}_{x'})\Big(z' F_k(1, h_{x'}, v_{x'}) + q_{x'}\Big) + \mu_{x'}\kappa' q_{x'}\right]\right\}\frac{\mathrm{d}b'_x}{\mathrm{d}\tau_x}. \tag{30}$$

We investigate the sign of $\mathrm{d}q_x/\mathrm{d}\tau_x$ in the following corollary.

**Corollary 3.** *If $\tilde{c}(\cdot, s)$ is nondecreasing and $u'(\tilde{c}(\cdot, s))\Big(zF_k(1, h(\cdot, s), v(\cdot, s)) + q(\cdot, s)\Big) + \mu(\cdot, s)\kappa q(\cdot, s)$—the payoff on capital in marginal utility units—is nonincreasing for all $s \in S$, then $\mathrm{d}q_x/\mathrm{d}\tau_x < 0$ for all $x \in \operatorname{int} X_u^\tau$.*

In general, it turns out that, numerically, $u'(\tilde{c}(\cdot, s))$ is strictly decreasing for all $s \in S$, and the payoff on capital in marginal utility units is strictly decreasing in $b$ for all $s \in S$, so that Corollary 3 applies. Specifically, since a one-shot positive marginal macroprudential tax causes an increase in next-period bond holdings $b'_x$ by Corollary 2, and thus a fall in net consumption $\tilde{c}_x$ and an increase in the current marginal utility of consumption $u'(\tilde{c}_x)$, but there is a decrease in the expected future payoff on capital in marginal utility units, the capital Euler equation (14) requires the equilibrium asset price $q_x$ to fall. In turn, the fall in the asset price translates to the fall in the value of collateral, and thus, the collateral constraint is more likely to bind for a given level of debt. This demonstrates a trade-off associated with macroprudential policy. On the one hand, it discourages borrowing, making the collateral constraint less likely to bind. On the other hand, it depresses asset prices, making the collateral constraint more likely to bind. Hence, its welfare benefits are in general ambiguous.

If $x \in X_c^\tau$, so that the collateral constraint is strictly binding, the effects of a one-shot tax change are more complex, unless $\theta = 0$. If $x \in X_c^\tau$ and $\theta = 0$, the allocation and the asset price are pinned down by (9), (11), (12), (14), and (15) independently of $\tau$, and the change $\mathrm{d}\tau_x$ affects only the Lagrange multiplier $\mu_x$ determined from (28). Hence, $\mathrm{d}b'_x/\mathrm{d}\tau_x = \mathrm{d}q_x/\mathrm{d}\tau_x = 0$.

If $x \in X_c^\tau$ and $\theta > 0$, the adjustment in next-period bond holdings $b'_x$ and the asset price $q_x$ in response to the tax change $\mathrm{d}\tau_x$ causes an adjustment in intermediate inputs $v_x$ given by the binding collateral constraint in (15). In turn, the change in inputs leads to a change in labor $h_x$ according to (11), net consumption $\tilde{c}_x$ given by (9), and the Lagrange multiplier $\mu_x$ consistent with (12). These changes lead to further adjustments in $b'_x$ and $q_x$ that satisfy (14) and (28), and so on. Our numerical analysis shows that Corollaries 2 and 3 continue to hold, indicating that, numerically, $\Psi_x^{q(\tau)} < 0$ and $\Psi_x^{b'(\tau)} > 0$, so that $\mathrm{d}b'_x/\mathrm{d}\tau_x > 0$ and $\mathrm{d}q_x/\mathrm{d}\tau_x < 0$.

## 4.2 Optimal time-consistent policy

We now characterize the optimal time-consistent tax on debt. We begin by replicating the analytical structure derived in Bianchi and Mendoza (2018) for the time-consistent optimal policy conditional on the debt tax instrument. We extend their analysis by characterizing the sign and role of the individual components of the policy function and showing how the nature of the optimal intervention changes with the presence of a working capital constraint ($\theta > 0$). In particular, we establish that when $\theta > 0$, the optimal policy generally involves taxing debt in slack states and subsidizing debt in binding states, but cannot implement the unconstrained allocation unless it is complemented by subsidies to intermediate inputs. We further show that when $\theta = 0$, a pure debt tax suffices to support the unconstrained allocation. Finally, we go beyond the Markov Perfect setting and construct a fully time-consistent policy that implements the unconstrained allocation as part of a dynamic regulated competitive equilibrium.

**Proposition 5** (Optimal time-consistent tax on debt)**.** *Consider an MPE of a game in which successive policymakers choose the tax function $\tau \in \mathcal{F}(X)$ to achieve the best regulated CE of Definition 6, taking the next-period decision rules as given. If $\theta = 0$, the set of such MPE is equivalent to the set of constrained-efficient MPE of Definition 3. If $\theta > 0$, the equilibrium $\tau$ induces one specific constrained-efficient MPE corresponding to $\mu \in \mathcal{F}(X)$ defined as*

$$\mu_x = \frac{u'(\tilde{c}_x)}{u'(\tilde{c}_x) - \mu_x^{SP}\kappa q_x \frac{u''(\tilde{c}_x)}{u'(\tilde{c}_x)}}\mu_x^{SP}, \tag{31}$$

*where $\mu_x^{SP}$ is the policymaker's shadow value of collateral. If $\lambda(A) > 0$, this MPE is not a first-best MPE of Definition 4.*

*For any $\theta \geq 0$, if a constrained-efficient MPE entails differentiable decision rules, the tax function that induces this MPE as a regulated CE satisfies*

$$\tau_x = \tau_x^{MP} + \tau_x^{EP}, \tag{32}$$

*where*

$$\tau_x^{MP} \equiv -\frac{\mathbb{E}_s\left(\mu_{x'}^{SP}\kappa' q_{x'}\frac{u''(\tilde{c}_{x'})}{u'(\tilde{c}_{x'})}\right)}{\mathbb{E}_s u'(\tilde{c}_{x'}) + \frac{\mu_x}{\beta R}} \geq 0 \tag{33}$$

*and*

$$\tau_x^{EP} \equiv \underbrace{\frac{\mu_x^{SP} - \mu_x}{\beta R \mathbb{E}_s u'(\tilde{c}_{x'}) + \mu_x}}_{\textit{risk sharing component}} + \underbrace{\mu_x^{SP}\frac{\kappa}{u'(\tilde{c}_x)}\frac{q_x u''(\tilde{c}_x) + \beta R \mathbb{E}_s \frac{\partial}{\partial b_x'}\left[u'(\tilde{c}_{x'})\Big(z' F_k(1, h_{x'}, v_{x'}) + q_{x'}\Big) + \mu_{x'}\kappa' q_{x'}\right]}{\beta R \mathbb{E}_s u'(\tilde{c}_{x'}) + \mu_x}}_{\textit{collateral externality component}}. \tag{34}$$

*Proof.* See Appendix A.6. ■

According to Proposition 5, in the absence of the working capital constraint ($\theta = 0$), the tax on debt can implement any constrained-efficient MPE of Definition 3, including the unconstrained allocation MPE of Definition 4 if it exists. In this case, the planner can vary the agent's shadow value of collateral $\mu$ when the collateral constraint binds, affecting the equilibrium asset price, and expanding the set of feasible allocations. (See the discussion preceding Remark 3.) If $\theta > 0$, however, the tax on debt can implement only one specific constrained-efficient MPE, and this MPE is not the unconstrained allocation, provided that the collateral constraint is relevant on the unconstrained ergodic set (i.e., $\lambda(A) > 0$). This is because the working capital constraint restricts the Lagrange multiplier function $\mu$ to be consistent with the DE optimal input condition (12), introducing a wedge relative to the unconstrained condition (16) and generating a one-to-one mapping between the planner's and agent's shadow values of collateral given by (31). Note that $u'(\tilde{c}_x)$ and $u'(\tilde{c}_x) - \mu_x^{\mathrm{SP}}\kappa q_x \frac{u''(\tilde{c}_x)}{u'(\tilde{c}_x)}$ are the agent's and planner's shadow values of income (Lagrange multipliers on the resource constraint), so (31) requires the equality between the agent's and planner's shadow values of collateral normalized by the corresponding shadow values of income.

It is important to emphasize that there is a single-valued mapping between a constrained-efficient MPE and a tax function that induces that MPE as a regulated CE. The tax is uniquely determined not only in the states in which the collateral constraint is slack but also in the binding states, unlike in the analysis of Bianchi (2011), Schmitt-Grohé and Uribe (2017), and Jeanne and Korinek (2019), who all find that the tax on debt cannot affect the allocation in the binding state—it can only affect the agent's shadow value of collateral. In our environment, if $\theta > 0$, the agent's shadow value of collateral $\mu$ is linked to the input allocation through (12), so when the collateral constraint binds, there is still room for optimal reallocation between bond holdings and inputs in (15) by adjusting the tax rate. If $\theta = 0$, *for given next-period decision rules*, the current tax rate $\tau_x$ maps to the current multiplier $\mu_x$ in (28) in the binding state, and the allocation and asset price are pinned down by the remaining constraints in the regulated CE of Definition 6. However, in an MPE, the current $\mu_x$ must be consistent with the function $\mu$ that affects the next-period payoff on capital in (14). Hence, in a given MPE, the value of $\mu_x$ is given, and (28) provides a unique value of $\tau_x$ that maps to the former. This property of the optimal time-consistent policy arises because the shadow value of collateral affects the equilibrium asset price, while there is no such effect in the alternative environments mentioned above.

If a constrained-efficient MPE entails differentiable decision rules, the tax function that implements this MPE is given by (32)–(34), comprising two components. The first component is a "macroprudential component" $\tau^{\mathrm{MP}}$ given by (33). This component reflects that greater current savings $b'_x$ relax the next-period resource constraint, having a positive effect on future net consumption $\tilde{c}_{x'}$ and asset price $q_{x'}$, and thus relaxing the collateral constraint in the next-period states $x' = (b'_x, s')$ in which the constraint is strictly binding ($\mu_{x'}^{\mathrm{SP}} > 0$). The macroprudential component is nonnegative and captures the planner's motive to subsidize savings (tax debt issuance) today to prevent or mitigate financial crises in the future. If the collateral constraint is slack in the current period, the agent's and planner's complementary slackness conditions require $\mu_x = \mu_x^{\mathrm{SP}} = 0$ in any MPE. In this case, the macroprudential component is the only component of the optimal tax.

**Corollary 4.** *Consider the optimal time-consistent tax on debt described in* (32)–(34)*. If the collateral constraint is slack at $x$, then*

$$\tau_x = \tau_x^{MP} = -\frac{\mathbb{E}_s\left(\mu_{x'}^{SP}\kappa' q_{x'}\frac{u''(\tilde{c}_{x'})}{u'(\tilde{c}_{x'})}\right)}{\mathbb{E}_s u'(\tilde{c}_{x'})}. \tag{35}$$

*Given* (A.16)*, $\tau^{MP}$ in* (35) *is equivalent to the "macroprudential debt tax" in Bianchi and Mendoza (2018, eq. (17), p. 605).*

The second component of the optimal time-consistent tax on debt is an "ex post component" $\tau^{\mathrm{EP}}$ given by (34). By Corollary 4, the ex post component is active only if the collateral constraint is binding. The ex post component is, in turn, a sum of two terms: a “risk sharing component” and a “collateral externality component.”

The risk sharing component is proportional to the difference between the planner’s and agent’s shadow values of collateral $(\mu_x^{\text{SP}} - \mu_x)$. If $\mu_x^{\text{SP}} > \mu_x$, the representative agent undervalues the utility benefit of increased consumption smoothing achieved by relaxing the binding collateral constraint, and the planner has an incentive to subsidize savings in the binding state. Conversely, if $\mu_x^{\text{SP}} < \mu_x$, the representative agent overvalues savings in the binding state, and the negative risk sharing component provides an incentive to tax savings (subsidize debt) when the collateral constraint binds. If $\theta > 0$, (31) implies $\mu_x^{\text{SP}} > \mu_x$ whenever $\mu_x^{\text{SP}} > 0$.

**Corollary 5.** *Consider the optimal time-consistent tax on debt described in (32)–(34). If $\mu$ is constrained by (31), the risk sharing component of $\tau^{EP}$ is positive in the states in which the collateral constraint is strictly binding.*

The collateral externality component arises due to the pecuniary externality (Dávila and Korinek, 2018) and reflects the effect of greater current savings $b'_x$ on the current asset price $q_x$. First, greater current savings crowd out current net consumption $\tilde{c}_x$ and have a negative effect on the asset price through the current marginal utility of consumption, which corresponds to $q_x u''(\tilde{c}_x) < 0$ in the numerator in (34). Second, a change in $b'_x$ affects the next-period state $x' = (b'_x, s')$, and thus the next-period payoff on capital in marginal utility units, i.e., $u'(\tilde{c}_{x'})\Big(z' F_k(1, h_{x'}, v_{x'}) + q_{x'}\Big) + \mu_{x'} \kappa' q_{x'}$. The latter effect depends on the monotonicity properties of the MPE decision rules.

**Corollary 6.** *Consider the optimal time-consistent tax on debt described in (32)–(34). If the MPE payoff on capital in marginal utility units is nonincreasing in $b$ for all $s \in S$, the collateral externality component of $\tau^{EP}$ is negative in the states in which the collateral constraint is strictly binding.*

The sufficient condition for a negative collateral externality in Corollary 6 is similar to the sufficient condition for a negative effect of a one-shot tax change on the the asset price in Corollary 3. It is satisfied if an increase in future consumption (a decrease in the marginal utility of consumption) dominates the effects through the changes in the future marginal product of capital, asset price, and agent’s shadow value of collateral. If an increase in current savings does decrease the current asset price, so that the collateral externality component is negative, the planner has an incentive to tax savings (subsidize debt issuance) in order to increase the current asset price and relax the binding collateral constraint.

A striking simplification of the optimal time-consistent tax on debt arises if it implements an unconstrained allocation MPE of Definition 4. In this case, the policymaker acts as if being subject to the resource constraint only, consistent with the unconstrained problem of Definition 2, while $\{q, \mu\}$ are set to satisfy the collateral constraint, asset pricing constraint, and the agent's complementary slackness conditions at the unconstrained allocation, as described in Proposition 3. Consequently, the collateral constraint can be dropped from the planner's best response problem, which implies $\mu_x^{\mathrm{SP}} = 0$ for all $x \in X$. Imposing the latter in (33) and (34), we observe that both the macroprudential and collateral externality components become zero, and the optimal tax is given by the nonpositive risk sharing component.

**Corollary 7.** *The optimal time-consistent tax on debt that implements a first-best MPE of Definition 4 is given by*

$$\tau_x = \tau_x^{EP} = \frac{-\mu_x}{u'\left(\tilde{c}_x^{FB}\right) + \mu_x} \in (-1, 0], \tag{36}$$

*where the denominator has been simplified with (17).*

The optimal time-consistent policy given by (36) is a debt *subsidy* that responds to the agent's shadow value of collateral $\mu$, closing the wedge in the DE Euler equation (13) introduced by the Lagrange multiplier. The optimal policy thus ensures that the representative agent can borrow as much as it is optimal to do at the unconstrained allocation, counteracting the distortion introduced by the collateral constraint.

By Proposition 5, the tax on debt can implement an unconstrained MPE only if $\theta = 0$. If $\theta > 0$, a wedge in the input optimality condition (12) prevents the implementation of the unconstrained equilibrium. Not surprisingly, if the planner can close that wedge by subsidizing inputs, the first-best MPE, if it exists, can be implemented.

