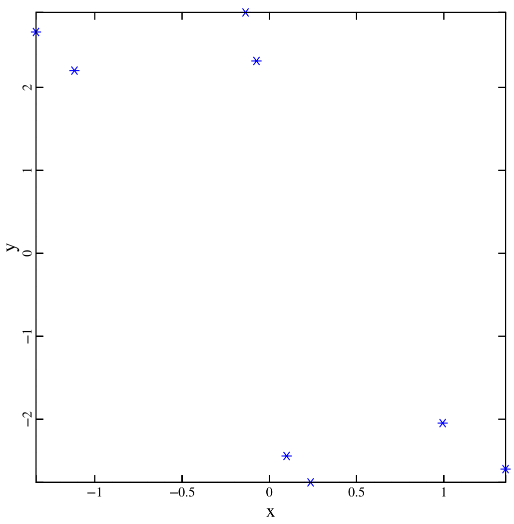
Problem 3-7: Shape identification

This problem deals with pattern recognition, a central problem in image processing (e.g. in aerial photography, self-driving cars, and recognition of pictures of cats). We will deal with a very simplistic problem.

This problem practises the solution of a linear least squares problem based on the normal-equation method, see [Lecture → Section 3.2].

A routine within the program of a self driving-car is tasked with the job of identifying road signs. The task is the following: given a collection of points $\mathbf{p}_i \in \mathbb{R}^2, i = 1, \dots, n$, we have to decide whether those point represent a stop sign, or a "priority road sign". In this exercise we consider $n = 8$.



A set of possible input points $\mathbf{p}^i$: corners detected in the photo.

Which traffic sign had been in the focus of the camera?

Fig. 11

Fig. 12

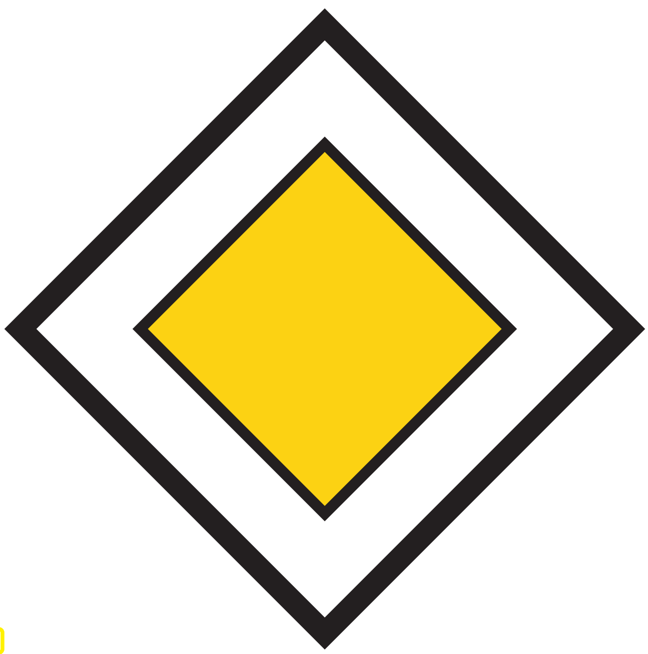


Fig. 13

The shape of the sign can be represented by a 2×8 matrix with the 8 coordinates in $\mathbb{R}^2$ defining the shape of the sign. We assume that the stop sign resp. the priority road sign are defined by the "model" points (known a priori) $\mathbf{x}^i_{stop} \in \mathbb{R}^2$ resp. $\mathbf{x}^i_{priority} \in \mathbb{R}^2$ for $i = 1, \dots, 8$.

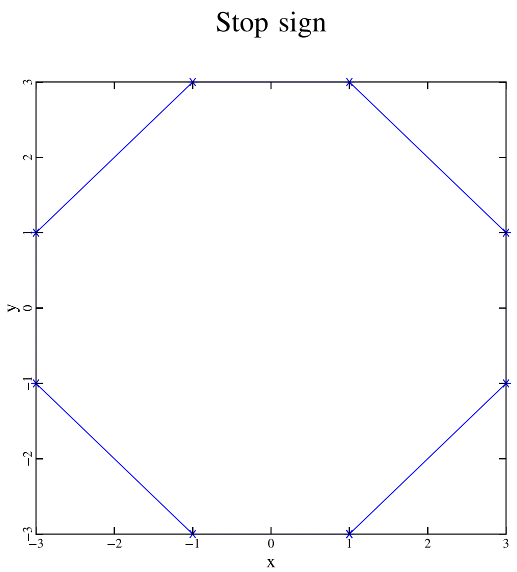


Fig. 14

The 8-points $\mathbf{x}^i_{stop}$ defining the model of a stop sign.