**Proposition 6** (Optimal time-consistent policy)**.** *Suppose, in addition to a tax on debt, a policymaker can tax intermediate inputs, so that the agent's budget constraint (2) becomes*

$$c_t + q_t k_{t+1} + \frac{b_{t+1}}{R_t(1+\tau_t)} \leq z_t F(k_t, h_t, v_t) - (1+\tau_t^v) p_v v_t + q_t k_t + b_t + T_t,$$

*and the government budget constraint is $T_t = \left(\frac{1}{1+\tau_t} - 1\right)\frac{b_{t+1}}{R_t} + \tau_t^v p_v v_t$. Then the set of MPE of a game in which successive policymakers choose $\{\tau, \tau^v\} \subset \mathcal{F}(X)$ to achieve the best regulated CE, taking the next-period decision rules as given, is equivalent to the set of constrained-efficient MPE of Definition 3. The policy that induces a given constrained-efficient MPE as a regulated CE satisfies (32)–(34) and*

$$\tau_x^v = \theta \left( \frac{\mu_x^{SP}}{u'(\tilde{c}_x) - \mu_x^{SP} \kappa q_x \frac{u''(\tilde{c}_x)}{u'(\tilde{c}_x)}} - \frac{\mu_x}{u'(\tilde{c}_x)} \right). \tag{37}$$

*The policy that implements an unconstrained MPE of Definition 4 is given by* (36) *and*

$$\tau_x^v = -\frac{\theta \mu_x}{u'\left(\tilde{c}_x^{FB}\right)} \leq 0. \tag{38}$$

*Proof.* See Appendix A.7. ∎

By Proposition 6, the tax on debt and inputs, jointly, can implement the whole set of constrained-efficient MPE of Definition 3. Consistent with Proposition 5, (37) implies that the tax on inputs is required only if $\theta > 0$ and only in the states in which the collateral constraint binds. In this case, the tax is proportional to the difference between the planner's and agent's shadow values of collateral normalized by the corresponding shadow values of income. If the representative agent underestimates the normalized shadow value of collateral, it is optimal to levy a positive input tax. Conversely, if the agent's normalized shadow value of collateral is greater than the planner's, it is optimal to subsidize inputs in the binding state. If the normalized shadow values are exactly equal, the optimal input tax is zero, and the tax on debt induces the specific MPE that satisfies (31).

Note that the optimal tax on debt in Proposition 6 is characterized by the same conditions (32)–(34) as in Proposition 5. In particular, Corollaries 4–7 continue to hold. However, if $\theta > 0$, to implement an unconstrained MPE of Definition 4, we need to augment the debt subsidy given by (36) with an input *subsidy* given by (38) that equalizes the effective marginal cost of inputs with its price, consistent with (16).

The tax rates in (36) and (38) have simple expressions, but they involve the agent's shadow value of collateral $\mu$ that is part of a first-best MPE. Propositions 2 and 3 and Corollary 1 could go as far as providing the necessary conditions for the existence of an unconstrained MPE and describing a candidate unconstrained MPE, lacking the sufficient conditions. However, if we drop the Markov perfection requirement, under the premise of Proposition 4, (27) provides a closed-form expression for a history-contingent $\{\mu_t\}_{t=0}^{\infty}$ that is part of a time-consistent constrained-efficient plan of Definition 5 that entails the unconstrained allocation. Using that expression together with (36) and (38), we can construct the corresponding time-consistent policy $\{(\tau_t, \tau_t^v)\}_{t=0}^{\infty}$ that implements the unconstrained allocation in a regulated CE.

# 5 Quantitative results

In this section, we discuss calibration and computation of the model economy in Section 2. We then quantitatively analyze some of the allocations discussed above. Finally, we delve into the sources of the welfare gains associated with the time-consistent macroprudential policy in BM.

## 5.1 Calibration and computation

The calibration is the same as in Bianchi and Mendoza (2018). We compute all equilibria using global nonlinear methods. The numerical algorithms are explained in Appendix B.

We compute the optimal time-consistent tax on debt $\tau$ of Proposition 5 that implements the constrained-efficient MPE of Definition 3 consistent with (31). When computing this MPE, we do not rely on the differentiability of decision rules. However, by construction, the piecewise linear decision rules we obtain are differentiable a.e. To obtain the optimal tax, we use the primal approach, backing out $\tau$ from (28). To decompose $\tau$ into its macroprudential and ex post components, we note that, under our calibration, the collateral constraint binds at the DE or SP allocations only if $\kappa = \kappa^L$ and $\Pr\left(\kappa' = \kappa^L \mid \kappa = \kappa^L\right) = 0$. Hence, $\Pr\left(\mu^{\mathrm{SP}}_{x'} > 0 \mid \mu^{\mathrm{SP}}_{x} > 0\right) = 0$, and (32)–(34) combined with Corollary 4 imply that $\tau_x = \tau^{\mathrm{MP}}_x$ when $\mu_x = \mu^{\mathrm{SP}}_x = 0$ and $\tau_x = \tau^{\mathrm{EP}}_x$ when $\mu_x > 0$ (equivalently, $\mu^{\mathrm{SP}}_x > 0$ due to (31)), where $\mu$ is backed out from (12).

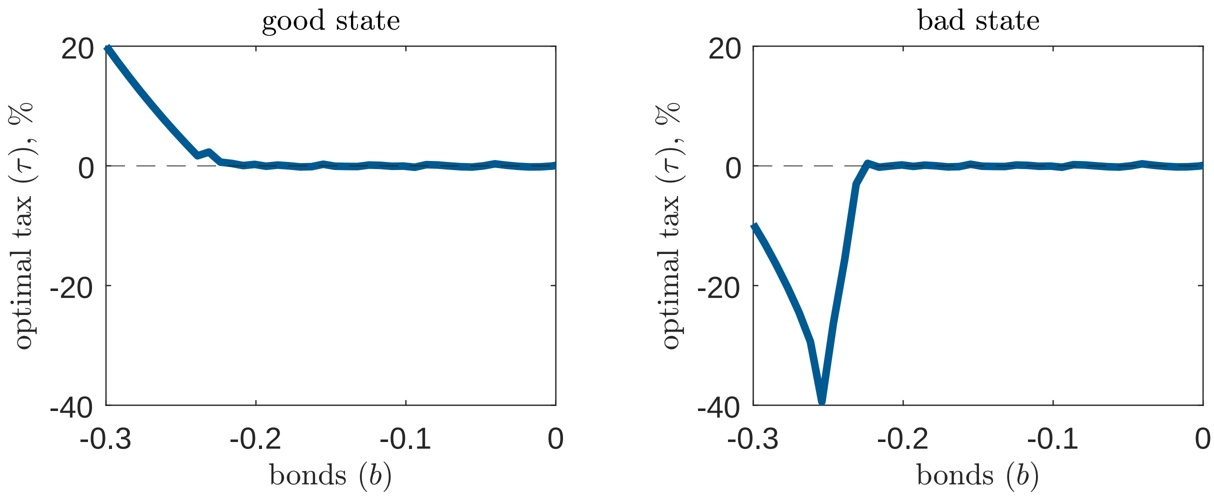
## 5.2 Optimal financial crisis management

Figure 1 plots the optimal time-consistent tax on debt $\tau$ of Proposition 5 as a function of bond holdings $b$ on the horizontal axis in the two exogenous states $s$ that both have average $z$ and high $R$ but differ in the value of $\kappa$: $\kappa = \kappa^H$ in the left panel ("good state") and $\kappa = \kappa^L$ in the right panel ("bad state").

In the good state, the collateral constraint is slack for all $b$ (see Figure 3). By Corollary 4, $\tau_x = \tau^{\mathrm{MP}}_x \geq 0$. For $b \leq -0.2$, the constraint may bind in the next period with a positive probability, i.e., $\Pr\left(\mu^{\mathrm{SP}}_{x'} > 0\right) > 0$, in which case $\tau_x = \tau^{\mathrm{MP}}_x > 0$, consistent with (35). In this region, as $b$ increases, next-period bond holdings $b'_x$ slightly trend upwards (Figure 3), and the marginal propensity to consume out of greater asset income is close to 1. By (28), a decrease in the marginal utility of consumption translates into a decrease in $\tau$ from around 20% to zero when $\Pr\left(\mu^{\mathrm{SP}}_{x'} > 0\right) = 0$. This is consistent with (35), since an increase in $b'_x$ induces a fall in the planner's next-period shadow value of collateral $\mu^{\mathrm{SP}}_{x'}$ in the states in which the constraint binds, implying a smaller tax.

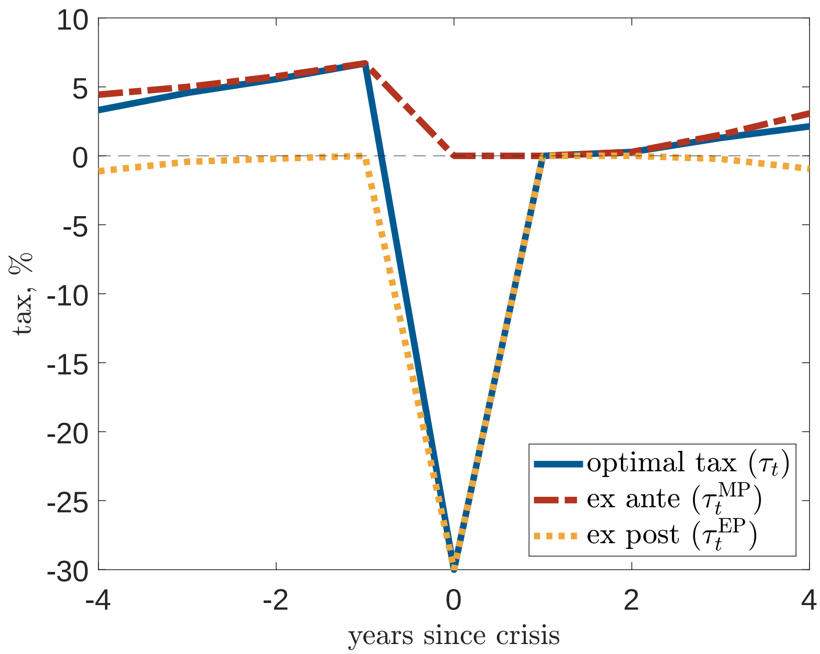
In the bad state, the collateral constraint binds for $b \leq -0.23$ (see Figure 4). Under this calibration, the constraint is slack in the next period, which implies $\tau^{\mathrm{MP}}_x = 0$. We observe from Figure 1 that $\tau_x = \tau^{\mathrm{EP}}_x < 0$ when the constraint binds. This means that the (negative) collateral externality component in (34) dominates the (positive, by Corollary 5) risk sharing

Figure 1: Optimal tax in good and bad state




Notes: "good state" = (average $z$, high $R$, high $\kappa$), "bad state" = (average $z$, high $R$, low $\kappa$).

Figure 2: Optimal tax around financial crises




Notes: Each line is an average across all financial crisis events. The optimal tax function and its "ex ante" (macroprudential) and ex post components given by (32)–(34) are evaluated at the DE states observed during the crises.

component of $\tau^{\mathrm{EP}}$. As $b$ increases from $-0.3$ to $-0.255$, an increase in the asset price and greater borrowing capacity allows the planner to issue more debt (Figure 4), which is induced in the regulated CE through a greater debt subsidy (i.e., $\mathrm{d}b'_x/\mathrm{d}\tau_x > 0$ and $\mathrm{d}q_x/\mathrm{d}\tau_x < 0$ in the context of Proposition 7) that reaches around 40% ($\tau$ falls to $-40\%$). As $b$ increases further from $-0.255$ to $-0.23$, the optimal debt issuance starts to fall ($b'_x$ increases), and so does the optimal subsidy, reaching zero when the collateral constraint becomes slack.

Figure 2 illustrates how the optimal tax on debt and its macroprudential and ex post components given by (32)–(34) are used around financial crisis episodes. Specifically, we simulate the DE for 101,000 periods, drop the first 1,000, and identify the dates at which the current account $ca_t = b_{t+1} - b_t$ exceeds its two standard deviations, i.e., $ca_t > \overline{ca} \equiv 2\hat{\sigma}(ca_t)$, which indicates a significant capital outflow (Bianchi and Mendoza, 2018). Each such date $t$ indicates a financial crisis event. We then extract the DE states $x_t = (b_t, s_t)$ in a four-year window around each crisis, evaluate the tax functions at these states, and compute an average over all crises paths. Hence, the paths provide the values of the tax that would have to be applied if the policymaker intervened under discretion at a specific date of the crisis window, directly reflecting the policy functions in Figure 1.

As the economy gets closer to a financial crisis, the macroprudential component increases by 2.3 percentage points, on average, from $\tau_{-4}^{\text{MP}} = 4.4\%$ to $\tau_{-1}^{\text{MP}} = 6.7\%$. When a financial crisis occurs, $\tau_0^{\text{MP}} = 0$, and the policy response is driven by the ex-post intervention that averages at $\tau_0^{\text{EP}} = -30\%$. Although a financial crisis occurs only if the collateral constraint is binding, the converse is not true: the constraint may be binding, but the capital outflow is not large enough to qualify as a crisis. Consequently, the ex post component can be slightly negative before and after a crisis, which explains a slight discrepancy between $\tau$ and $\tau^{\text{MP}}$ at the start and end of the crisis window.

## 5.3 Restricted optimal time-consistent policy

In this section, we compute a restricted optimal time-consistent tax on debt under the additional constraint

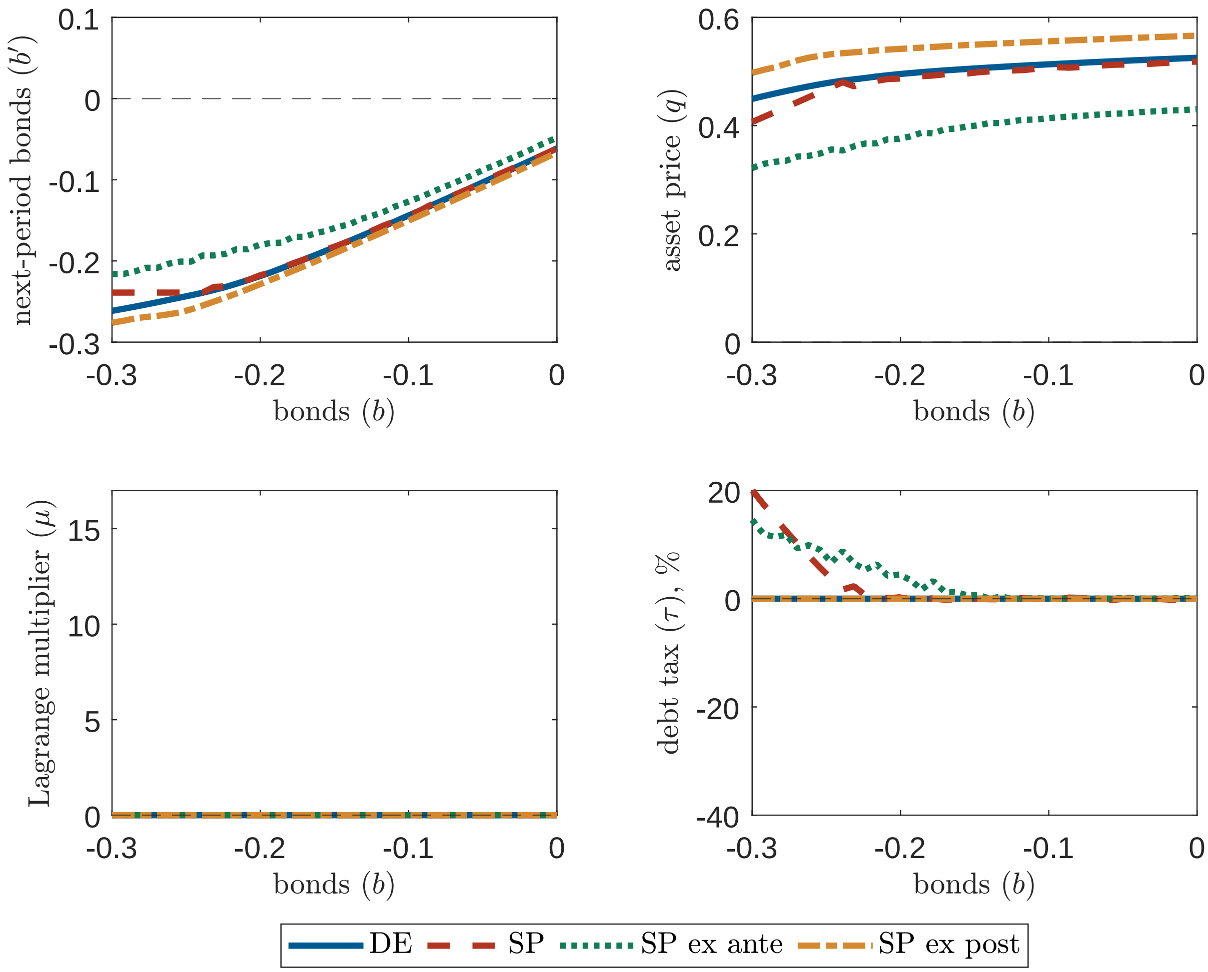
$$\underline{\tau} \leq \tau(x) \leq \overline{\tau}, \qquad \text{for all } x \in X, \tag{39}$$

where $\underline{\tau} \leq 0$ potentially restricts the ex post (subsidy) component, while $\overline{\tau} \geq 0$ may restrict the macroprudential (tax) component. We thus study the MPE of a policy game in Proposition 5 where policymakers are, in addition, constrained by (39). The latter implies that the resulting MPE is *not* constrained efficient, provided (39) binds in some states. Since $\theta > 0$, the resulting MPE, if it exists, is generally unique. Numerically, we impose (39) in the policymaker's best response by backing out $\tau$ from (28).

### 5.3.1 Policy functions and financial crises dynamics

First, we compare the policy functions for next-period bond holdings $b'$, the asset price $q$, agent's Lagrange multiplier $\mu$, and tax on debt $\tau$ in the good (Figure 3) and bad (Figure 4) states (defined as in Figure 1) across the DE, baseline optimal time-consistent policy ("SP"), the optimal policy that allows only a nonnegative tax $\tau_x \geq 0$ ("SP ex ante"), and optimal policy that allows only a subsidy $\tau_x \leq 0$ ("SP ex post").

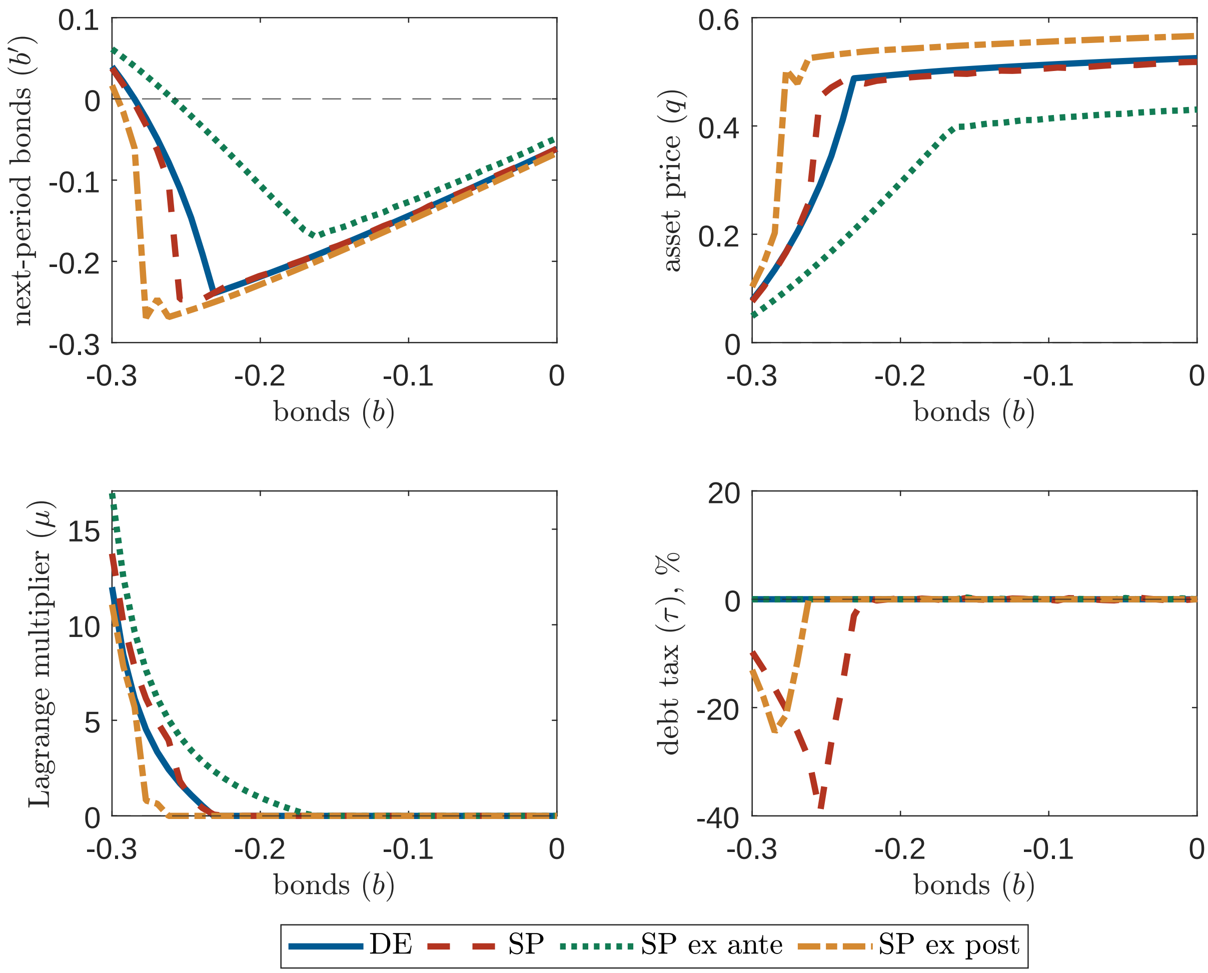
Figure 3: Policy functions in the good state




Notes: "good state" = (average $z$, high $R$, high $\kappa$). "SP" corresponds to the unrestricted optimal time-consistent policy ($\underline{\tau} = -\infty$ and $\overline{\tau} = +\infty$), "SP ex ante" to $\underline{\tau} = 0$ and $\overline{\tau} = +\infty$, and "SP ex post" to $\underline{\tau} = -27\%$ and $\overline{\tau} = 0$, where $-27\%$ is the lowest $\underline{\tau}$ (up to 1%) such that the MPE exists.

Consider the good state (Figure 3). In all equilibria, the collateral constraint is slack independently of the level of bond holdings. There is slightly more saving in the SP allocation compared to the DE when current debt is sufficiently large, induced by the positive debt tax. There is significantly more saving in the "SP ex ante" equilibrium, induced by a broader application of the tax on debt in an effort to decrease the probability of a binding constraint in the next period, reflecting the nonavailability of the debt subsidy in the bad state. Conversely, there is more borrowing in the "SP ex post" equilibrium, induced by the nonavailability of a (positive) debt tax in the current state and the use of debt subsidy in

Figure 4: Policy functions in the bad state



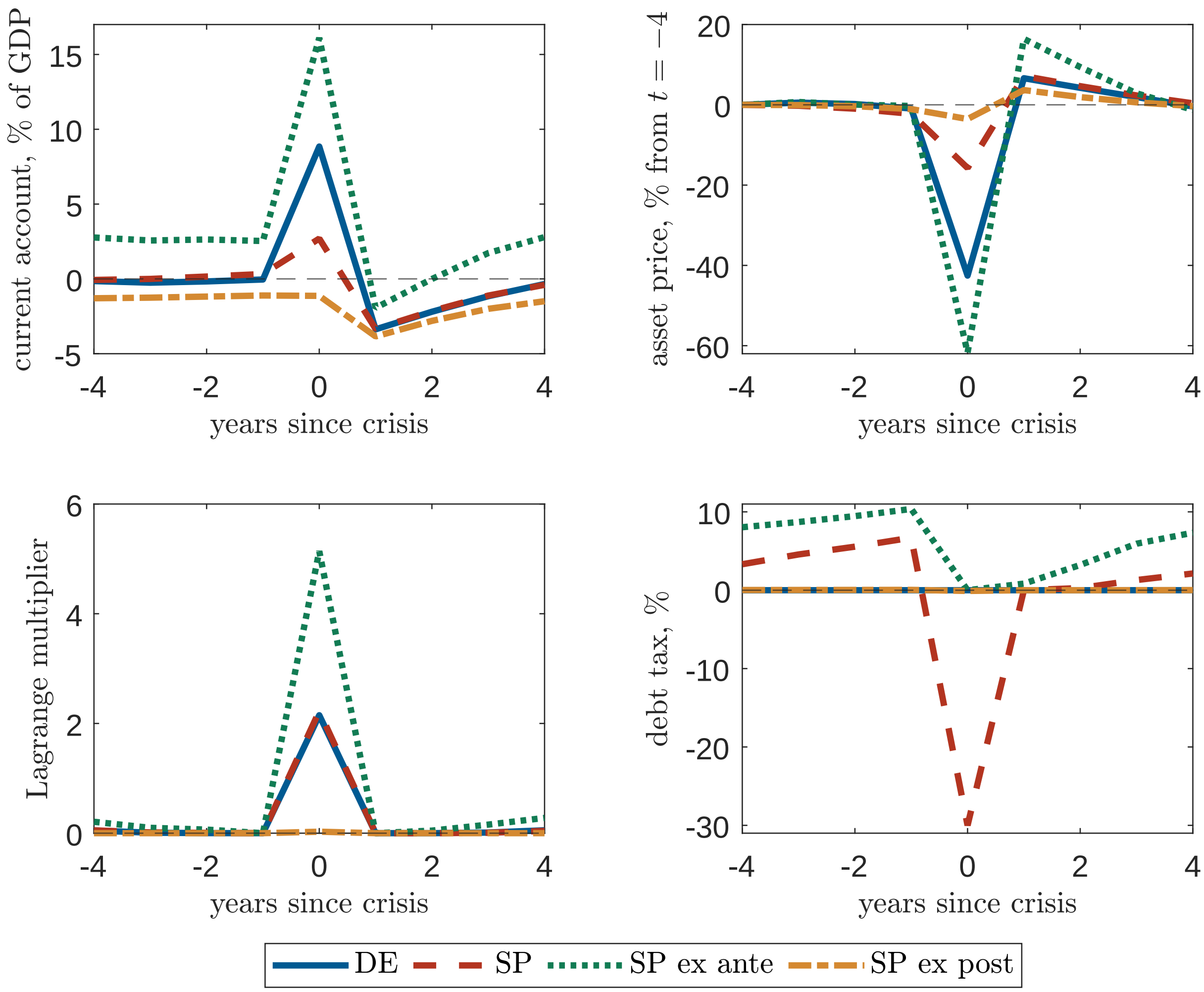

Notes: "bad state" = (average $z$, high $R$, low $\kappa$). "SP" corresponds to the unrestricted optimal time-consistent policy ($\underline{\tau} = -\infty$ and $\overline{\tau} = +\infty$), "SP ex ante" to $\underline{\tau} = 0$ and $\overline{\tau} = +\infty$, and "SP ex post" to $\underline{\tau} = -27\%$ and $\overline{\tau} = 0$, where $-27\%$ is the lowest $\underline{\tau}$ (up to 1%) such that the MPE exists.

the bad states that may occur in next periods. Corresponding to the differences in next-period bond holdings $b'$ are the differences in the asset price functions: more saving in the SP and "SP ex ante" equilibria compared to the DE is associated with lower asset prices in the former compared to the latter. Conversely, asset prices are higher in the "SP ex post" equilibrium compared to the DE. This is consistent with the discussion following Proposition 7, and in particular with Corollaries 2 and 3.

Consider now the bad state (Figure 4). The unconstrained policy (SP) subsidizes debt issuance in the binding region, which is accompanied by higher asset prices due to lower marginal utility of consumption. There is even more debt and higher asset prices in the "SP ex post" equilibrium, but the debt subsidy has smaller magnitude than in the SP equilibrium. The latter follows from the fact that positive taxes are not allowed in the good states, and both debt and asset prices are high in those states. Due to higher asset prices, the binding region is the smallest in the "SP ex post" equilibrium. Conversely, lack of debt subsidies in the "SP ex ante" equilibrium leads to lower debt, lower asset prices, and a larger binding region than in the DE.

Figure 5 compares the dynamics of the current account (in % of output), asset price, agent's Lagrange multiplier, and tax on debt across alternative equilibria around financial crises. The ex post debt subsidy provided in the SP equilibrium mitigates financial crises by

Figure 5: Financial crises




Notes: Each line is an average across all financial crisis events. "SP" corresponds to the unrestricted optimal time-consistent policy ($\underline{\tau} = -\infty$ and $\overline{\tau} = +\infty$), "SP ex ante" to $\underline{\tau} = 0$ and $\overline{\tau} = +\infty$, and "SP ex post" to $\underline{\tau} = -27\%$ and $\overline{\tau} = 0$, where $-27\%$ is the lowest $\underline{\tau}$ (up to 1%) such that the MPE exists. The policy functions are evaluated at the DE states observed during the crises.

significantly decreasing the capital outflow and the fall in the asset price compared to the DE. The probability of a financial crisis decreases from 4.02% in the DE to 0.01% in the SP equilibrium (see Table 1). Conversely, in the "SP ex ante" equilibrium, the nonavailability of the debt subsidy exacerbates financial crises, leading to a larger capital outflow and greater fall in the asset price compared to the DE. The financial crisis probability decreases, but only to 0.25%. Interestingly, financial crises are almost nonexistent in the "SP ex post" equilibrium at the DE crises states: the collateral constraint is rarely binding in those states, consistent with the smaller binding region in Figure 4, there is virtually no fall in the asset price, no capital outflow, and no need to subsidize debt. Financial crises do occur in the "SP ex post" equilibrium but at greater levels of debt that are not commonly observed in the DE. The financial crisis probability decreases to 1.88%.

### 5.3.2 Welfare gains

Table 1 reports the welfare gains from alternative equilibria relative to the DE in terms of permanent changes in net consumption, provides selected moments of the corresponding tax functions, and reports the financial crisis probabilities. The unconstrained optimal time-

Table 1: Statistics

| | DE | SP | SP+ | SP− | SP+ best | SP− best |
|---|---|---|---|---|---|---|
| Average welfare gain, initial $\pi$ | 0 | 0.61 | −0.14 | −0.08 | 0.03 | 0.03 |
| Average welfare gain, final $\pi$ | 0 | 0.62 | −0.03 | 0.01 | 0.04 | 0.03 |
| $\min(\tau)$ | 0 | −42.2 | 0 | −27 | 0 | −6 |
| $\max(\tau)$ | 0 | 29.8 | 20.1 | 0 | 0.8 | 0 |
| $\mathbb{E}(\tau)$ | 0 | 2.4 | 2.7 | −0.7 | 0.8 | −0.3 |
| $\Pr(\tau < 0)$ | 0 | 8.1 | 0 | 30.4 | 0 | 5.1 |
| $\Pr\left(ca_t > \overline{ca}^{\text{DE}}\right)$ | 4.02 | 0.01 | 0.25 | 1.88 | 3.7 | 3.57 |

Notes: All statistics are in %. "SP" corresponds to the unrestricted optimal time-consistent policy ($\underline{\tau} = -\infty$ and $\overline{\tau} = +\infty$), "SP+" to $\underline{\tau} = 0$ and $\overline{\tau} = +\infty$ ("SP ex ante"), "SP−" to $\underline{\tau} = -27\%$ and $\overline{\tau} = 0$ ("SP ex post"), "SP+ best" to $\underline{\tau} = 0$ and $\overline{\tau} = 0.8\%$ (the maximum in the right panel of Figure 7), and "SP− best" to $\underline{\tau} = -6\%$ and $\overline{\tau} = 0$ (the maximum in the left panel of Figure 7). Welfare gains are in permanent net consumption equivalents. "initial $\pi$" is the DE ergodic distribution, while "final $\pi$" is the ergodic distribution under the corresponding alternative equilibrium. The moments of $\tau$ are with respect to the "final $\pi$." $\overline{ca}^{\text{DE}}$ is the DE financial crisis threshold.