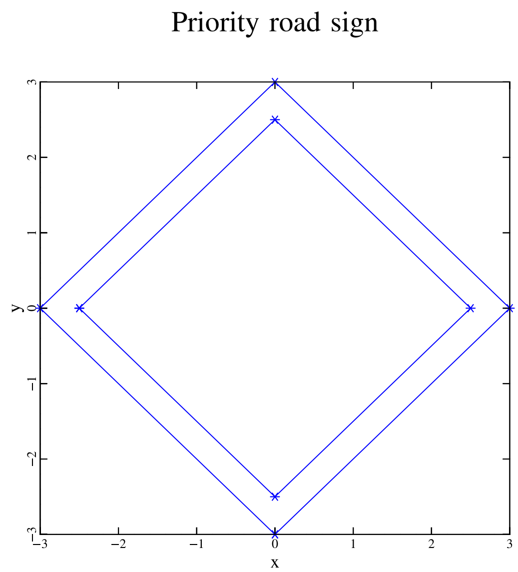


Fig. 15

The 8-points $\mathbf{x}^i_{priority}$ defining the model of a priority road sign.

However, in a real case scenario, one can imagine that the photographed shape is not exactly congruent to the one specified by the "model" points. Instead, one can assume that the points on the photo ($\mathbf{p}^i \in \mathbb{R}^2$) are the result of a *linear transformation* of the original points $\mathbf{x}^i \in \mathbb{R}^2$ for $i = 1, \dots, 8$; i.e. we can assume that there exists a matrix $\mathbf{A} \in \mathbb{R}^{2,2}$, s.t.

$$\mathbf{p}^i = \mathbf{A}\mathbf{x}^i, i = 1, \dots, n \,. \tag{3.7.1}$$

We do not know whether $\mathbf{x}^i = \mathbf{x}^i_{stop}$ or $\mathbf{x}^i = \mathbf{x}^i_{priority}$.

Moreover, we have some error in the data, i.e. our points do not represent exactly one of the linearly transformed shapes (it is plausible to imagine that there is some measurement error, and that the photographed shape is not exactly the same as our "model" shape), i.e. (3.7.1) is satisfied only "approximately".

With this problem, we will try to use the least square method to find the matrix $\mathbf{A}$ and to *classify* our points, i.e. to decide whether they represent a stop or a priority road sign.

(3-7.a) ⚃ (20 min.) For the moment, assume we know the points $\mathbf{x}^i$ (i.e. we know the shape of our sign). Explicitly write (3.7.1) as an overdetermined linear system:

$$\mathbf{B}\mathbf{v} = \mathbf{w} \,,$$

whose least-squares solution $\mathbf{v}^*$ will allow to determine the "best" linear transformation $\mathbf{A}$ (in the least-squares sense). How is the vector of unknowns related to $\mathbf{A}$?

SOLUTION for (3-7.a) → `3-7-1-0:ShapeIdent1.pdf` ▲

(3-7.b) ⚂ (10 min.) When does the matrix $\mathbf{B}$ have full rank? Give a geometric interpretation of this condition.

SOLUTION for (3-7.b) → `3-7-2-0:si2s.pdf` ▲

(3-7.c) ⚁ (15 min.) [depends on Sub-problem (3-7.a)]

In the file `shape_ident.hpp` implement a C++ function

```
Eigen::MatrixXd shape_ident_matrix(const Eigen::MatrixXd & X);
```

that returns the matrix $\mathbf{B}$. Pass the vectors $\mathbf{x}^i$ in the columns of a $2 \times n$ EIGEN matrix X.

SOLUTION for (3-7.c) → `3-7-3-0:si3s.pdf` ▲

(3-7.d) (30 min.) [depends on Sub-problem (3-7.c)]

In the file `shape_ident.hpp` implement a C++ function

```
double solve_lsq(const Eigen::MatrixXd & X, const Eigen::MatrixXd
  & P,
                 Eigen::MatrixXd & A);
```

that computes the matrix $\mathbf{A}$ by solving the least squares problem based on the normal equations, see [Lecture → Section 3.2]. It should return the Euclidean norm of the residual of the least square solution. Pass the vectors $\mathbf{x}^i$ and the vectors $\mathbf{p}^i$ as a $2 \times n$ EIGEN matrix `X` resp. `P`.

SOLUTION for (3-7.d) → `3-7-4-0:si4s.pdf` ▲

(3-7.e) (30 min.) [depends on Sub-problem (3-7.d)]

Explain how the norm of the residual of the least square solution can be used to identify the shape defined by the points $\mathbf{p}^i$. Implement a function

```
enum Shape { Stop, Priority };
Shape identify(const Eigen::MatrixXd Xstop,
               const Eigen::MatrixXd Xpriority,
               const Eigen::MatrixXd & P,
               Eigen::MatrixXd & A);
```

that identifies the shape (either Stop or Priority sign) of the input points $\mathbf{p}^i$. The function returns an `enum` that classifies the shape of points specified by `P`. Return the linear transformation in the matrix `A`. The "model points" $\mathbf{x}^i_{stop}$ resp. $\mathbf{x}^i_{priority}$ are passed through `Xstop` resp. `Xpriority`.

SOLUTION for (3-7.e) → `3-7-5-0:si5s.pdf` ▲

End Problem 3-7 , 105 min.