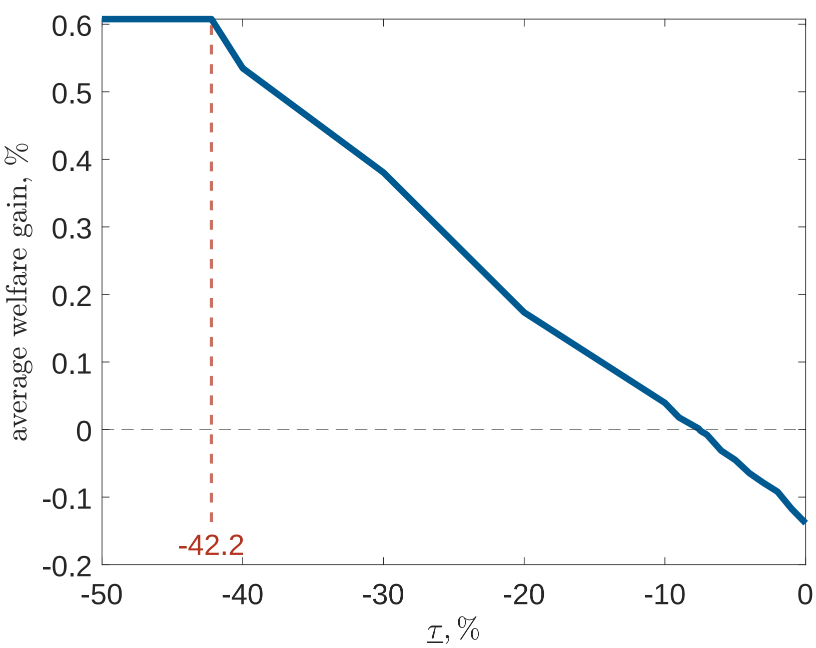
consistent policy ("SP") induces significant welfare gains of more than 0.6%, both with respect to the DE and SP ergodic distributions. The welfare gains are achieved by taxing debt in "good times" and subsidizing debt in "bad times." If either (positive) taxes or subsidies are not available to the policymaker, the optimal time-consistent policy is counterproductive and leads to welfare losses: -0.14% in the "SP ex ante" equilibrium and -0.08% in the "SP ex post equilibrium" with respect to the DE ergodic distribution. The welfare losses are smaller if computed with respect to the ergodic distributions of the corresponding equilibria, and there is a marginal welfare gain of 0.01% from the "SP ex post equilibrium" in this case.

Using the ergodic distribution to compute these moments, both SP+ and SP- generate welfare losses, but SP+ are slightly larger. Using the alternative equilibrium distribution the ranking are essentially the same with the SP+ now doing slightly better and than the SP-

In both SP+ and SP- the intensity of the subsidy or tax is lower than in the SP, The average tau in SP and SP+ is similar. The subsidy is used much more frequently. Crisis probabilities are highest under SP+.

How do we get similar welfare gains, trade-off between frequency and severity of the crisis? Figure 6 further illustrates the role of the ex post debt subsidy for welfare gains. In this figure, we plot the welfare gains from the optimal time-consistent policy constrained by $\tau(x) \geq \underline{\tau}$ for different values of $\underline{\tau}$ on the horizontal axis. Hence, only the subsidy component is restricted, but taxes can be set as high as needed, i.e., $\overline{\tau} = +\infty$ in (39). If $\underline{\tau} < -42.2\%$ (i.e.,

Figure 6: Welfare gains from optimal time-consistent policy constrained by $\tau(x) \geq \underline{\tau}$

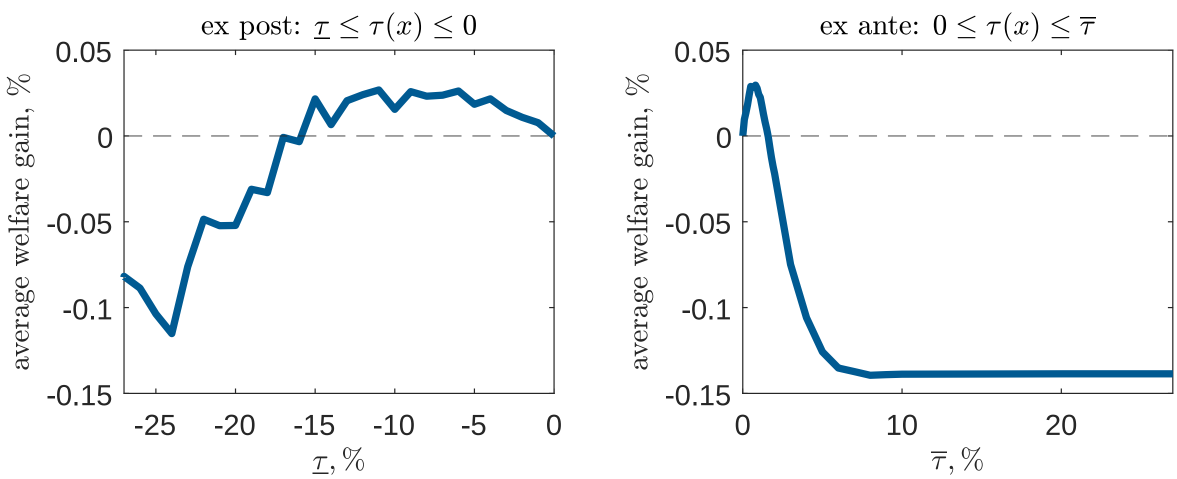



Notes: If $\underline{\tau} < -42.2\%$, the constraint $\tau(x) \geq \underline{\tau}$ is slack, and we obtain the "SP" equilibrium.

min($\tau$) for "SP" in Table 1), the constraint $\tau(x) \geq \underline{\tau}$ is slack, and we obtain the unconstrained "SP" equilibrium. As the tax lower bound $\underline{\tau}$ increases from $-42.2\%$, welfare gains decrease and become negative when $\underline{\tau} \approx -7.5\%$. Hence, substantial ex post interventions are essential for welfare gains from a time-consistent policy.

Interestingly, we find that optimal time-consistent ex ante and ex post policies can be welfare enhancing if we restrict the magnitude of these policies. The left panel of Figure 7

Figure 7: Welfare gains from optimal time-consistent ex post and ex ante policies




Notes: The left panel corresponds to optimal time-consistent policies constrained by $\underline{\tau} \leq \tau(x) \leq 0$ for different values of $\underline{\tau}$. In the right panel, the constraint is $0 \leq \tau(x) \leq \overline{\tau}$ for different values of $\overline{\tau}$.

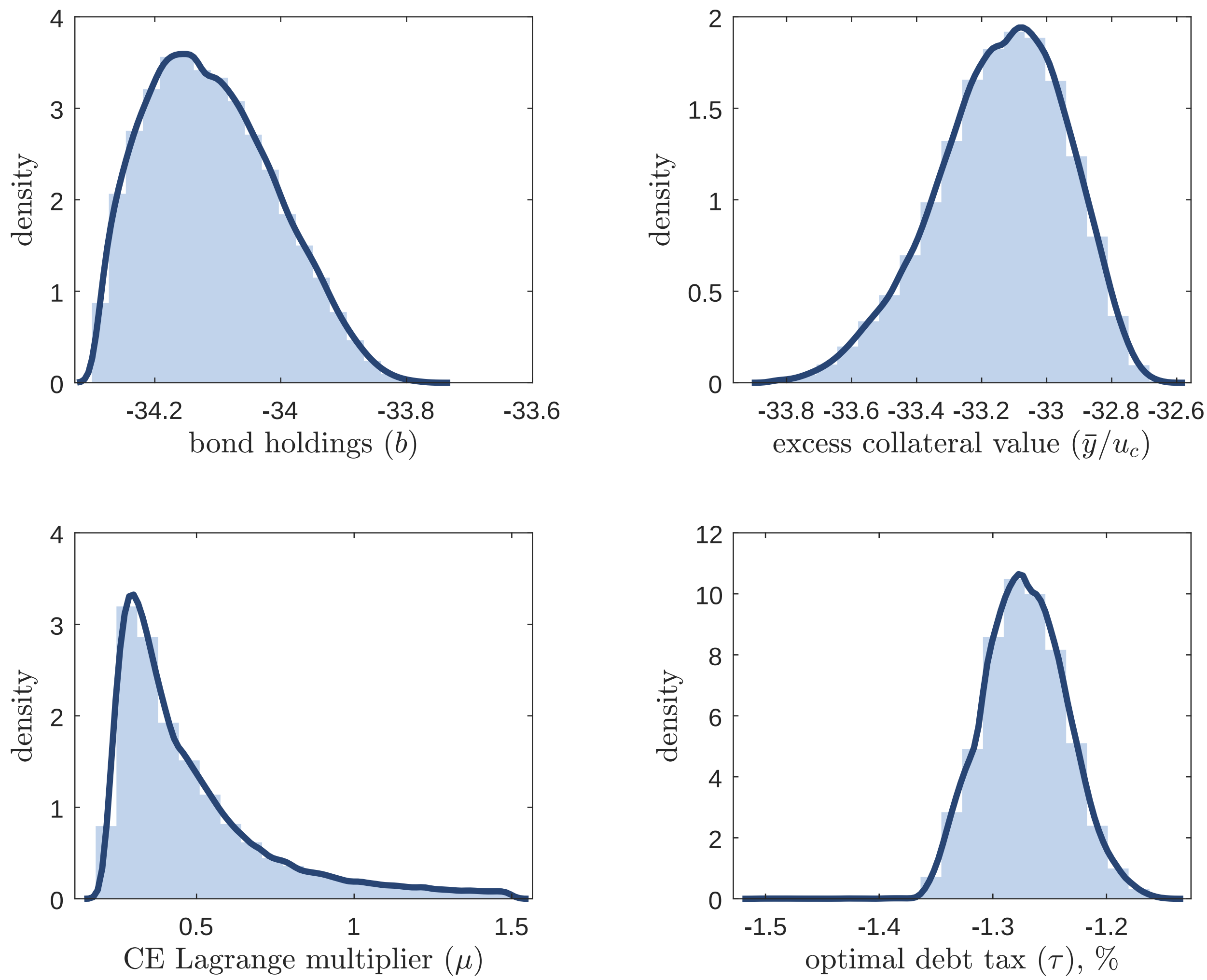
plots welfare gains from optimal time-consistent policies constrained by $\underline{\tau} \leq \tau(x) \leq 0$ for different values of $\underline{\tau}$ on the horizontal axis. These policies do not allow positive taxes and have an interpretation of "ex post" policies. For $\underline{\tau} \geq -15\%$, there is a welfare gain from an ex post policy, with the maximum welfare gain of around 0.03% when $\underline{\tau} = -6\%$ ("SP$-$ best" in Table 1). Symmetrically, the right panel of Figure 7 considers optimal time-consistent policies constrained by $0 \leq \tau(x) \leq \overline{\tau}$ for different values of $\overline{\tau}$ on the horizontal axis. For $\overline{\tau} \leq 1.5\%$, there is a welfare gain from such "ex ante" policies, with the maximum of also around 0.03% when $\overline{\tau} = 0.8\%$ ("SP+ best" in Table 1). Hence, there is a wider range of welfare-enhancing ex post policies, but welfare gains from such policies are significantly smaller than from the unrestricted optimal time-consistent policy.

## 5.4 First-best policy

In this section, we use (27), (36), and (38) to construct numerically the optimal time-consistent policy $\{(\tau_t, \tau_t^v)\}$ that implements the first-best allocation. For simplicity, we focus on the productivity process $\{z_t\}$ as a source of uncertainty and set $R_t = 1.012$ (the intermediate state of the 3-state process) and $\kappa_t = 0.886$ (the mean of the 2-state process with respect to its stationary distribution).

Figure 8 displays the empirical distributions—both histograms and kernel density function estimates—of the first-best bond holdings $b_t^{\mathrm{FB}}$, excess collateral value $\kappa_t q_t^{\mathrm{FB}} + b_{t+1}^{\mathrm{FB}}/R_t - \theta p_v v_t^{\mathrm{FB}}$, agent's Lagrange multiplier $\mu_t$ given by (27), and optimal time-consistent tax on debt $\tau_t$ given by (36).

Figure 8: First-best policy: empirical distributions




Notes: The empirical distributions are based on a 100,000-period simulation with a 1,000-period burn-in. The bars represent the histograms with a density normalization on the y-axis. The solid lines are the probability density function estimates based on a normal kernel. The Lagrange multiplier distribution is heavily right skewed, so a truncated distribution is plotted.

The collateral constraint evaluated at the first-best allocation is always violated on the first-best ergodic set $X^{\mathrm{FB}}$, since the excess collateral value is always negative. Hence, $\lambda(A) = 1$, consistent with the premise of Proposition 4. The condition (22) is always satisfied strictly, so that $\mu_t > 0$ and $\tau_t < 0$ for all $t$. There is *underborrowing* in the DE compared to the first best. Agents borrow close to the natural borrowing limit, and the distribution of bond holdings is right skewed, reflecting the former. The distribution of the optimal time-consistent tax on debt is left skewed but has a relatively low variance, with most of its mass concentrated in the $[-1.4\%, -1.1\%]$ interval. Hence, the policymaker uses a moderate subsidy in the ergodic distribution. By Corollary 7, the optimal subsidy is driven entirely by the risk sharing component, while both the macroprudential and collateral externality components are zero.

# 6 Conclusion

**TBC**

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# Appendices

## A Proofs

### A.1 Proposition 1

Define an operator $T : \mathcal{F}(X) \to \mathcal{F}(X)$ as

$$T(V)(b,s) = \max_{\hat{h},\hat{v},\hat{b}} \left[ u\left( zF(1,\hat{h},\hat{v}) - p_v\hat{v} - g(\hat{h}) + b - \frac{\hat{b}}{R} \right) + \beta \mathbb{E}_s V(\hat{b}, s') \right],$$

for all $(b,s) = (b,(z,R,\kappa)) \in X$. Since $u$ is strictly increasing,

$$T(V)(b,s) = \max_{\hat{b}} \left[ u\left( \max_{\hat{h},\hat{v}} \Big\{ zF(1,\hat{h},\hat{v}) - p_v\hat{v} - g(\hat{h}) \Big\} + b - \frac{\hat{b}}{R} \right) + \beta \mathbb{E}_s V(\hat{b}, s') \right].$$

Consider the inner maximization problem

$$f(z) = \max_{\hat{h},\hat{v}} \Big\{ zF(1,\hat{h},\hat{v}) - p_v\hat{v} - g(\hat{h}) \Big\}.$$

Since $F$ is concave and Cobb—Douglas, it is strictly concave in labor and inputs. Also, $g$ is convex. Hence, the unique maximum is described by the first-order conditions (11) and (16). Clearly, the functions $h$ and $v$ depend on $z$ only. To see that they are strictly increasing, suppose for a moment that $z$ can be varied continuously. Then, applying the implicit function theorem,

$$\begin{aligned}
\frac{\mathrm{d}h_x}{\mathrm{d}z} &= \frac{F_h(1,h_x,v_x) - \frac{F_{hv}(1,h_x,v_x)}{F_{vv}(1,h_x,v_x)} F_v(1,h_x,v_x)}{g''(h_x) - z\left( F_{hh}(1,h_x,v_x) - \frac{F_{hv}(1,h_x,v_x)^2}{F_{vv}(1,h_x,v_x)} \right)} > 0, \\
\frac{\mathrm{d}v_x}{\mathrm{d}z} &= -\frac{F_v(1,h_x,v_x)}{zF_{vv}(1,h_x,v_x)} - \frac{F_{hv}(1,h_x,v_x)}{F_{vv}(1,h_x,v_x)} \frac{\mathrm{d}h_x}{\mathrm{d}z} > 0,
\end{aligned}$$

where the signs follow from our assumptions on $F$ and $g$ that imply $F_h, F_v, F_{hv} > 0$, $F_{hh}, F_{vv} < 0$, $g'' \geq 0$, and $F_{hh} - \frac{F_{hv}^2}{F_{vv}} < 0$, where the latter is due to strict concavity of $F$ in the last two arguments and its Cobb—Douglas form. Note that $f$ is also strictly increasing by the envelope theorem.

The operator $T$ simplifies to

$$T(V)(b,s) = \max_{\hat{b}\in\Gamma(b,s)} \left[ u\left( f(z) + b - \frac{\hat{b}}{R} \right) + \beta \mathbb{E}_s V(\hat{b}, s') \right],$$

where $\Gamma(b,s) = (-\infty, R(f(z)+b)] \cap B$, so that $\hat{c} \geq 0$. We have $\Gamma(b,s) \neq \emptyset$ for all $(b,s) \in X$ if and only if $\underline{b} \leq R(f(\underline{z}) + \underline{b})$ for all $R \in [\underline{R}, \overline{R}]$, which is if and only if $\underline{b} \geq -\frac{\overline{R}}{\overline{R}-1} f(\underline{z})$ if $\overline{R} > 1$ and $\underline{b} \leq \frac{\underline{R}}{1-\underline{R}} f(\underline{z})$ if $\underline{R} < 1$. Let $\underline{b}$ satisfy these inequalities strictly, so that $\Gamma(b,s)$ is infinite for all $(b,s) \in X$. Clearly, $\Gamma$ is compact-valued and continuous. Since $S$ is finite, $X = B \times S$ is compact, so any continuous function on $X$ is bounded by the extreme value theorem. Since $u$ is continuous, it then follows from the maximum theorem that $T : C(X) \to C(X)$, where $C(X) \subset \mathcal{F}(X)$ is the set of all continuous (and thus bounded) functions on $X$. Then by Theorem 9.6 in Stokey et al. (1989), $T$ has a unique fixed point $V$—the solution to the Bellman equation.

Since $u$ is strictly concave and continuously differentiable and $\Gamma$ is convex in $b$, it follows that $V$ is strictly concave in $b$, the policy function $b'$ is single-valued and continuous, and $V$ is continuously differentiable in $b$ at interior points whenever $b'(b,s)$ is interior in $\Gamma(b,s)$ (Stokey et al., 1989, Theorems 9.8 and 9.10). Specifically,

$$V_b(b,s) = u'(\tilde{c}(b,s)), \tag{A.1}$$

where $\tilde{c}(b,s) = f(z) + b - \frac{b'(b,s)}{R}$ is the implied policy function for net consumption, and the first-order condition for an interior maximum is

$$u'(\tilde{c}(b,s)) = \beta R \mathbb{E}_s V_b(b'(b,s), s'). \tag{A.2}$$

The Euler equation (17) is obtained by substituting (A.1) into (A.2). Note that $b'(b,s) = R(f(z)+b)$ cannot be optimal since it implies $u'(\tilde{c}(b,s)) = \infty$. Hence, if $b'(b,s)$ is at the boundary of $\Gamma(b,s)$, then it must be at the boundary of $B$.

Clearly, $\tilde{c}$ is strictly increasing in $b$ if $b'(b,s)$ is at the boundary of $B$. Since $V$ is strictly concave in $b$ and $u$ is strictly concave, (A.1) implies that $\tilde{c}$ is strictly increasing in $b$ if $b'(b,s) \in \operatorname{int} B$, and then (A.2) implies that $b'$ is injective in $b$ in the same region. Since $b'$ is continuous, it then must be strictly monotone in $b$ whenever $b'(b,s) \in \operatorname{int} B$. It is clear from (A.2) that $b'$ cannot be strictly decreasing in $b$, hence it is strictly increasing in $b$.

## A.2 Proposition 2

In an MPE, $\mu_x \geq 0$ for all $x \in X$. Hence, given the first-best allocation, (14) and (18) imply $q_x \geq q_x^{\mathrm{FB}}$ for all $x \in X$. Substituting (20) into the collateral constraint (10) evaluated at the first-best allocation, we obtain $-b_x^{\mathrm{FB}}/R + \theta p_v v_x^{\mathrm{FB}} = \kappa q_x^A \leq \kappa q_x$, which is equivalent to $q_x \geq q_x^A$. Hence, the complementary slackness condition in (15) evaluated at the first-best allocation is equivalent to $\mu_x \left(q_x - q_x^A\right) = 0$. Now let $x \in A$ and suppose $x(s^t) \in \operatorname{int} A^c$ for all $t \geq 1$ and $s^t \in S^t$. Then due to (19), we have $q^A(x(s^t)) < q^{\mathrm{FB}}(x(s^t)) \leq q(x(s^t))$, which implies $\mu(x(s^t)) = 0$ for all $t \geq 1$ and $s^t \in S^t$, and thus $q_x = q_x^{\mathrm{FB}}$. But since $x \in A$, we have $q_x = q_x^{\mathrm{FB}} < q_x^A$, which is a contradiction.

## A.3 Proposition 3

Since $\left\{\tilde{c}^{\mathrm{FB}}, h^{\mathrm{FB}}, v^{\mathrm{FB}}, b^{\mathrm{FB}}\right\}$ satisfy (9), $\left\{\tilde{c}^{\mathrm{FB}}, h^{\mathrm{FB}}, v^{\mathrm{FB}}, b^{\mathrm{FB}}, q, \mu\right\} \subset \mathcal{F}(X)$ is a constrained-efficient MPE of Definition 3 if and only if

$$-\frac{b_x^{\mathrm{FB}}}{R} + \theta p_v v_x^{\mathrm{FB}} \leq \kappa q_x, \tag{A.3}$$

$$q_x u'(\tilde{c}_x^{\mathrm{FB}}) = \beta \mathbb{E}_s \left[ u'(\tilde{c}_{x'}^{\mathrm{FB}}) \Big( z' F_k(1, h_{x'}^{\mathrm{FB}}, v_{x'}^{\mathrm{FB}}) + q_{x'} \Big) + \mu_{x'} \kappa' q_{x'} \right], \tag{A.4}$$

$$0 = \mu_x \left( \kappa q_x + \frac{b_x^{\mathrm{FB}}}{R} - \theta p_v v_x^{\mathrm{FB}} \right), \qquad \mu_x \geq 0, \tag{A.5}$$

with $x' = (b_x^{\mathrm{FB}}, (z', R', \kappa'))$, for all $x = (b, s) = (b, (z, R, \kappa)) \in X$. The "only if" is by definition, while "if" is since $\left\{\tilde{c}^{\mathrm{FB}}, h^{\mathrm{FB}}, v^{\mathrm{FB}}, b^{\mathrm{FB}}\right\}$ is optimal in the less constrained first-best problem of Definition 2.

Proposition 2 requires $q_x \geq q_x^A > q_x^{\mathrm{FB}}$ for all $x \in A$, which requires $\mu(x(s^t)) > 0$ for some $t \geq 1$ and $s^t \in S^t$. In turn, by Proposition 2, $\mu(x(s^t)) > 0$ implies $q(x(s^t)) = q^A(x(s^t))$. We can, therefore, set $q_x = q_x^A$ for all $x \in A$, which makes (A.3) hold with equality for all $x \in A$. Let $\mathcal{F}_A(X) = \{q \in \mathcal{F}(X) \mid q_x = q_x^A \text{ for all } x \in A\}$. If $\mu_x \geq 0$ for all $x \in X$, any fixed point $q \in \mathcal{F}_A(X)$ of (A.4) makes (A.3) hold for all $x \in A^c$, which follows from $q_x \geq q_x^{\mathrm{FB}} \geq q_x^A$ for all $x \in A^c$ by Proposition 2 and (19). We are left to construct $\mu \in \mathcal{F}(X)$ consistent with (A.5) for all $x \in X$ and (A.4) for all $x \in A$.

Given $q \in \mathcal{F}_A(X)$, define $\mu^q \in \mathcal{F}(X)$ such that $\left\{\left\{\mu^q_{x(s^t)}\right\}_{s^t \in S^t}\right\}_{t=1}^n$ satisfies

$$
\begin{aligned}
q^A_x = \sum_{t=1}^{n} \beta^t \sum_{s^t \in S^t} \Pr\big(s^t \mid s\big) \prod_{i=1}^{t-1} \mathbf{1}_{A^c}(x(s^i)) \frac{u'\left(\tilde{c}^{\mathrm{FB}}_{x(s^t)}\right)}{u'\left(\tilde{c}^{\mathrm{FB}}_x\right)} \\
\times \left[ z_t F_k \left(1, h^{\mathrm{FB}}_{x(s^t)}, v^{\mathrm{FB}}_{x(s^t)}\right) + \mathbf{1}_A(x(s^t)) \left(1 + \frac{\mu^q_{x(s^t)} \kappa_t}{u'\left(\tilde{c}^{\mathrm{FB}}_{x(s^t)}\right)}\right) q^A_{x(s^t)} \right] \\
+ \beta^n \sum_{s^n \in S^n} \Pr(s^n \mid s) \prod_{i=1}^{n} \mathbf{1}_{A^c}(x(s^i)) \frac{u'\left(\tilde{c}^{\mathrm{FB}}_{x(s^n)}\right)}{u'\left(\tilde{c}^{\mathrm{FB}}_x\right)} q_{x(s^n)},
\end{aligned}
\tag{A.6}
$$

for all $x = (b, s) \in A$, where $n \in [1, \infty)$ is such that

$$
\sum_{t=1}^{n} \left| \left\{ s^t \in S^t \mid x(s^t) \in A \text{ and } x(s^i) \in A^c \text{ for all } i \in [1, t) \right\} \right| \geq \left| \left\{ \hat{x} \in A \mid b^{\mathrm{FB}}(\hat{x}) = b^{\mathrm{FB}}(x) \right\} \right|,
$$

where $|\cdot|$ denotes the cardinality of a set, and $\mu^q(\cdot) = 0$ otherwise. Note that (A.6) is (A.4) iterated forward.

Several comments are in order. First, since $\lambda(A) > 0$, we are guaranteed to have an infinite number of $t \geq 1$ and $s^t \in S^t$ such that $x(s^t) \in A$. Therefore, we can indeed affect the right-hand side of (A.6) by varying $\mu^q$. Second, for each $x \in A$, there may be several $\hat{x} \in A$ such that $b^{\mathrm{FB}}(\hat{x}) = b^{\mathrm{FB}}(x)$. Therefore, making $\mu^q$ consistent with (A.6) at $x \in A$ requires solving a system of equations corresponding to $\left\{\hat{x} \in A \mid b^{\mathrm{FB}}(\hat{x}) = b^{\mathrm{FB}}(x)\right\}$. Since $b^{\mathrm{FB}}$ is strictly increasing in $b$ by Proposition 1, $\left|\left\{\hat{x} \in A \mid b^{\mathrm{FB}}(\hat{x}) = b^{\mathrm{FB}}(x)\right\}\right| \leq |S| < \infty$, so there is a finite number of equations. The system is linear in $\left\{\mu^q_{x(s^t)}\right\}$. For this linear system to have a solution, $n$ needs to be large enough. Due to ergodicity (Assumption 1) and continuity of $\left\{\tilde{c}^{\mathrm{FB}}, h^{\mathrm{FB}}, v^{\mathrm{FB}}, b^{\mathrm{FB}}\right\}$ (Proposition 1), we can choose $n$ such that the number of variables $\left\{\mu^q_{x(s^t)}\right\}$ is at least as great as the number of equations and the rank condition of Rouché—Capelli theorem holds. Third, since $b^{\mathrm{FB}}$ is strictly increasing in $b$, as we vary $x \in A$, the set $\left\{\mu^q_{x(s^t)}\right\}$ will vary as well, allowing to construct a function $\mu^q \in \mathcal{F}(X)$. By imposing $\mu^q(\cdot) = 0$ at all other points unrestricted by the construction above, we ensure that (A.5) holds at all $x \in A^c$.

Define an operator $T : \mathcal{F}_A(X) \to \mathcal{F}_A(X)$ as

$$T(q)(x) = \mathbf{1}_A(x) q_x^A \\ + \mathbf{1}_{A^c}(x) \mathbb{E}_s \left\{ \beta \frac{u'\left(\tilde{c}_{x'}^{\mathrm{FB}}\right)}{u'\left(\tilde{c}_x^{\mathrm{FB}}\right)} \left[ z' F_k\left(1, h_{x'}^{\mathrm{FB}}, v_{x'}^{\mathrm{FB}}\right) + \left(1 + \frac{\max\{\mu_{x'}^q, 0\}\kappa'}{u'\left(\tilde{c}_{x'}^{\mathrm{FB}}\right)}\right) q_{x'} \right] \right\},$$

with $x' = (b_x^{\mathrm{FB}}, (z', R', \kappa'))$, for all $x = (b, s) \in X$. Let $q = T(q) \in \mathcal{F}_A(X)$ with the corresponding $\mu^q \in \mathcal{F}(X)$ and define $\mu \in \mathcal{F}(X)$ as $\mu_x = \max\{\mu_x^q, 0\}$. Such $\{q, \mu\}$ make (A.3) and (A.5) hold on $X$ and (A.4) on $A^c$. If $\mu_x^q \geq 0$ for all $x \in A$, then (A.4) holds on $A$, otherwise holding approximately on $A$. Hence, $\left\{\tilde{c}^{\mathrm{FB}}, h^{\mathrm{FB}}, v^{\mathrm{FB}}, b^{\mathrm{FB}}, q, \mu\right\}$ is a candidate first-best MPE of Definition 4, and it is a first-best MPE if $\mu_x^q \geq 0$ for all $x \in A$.

If $\lambda(A) = 1$, a significant simplification is achieved if we restrict attention to $X = X^{\mathrm{FB}}$. In this case, there is no need to find a fixed point $q$ of $T$, since $\lambda(A^c) = 0$, so that (A.6) simplifies to (21), and $q = q^A$ on $X^{\mathrm{FB}}$.

## A.4 Proposition 4

The first-best allocation $\left\{\left(\tilde{c}_t^{\mathrm{FB}}, h_t^{\mathrm{FB}}, v_t^{\mathrm{FB}}, b_{t+1}^{\mathrm{FB}}\right)\right\}_{t=0}^{\infty}$ satisfies (23) with equality. Since $x_0 \in X^{\mathrm{FB}}$, $x(s^t) \in X^{\mathrm{FB}}$ for all $t \geq 0$ and $s^t \in S^t$ due to Assumption 1. Since $\lambda(A) = 1$, (24) is violated at the first-best allocation when prices are $\left\{q_t^{\mathrm{FB}}\right\}_{t=0}^{\infty}$ a.e. on $X^{\mathrm{FB}}$. Setting $\{q_t\}_{t=0}^{\infty} = \left\{q_t^A\right\}_{t=0}^{\infty}$ makes (24) hold with equality on $X^{\mathrm{FB}}$. Setting $\{\mu_t\}_{t=0}^{\infty}$ according to (27) makes (25) hold given the allocation and prices. Finally, $\mu_0 = 0$ and (22) imply $\mu_t(s^t) \geq 0$ for all $t \geq 0$ and $s^t \in S^t$, satisfying (26). Therefore, the proposed plan is feasible in the planning problem of Definition 5. Since the proposed plan entails the first-best allocation, it is optimal. Proposition 1 implies that the first-best plan is unique, so any constrained-efficient plan entails the first-best allocation.

Consider restarting the planning problem of Definition 5 at some $\tau > 0$ and $s^\tau \in S^\tau$. The continuation of the original plan is feasible, and it is optimal to choose a plan that entails the first-best allocation. There is no incentive to deviate from $\left\{\left\{\left(q_t^A(s^t), \mu_t(s^t)\right)\right\}_{s^t \in S^t | s^\tau}\right\}_{t=\tau}^{\infty}$. Consequently, any constrained-efficient plan is time consistent.

## A.5 Proposition 7

**Proposition 7** (One-shot tax change)**.** *Consider a regulated CE of Definition 6 given $\tau$. Let $X_u^\tau = \{x \in X \mid \mu_x = 0\}$ and $X_c^\tau = X \setminus X_u^\tau$. Let $\mathbf{1}_{X_c^\tau}$ denote the indicator function of $X_c^\tau$. Consider a one-shot change $\mathrm{d}\tau_x$ in the tax rate at the state $x \in \operatorname{int} X_u^\tau \cup X_c^\tau$. (Hence,* *the initial $\tau$ is used again starting from the next period.) The response in the asset price is*

$$\frac{\mathrm{d}q_x}{\mathrm{d}\tau_x} = \Psi_x^{q(\tau)} \frac{\mathrm{d}b'_x}{\mathrm{d}\tau_x},$$

*where*

$$\Psi_x^{q(\tau)} \equiv \frac{\frac{1}{R} q_x u''(\tilde{c}_x)\left(1 - \frac{\mu_x}{u'(\tilde{c}_x)}\right) + \beta \mathbb{E}_s \frac{\partial}{\partial b'_x}\left[u'(\tilde{c}_{x'})\Big(z' F_k(1, h_{x'}, v_{x'}) + q_{x'}\Big) + \mu_{x'} \kappa' q_{x'}\right]}{u'(\tilde{c}_x) + \mu_x \kappa q_x \frac{u''(\tilde{c}_x)}{u'(\tilde{c}_x)}},$$

*and the response in next-period bond holdings is*

$$\frac{\mathrm{d}b'_x}{\mathrm{d}\tau_x} = \frac{u'(\tilde{c}_x)}{\Psi_x^{b'(\tau)}(1+\tau_x)^2},$$

*where*

$$\begin{aligned}\Psi_x^{b'(\tau)} \equiv &-\frac{u''(\tilde{c}_x)}{R}\left(\frac{1}{1+\tau_x} - \frac{\mu_x}{u'(\tilde{c}_x)}\right) - \beta R \mathbb{E}_s\left(u''(\tilde{c}_{x'})\frac{\partial \tilde{c}_{x'}}{\partial b'_x}\right) + \mathbf{1}_{X_c^\tau}(x)\left(\kappa \Psi_x^{q(\tau)} + \frac{1}{R}\right) \\ &\times \left\{\mu_x \frac{u''(\tilde{c}_x)}{u'(\tilde{c}_x)}\left(\frac{1}{1+\tau_x} - \frac{\mu_x}{u'(\tilde{c}_x)}\right) - \frac{u'(\tilde{c}_x)}{(\theta p_v)^2}\left[\frac{(z F_{hv}(1, h_x, v_x))^2}{g''(h_x) - z F_{hh}(1, h_x, v_x)} + z F_{vv}(1, h_x, v_x)\right]\right\},\end{aligned}$$

*where $\Psi_x^{b'(\tau)}$ should be replaced with $\lim_{\theta \to 0} \Psi_x^{b'(\tau)}$ if $\theta = 0$.*

*Proof.* For any $x \in \operatorname{int} X_u^\tau \cup X_c^\tau$, (9), (11), and (12) imply

$$\mathrm{d}\tilde{c}_x + \frac{1}{R}\,\mathrm{d}b'_x = \frac{\theta p_v \mu_x}{u'(\tilde{c}_x)}\,\mathrm{d}v_x\,, \tag{A.7}$$

$$g''(h_x)\,\mathrm{d}h_x = z F_{hh}(1, h_x, v_x)\,\mathrm{d}h_x + z F_{hv}(1, h_x, v_x)\,\mathrm{d}v_x\,, \tag{A.8}$$

$$\theta p_v\left(\frac{1}{u'(\tilde{c}_x)}\,\mathrm{d}\mu_x - \mu_x \frac{u''(\tilde{c}_x)}{u'(\tilde{c}_x)^2}\,\mathrm{d}\tilde{c}_x\right) = z F_{hv}(1, h_x, v_x)\,\mathrm{d}h_x + z F_{vv}(1, h_x, v_x)\,\mathrm{d}v_x\,. \tag{A.9}$$

Moreover, (14) and (28) imply

$$q_x u''(\tilde{c}_x)\,\mathrm{d}\tilde{c}_x + u'(\tilde{c}_x)\,\mathrm{d}q_x = \beta \mathbb{E}_s \frac{\partial}{\partial b'_x}\left[u'(\tilde{c}_{x'})\Big(z' F_k(1, h_{x'}, v_{x'}) + q_{x'}\Big) + \mu_{x'} \kappa' q_{x'}\right]\mathrm{d}b'_x\,, \tag{A.10}$$

$$u''(\tilde{c}_x)\,\mathrm{d}\tilde{c}_x = (1+\tau_x)\left[\beta R \mathbb{E}_s\left(u''(\tilde{c}_{x'})\frac{\partial \tilde{c}_{x'}}{\partial b'_x}\right)\mathrm{d}b'_x + \mathrm{d}\mu_x\right] + \frac{u'(\tilde{c}_x)}{1+\tau_x}\,\mathrm{d}\tau_x\,. \tag{A.11}$$

**Unconstrained region** If $x \in \operatorname{int} X_u^\tau$, we have $\mathrm{d}\mu_x = \mu_x = 0$, (A.8) and (A.9) imply $\mathrm{d}h_x = \mathrm{d}v_x = 0$, (A.7) implies $\mathrm{d}\tilde{c}_x = -\frac{1}{R}\,\mathrm{d}b'_x$, (A.11) implies

$$\frac{\mathrm{d}b'_x}{\mathrm{d}\tau_x} = \left[-\frac{u''(\tilde{c}_x)}{R}\frac{1}{1+\tau_x} - \beta R \mathbb{E}_s\left(u''(\tilde{c}_{x'})\frac{\partial \tilde{c}_{x'}}{\partial b'_x}\right)\right]^{-1}\frac{u'(\tilde{c}_x)}{(1+\tau_x)^2},$$

and (A.10) implies

$$\frac{\mathrm{d}q_x}{\mathrm{d}\tau_x} = \frac{1}{u'(\tilde{c}_x)}\left\{\frac{1}{R}q_x u''(\tilde{c}_x) + \beta\mathbb{E}_s\frac{\partial}{\partial b'_x}\left[u'(\tilde{c}_{x'})\Big(z'F_k(1,h_{x'},v_{x'}) + q_{x'}\Big) + \mu_{x'}\kappa' q_{x'}\right]\right\}\frac{\mathrm{d}b'_x}{\mathrm{d}\tau_x}.$$

**Constrained region** If $x \in X_c^\tau$, (15) implies

$$0 = \kappa\,\mathrm{d}q_x + \frac{1}{R}\,\mathrm{d}b'_x - \theta p_v\,\mathrm{d}v_x\,. \tag{A.12}$$

If $\theta = 0$, analogous to the unconstrained case, (A.8) and (A.9) imply $\mathrm{d}h_x = \mathrm{d}v_x = 0$, and (A.7) implies $\mathrm{d}\tilde{c}_x = -\frac{1}{R}\,\mathrm{d}b'_x$. Then (A.10) and (A.12) jointly imply $\mathrm{d}b'_x = \mathrm{d}q_x = 0$, and thus $\mathrm{d}\tilde{c}_x = 0$. Hence, a one-shot change $\mathrm{d}\tau_x$ affects only the Lagrange multiplier, with (A.11) implying

$$\frac{\mathrm{d}\mu_x}{\mathrm{d}\tau_x} = -\frac{u'(\tilde{c}_x)}{(1+\tau_x)^2} < 0.$$

If $\theta > 0$, (A.12), (A.8), and (A.7) imply

$$\mathrm{d}v_x = \frac{1}{\theta p_v}\left(\kappa\,\mathrm{d}q_x + \frac{1}{R}\,\mathrm{d}b'_x\right), \qquad \mathrm{d}h_x = \frac{zF_{hv}(1,h_x,v_x)}{g''(h_x) - zF_{hh}(1,h_x,v_x)}\,\mathrm{d}v_x\,,$$
$$\mathrm{d}\tilde{c}_x = \frac{\mu_x}{u'(\tilde{c}_x)}\left(\kappa\,\mathrm{d}q_x + \frac{1}{R}\,\mathrm{d}b'_x\right) - \frac{1}{R}\,\mathrm{d}b'_x\,.$$

Then (A.9) implies

$$\mathrm{d}\mu_x = \mathbf{1}_{X_c^\tau}(x)\frac{u'(\tilde{c}_x)}{(\theta p_v)^2}\left[\frac{(zF_{hv}(1,h_x,v_x))^2}{g''(h_x) - zF_{hh}(1,h_x,v_x)} + zF_{vv}(1,h_x,v_x)\right]\left(\kappa\,\mathrm{d}q_x + \frac{1}{R}\,\mathrm{d}b'_x\right) + \mu_x\frac{u''(\tilde{c}_x)}{u'(\tilde{c}_x)}\,\mathrm{d}\tilde{c}_x\,.$$

The expressions for $\mathrm{d}\tilde{c}_x$ and $\mathrm{d}\mu_x$ remain correct when $x \in \operatorname{int} X_u^\tau$, in which case $\mu_x = \mathbf{1}_{X_c^\tau}(x) = 0$ (if $\theta = 0$, consider $\lim_{\theta\to 0}\mathrm{d}\mu_x$). Consequently, (A.10) implies a general formula

$$\frac{\mathrm{d}q_x}{\mathrm{d}\tau_x} = \Psi_x^{q(\tau)}\frac{\mathrm{d}b'_x}{\mathrm{d}\tau_x},$$

where

$$\Psi_x^{q(\tau)} \equiv \frac{\frac{1}{R} q_x u''(\tilde{c}_x)\left(1 - \frac{\mu_x}{u'(\tilde{c}_x)}\right) + \beta \mathbb{E}_s \frac{\partial}{\partial b'_x}\left[u'(\tilde{c}_{x'})\Big(z' F_k(1, h_{x'}, v_{x'}) + q_{x'}\Big) + \mu_{x'}\kappa' q_{x'}\right]}{u'(\tilde{c}_x) + \mu_x \kappa q_x \frac{u''(\tilde{c}_x)}{u'(\tilde{c}_x)}},$$

and (A.11) implies

$$\frac{\mathrm{d}b'_x}{\mathrm{d}\tau_x} = \frac{u'(\tilde{c}_x)}{\Psi_x^{b'(\tau)}(1+\tau_x)^2},$$

where

$$\begin{aligned}\Psi_x^{b'(\tau)} \equiv & -\frac{u''(\tilde{c}_x)}{R}\left(\frac{1}{1+\tau_x} - \frac{\mu_x}{u'(\tilde{c}_x)}\right) - \beta R \mathbb{E}_s\left(u''(\tilde{c}_{x'})\frac{\partial \tilde{c}_{x'}}{\partial b'_x}\right) + \mathbf{1}_{X_c^\tau}(x)\left(\kappa \Psi_x^{q(\tau)} + \frac{1}{R}\right) \\ & \times\left\{\mu_x \frac{u''(\tilde{c}_x)}{u'(\tilde{c}_x)}\left(\frac{1}{1+\tau_x} - \frac{\mu_x}{u'(\tilde{c}_x)}\right) - \frac{u'(\tilde{c}_x)}{(\theta p_v)^2}\left[\frac{(zF_{hv}(1,h_x,v_x))^2}{g''(h_x) - zF_{hh}(1,h_x,v_x)} + zF_{vv}(1,h_x,v_x)\right]\right\}.\end{aligned}$$

If $\theta = 0$, $\Psi_x^{b'(\tau)}$ should be replaced with $\lim_{\theta\to 0}\Psi_x^{b'(\tau)}$. Indeed, if $x \in \operatorname{int} X_u^\tau$ and $\theta = 0$, then $\mathbf{1}_{X_c^\tau}(x) = 0$ as $\theta \to 0$, so the limit of the product involving $\mathbf{1}_{X_c^\tau}(x)$ converges to 0, resulting in the expression we obtained for the unconstrained region. If $x \in X_c^\tau$ and $\theta = 0$, then $\lim_{\theta\to 0}\Psi_x^{b'(\tau)} = \pm\infty$, which implies $\mathrm{d}b'_x/\mathrm{d}\tau_x = 0$, and thus $\mathrm{d}q_x/\mathrm{d}\tau_x = 0$, as argued above.

■

## A.6 Proposition 5

The regulated CE of Definition 6 is described by (9)–(12), (14), (15), and (28). Clearly, (28) can be used to back out $\tau$ given the other functions. We will show that (11), (12), and (15) are slack as constraints in the current policymaker's best response to the future policymaker's decision rules. Similar to Bianchi and Mendoza (2018, Appendix A.1, Proposition II), who assume $\theta > 0$, consider the best response in a relaxed problem given by

$$V(b,s) = \max_{\hat{c},\hat{h},\hat{v},\hat{b},\hat{q}}\left[u(\hat{c}) + \beta\mathbb{E}_s V(\hat{b}, s')\right]$$

subject to

$$\begin{aligned}\hat{c} + \frac{\hat{b}}{R} &\le zF(1,\hat{h},\hat{v}) - p_v\hat{v} - g(\hat{h}) + b, \\ -\frac{\hat{b}}{R} + \theta p_v \hat{v} &\le \kappa\hat{q}, \\ \hat{q}u'(\hat{c}) &= \beta\mathbb{E}_s\left[u'(\tilde{c}_{x'})\Big(z'F_k(1,h_{x'},v_{x'}) + q_{x'}\Big) + \mu_{x'}\kappa' q_{x'}\right],\end{aligned}$$

with $x' = (\hat{b}, (z', R', \kappa'))$, for all $x = (b, s) = (b, (z, R, \kappa)) \in X$. The corresponding Lagrangian is

$$\mathcal{L} = u(\hat{c}) + \beta \mathbb{E}_s V(\hat{b}, s') + \hat{\lambda} \left( zF(1, \hat{h}, \hat{v}) - p_v \hat{v} - g(\hat{h}) + b - \hat{c} - \frac{\hat{b}}{R} \right)$$
$$+ \hat{\mu}^{\mathrm{SP}} \left( \kappa \hat{q} + \frac{\hat{b}}{R} - \theta p_v \hat{v} \right) + \hat{\xi} \left\{ \beta \mathbb{E}_s \left[ u'(\tilde{c}_{x'}) \Big( z' F_k(1, h_{x'}, v_{x'}) + q_{x'} \Big) + \mu_{x'} \kappa' q_{x'} \right] - \hat{q} u'(\hat{c}) \right\}.$$

The first-order conditions for $\hat{c}$, $\hat{h}$, $\hat{v}$, $\hat{q}$ and the complementary slackness conditions in an MPE are, respectively,

$$\lambda_x = u'(\tilde{c}_x) - \xi_x q_x u''(\tilde{c}_x), \tag{A.13}$$
$$0 = \lambda_x \Big( z F_h(1, h_x, v_x) - g'(h_x) \Big), \tag{A.14}$$
$$0 = \lambda_x \Big( z F_v(1, h_x, v_x) - p_v \Big) - \mu_x^{\mathrm{SP}} \theta p_v, \tag{A.15}$$
$$\xi_x = \mu_x^{\mathrm{SP}} \frac{\kappa}{u'(\tilde{c}_x)}, \tag{A.16}$$
$$0 = \mu_x^{\mathrm{SP}} \left( \kappa q_x + \frac{b'_x}{R} - \theta p_v v_x \right), \qquad \mu_x^{\mathrm{SP}} \geq 0. \tag{A.17}$$

Since $\mu_x^{\mathrm{SP}} \geq 0$ by (A.17), $\xi_x \geq 0$ by (A.16), and $\lambda_x > 0$ by (A.13). Hence, (A.14) is equivalent to (11).

If $\theta = 0$, (A.15) is equivalent to (12), which means that the set of MPE in the optimal policy problem is equivalent to the set of constrained-efficient MPE of Definition 3. Specifically, the current policymaker, taking as given the future policymaker's decision rule $\mu$, can set $\hat{\mu}_x = \mu_x$ today to satisfy (15). There generally exist multiple such $\mu$, and thus multiple MPE (Remark 3).

If $\theta > 0$, rearranging (A.15), we obtain

$$\left( 1 + \theta \frac{\mu_x^{\mathrm{SP}}}{\lambda_x} \right) p_v = z F_v(1, h_x, v_x). \tag{A.18}$$

If

$$\mu_x = \frac{u'(\tilde{c}_x)}{\lambda_x} \mu_x^{\mathrm{SP}},$$

(A.18) is equivalent to (12) and (A.17) is equivalent to (15). This selects a specific MPE of Definition 3 that corresponds to $\mu$ satisfying (31), having used (A.13) and (A.16). If it were a first-best MPE of Definition 4, (16) would have to hold, requiring $\mu_x^{\mathrm{SP}} = \mu_x = 0$ for all $x \in X$, and thus $q = q^{\mathrm{FB}}$, which could be consistent with (10) only if $\lambda(A) = 0$.

If the MPE decision rules are differentiable, the first-order condition for $\hat{b}$ in an MPE is

$$0 = \beta \mathbb{E}_s V_b(b'_x, s') - \frac{\lambda_x}{R} + \frac{\mu_x^{\mathrm{SP}}}{R} + \xi_x \beta \mathbb{E}_s \frac{\partial}{\partial b'_x} \left[ u'(\tilde{c}_{x'}) \Big( z' F_k(1, h_{x'}, v_{x'}) + q_{x'} \Big) + \mu_{x'} \kappa' q_{x'} \right],$$

with $x' = (b'_x, (z', R', \kappa'))$. The envelope condition in an MPE is $V_b(b, s) = \lambda(b, s)$. Combining this with (A.13) and (A.16), we obtain

$$\begin{aligned} u'(\tilde{c}_x) &= \beta R \mathbb{E}_s \left( u'(\tilde{c}_{x'}) - \mu_{x'}^{\mathrm{SP}} \kappa' q_{x'} \frac{u''(\tilde{c}_{x'})}{u'(\tilde{c}_{x'})} \right) \\ &+ \mu_x^{\mathrm{SP}} \left\{ 1 + \kappa q_x \frac{u''(\tilde{c}_x)}{u'(\tilde{c}_x)} + \frac{\kappa \beta R}{u'(\tilde{c}_x)} \mathbb{E}_s \frac{\partial}{\partial b'_x} \left[ u'(\tilde{c}_{x'}) \Big( z' F_k(1, h_{x'}, v_{x'}) + q_{x'} \Big) + \mu_{x'} \kappa' q_{x'} \right] \right\}. \end{aligned} \tag{A.19}$$

The tax $\tau_x$ that makes (A.19) equivalent to (28) is given by (32)–(34).

## A.7 Proposition 6

The corresponding regulated CE is described by the same conditions as in Definition 6, except (12) is replaced by

$$\left( 1 + \tau_x^v + \theta \frac{\mu_x}{u'(\tilde{c}_x)} \right) p_v = z F_v(1, h_x, v_x). \tag{A.20}$$

The policymaker's best response is characterized by exactly the same relaxed problem as in Appendix A.6. Indeed, the solution to that problem implies that (11) holds in an MPE, (A.20) can be used to back out $\tau^v$ given the other functions, and $\mu$ can be chosen to satisfy (15) as explained in Appendix A.6 for the case $\theta = 0$ (in the current problem, the argument applies to any $\theta \geq 0$). The latter implies that the set of MPE in the current optimal policy problem is equivalent to the set of constrained-efficient MPE of Definition 3.

Since the relaxed policy problem is equivalent to that in Appendix A.6, so is the generalized Euler equation (A.19) that describes the MPE allocation of bond holdings, and thus (32)–(34) continue to hold. The tax $\tau_x^v$ that makes (A.18) equivalent to (A.20) is given by (37), having used (A.13) and (A.16). The policy that implements a first-best MPE of Definition 4 is obtained by imposing $\mu_x^{\mathrm{SP}} = 0$ for all $x \in X$ in (32)–(34) and (37), which gives (36) and (38).

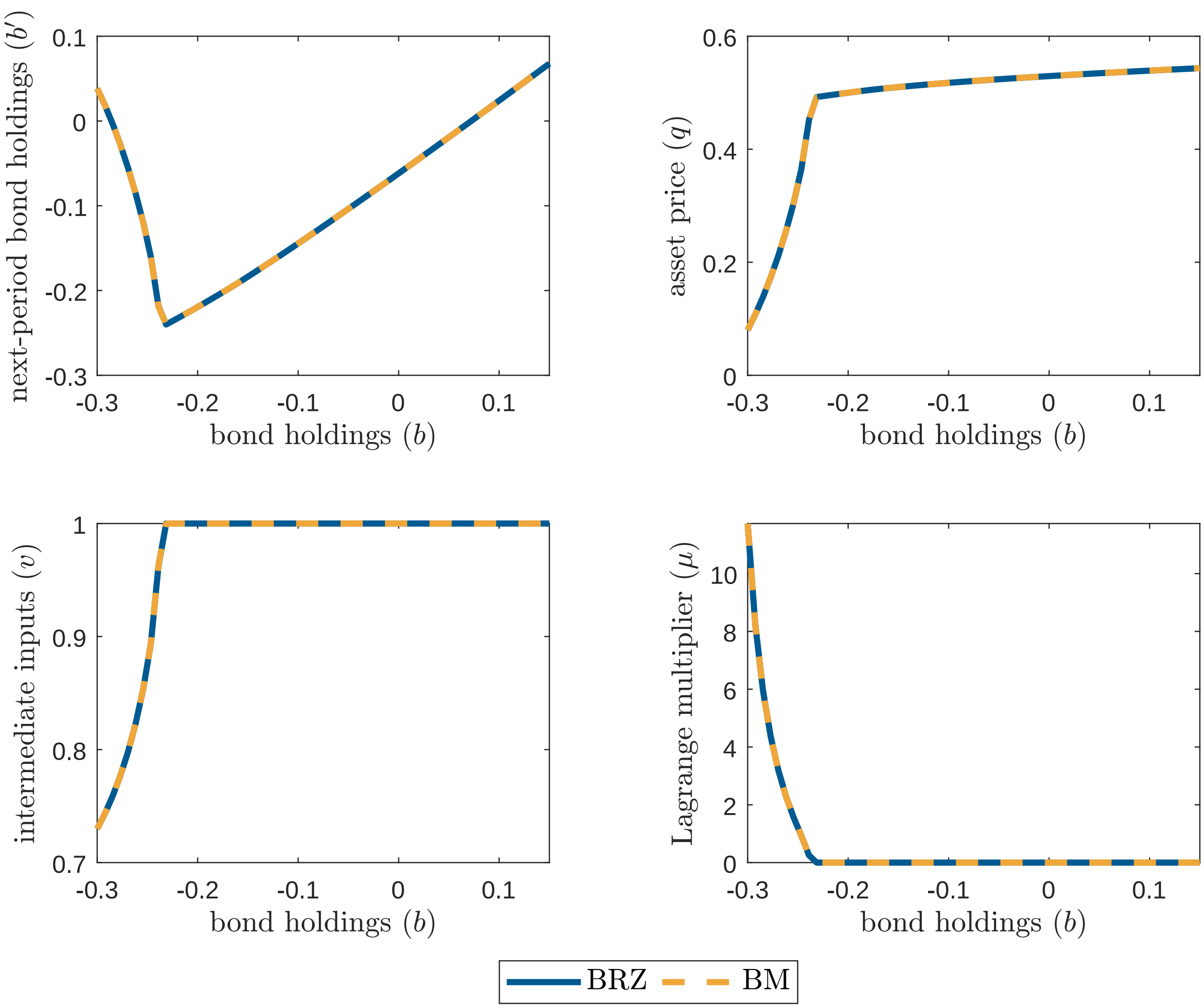
# B Computation

## B.1 Numerical differences in replicating BM

The DE allocation we obtain is identical to that in the BM replication package (Figures B.1 and B.2). However, there are significant differences between the SP allocations. Specifically, we find mostly underborrowing rather than overborrowing when the collateral constraint is binding (Figure B.3), and overborrowing is quantitatively much smaller when the collateral constraint is slack (Figure B.4), consistent with the fact that the ex-post component of the optimal policy is critical in this economy. As a result, there are twice as large welfare gains from the SP allocation in our computation compared to the BM replication package (0.6% versus 0.3%), as shown in Table 1.

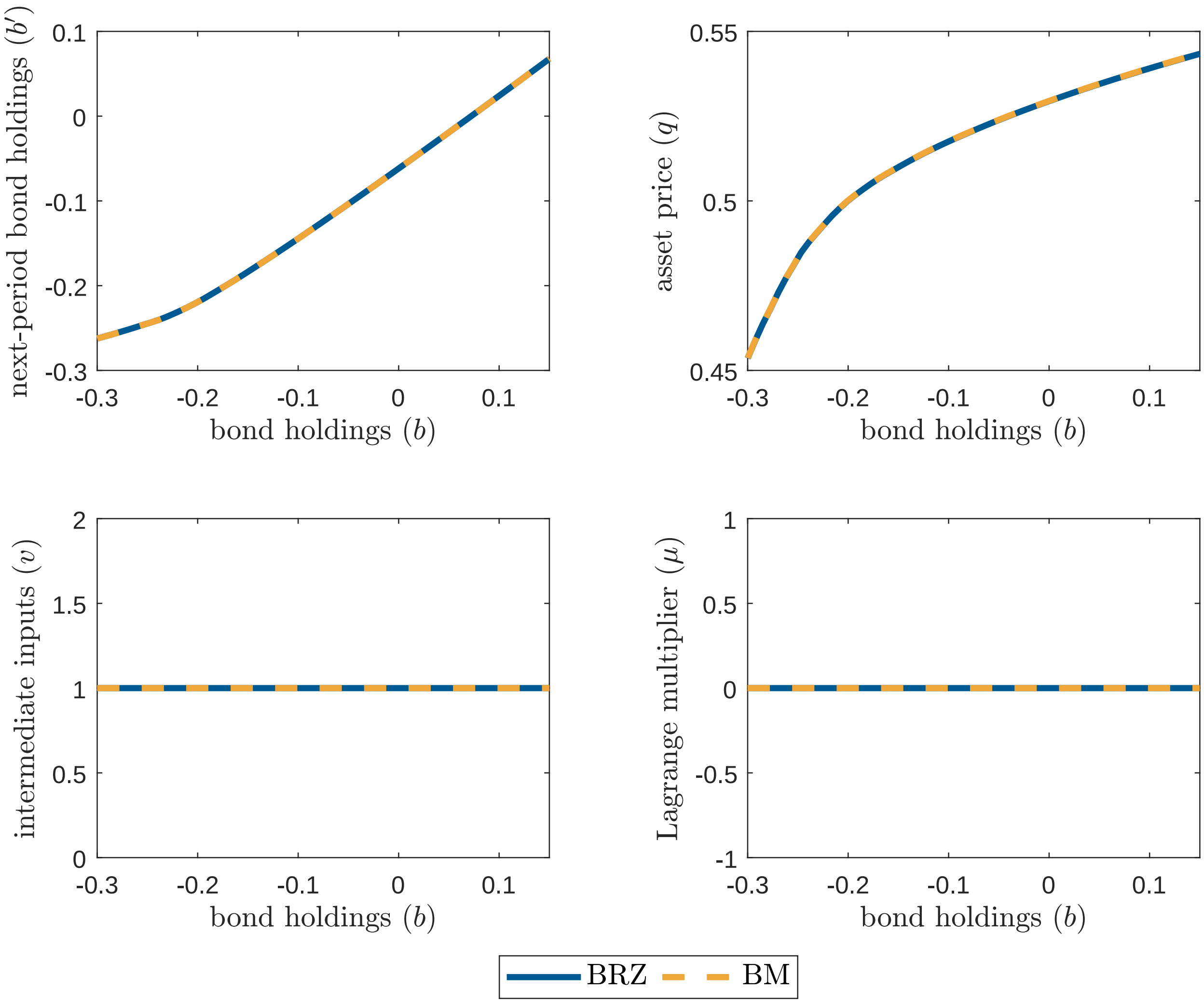
These differences can arise because the BM Fortran code does not fully account for the nonconvexity of the planner's feasible set when the collateral constraint is binding. First, as illustrated in Figure B.5, the planner's best response function may have multiple local maxima, and the Fortran routine `mnbrak` may fail to bracket the global optimum. Second, as illustrated in Figure B.6, for a given level of next-period bond holdings, there may be multiple intermediate input/asset price pairs that satisfy the binding collateral constraint and the asset pricing equation, and the Fortran routine `zbrac` may fail to bracket the welfare-maximizing root.

Figure B.1: DE policy functions in the tight credit regime



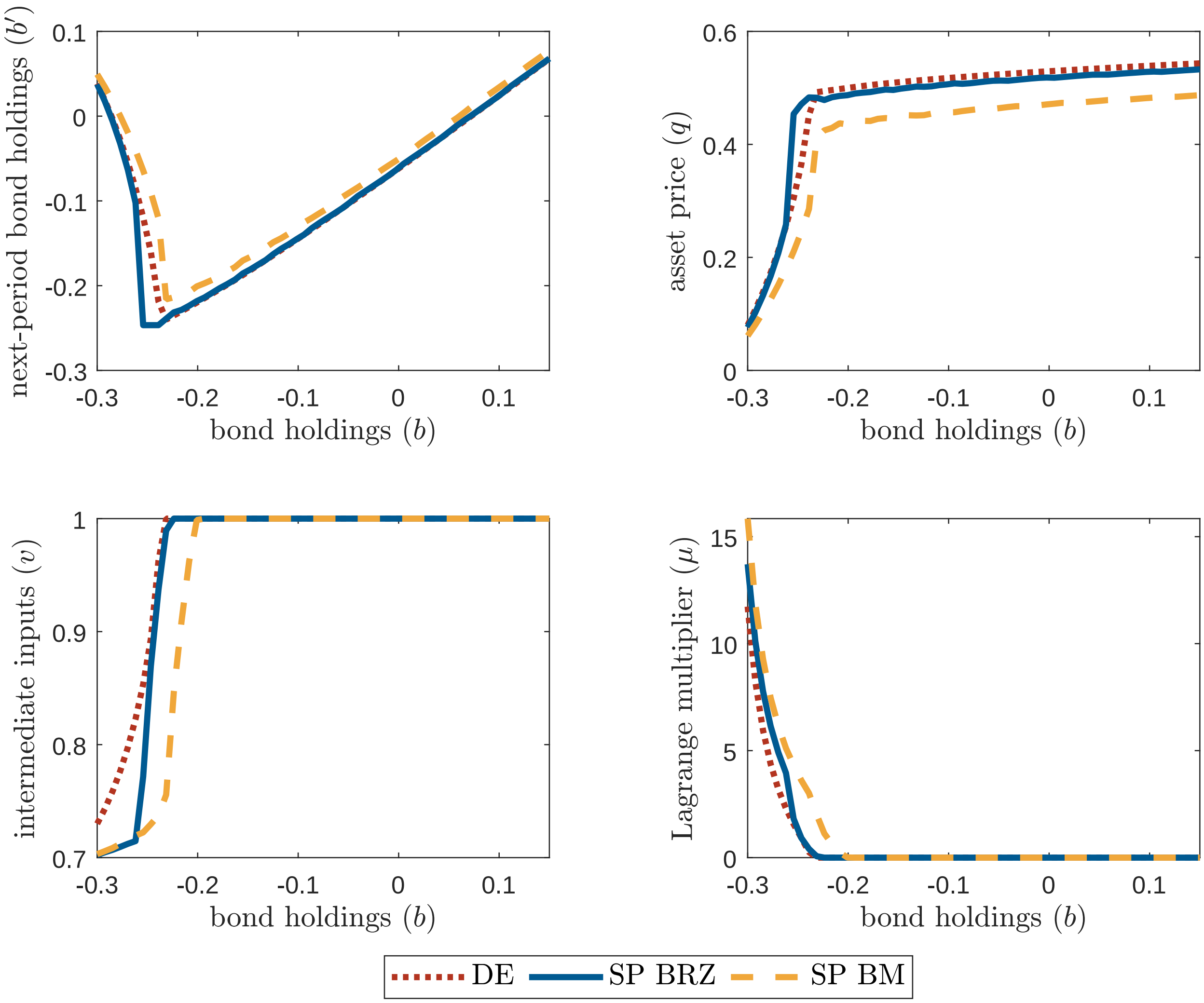

Notes: The policy functions are conditional on average $z$, high $R$, and tight credit regime ($\kappa = \kappa^L$), as in Fig. 2 and Fig. 3 in BM.

Figure B.2: DE policy functions in the normal credit regime



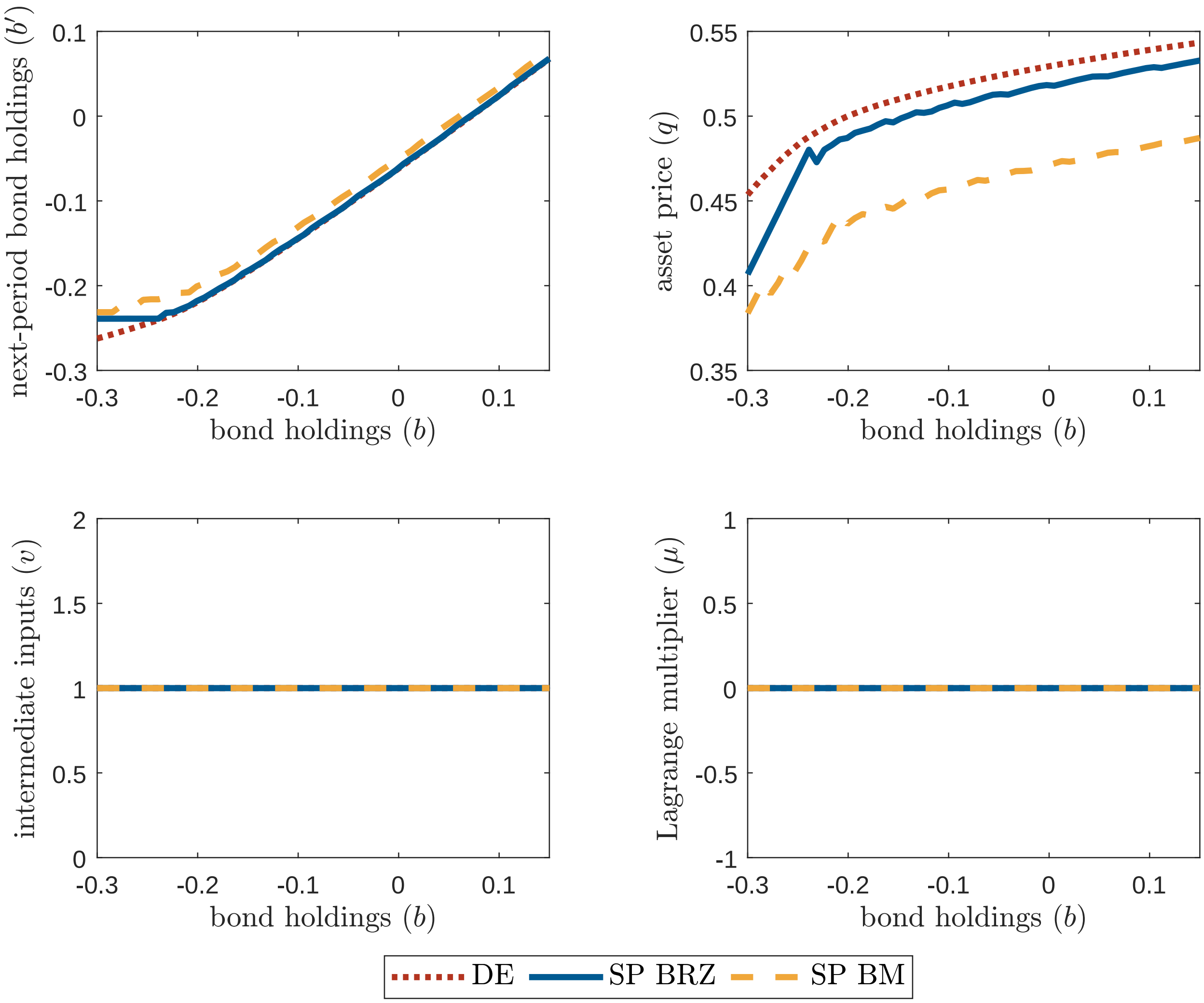

Notes: The policy functions are conditional on average $z$, high $R$, and normal credit regime ($\kappa = \kappa^H$), consistent with Fig. 5A in BM.

Figure B.3: DE and SP policy functions in the tight credit regime



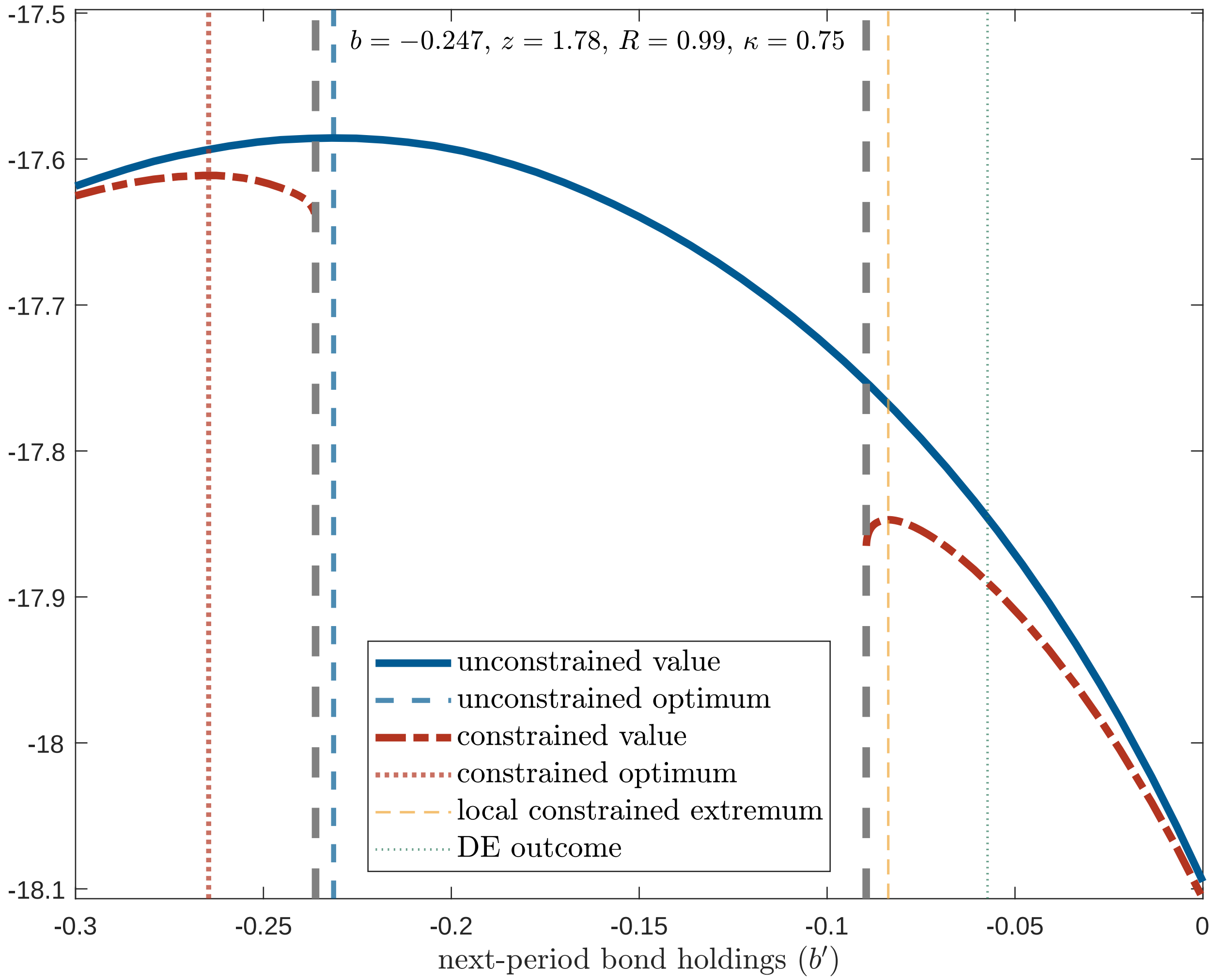

Notes: The policy functions are conditional on average $z$, high $R$, and tight credit regime ($\kappa = \kappa^L$), as in Fig. 2 and Fig. 3 in BM.

Figure B.4: DE and SP policy functions in the normal credit regime



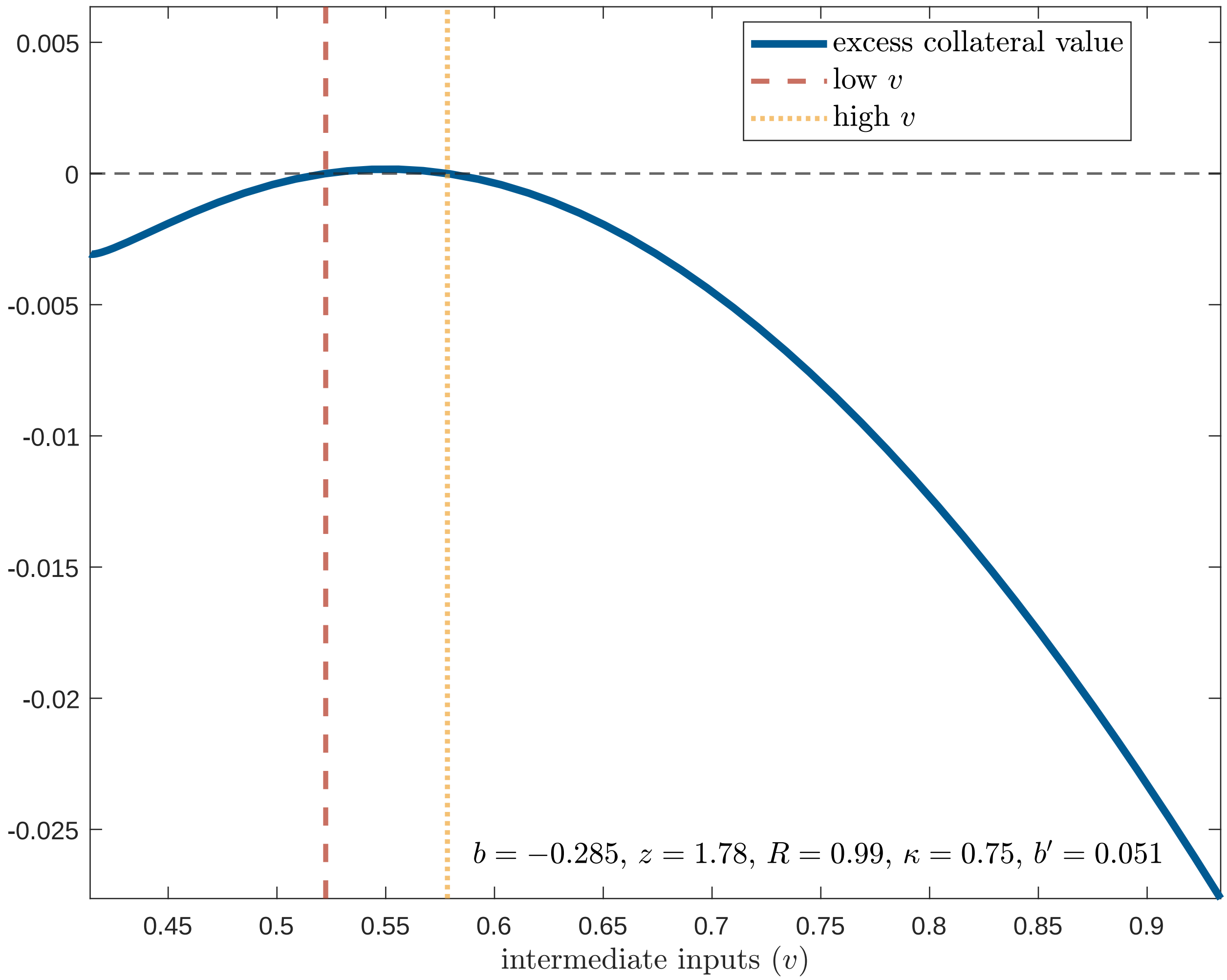

Notes: The policy functions are conditional on average $z$, high $R$, and normal credit regime ($\kappa = \kappa^H$), consistent with Fig. 5A in BM.

Figure B.5: SP best response to DE with multiple constrained local extrema




Notes: The planner's values are the maximized expressions in the Bellman equation represented as functions of next-period bond holdings $b'$. These values are conditional on the state $b = -0.247$, $z = 1.78$, $R = 0.99$, $\kappa = 0.75$, and next-period policy functions being the DE policy functions. The unconstrained value ignores the collateral constraint, and the unconstrained optimum violates the collateral constraint, so it is an infeasible choice for the planner. The constrained value is conditional on the binding collateral constraint. The constrained feasible set is nonconvex (disconnected): conditional on $b'$ being in the region between the two dashed gray vertical lines, there does not exist intermediate inputs ($v$) and asset price ($q$) that satisfy the binding collateral constraint and the asset Euler equation. If one were to use the bracketing routine `mnbrak`, starting from the neighborhood of the DE outcome, as in the BM code, the routine would bracket the suboptimal local constrained extremum.

Figure B.6: Multiple constrained roots




Notes: The excess collateral value $\kappa q + \frac{b'}{R} - \theta p_v v$ is conditional on the state $b = -0.285$, $z = 1.78$, $R = 0.99$, $\kappa = 0.75$, next-period bond holdings $b' = 0.051$, and next-period policy functions being the DE policy functions. The current asset price $q$ is given by the asset Euler equation. If one were to use the bracketing routine `zbrac`, starting from the neighborhood of the unconstrained level of intermediate inputs, as in the BM code, the routine would fail to bracket the roots. The largest root is welfare-maximizing.